On primes of the form $\lfloor \alpha p + \beta \rfloor$ in Piatetski-Shapiro sequence

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Abstract

In this paper, we investigate whether there are infinitely many primes of the form $\lfloor \alpha p + \beta \rfloor$ in Piatetski-Shapiro sequence for almost all irrationals $\alpha > 0$ (in the sense of Lebesgue measure). Our result provides an affirmative answer to this question by utilizing elementary and analytic methods, especially the 'Partition of Unity' and the estimations of exponential sums.

Keywords: Goldbach conjecture, Twin prime conjecture, Selberg's sieve method, Piatetski-Shapiro primes, Partition of Unity

2020 MSC: 11N05, 11N35, 11L07

1. Introduction

In 2009, motivated by the Goldbach conjecture and the twin prime conjecture, H. Li and H. Pan [9] proposed the following conjecture:

Conjecture 1. Let $\alpha > 0$ be an irrational number, and β be a real number. Then there exist infinitely many primes p such that $\lfloor \alpha p + \beta \rfloor$ is also a prime.

Concerned with the above conjecture, H. Li and H. Pan [9] proved a weaker theorem (Theorem 1 in [9]) as follows:

Theorem 1. *Let β be a real number. Then for almost all irrational $\alpha > 0$ (in the sense of Lebesgue measure),*

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$$\limsup_{x\to\infty} \frac{\pi_{\alpha,\beta}(x)(\log^2 x)}{x} \geq 1,$$

where

$$\pi_{\alpha,\beta}(x) = \{p \leq x : \textit{both } p \textit{ and } \lfloor \alpha p + \beta \rfloor \textit{ are primes}\}.$$

On the other hand, The Piatetski-Shapiro sequence is a fundamental object in analytic number theory, named after the Soviet mathematician Israel Moiseevich Piatetski-Shapiro. It refers to sequences of the form $\{\lfloor n^c \rfloor\}_{n=1}^{\infty}$, where $c > 1$ is a real number. The theory of Piatetski-Shapiro prime play an important role in analytic number theory. In 1953 Piatetski-Shapiro [13] give an asymptotic formula for the counting function of Piatetski-Shapiro prime for $c \in (1, 12/11)$. It is worth emphasizing that the range $c \in (1, 12/11)$ was improved by Kolesnik [7], Heath-Brown [6], Kolesnik [8] and Liu, Rivat [11]. The best known result in this direction is due to Rivat and Sargos [14], which give an asymptotic formula for the counting function of Piatetski-Shapiro prime for $c \in (1, 2817/2426)$. We remark that Rivat and Wu [15] give the lower bound for the counting function for $c \in (1, 243/205)$. This result is the best regarding the infinitude of the Piatetski-Shapiro primes.

It is worth noting that some authors generalize the Piatetski-Shapiro primes, interested readers can see [1, 2, 5, 10] for examples.

Considering the above discussions of the two lines of research, we proposed the following conjecture:

Conjecture 2. Let $\alpha > 0$ be an irrational number, and β be a real number. Then there exist infinitely many primes p such that $\lfloor \alpha p + \beta \rfloor$ are primes in Piatetski-Shapiro sequence.

We remark that Guo [4] was the first to consider Piatetski-Shapiro Primes in Beatty sequence, the Conjecture 2 can be seen as the generalization of Guo's work [4]. Concerned with the above conjecture, we proved a weaker theorem as follows:

Theorem 2. *Let α, β and $\frac{12}{13} < \gamma < 1$ be real numbers, and $P = \{\lfloor n^{\frac{1}{\gamma}} \rfloor : n \in$ $\mathbf{N}\}$. Then for almost all irrational $\alpha > 0$ (in the sense of Lebesgue measure),*

$$\limsup_{x\to\infty} \frac{\pi^{\gamma*}_{\alpha,\beta}(x)(\log^2 x)}{x^\gamma} \geq 1,$$

where

$$\pi^{\gamma*}_{\alpha,\beta}(x) = \{p \leq x :\ p \textit{ is a prime},\ [\alpha p + \beta] \in P \textit{ is a prime}\}.$$

Remark. In this remark, we summarize the proof strategy of Theorem 2 as follows. Following Li-Pan [9], the approach is to reduce the problem to a sieve upper bound $\pi^{\gamma*}_{\alpha,\beta}(x) \ll x^\gamma/\alpha\log^2 x$ (Lemma 7). In our case, the first additional complexity arises from the formula for the characteristic function of Piatetski-Shapiro primes (as presented in Lemma 2), while the second lies in estimating the exponential sum involving floor functions. To address these challenges, we employ the 'Partition of Unity' technique introduced in Lemma 4.

Some Notations. Let p always denote a prime. Let $mes(A)$ denote the Lebesgue measure of A. We shall frequently use ε with a slight abuse of notation to mean a small positive number, possibly a different one each time. Given a real number t, we write $e(t) = e^{2\pi i t}$, $\{\alpha\}$ for the fractional part of α, $\lfloor\alpha\rfloor$ for the largest integer not exceeding α. We recall that for functions F and real nonnegative G the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leq \alpha G$ holds for some constant $\alpha > 0$. We also write $F \sim G$ to indicate that $F \ll G$ and $G \ll F$.

2. Auxiliary results

For readers' convenience, we list some auxiliary results before proving Theorem 2. Firstly, we need the following approximation of Vaaler [17].

Lemma 1. *Let $\psi(x) = x - \lfloor x\rfloor - 1/2$, then for any $H \geq 1$ there are numbers a_h, b_h such that*

$$\left|\psi(t) - \sum_{0<|h|\leq H} a_h e(th)\right| \leq \sum_{|h|\leq H} b_h e(th),$$

where

$$a_h \ll \frac{1}{|h|}, b_h \ll \frac{1}{H}.$$

We need the following lemma, which is Lemma 4 in [16].

Lemma 2. *A natural number m has the form $\lfloor n^c \rfloor$ if and only if $\mathcal{X}(m) = 1$, where*

$$\mathcal{X}(m) = \gamma m^{\gamma-1} + \psi(-m^\gamma) - \psi(-(m+1)^\gamma) + O(m^{\gamma-2}),$$

and $\gamma = 1/c$.

Lemma 3. *Let f be three times continuously differentiable on a subinterval I of $(N, 2N]$. Suppose that for some $\lambda > 0$, the inequalities*

$$\lambda \ll |f^{'''}(x)| \ll \lambda.$$

hold, where the implied constants are independent of f and λ. Then

$$\sum_{n \in I} e(f(n)) \ll N\lambda^{1/6} + N^{3/4} + N^{1/4}\lambda^{-1/4}.$$

Proof. This is Theorem 2.6 in [3]. □

The following lemma is the so-called 'Partition of Unity'.

Lemma 4. *For $Z \in \mathbb{N}$, there exists periodic functions $g_z(t)$ $(0 \le z \le Z)$, such that*

$$\begin{aligned}
&0 < g_z(t) \le 1, \textit{if } \left|t - \frac{z}{2Z}\right| < \frac{1}{2Z},\\
&g_z(t) = 0, \textit{if } \frac{1}{2Z} \le \left|t - \frac{z}{2Z}\right| < \frac{1}{2},\\
&g_z(t) = \sum_{|n| \le Z(\log x)^4} \beta_n^{(z)} e(nt) + O(x^{-\log\log x}),
\end{aligned}$$

where

$$|\beta_n^{(z)}| \le \frac{1}{2Z}.$$

We also have

$$\sum_{z=0}^{2Z-1} g_z(t) = 1.$$

Proof. This is Lemma 2 in [16]. $\square$

Lemma 5. *We have*

$$\sum_{a=0}^{d-1} e\left(\frac{ma}{d}\right) = \begin{cases} d, & \text{if } d \mid m; \\ 0, & \text{otherwise}, \end{cases}$$

Proof. The proof of this lemma is easy, so we left it to readers. $\square$

3. Some lemmas

Lemma 6. *Suppose that $b_2 > b_1 > 0$, $12/13 < \gamma < 1$, α, β and are arbitrary real numbers. For each $0 < \varepsilon < 1/16800$ and all sufficiently large real number x, there exists an exceptional set $J_E \subseteq (b_1, b_2)$ with $mes(J_E) = O(x^{-\varepsilon})$ such that for any square-free $d \le x^{\frac{13\gamma}{12}-1}$ and irrational $\alpha \in (b_1, b_2) \setminus J_E$,*

$$|\{1 \le n \le x : d|n\lfloor \alpha n+\beta\rfloor,\ \ \lfloor \alpha n+\beta\rfloor = \lfloor m^{\frac{1}{\gamma}}\rfloor\}| = \frac{\alpha^{\gamma-1}x^{\gamma}}{d}\prod_{p|d}\left(2-\frac{1}{p}\right)+O\left(\frac{x^{\gamma-\varepsilon}}{d}\right).$$

Proof. Let

$$S := \sum_{\substack{\lfloor \alpha n+\beta\rfloor = \lfloor m^{1/\gamma}\rfloor \\ d|n\lfloor \alpha n+\beta\rfloor}} 1, \tag{1}$$

and

$$E_{\alpha,\beta,d}(n) = \begin{cases} 1, & \text{if } d \mid \lfloor \alpha n+\beta\rfloor; \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 2, we have

$$\mathcal{X}_\gamma(m) = \gamma m^{\gamma-1} + \psi(-m^\gamma) - \psi(-(m+1)^\gamma) + O(m^{\gamma-2}). \tag{2}$$

Let

$$P = \{1 \le n \le x : d|n\lfloor \alpha n+\beta\rfloor,\ \ \lfloor \alpha n+\beta\rfloor = \lfloor m^{\frac{1}{\gamma}}\rfloor\}. \tag{3}$$

So by (1) and (3), we have

$$
\begin{aligned}
S =& |P| \\
=& \sum_{s|d} |\{1 \le n \le x/s : \lfloor \alpha sn + \beta \rfloor \equiv 0(\bmod d/s), (n, d/s) = 1, ns \in P\}| \\
=& \sum_{s|d} \sum_{t|(d/s)} \mu(t) |\{1 \le n \le x/s : \lfloor \alpha sn + \beta \rfloor \equiv 0(\bmod d/s), t|n, ns \in P\}| \\
=& \sum_{s|d, t|s} \mu(t) |\{1 \le n \le x/s : \lfloor \alpha sn + \beta \rfloor \equiv 0(\bmod td/s), ns \in P\}| \\
=& \sum_{s|d, t|s} \mu(t) \sum_{\substack{n \le x/s \\ td/s | \lfloor \alpha ns + \beta \rfloor}} (\gamma \lfloor \alpha ns + \beta \rfloor^{\gamma - 1} + \psi(-\lfloor \alpha ns + \beta \rfloor^{\gamma}) - \psi(-(\lfloor \alpha ns + \beta \rfloor + 1)^{\gamma}) \\
& + O(\lfloor \alpha ns + \beta \rfloor^{\gamma - 2})) \\
=& S_1 + S_2 + O(x^{\gamma - 1 + \epsilon}),
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
S_1 =& \sum_{s|d, t|s} \mu(t) \sum_{\substack{n \le x/s \\ td/s | \lfloor \alpha ns + \beta \rfloor}} \gamma \lfloor \alpha sn + \beta \rfloor^{\gamma - 1}, \\
S_2 =& \sum_{s|d, t|s} \mu(t) \sum_{\substack{n \le x/s \\ td/s | \lfloor \alpha ns + \beta \rfloor}} (\psi(-\lfloor \alpha sn + \beta \rfloor^{\gamma}) - \psi(-(\lfloor \alpha sn + \beta \rfloor + 1)^{\gamma})).
\end{aligned}
$$

By Lemma 7 in [9], we have for almost all $\alpha > 0$

$$
\sum_{n \le x/s} E_{\alpha, \beta, td/s}(ns) = \frac{x}{td}(1 + O(x^{-2\epsilon})). \tag{5}
$$

For S_1, by summation by parts we have

$$
S_1 = \frac{\alpha^{\gamma - 1} x^{\gamma}}{d} \prod_{p|d} \left(2 - \frac{1}{p}\right) + O\left(\frac{x^{\gamma - \varepsilon}}{d}\right). \tag{6}
$$

For S_2, it's enough to prove

$$
S_2^{'} := \sum_{\substack{n \le x/s \\ td/s | \lfloor \alpha ns + \beta \rfloor}} (\psi(-\lfloor \alpha sn + \beta \rfloor^{\gamma}) - \psi(-(\lfloor \alpha sn + \beta \rfloor + 1)^{\gamma})) \ll \frac{x^{\gamma - \epsilon}}{d}. \tag{7}
$$

For $S_2^{'}$, by Lemma 5, we have

$$\begin{aligned}S_2^{'} &= \sum_{n\le x/s} E_{\alpha,\beta,td/s}(ns)(\psi(-\lfloor\alpha sn+\beta\rfloor^\gamma)-\psi(-(\lfloor\alpha sn+\beta\rfloor+1)^\gamma))\\ &= \sum_{n\le x/s}(\psi(-\lfloor\alpha sn+\beta\rfloor^\gamma)-\psi(-(\lfloor\alpha sn+\beta\rfloor+1)^\gamma))\frac{1}{td/s}\sum_{a=1}^{dt/s} e\left(\frac{as\lfloor\alpha sn+\beta\rfloor}{td}\right).\end{aligned}\tag{8}$$

By Lemma 1 with $H=dx^{1-\gamma+\epsilon}$, we have

$$\begin{aligned}&\psi(-\lfloor\alpha sn+\beta\rfloor^\gamma)-\psi(-(\lfloor\alpha sn+\beta\rfloor+1)^\gamma)\\ =&\sum_{0<|h|\le H} a_h(e(h\lfloor\alpha sn+\beta\rfloor^\gamma)-e(h(\lfloor\alpha sn+\beta\rfloor+1)^\gamma))\\ &+O\left(\sum_{|h|\le H} b_h(e(h\lfloor\alpha sn+\beta\rfloor^\gamma)+e(h(\lfloor\alpha sn+\beta\rfloor+1)^\gamma))\right).\end{aligned}$$

So we have

$$\begin{aligned}S_2^{'}\ll&\frac{s}{td}\sum_{a=1}^{dt/s}\sum_{0<|h|\le H}\frac{1}{H}\left|\sum_{n\le x/s}(e(h\lfloor\alpha sn+\beta\rfloor^\gamma)\right|\\ &+\frac{s}{td}\sum_{a=1}^{dt/s}\sum_{0<|h|\le H}\frac{1}{h}\left|\sum_{n\le x/s} e\left(\frac{as\lfloor\alpha sn+\beta\rfloor}{td}+h\lfloor\alpha sn+\beta\rfloor^\gamma\right)\right|\\ :=&T_1+T_2+O\left(\frac{x^{\gamma-\varepsilon}}{d}\right).\end{aligned}\tag{9}$$

We only treat T_2, it is enough to prove

$$\sum_{n\le x/s} e\left(\frac{as\lfloor\alpha sn+\beta\rfloor}{td}+h\lfloor\alpha sn+\beta\rfloor^\gamma\right)\ll\frac{x^{\gamma-\epsilon}}{d}.$$

By Lemma 4 with $Z = d^2 x^{2-2\gamma+\epsilon}$, we have

$$
\begin{aligned}
&\sum_{n\le x/s} e\left(\frac{as\lfloor \alpha sn+\beta\rfloor}{td} + h\lfloor \alpha sn+\beta\rfloor^{\gamma}\right)\\
=&\sum_{n\le x/s} e\left(\frac{as\lfloor \alpha sn+\beta\rfloor}{td} + h\lfloor \alpha sn+\beta\rfloor^{\gamma}\right)\sum_{z=0}^{2Z-1} g_z(\alpha sn+\beta)\\
=&\sum_{z=0}^{2Z-1}\sum_{n\le x/s}\left(e\left(\frac{as^2\alpha n+as\beta}{td} - \frac{asz}{2tdZ} + h\left(\alpha sn+\beta-\frac{z}{2Z}\right)^{\gamma}\right) + O\left(\frac{x^{\gamma-1}h}{Z}\right)\right)\\
&\times g_z(\alpha sn+\beta)\\
=&\sum_{z=0}^{2Z-1}\sum_{n\le x/s}\left(e\left(\frac{\alpha as^2 n+as\beta}{td} + h\left(\alpha sn+\beta-\frac{z}{2Z}\right)^{\gamma}\right)\right) g_z(\alpha sn+\beta)\\
&+O\left(\frac{x^{\gamma-\varepsilon}}{d}\right)\\
\ll&\sum_{z=0}^{2Z-1}\sum_{|r|\le Z(\log x)^4}|\beta_r^{(z)}|\left|\sum_{n\le x/s}\left(e\left(\frac{\alpha as^2 n+as\beta}{td} + h\left(\alpha sn+\beta-\frac{z}{2Z}\right)^{\gamma} + r(\alpha sn+\beta)\right)\right)\right|\\
&+O\left(\frac{x^{\gamma-\varepsilon}}{d}\right).
\end{aligned}
$$

For the innermost sum, in the interval $[1, \frac{x^{\gamma-\epsilon}}{d}]$ we estimate the innermost sum trivially, and next we divide the interval $[\frac{x^{\gamma-\epsilon}}{d}, \frac{x}{s}]$ of the innermost sum into intervals of the form $[\frac{x/s}{2^{j+1}}, \frac{x/s}{2^j}]$ and apply Lemma 4 to the innermost sum, we have

$$
\begin{aligned}
&\sum_{n\le x/s} e\left(\frac{as\lfloor \alpha sn+\beta\rfloor}{td} + h\lfloor \alpha sn+\beta\rfloor^{\gamma}\right)\\
\ll& Z(x^{1/2+\gamma/6}h^{1/6}d^{1/2} + x^{3/4} + x^{1-\gamma/4}h^{-1/4})\log x + O\left(\frac{x^{\gamma-\varepsilon}}{d}\right)\\
\ll&\frac{x^{\gamma-\varepsilon}}{d}.
\end{aligned}
\tag{10}
$$

By (7), (9) and (10), we have

$$S_2 \ll \frac{x^{\gamma-\varepsilon}}{d},$$

this together with (6) completes the proof of this lemma. □

Lemma 7. *Suppose that $b_2 > b_1 > 0$, $12/13 < \frac{1}{c} = \gamma < 1$, and α, β are arbitrary real numbers. And we denote P the set $\{p : \lfloor n^{\frac{1}{\gamma}}\rfloor = \lfloor \alpha p +$*

$\beta\rfloor, \quad p$ is a prime$\}$*, For all sufficiently large real number x, there exists an exceptional set $J_E \subseteq (b_1, b_2)$ with $mes(J_E) = O(x^{-\varepsilon})$ such that for irrational $\alpha \in (b_1, b_2) \diagdown J_E$, we have*

$$|\{1 \le p \le x, p \in P : p \text{ is a prime}\}| \ll \frac{x^{\gamma}}{\log^2 x}.$$

Proof. Let d be a squarefree number such that $d \le x^{\frac{13\gamma}{12}-1}$. Let $z = d^{1/2}$,

$$P(z) = \prod_{p<z, p \text{ prime}} p,$$

and

$$S(A, z) = \{a \in A : (a, P(z)) = 1\}.$$

Let $A(\alpha) = \{n\lfloor \alpha n + \beta \rfloor : 1 \le n \le x\}$. We find

$$\{p\lfloor \alpha p + \beta\rfloor : \alpha^{-1}(z - \beta + 1) < p < x, p \in P, p \textit{ and } \lfloor \alpha p + \beta \rfloor \text{ are primes}\}$$

is a subset of $S(A(\alpha), z)$. Then by Lemma 7, we know that there exists an exceptional set $J_E \subseteq (b_1, b_2)$ with $mes(J_E) = O(x^{-\varepsilon})$ such that for any squarefree $d \le x^{\frac{13\gamma}{4}-3}$ and any irrational $\alpha \in (b_1, b_2) \diagdown J_E$,

$$|A_d(\alpha)| = \frac{\alpha^{\gamma-1}x^{\gamma}}{d} \prod_{p|d} \left(2 - \frac{1}{p}\right) + O\left(\frac{x^{\gamma-\varepsilon}}{d}\right),$$

where $A_d(\alpha) = \{x \in A(\alpha) : d|x\}$. We define a completely multiplicative function such that $g(p) = \frac{2}{p} - \frac{1}{p^2}$ for each prime p. Define $G(z) = \sum_{m<z, m|P(z)} g(m)$. By Selberg's sieve method, we have

$$|S(A(\alpha), z)| \le \frac{|A(\alpha)|}{G(z)} + O\left(\sum_{d<z^2} 3^{\omega(d)} x^{\gamma-\varepsilon}/d\right),$$

where $\omega(d)$ denotes the number of distinct prime divisors of d. So we have $3^{\omega(d)} \ll d^{\varepsilon}$. So the error term is negligible. By the prime number theorem and Theorem 7.14 of [12], we have

$$G(z) \gg \prod_{p<z} (1 - g(p))^{-1} \gg \log^2 z.$$

This completes the proof of this lemma. $\square$

4. Proof of the Theorem 2

The proof of this theorem is similar to the proof of the Theorem 1 in [9]. The only difference is the formula (4) in [9]. For completeness, we give the whole proof of this theorem.

Suppose that $b_2 > b_1 > 0$. Let

$$\mathcal{F} = \{\alpha \in (b_1, b_2) : \limsup_{x\to\infty} \pi^{\gamma *}_{\alpha,\beta}(x) \log^2 x / x^\gamma < 1\},$$

and

$$\mathcal{F}_n = \{\alpha \in (b_1, b_2) : \limsup_{x\to\infty} \pi^{\gamma *}_{\alpha,\beta}(x) \log^2 x / x^\gamma < 1 - 1/n\}.$$

Obviously $\mathcal{F} = \bigcup_{n>1} \mathcal{F}_n$. So it suffices to show $mes(\mathcal{F}_n) = 0$ for every $n > 1$.

Assume on the contrary that there exists $n > 1$ such that $mes(\mathcal{F}_n) > 0$. Let $I := (c_1, c_2)$ be an arbitrary sub-interval of (b_1, b_2), and let P denotes the set of Piatetski-Shapiro primes, we have

$$\begin{aligned}
\int_{c_1}^{c_2} \pi^{\gamma *}_{\alpha,\beta}(x) d\alpha &= \int_{c_1}^{c_2} \left(\sum_{\substack{p \le x \\ p \in P}} \sum_{\substack{\alpha p + \beta - 1 < q \le \alpha p + \beta \\ q \ is \ a \ prime}} 1 \right) d\alpha \\
\geq & \sum_{\substack{p \le x \\ p \in P}} \sum_{\substack{c_1 p + \beta - 1 < q \le c_2 p + \beta \\ q \ is \ a \ prime}} mes([(q-\beta)/p, (q+1-\beta)/p) \cap [c_1, c_2]) \\
\geq & \sum_{\substack{p \le x \\ p \in P}} \sum_{\substack{c_1 p + \beta - 1 < q \le c_2 p + \beta \\ q \ is \ a \ prime}} \frac{1}{p} \\
= & \ (c_1 - c_2) \sum_{\substack{p \le x \\ p \in P}} \frac{1}{\log p} \left(1 + O\left(\frac{1}{\log(c_1 p)} \right) \right) \\
\geq & \ (c_1 - c_2) \frac{x^\gamma}{(\log x)^2} \left(1 + O\left(\frac{1}{\log x} \right) \right),
\end{aligned} \tag{11}$$

Suppose that $C > 1$ is the implied constant in Lemma 8. Let $\mathcal{L}_I = \mathcal{F}_n \cap I$ and

$$\mathcal{L}_{I,\delta}(x) = \{\alpha \in I : \pi^{\gamma *}_{\alpha,\beta}(x) \log^2 x / x^\gamma < 1 - \delta\}.$$

For any two primes p, q,

$$J_{p.q} := \{\alpha \in I : \lfloor \alpha p + \beta \rfloor = q\}$$

is an interval or an empty set. Let

$$\mathcal{H}_{I,\delta}(x) = \left(\bigcup_{\substack{k>(1-\delta)x^\gamma/(\log x)^2 \\ p_1,\cdots,p_k \le x \ are\ distinct\ \ and\ p_i \in P \\ q_1,\cdots,q_k\ are\ primes}} \bigcap_{j=1}^{k} J_{p_j,q_j} \right).$$

So we have

$$\mathcal{L}_{I,\delta}(x) = I \setminus \mathcal{H}_{I,\delta}(x)$$

is measurable.

By Lemma 8,

$$\int_{c_1}^{c_2} \pi_{\alpha,\beta}^{\gamma*}(x) d\alpha \le mes(\mathcal{L}_{I,\delta}(x)) \frac{(1-\delta)x^\gamma}{(\log x)^2} + (c_2 - c_1 - mes(\mathcal{L}_{I,\delta}(x))) \frac{Cx^\gamma}{(\log x)^2}.$$

Combining the above inequality with (12), we have

$$mes(\mathcal{L}_{I,\delta}(x)) \le \frac{C-1}{C-1+\delta} mes(I). \tag{12}$$

We claim that

$$\mathcal{L}_I = \bigcap_{m>n} \bigcup_{\substack{y \ge 1 \\ y \in \mathbf{Z}}} \bigcap_{\substack{x \ge y \\ x \in \mathbf{Z}}} \mathcal{L}_{I,1/n-1/m}(x). \tag{13}$$

In fact, for any $m > n$, if

$$\limsup_{x\to\infty} \pi_{\alpha,\beta}^{\gamma*}(x) \log^2 x / x^\gamma < 1 - 1/n + 1/m,$$

then there exists y_0 such that for any $x \ge y_0$

$$\pi_{\alpha,\beta}^{\gamma*}(x) \le (1 - 1/n + 1/m) x^\gamma / (\log^2 x).$$

On the other hand, if $\alpha \in \cup_y \cap_{x \ge y} \mathcal{L}_{I,1/n-1/m}(x)$, clearly we have

$$\limsup_{x\to\infty} \pi_{\alpha,\beta}^{\gamma*}(x) \log^2 x / x^\gamma < 1 - 1/n + 1/m.$$

By (13) and (14), we get

$$mes(\mathcal{L}_I) \le \limsup_{x\to\infty} mes(\mathcal{L}_{1,1/3n}(x)) \le \frac{C-1}{C-1+1/3n} mes(I).$$

Since $mes(\mathcal{F}_n) > 0$, there exist open intervals I_1, I_2, $\cdots \subseteq (b_1, b_2)$ such that

$$\mathcal{F}_n \subseteq \bigcup_{k=1}^{\infty} I_k$$

and

$$\sum_{k=1}^{\infty} mes(I_k) \leq \frac{C-1+1/4n}{C-1} mes(\mathcal{F}_n).$$

But by (13),

$$mes(\mathcal{F}_n) \leq \sum_{k=1}^{\infty} mes(\mathcal{L}_{I_k}) \leq \frac{C-1}{C-1+1/3n} \sum_{k=1}^{\infty} mes(I_k) \leq \frac{C-1+1/4n}{C-1+1/3n} mes(\mathcal{F}_n).$$

This leads to a contradiction.

5. Acknowledgments

We thank the anonymous referee for his or her very useful comments on this paper.

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