

DECISION PROBLEMS FOR QUASIVARIETIES

M. V. SCHWIDEFSKY

ABSTRACT. We prove that, under certain circumstances, a quasivariety contains continuum many Q -universal subquasivarieties for which the finite membership problem and the quasi-equational theory are undecidable.

UDC 512.58

1. INTRODUCTION

When certain sufficient conditions are satisfied by a quasivariety \mathbf{K} , \mathbf{K} turns out to have a highly complicated inner structure. First conditions of this type were offered by M. E. Adams and W. Dziobiak [1] and similar conditions were found by V. A. Gorbunov [2]. Later, stronger conditions than those in [1] were found in [3]; under these conditions, many complexity results for quasivarieties were established in [3]–[6] as well as in other papers by the same authors. In particular, it was established in [4] that some algorithmic problems are undecidable in a quasivariety \mathbf{K} once \mathbf{K} satisfies the conditions from [3].

In [7], more general conditions than those in [3] were introduced, see Definition 2.3. These conditions still imply the conditions of M. E. Adams and W. Dziobiak [1]. Moreover, under these conditions, the results of [3, 5, 6] still hold, as demonstrated in [7]–[9].

Here, we generalize some results from [4] and show that Theorem 6.5 from [4] has a much wider application area than just Corollary 7.8 in [4]. In particular, Theorem 4.1, presents new examples of quasivarieties where certain algorithmic problems are undecidable. For the notation and notions not defined here, we refer to V. A. Gorbunov [2] as well as to [1, 7].

2. PRELIMINARIES

In this section, we present known results on quasivarieties containing countable subclasses possessing particular properties.

2.1. Adams–Dziobiak classes. The following definition is due to M. E. Adams and W. Dziobiak [1]. In the present form, it appears in [10].

Definition 2.1. Let a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{fin}(\omega)\}$ of structures of finite similarity type σ possess the following properties.

- (P₀) For each $X \in \mathcal{P}_{fin}(\omega)$, the structure \mathcal{A}_X is l -projective in $\mathbf{Q}(\mathbf{A})$ and the trivial congruence $\Delta_{\mathcal{A}_X}$ is a dually compact element in $\text{Con}_{\mathbf{Q}(\mathbf{A})} \mathcal{A}_X$.
- (P₁) \mathcal{A}_\emptyset is a trivial structure.

Date: September 5, 2025.

2000 Mathematics Subject Classification. 08C15, 03C60, 03D35, 08A30.

Key words and phrases. Decision problem; Finite membership problem; Quasi-equational theory; Quasivariety.

The research was carried out under the support of the Russian Science Foundation, project number 24-21-00075.

- (P₂) If $X = Y \cup Z$ in $\mathcal{P}_{fin}(\omega)$ then $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y, \mathcal{A}_Z)$.
- (P₃) If $\emptyset \neq X \in \mathcal{P}_{fin}(\omega)$ and $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y)$ then $X = Y$.
- (P₄) If $\mathcal{A}_X \leq \mathcal{B}_0 \times \mathcal{B}_1$ for some $\mathcal{B}_0, \mathcal{B}_1 \in \mathbf{Q}(\mathbf{A})$ then there are $Y_0, Y_1 \in \mathcal{P}_{fin}(\omega)$ such that $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0), \mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_1)$, and $X = Y_0 \cup Y_1$.

In this case \mathbf{A} is called an *AD-class*. If each structure from \mathbf{A} is finite then \mathbf{A} is called a *finite AD-class*.

For statement (i) in the following theorem, we refer to M. E. Adams and W. Dziobiak [1, Theorem 3.3]. For statement (ii) — to Corollary 3.5 and Theorem 4.4 in [10].

Theorem 2.2. *Let a quasivariety \mathbf{K} contain an AD-class. Then*

- (i) \mathbf{K} is *Q-universal*;
- (ii) \mathbf{K} contains continuum many classes $\mathbf{K}' \subseteq \mathbf{K}$ such that the set of [isomorphism types] of finite sublattices of the \mathbf{K}' -quasivariety lattice $\text{Lq}(\mathbf{K}')$ is not computably enumerable.

2.2. B*-classes.

Definition 2.3. [7] Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite similarity type σ and let $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ be a variety. A class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ is a *B*-class with respect to \mathbf{M} and \mathbf{V}* , if \mathbf{A} satisfies the following conditions.

- (B₀) For each nonempty $F \in \mathcal{P}_{fin}(\omega)$, the structure \mathcal{A}_F is finitely presented in \mathbf{M} ; \mathcal{A}_\emptyset is a trivial structure.
- (B₁) If $F = G \cup H$ in $\mathcal{P}_{fin}(\omega)$ then $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathcal{A}_H)$.
- (B₂^{*}) For each $F, G \in \mathcal{P}_{fin}(\omega)$, if $F \neq \emptyset$ and $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathbf{V})$ then $F = G$.
- (B₃^{*}) For every $F \in \mathcal{P}_{fin}(\omega)$ and every $i < \omega$, if $f \in \text{Hom}(\mathcal{A}_F, \mathcal{A}_{\{i\}})$ then either $f(\mathcal{A}_F) \in \mathbf{V}$ or $i \in F$.
- (B₄^{*}) $\mathbf{H}(\mathbf{A}) \cap (\mathbf{M} \setminus \mathbf{V}) \subseteq \mathbf{A}$.
- (B^{*}) For all $F, G \in \mathcal{P}_{fin}(\omega)$ such that $\emptyset \neq G \subseteq F$, for all $\mathcal{B} \in \mathbf{V}$, and for all homomorphisms $f \in \text{Hom}(\mathcal{A}_F, \mathcal{B})$ and $g \in \text{Hom}(\mathcal{A}_G, \mathcal{B})$, there is a homomorphism $h \in \text{Hom}(\mathcal{A}_G, \mathcal{B})$ such that $f = hg$.

If \mathcal{A}_F is a finite structure for every nonempty set $F \in \mathcal{P}_{fin}(\omega)$ then \mathbf{A} a *finite B*-class with respect to \mathbf{M} and \mathbf{V}* .

Remark 2.4. The B-classes with respect to a quasivariety $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ introduced in [3] are nothing else than the B*-classes with respect to \mathbf{M} and the trivial variety \mathbf{T} .

Moreover, each finite B*-class with respect to a variety and a quasivariety is a finite AD-class according to [7, Proposition 3.3].

2.3. C-classes. The following two definitions appear in [4].

Definition 2.5. [4, Definition 5.1] Let σ be a finite similarity type, let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety, and let $\mathbf{C} = \{\mathcal{C}_n \mid n < \omega\} \subseteq \mathbf{M}$ be a class of nontrivial structures with the following properties.

- (C₁) For $m, n < \omega$, the structure \mathcal{C}_m embeds into the structure \mathcal{C}_n if and only if $m = n$.
- (C₂) For each $n < \omega$, the structure \mathcal{C}_n is \mathbf{M} -subdirectly irreducible and finitely presented in \mathbf{M} .

Then \mathbf{C} is called a *C-class with respect to \mathbf{M}* . If the structure \mathcal{C}_n is finite for all $n < \omega$ then \mathbf{C} is a *finite C-class with respect to \mathbf{M}* .

We note that the definition of *computable sequence*, a notion used below, can be found in the second paragraph of Section 6 in [4].

Definition 2.6. [4, Definition 6.1] Let \mathbf{M} be a quasivariety and let \mathbf{C} be a C-class with respect to \mathbf{M} . Then \mathbf{C} is an *effective C-class with respect to \mathbf{M}* if there is a computable sequence $\Delta = \{(X_n, \Delta_n, A_n) \mid n < \omega\}$, where for all $n < \omega$, $X_n = \{x_0, \dots, x_{l_n}\}$, $\mathcal{C}_n \cong \mathcal{F}_{\mathbf{M}}(X_n, \Delta_n)$, and $A_n(x_0, \dots, x_{l_n})$ is a monolith-generating atomic formula for \mathcal{C}_n in \mathbf{M} .

The following statement is a slight modification of [4, Theorem 6.5] and follows from its proof.

Theorem 2.7. *Let σ be a finite similarity type, let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety, and let $\mathbf{C} = \{\mathcal{C}_n \mid n < \omega\} \subseteq \mathbf{M}$ be a finite C-class with respect to \mathbf{M} . The following statements hold.*

- (i) *There are continuum many quasivarieties $\mathbf{K} \subseteq \mathbf{M}$ which have an independent quasi-equational basis relative to \mathbf{M} .*
- (ii) *If \mathbf{C} is a finite effective C-class with respect to \mathbf{M} then there are continuum many quasivarieties $\mathbf{K} \subseteq \mathbf{M}$ such that the finite membership problem is undecidable for \mathbf{K} , the quasi-equational theory of \mathbf{K} is undecidable, and \mathbf{K} has an independent quasi-equational basis relative to \mathbf{M} .*

3. MAIN RESULTS

The following statement generalizes Lemma 5.2 in [4].

Proposition 3.1. *Let σ be a finite type and let $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{K}(\sigma)$ be a finite AD-class possessing the following property*

- (P₅) *If $\alpha: \mathcal{A}_{\{n\}} \rightarrow \prod_{i \in X} \mathcal{A}_{\{i\}}$ is an embedding for some $n \in \omega$ and some finite nonempty set $X \subseteq \omega$ then $n \in X$ and $\pi_n \alpha$ is an embedding.*

Then $\mathbf{C} = \{\mathcal{A}_{\{n\}} \mid n < \omega\}$ is a finite C-class with respect to $\mathbf{Q}(\mathbf{A})$.

Proof. It follows from conditions (P₁) and (P₃) in Definition 2.1 that \mathbf{C} consists of finite (and thus finitely presented in $\mathbf{Q}(\mathbf{A})$) nontrivial structures. Condition (C₁) in Definition 2.5 follows from condition (P₃) in Definition 2.1.

In order to prove (C₂), we fix $n < \omega$ and suppose that $\mathcal{A}_{\{n\}} = \mathcal{C}_n \leq_s \prod_{i \in I} \mathcal{B}_i$ is a subdirect decomposition of \mathcal{C}_n such that $\mathcal{B}_i \in \mathbf{Q}(\mathbf{A})$ for all $i \in I$. Since \mathcal{C}_n is a finite structure of finite type, $\mathcal{C}_n \leq_s \prod_{i \in F} \mathcal{B}_i$ for some finite nonempty set $F \subseteq I$. This means that, for each $i \in F$, there is an onto homomorphism $\alpha_i: \mathcal{C}_n \rightarrow \mathcal{B}_i$ such that $\Delta_{\mathcal{C}_n} = \bigcap_{i \in F} \ker \alpha_i$. Without loss of generality, we may assume that F is a minimal set with this property. Then we also have that, for all $i \in F$, \mathcal{B}_i is a finite nontrivial structure of finite type. Since $\mathcal{B}_i \in \mathbf{Q}(\mathbf{A})$, we conclude by condition (P₂) of Definition 2.1 that $\mathcal{B}_i \in \mathbf{Q}(\mathbf{C})$ for all $i \in F$. Hence, there are finite nonempty sets $Z_i \subseteq \omega$, $i \in F$, such that $\mathcal{B}_i \leq \prod_{t \in Z_i} \mathcal{C}_t$. We may again assume that the set Z_i is minimal with respect to this property. Therefore, for each $i \in F$ and each $t \in Z_i$, there is an onto homomorphism $\beta_{it}: \mathcal{B}_i \rightarrow \mathcal{D}_t \leq \mathcal{C}_t$ such that

$$\Delta_{\mathcal{C}_n} = \bigcap \{\ker \beta_{it} \alpha_i \mid i \in F, t \in Z_i\}.$$

In particular, the map $\beta_{it} \alpha_i: \mathcal{C}_n \rightarrow \mathcal{C}_t$ is a homomorphism for all $i \in F$ and all $t \in Z_i$. Applying (P₅), we obtain that $n \in Z_i$ for some $i \in F$ and that $\beta_{in} \alpha_i: \mathcal{C}_n \rightarrow \mathcal{C}_n$ is an embedding whence $\alpha_i: \mathcal{C}_n \rightarrow \mathcal{B}_i$ is an isomorphism. This proves the subdirect irreducibility of \mathcal{C}_n in $\mathbf{Q}(\mathbf{C})$. \square

Lemma 3.2. *If a quasivariety $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ contains a finite B^* -class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\}$ with respect to \mathbf{M} and a variety $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ then \mathbf{A} is a finite AD-class satisfying the condition (P_5) . In particular, $\mathbf{C} = \{\mathcal{A}_{\{n\}} \mid n < \omega\}$ is a finite C-class with respect to \mathbf{M} .*

Proof. It follows from [7, Proposition 3.3] that \mathbf{A} is a finite AD-class with respect to \mathbf{M} . For all $i < \omega$, we put $\mathcal{C}_i = \mathcal{A}_{\{i\}}$ for the sake of brevity. Furthermore, let $n < \omega$ and a finite nonempty set $F \subseteq \omega$ be such that $\alpha: \mathcal{C}_n \rightarrow \prod_{i \in F} \mathcal{C}_i$ is an embedding. Since $\mathcal{C}_n \notin \mathbf{V}$ by [7, Lemma 1.3(i)], we conclude that there is $i \in F$ such that $\pi_i \alpha(\mathcal{C}_n) \notin \mathbf{V}$. By condition (P_5) , this means that $\pi_i \alpha(\mathcal{C}_n) = \pi_i \alpha(\mathcal{A}_{\{n\}}) \cong \mathcal{A}_G$ for some finite nonempty set $G \subseteq \omega$. Then the inclusion $G \subseteq \{n\}$ follows by [7, Lemma 1.3(iii)]. Hence, $G = \{n\}$. Since \mathcal{C}_n is a finite structure, we conclude that $\pi_i \alpha$ is an isomorphism and condition (P_5) follows.

The last statement follows from Proposition 3.1. \square

4. APPLICATIONS

We present in this section some applications of Theorem 2.7 and Proposition 3.1.

Theorem 4.1. *Let \mathbf{K} be one of the following classes:*

- (a) *the quasivariety of pseudocomplemented semilattices generated by the 3-element chain;*
- (b) *the locally variety \mathbf{A}_i , where $i < \omega$, of $(0, 1)$ -lattices constructed in [11];*
- (c) *the variety of idempotent semigroups.*

Then there is a quasivariety $\mathbf{M} \subseteq \mathbf{K}$ which contains continuum many Q -universal subquasivarieties $\mathbf{M}' \subseteq \mathbf{M}$ such that the finite membership problem is undecidable for \mathbf{M}' , the quasi-equational theory of \mathbf{M}' is undecidable, and \mathbf{M}' has an independent quasi-equational basis relative to \mathbf{M} .

Proof. For each of the classes listed in (a)–(c), an effective procedure was given by M. E. Adams *et al.* in the papers [12], [11], [13] respectively to construct a finite AD-class \mathbf{A} . That this AD-class \mathbf{A} possesses the property (P_5) follows from [12, Proposition 4.3], [11, Lemma 3.6], [13, Lemma 4.5] respectively. By Proposition 3.1, the quasivariety $\mathbf{Q}(\mathbf{A})$ contains an effective C-class with respect to $\mathbf{Q}(\mathbf{A})$. The desired statement follows from Theorem 2.2 and Theorem 2.7. \square

Remark 4.2. In view of Proposition 3.1 and Lemma 3.2, the results of [4, Corollary 7.8] and from [7] follow from Theorem 2.7 in a uniform way.

Other examples where Proposition 3.1 and Lemma 3.2 apply to obtain the conclusion of Theorem 4.1 can be found in M. E. Adams, W. Dziobiak [14, 15] as well as in V. Koubek, J. Sichler [16]–[19].

REFERENCES

- [1] M. E. Adams, W. Dziobiak, *Q-universal quasivarieties of algebras*, Proc. Amer. Math. Soc. **120**, no. 4 (1994), 1053–1059.
- [2] V. A. Gorbunov, *Algebraic Theory of Quasivarieties*, Plenum, New York, 1998.
- [3] A. V. Kravchenko, A. M. Nurakunov, M. V. Schwidefsky, *Structure of quasivariety lattices. I. Independent axiomatizability*, Algebra and Logic **57**, no. 6 (2018), 445–462.
- [4] A. V. Kravchenko, A. M. Nurakunov, M. V. Schwidefsky, *Structure of quasivariety lattices. II. Undecidable problems*, Algebra and Logic **58**, no. 2 (2019), 123–136.
- [5] A. V. Kravchenko, A. M. Nurakunov, M. V. Schwidefsky, *Structure of quasivariety lattices. III. Finitely partitionable bases*, Algebra and Logic **59**, no. 3 (2020), 222–229.
- [6] A. V. Kravchenko, A. M. Nurakunov, M. V. Schwidefsky, *On the structure of quasivariety lattices. IV. Nonstandard quasivarieties*, Siberian Math. J. **62**, no. 5 (2021), 850–858.

- [7] M. V. Schwidefsky, *Existence of independent quasi-equational bases*, Algebra and Logic **58**, no. 6 (2020), 514–537.
- [8] M. V. Schwidefsky, *The complexity of quasivariety lattices. II*, Siberian Electronic Mathematical Reports **20**, no. 1 (2023), 501–513.
- [9] M. V. Schwidefsky, *On the existence of independent quasi-equational bases. II*, Algebra and Logic **62**, no. 6 (2023), 516–531.
- [10] M. V. Schwidefsky, *On the complexity of quasivariety lattices*, Algebra and Logic **54**, no. 3 (2015), 245–257.
- [11] M. E. Adams, W. Dziobiak, *Q-universal varieties of bounded lattices*, Algebra Universalis **48** (2002), 333–356.
- [12] M. E. Adams, W. Dziobiak, M. Gould, J. Schmid, *Quasivarieties of pseudocomplemented semilattices*, Fund. Math. **146** (1995), 295–312.
- [13] M. E. Adams, W. Dziobiak, *Quasivarieties of idempotent semigroups*, Internat. J. Algebra Comput. **13**, no. 6 (2003), 733–752.
- [14] M. E. Adams, W. Dziobiak, *Lattices of quasivarieties of 3-element algebras*, J. Algebra **166**, no. 1 (1994), 181–210.
- [15] M. E. Adams, W. Dziobiak, *Joins of minimal quasivarieties*, Studia Logica **54** (1995), 371–389.
- [16] V. Koubek, J. Sichler, *Universality of small lattice varieties*, Proc. Amer. Math. Soc. **91**, no. 1 (1984), 19–24.
- [17] V. Koubek, J. Sichler, *On almost universal varieties of modular lattices*, Algebra Universalis **45**, no. 2–3 (2001), 191–210.
- [18] V. Koubek, J. Sichler, *Almost ff-universal and Q-universal varieties of modular 0-lattices*, College Math. J. **101**, no. 2 (2004), 161–182.
- [19] V. Koubek, J. Sichler, *Finitely generated almost universal varieties of 0-lattices*, Comment. Math. Univ. Carolin. **46**, no. 2 (2005), 304–325.

(M. V. Schwidefsky) NOVOSIBIRSK STATE UNIVERSITY, PIROGOVA STR. 1, 630090 NOVOSIBIRSK, RUSSIA

(M. V. Schwidefsky) SOBOLEV INSTITUTE OF MATHEMATICS SB RAS, KOPTYUGA AVE. 4, 630090 NOVOSIBIRSK, RUSSIA

E-mail address: m.schwidefsky@g.nsu.ru