ON THE GENUS OF THE NON-NILPOTENT GRAPHS OF NON-WEAKLY NILPOTENT GROUPS

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Abstract. In this paper, we focus on a topological aspect, namely, the genus of the non-nilpotent graphs associated with non-weakly nilpotent groups. We determine the genus of the non-nilpotent graphs of some classes of finite non-nilpotent groups. We also classify all non-weakly nilpotent groups whose non-nilpotent graphs are planar, toroidal, double-toroidal, triple-toroidal, quadruple-toroidal or pentuple-toroidal.

1. Introduction

Let G be a group and $nil(G) = \{x \in G \mid < x, y > \text{ is nilpotent for all } y \in G\}$. It is still unknown whether the subset $nil(G)$ of G is a subgroup of G or not. In the case when G is finite, $nil(G)$ equals the hypercenter $Z^*(G)$ of G (see [8, Lemma 3.1]).

Associate with a group G, a (simple) graph $\mathfrak{R}_G$ as follows: the vertex set $V(\mathfrak{R}_G)$ is $G \setminus nil(G)$ and two distinct vertices x and y are adjacent if and only if $< x, y >$ is not nilpotent. We call $\mathfrak{R}_G$, the non-nilpotent graph of G. The non-nilpotent graph can be seen as a generalization of the non-commuting graph, considered in [1] and [7]. Recently, in [3] and [14], the group and graph properties of the non-nilpotent graphs of groups were examined. For a finite non-nilpotent group G, in [3], Abdollahi and Zarrin, show that $\mathfrak{R}_G$ is planar if and only if $G \cong S_3$ and in [14], Nongsiang and Saikia show that $\mathfrak{R}_G$ is not toroidal.

In this paper, we examine a topological feature, specifically the genus of non-nilpotent graphs associated with non-weakly nilpotent groups. In Section 2, we obtained some properties of the non-nilpotent graphs. We show that, if G is a non-weakly nilpotent group (a group G is said to be weakly nilpotent if every two generated subgroup of G is nilpotent) and if $\gamma(\mathfrak{R}_G) = n$, where n is a non-negative integer, then G

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is finite. We also show that, for any non-negative integer n, there can be only finitely many non-weakly nilpotent groups whose non-nilpotent graphs are of genus n.

In Section 4, we determine the genus of the non-nilpotent graphs of non-nilpotent groups of some well-known classes of finite groups and in Section 5, we classify all non-weakly nilpotent groups whose non-nilpotent graphs are planar, toroidal, double-toroidal, triple-toroidal, quadruple-toroidal or pentuple-toroidal.

2. Some prerequisites

In this section, we review specific group-theoretic and graph-theoretic terminology ([6], [15] and [16]) along with some well-established results that will be referenced in the coming sections.

The *nilpotent graph* of a group G ([8]), denoted by $\Gamma_{nil}(G)$, is a simple undirected graph whose vertex set is $G \setminus nil(G)$ and any two distinct vertices x and y are adjacent if and only if $< x, y >$ is nilpotent. Let $Sol(G) = \{x \in G : < x, y > \text{ is solvable for all } y \in G\}$. The *non-solvable graph* of G ([5]), denoted by $\mathcal{S}_G$, is a simple undirected graph whose vertex set is $G \setminus Sol(G)$ and any two distinct vertices x and y are adjacent if and only if $< x, y >$ is not solvable. The *non-commuting graph* of G ([1]), denoted by Γ_G, is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and any two distinct vertices x and y are adjacent if and only if x and y do not commute.

A graph is said to be *complete* if there exist an edge between every pair of distinct vertices in Γ. We denote the complete graph with n vertices by K_n. The *complete bipartite graph* and the *complete tripartite graph* are the one whose vertex set can be partitioned into two disjoint parts and three disjoint parts, respectively, and two vertices are adjacent if and only if they lie in different parts. Similarly, a *complete multipartite graph* or a *complete $k-$partite graph* is a simple graph whose vertices can be partitioned into k sets so that two distinct vertices u and v are adjacent if and only if u and v belong to different sets of the partition. We write $K_{n_1,\dots,n_k}$ for the complete $k-$partite graph with partite sets of sizes $n_1, \dots, n_k$. If, $n_1 = \cdots = n_k = m$, then we get the regular complete $k-$partite graph $K_{m,m,\dots,m}$ and it is denoted by $K_{k(m)}$. If $n_1 = \cdots = n_{l_1} = m_1, \dots, n_{l+1} = \cdots = n_{l+l_r} = m_r, (l = l_1 + l_2 + \cdots + l_{r-1})$, then we denote the complete $k-$partite graph $K_{n_1,\dots,n_k}$ by $K_{l_1(m_1),\dots,l_r(m_r)}$. Given a graph Γ, let U be a non-empty subset of $V(\Gamma)$. Then the *induced subgraph* of Γ on U is defined to be the graph $\Gamma[U]$ in which the vertex set is U and the edge set consists precisely of those edges in Γ whose endpoints lie in U. The graph obtained by taking the union of graphs Γ_1 and Γ_2 with disjoint vertex sets is the disjoint union or sum, written $\Gamma_1 + \Gamma_2$. In general, $m\Gamma$ is the graph consisting of m pairwise disjoint copies of Γ. The join of two graphs Γ_1 and Γ_2, denoted by $\Gamma_1 \vee \Gamma_2$, is the graph obtained from $\Gamma_1 + \Gamma_2$ by joining each vertex of Γ_1 to each vertex of Γ_2. Further, given a graph Γ, its *complement* is defined to be the graph in which the vertex set is the same as the one in Γ and two distinct vertices are adjacent if and only if they are not adjacent vertices in Γ.

The *genus* of a graph Γ, denoted by $\gamma(\Gamma)$, is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. Clearly, if $\widetilde{\Gamma}$ is a subgraph of Γ, then $\gamma(\widetilde{\Gamma}) \leq \gamma(\Gamma)$. The sphere with one, two, three, four and five handles are the *torus, double-torus, triple-torus, quadruple-torus and pentuple-torus*, respectively. The graphs embeddable on the surfaces of genus 0, 1, 2, 3, 4 and 5 are the *planar, toroidal, double-toroidal, triple-toroidal, quadruple-toroidal and pentuple-toroidal graphs*, respectively. A *block* of a graph Γ is a connected subgraph B of Γ that is maximal with respect to the property that removal of a single vertex (and the incident edges) from B does not make it disconnected, that is, the graph $B \setminus \{v\}$ is connected for all $v \in V(B)$. Given a graph Γ, there is a unique finite collection $\mathfrak{B}$ of blocks of Γ, such that $\Gamma = \bigcup\limits_{B \in \mathfrak{B}} B$. The collection $\mathfrak{B}$ is called the *block decomposition* of Γ. In [4, Theorem 1], it has been proved that the genus of a graph is the sum of the genera of its blocks. Some of the important helpful results are listed below:

Lemma 2.1. [16, Theorem 6-38]. *If $n \geq 3$, then*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Lemma 2.2. [16, Theorem 6-37]. *If $m, n \geq 2$, then*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Lemma 2.3. [9, Theorem 6.1]. *For all $n \neq 2$,*

$$\gamma(K_{2n,n,n,n}) = \left\lceil \frac{(n-1)(3n-2)}{2} \right\rceil.$$

Lemma 2.4. [16, Corollary 6-14]. *If G is connected, with p number of vertices, $p \geq 3$ and q number of edges, then, $\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1$. Furthermore, equality holds if and only if a triangular imbedding can be found for G.*

3. Some properties of non-nilpotent graphs

Given a group G with $x \in G$, the nilpotentizer of x is defined as $nil_G(x) = \{y \in G \mid < x, y > \text{ is nilpotent}\}$. As in [3], a group G is said to be an $\mathfrak{n}$-group if $nil_G(x)$ is a subgroup of G for all $x \in G$ and a group G is said to be an $\mathfrak{nn}$-group (see [8]), if $nil_G(x)$ is a nilpotent subgroup of G for all $x \in G \setminus nil(G)$. We call G an $\mathfrak{n}_p$-group ([3]) if $nil_G(x)$ is a subgroup of G for all p-elements x of G. A group G is said to be an AC-group if the centralizer $C_G(x)$ of every non-central element x of G is an abelian subgroup of G.

For any group G, the upper central series of a group G is given by $\{e\} = Z_0(G) \leq Z_1(G) \leq \dots$, where $Z_{n+1}(G)/Z_n(G)$ is the center of $G/Z_n(G)$ for all $n \geq 0$. The subgroup $Z_n(G)$ is called the $n^{th}-$center of G. Clearly, $Z_1(G) = Z(G)$, the center of G. This series need not reach G, but if G is finite then this series terminates at a subgroup called the hypercenter of G, denoted by $Z^*(G)$.

We start this section with the following result, which allows us to use $Z^*(G)$ and $nil(G)$ interchangeably, whenever the group G is finite.

Lemma 3.1. [8, Lemma 3.1]. *Let G be a finite group. Then, the following assertions hold:*

1. *$< x, Z^*(G) >$ is nilpotent for every $x \in G$.*
2. *$Z^*(G) = nil(G)$.*

Lemma 3.2. *A periodic finitely generated abelian group is finite.*

Proof. Let G be a periodic finitely generated abelian group. Let $g_i, 1 \leq i \leq k$ be the generators of G of order n_i. Every element of G can be written as a product of powers of these generators i.e., if $x \in G$, then $x = g_1^{a_1} g_2^{a_2} \dots . g_k^{a_k}$, where $0 \leq a_i \leq n_i$, for $1 \leq i \leq k$. The number of such possible products is finite as G is abelian and each a_i can only take finitely many values. Thus, G is finite. □

Lemma 3.3. *Let G be a periodic finitely generated nilpotent group. Then, G is finite.*

Proof. Let m be the nilpotency class of G. We will prove the result by induction on the nilpotency class. If nilpotency class of G is 1, then G is abelian. By Lemma 3.2, G is finite. Suppose the result holds for all groups of nilpotency class at most $m - 1$. Let the upper central series of G be $Z_0(G) \leq Z_1(G) \leq \dots \leq Z_m(G) = G$. Let $G_i = Z_{i+1}(G)/Z_1(G)$, $i = 0, 1, 2, \dots, m-1$. Then $G_0 \leq G_1 \leq \dots \leq G_{m-1}$ forms a central series of $\bar{G} = G/Z_1(G)$ as $[\bar{G}, G_{i+1}] \leq G_i$ for all $i = 0, 1, 2, \dots, m-2$. It follows that $\bar{G}$ is nilpotent and its nilpotency class is at most $m - 1$. By hypothesis,

$\bar{G}$ is finite. Since G is periodic finitely generated, we have $Z(G)$ is finitely generated, periodic and abelian. By Lemma 3.2, $Z(G)$ is finite. Thus, G is finite. □

Lemma 3.4. *Let G be a non-weakly nilpotent group and $x \in G \setminus nil(G)$. If $deg(x)$ in $\mathfrak{R}_G$ is finite, then the order of x is finite.*

Proof. Suppose x has infinite order, that is, the order of the subgroup $< x >$ is infinite. Since $x \in G \setminus nil(G)$, there exist $y \in G \setminus nil(G)$ such that $< x, y >$ is non-nilpotent. We will show that $< x, x^i y >$ is non-nilpotent for all $i \in \mathbb{Z}$. Clearly, $x \in < x, x^i y >$ and so $y \in < x, x^i y >$. Thus, $< x, y > = < x, x^i y >$ and so $< x, x^i y >$ is non-nilpotent for all $i \in \mathbb{Z}$. Hence, $deg(x)$ is infinite, which is a contradiction. □

Lemma 3.5. *Let G be a non-weakly nilpotent group. Then $\mathfrak{R}_G$ has at most one planar connected component.*

Proof. If $\mathfrak{R}_G$ has no planar connected component, then there is nothing to prove. So suppose $\mathfrak{R}_G$ has a planar connected component. Let Γ be the subgraph of $\mathfrak{R}_G$ consisting of all the planar connected components of $\mathfrak{R}_G$.

Firstly, we will show that $\circ(x) \leq 4$, for all $x \in V(\Gamma)$. Suppose that $x \in V(\Gamma)$ such that $\circ(x) > 4$. Since $x \notin nil(G)$, there exist $y \in G \setminus nil(G)$ such that $< x, y >$ is non-nilpotent. Note that $y \in V(\Gamma)$.

Suppose $\circ(y) > 2$. Clearly, $x^{-1}, y^{-1}, xy, yx, xy^{-1} \notin nil(G)$. Suppose $|xy| = 2$. Then $yx = x^{-1}y^{-1}$ and so the element xy^{-1} is distinct from $x, x^{-1}, y, y^{-1}, xy, yx$. Let $H = \{x, x^{-1}, y, y^{-1}, xy, yx, xy^{-1}\}$ and $\Gamma' = \mathfrak{R}_G[H]$. Then, Γ' contains $K_{3,2,2}$ as a subgraph with partite sets $\{x, x^{-1}\}$, $\{y, y^{-1}\}$ and $\{xy, yx, xy^{-1}\}$. Since $x \in V(\Gamma')$ and Γ' is connected, we have Γ' is a subgraph of Γ, which is a contradiction. So, suppose $|xy| > 2$. Now $(xy)^{-1} \notin nil(G)$. If $(xy)^{-1} \neq yx$, then Γ contains $K_{3,2,2}$ as a subgraph with partite sets $\{x, x^{-1}\}$, $\{y, y^{-1}\}$ and $\{xy, yx, (xy)^{-1}\}$, which is not planar. If $(xy)^{-1} = yx$, then $y^2 = x^{-2}$ and if $x^{-1}y = yx$, then $y = xyx$ and so $y^2 = xyx^2yx = xy(y^{-1})^2yx = x^2$, that is $x^4 = e$, a contradiction. It follows that, $x^{-1}y \neq yx$. Since $x^{-1}y \notin nil(G)$ is distinct from x, x^{-1}, y, y^{-1}, xy, we have Γ contains $K_{3,2,2}$ as a subgraph with partite sets $\{x, x^{-1}\}$, $\{y, y^{-1}\}$ and $\{xy, yx, x^{-1}y\}$, which is not planar. Thus $\circ(y) > 2$ is not possible.

Suppose $\circ(y) = 2$. If $|xy| > 2$, then Γ contains $K_{3,2,1}$ as a subgraph with partite sets $\{x, x^{-1}\}$, $\{y\}$ and $\{xy, yx, (xy)^{-1}\}$, which is not planar. If $|xy| = 2$, then $< x, y > \cong D_{2|x|}$ and either $D_{2|x|}$ is nilpotent or Γ contains $\mathfrak{R}_{D_{2|x|}}$ which is a connected non-planar graph, a contradiction. Hence, $|x| \leq 4$, $\forall\ x \in V(\Gamma)$.

Secondly, we will prove the existence of 2-elements and 3-elements in $V(\Gamma)$. Let $x, y \in V(\Gamma)$ such that $< x, y >$ is non-nilpotent. Suppose $V(\Gamma))$ consists of only 2-elements. If $|x| = |y| = |xy| = 2$, then $xy = yx$, a contradiction. If $|x| = |y| = 2$ and $|xy| = 4$, then $< x, y >\cong D_8$, which is nilpotent. If $|x| = 2$, $|y| = 4$ and $|xy| = 2$, then again $< x, y >\cong D_8$, which is nilpotent. If $|x| = 2$, $|y| = 4$, $|xy| = 4$ and if $yx = xy^{-1}$, then $< x, y >\cong D_8$. If $|x| = 2$, $|y| = 4$, $|xy| = 4$ and if $yx \neq xy^{-1}$, then Γ contains $K_{3,2,1}$ as a subgraph with partite sets $\{x\}$, $\{y, y^{-1}\}$ and $\{xy, yx, xy^{-1}\}$, which is not planar. If $|x| = |y| = 4$ and $|xy| = 2$, then $yx = x^{-1}y^{-1}$ and so $yx \neq xy^{-1}$. Thus Γ contains $K_{3,2,2}$ as a subgraph with partite sets $\{x, x^{-1}\}$, $\{y, y^{-1}\}$ and $\{xy, yx, xy^{-1}\}$, which is not planar. If $|x| = |y| = |xy| = 4$ and $(xy)^{-1} = x^{-1}y = xy^{-1} = yx$, then $\langle x, y\rangle \cong Q_8$, which is a contradiction. Thus Γ contains $K_{3,2,2}$ as a subgraph with partite sets $\{x, x^{-1}\}$, $\{y, y^{-1}\}$ and $\{xy, yx, u\}$, where $u \in \{(xy)^{-1}, x^{-1}y, xy^{-1}\}, u \neq yx$, which is a contradiction. Thus, there exists a 3-element in $V(\Gamma)$. If $V(\Gamma))$ consists of only 3-elements i.e., if $|x| = |y| = |xy| = 3$, then Γ contains $K_{3,2,2,}$ as a subgraph with partite sets $\{x, x^{-1}\}$, $\{y, y^{-1}\}$ and $\{xy, yx, y^{-1}x^{-1}\}$, which is not planar. Thus, there exists a 2-element in $V(\Gamma)$.

Now, we will show that Γ is connected. Suppose $x, y \in V(\Gamma)$ such that $< x, y >$ is nilpotent. Then, by Lemma 3.3, $< x, y >$ is finite. If x is a 2-element and y is a 3-element, then since $gcd(|x|, |y|) = 1$, we have $< x, y >=< xy >$ (by [17, Lemma 2.2]). Since $x, y \notin nil(G)$, there exist $u, v \in G \setminus nil(G)$ such that $< x, u >$ and $< y, v >$ are non-nilpotent and thus $u, v \in V(\Gamma)$. Now, $< x, u >\subseteq< xy, u >$ and $< y, v >\subseteq< xy, v >$. Thus, xy is adjacent to both u and v and $x - u - xy - v - y$ is the path connecting x and y. If x and y are both 2-elements then by the previous argument $\exists$ a 3-element z which is connected to x and y and thus there exists a path connecting x and y. If x and y and both 3-elements, then by the same argument, $\exists$ a 2-element z which is connected to x and y and thus there exists a path connecting x and y. Thus, Γ is connected. Hence $\mathfrak{R}_G$ has at most one planar connected component. □

Theorem 3.6. *Let G be a non-weakly nilpotent group and $n \in \mathbb{N}\cup\{0\}$. If $\gamma(\mathfrak{R}_G) = n$, then G is finite.*

Proof. Since $\gamma(\mathfrak{R}_G) = n$, therefore $\mathfrak{R}_G$ has an embedding in S_n. By [16, Def 6-10], the embedding is a minimal embedding. If $\mathfrak{R}_G$ is connected, then by [16, Theorem 6-11], the embedding of $\mathfrak{R}_G$ is a 2-cell embedding or a cellular embedding. By [13, Proposition 3.1], $\mathfrak{R}_G$ must be finite and hence by [14, Proposition 4.2] G is finite. If $\mathfrak{R}_G$ is disconnected, then the number of non-planar connected components of $\mathfrak{R}_G$ is finite (since $\gamma(\mathfrak{R}_G) = n$) and by Lemma 3.5, there is at most one planar connected component. Each of these connected components of $\mathfrak{R}_G$ is also finite, otherwise it would contradict [13, Proposition 3.1]. Hence, $\mathfrak{R}_G$ is finite which implies that G is finite by [14, Proposition 4.2]. □

Proposition 3.7. *Let n be a non-negative integer. Then, there are finitely many non-weakly nilpotent groups whose non-nilpotent graphs are of genus n.*

Proof. Let G be a non-weakly nilpotent group such that $\gamma(\mathfrak{R}_G) = n$. Then by Theorem 3.6, G is a finite group.

If $n = 0$, then by [3, Theorem 6.1], there is only one finite non-nilpotent group whose non-nilpotent graph is planar, namely S_3.

So suppose $n \geq 1$. Let $h = \lfloor \frac{7+\sqrt{1+48n}}{2} \rfloor$ and $\omega(\mathfrak{R}_G) = m$. By Heawood's Formula [15, Theorem 6.3.25], we have $m \leq \chi(\mathfrak{R}_G) \leq h$. If G is a solvable group, then by [3, Theorem 4.4], $|G/Z^*(G)| < m^{m^4} \leq h^{h^4}$ and thus by [14, Lemma 5.1], we have $|G| < h^{h^4}(\sqrt{n}+1)$. If G is a non-solvable group, then $\mathcal{S}_G$ is a subgraph of $\mathfrak{R}_G$. This implies that $\gamma(\mathcal{S}_G) \leq n$. By [2, Corollary 2.4], $|G/Sol(G)| < c^{2m^2[log_{21}m]}[log_{21}m]!$, where c is a constant. By [5, Proposition 38], $|Sol(G)| \leq \sqrt{2\gamma(\mathcal{S}_G)} + 2$. Thus, we have $|G| < c^{2h^2[log_{21}h]}[log_{21}h]!(\sqrt{2n}+2)$. Hence, the result. □

Proposition 3.8. *Let G be a non-weakly nilpotent group such that $\gamma(\mathfrak{R}_G) = n$, then*

$$|E(\mathfrak{R}_G)| \leq 3(2n + |G| - 3)$$

Proof. By Theorem 3.6, G is a finite group. From Lemma 2.4, we have $\gamma(\mathfrak{R}_G) \geq \frac{q}{6} - \frac{p}{2} + 1$, where p and q denote the number of vertices and edges of $\mathfrak{R}_G$, respectively. Thus it follows that $q \leq 6n + 3p - 6$. Since $p = |G| - |Z^*(G)| \leq |G| - 1$, the result follows. □

Proposition 3.9. *Let G be a finite non-nilpotent group and k be the number of conjugacy classes of G. Then,*

$$|E(\mathfrak{R}_G)| \leq (|G|^2 - k|G|)/2$$

Proof. By [1, Lemma 3.27], $2|E(\Gamma_G)| = |G|^2 - k|G|$, where Γ_G is the non-commuting graph of a group G. Since $\mathfrak{R}_G$ is a subgraph of Γ_G, the result follows. □

Proposition 3.10. *Let G be a non-weakly nilpotent group. Then $\mathfrak{R}_G$ is not a complete graph. If G is finite, then $\mathfrak{R}_G$ is neither a bipartite graph nor a tripartite graph.*

Proof. The first part follows from [3, Proposition 4.8] and the second part follows from [3, Theorem 4.2]. □

4. Genus of non-nilpotent graphs of some groups

In this section, we compute the genus of non-nilpotent graphs of some non-nilpotent finite groups. We need the following result in this section.

Proposition 4.1. [14, Proposition 3.12]. *Let G be a finite non-nilpotent group. Then $\mathfrak{R}_G$ is a complete multi-partite graph if and only if G is an nn-group. In particular, $\mathfrak{R}_G \cong K_{|X_1|,\ldots|X_n|}$, where $\mathfrak{P} = \{nil_G(u) \setminus nil(G) \mid u \in G \setminus nil(G)\} = \{X_1, X_2, \ldots\ldots X_n\}$.*

Proposition 4.2. *If $t \geq 1$, $m \geq 3$ and m is odd then $\mathfrak{R}_{D_{2^{t+1}m}} \cong \mathfrak{R}_{Q_{2^{t+1}m}}$ and they are isomorphic to the $(m+1)-$partite graph $K_{2^t m-2^t, m(2^t)}$. In Particular*

$$\gamma(\mathfrak{R}_{D_{2^{t+1}m}}) = \gamma(\mathfrak{R}_{Q_{2^{t+1}m}}) = \gamma(K_{2^t m-2^t, m(2^t)}).$$

Proof. If G denotes any of the groups $D_{2^{t+1}m} = \langle x, y : y^{2^t m} = x^2 = e, x^{-1}yx = y^{-1}\rangle$ or $Q_{2^{t+1}m} = \langle x, y : y^{2^t m} = x^4 = e, y^{2^{t-1}m} = x^2, x^{-1}yx = y^{-1}\rangle$, then G is a non-nilpotent $\mathfrak{nn}$-group ([8]) with $Z^*(G) = \langle y^m\rangle$, for $m \nmid i$ we have $nil_G(y^i) =< y >$ and for $1 \leq i \leq 2^t m$ we have $nil_G(xy^i) = Z^*(G) \cup xy^i Z^*(G)$. Thus, the distinct nilpotentizers of the non-hypercenter elements of G are $< y >$ and $Z^*(G) \cup xy^i Z^*(G)$, $1 \leq i \leq m$. Hence, by Proposition 4.1, $\mathfrak{R}_{D_{2^{t+1}m}} \cong \mathfrak{R}_{Q_{2^{t+1}m}} \cong K_{2^t m-2^t, m(2^t)}$ and the result follows. □

Corollary 4.3. *If $m = 3$, then $\mathfrak{R}_{D_{2^{t+1}3}} \cong \mathfrak{R}_{Q_{2^{t+1}3}} \cong K_{2^{t+1}, 2^t, 2^t, 2^t}$ and $\gamma(\mathfrak{R}_{D_{2^{t+1}3}}) = \gamma(\mathfrak{R}_{Q_{2^{t+1}3}}) = \lceil (2^t - 1)(3.2^{t-1} - 1) \rceil$.*

Proof. Follows from Lemma 2.3. □

Proposition 4.4. *If m is odd, then $\mathfrak{R}_{D_{2m}}$ is isomorphic to the $(m+1)-$partite graph $K_{m-1, m(1)}$. In particular,*

$$\gamma(\mathfrak{R}_{D_{2m}}) = \gamma(K_{m-1, m(1)}).$$

Proof. The group $G \cong D_{2m} = \langle x, y : y^m = x^2 = e, x^{-1}yx = y^{-1}\rangle$ is a non-nilpotent $\mathfrak{nn}$-group with $Z^*(G) = \{e\}$, for $1 \leq i < m$, we have $nil_G(y^i) =< y >$ and for $1 \leq i \leq m$, we have $nil_G(xy^i) = \{e, xy^i\}$. Thus, the distinct nilpotentizers of the non-hypercenter elements are $< y >$ and $\{e, xy^i\}$, $1 \leq i \leq m$. So, in view of Proposition 4.1, $\mathfrak{R}_{D_{2m}} \cong K_{m-1, m(1)}$ and the result follows. □

Corollary 4.5. *Let $m = 3^q(2^p + 1/2) + 3/2$, where $p \geq 3$, $q \geq 1$ and q is odd. Then,*

$$\gamma(\mathfrak{R}_{D_{2m}}) = \gamma(K_{m-1,m}) = \left\lceil \frac{(m-3)(m-2)}{4} \right\rceil.$$

Proof. Here m is odd and so $\mathfrak{R}_{D_{2m}} \cong K_{m-1,m(1)} \cong \overline{K_{m-1}} \vee K_m$. By [9, Theorem 5.6], the result follows. □

Proposition 4.6. *If N is a nilpotent group of order n, $t \geq 1$, $m \geq 3$ and m is odd, then $\mathfrak{R}_{N\times D_{2^{t+1}m}} \cong \mathfrak{R}_{N\times Q_{2^{t+1}m}}$ and they are isomorphic to the $(m+1)-$partite graph $K_{2^tmn-2^tn,m(2^tn)}$. In particular*

$$\gamma(\mathfrak{R}_{N\times D_{2^{t+1}m}}) = \gamma(\mathfrak{R}_{N\times Q_{2^{t+1}m}}) = \gamma(K_{2^tmn-2^tn,m(2^tn)}).$$

Proof. If G denotes the group $N \times K$, where K is any of the groups $D_{2^{t+1}m} = \langle x, y : y^{2^tm} = x^2 = e, x^{-1}yx = y^{-1}\rangle$ or $Q_{2^{t+1}m} = \langle x, y : y^{2^tm} = x^4 = e, y^{2^{t-1}m} = x^2, xyx^{-1} = y^{-1}\rangle$, then G is a non-nilpotent $\mathfrak{nn}$-group with $Z^*(G) = N \times \langle y^m\rangle$, for $m \nmid i$ we have $nil_G((a, y^i)) = N \times \langle y\rangle$ and for $1 \leq j \leq 2^tm$ we have $nil_G((a, xy^j)) = Z^*(G) \cup (a, xy^j)Z^*(G)$. Thus, the distinct nilpotentizers of the non-hypercenter elements of G are $N \times \langle y\rangle$ and $Z^*(G) \cup (e', xy^i)Z^*(G)$, $1 \leq i \leq m$, where e' is the identity element of N. Hence, $\mathfrak{R}_G \cong K_{2^tmn-2^tn,m(2^tn)}$ and the result follows. □

Corollary 4.7. *If $m = 3$, $\gamma(\mathfrak{R}_{N\times D_{2^{t+1}3}}) = \gamma(\mathfrak{R}_{N\times Q_{2^{t+1}3}}) = \lceil (2^tn - 1)(3.2^{t-1}n - 1)\rceil$.*

Proof. Follows from Lemma 2.3. □

Proposition 4.8. *If N is a nilpotent group of order n, $m \geq 3$ and m is odd, then $\mathfrak{R}_{N\times D_{2m}}$ is isomorphic to the $(m+1)-$partite graph $K_{mn-n,m(n)}$. In particular,*

$$\gamma(\mathfrak{R}_{N\times D_{2m}}) = \gamma(K_{mn-n,m(n)}).$$

Proof. If $G = N \times \langle x, y : y^m = x^2 = e, x^{-1}yx = y^{-1}\rangle$, then G is a non-nilpotent $\mathfrak{nn}$-group with $Z^*(G) = N \times \{e\}$. Also, for each $i, 1 \leq i < m$, $nil_G((a, y^i)) = N \times \langle y\rangle$ and $nil_G((a, xy^i)) = N \times \{e, xy^i\}$. Thus, the distinct nilpotentizers of the non-hypercenter elements are $N \times \langle y\rangle$ and $N \times \{e, xy^i\}$, $1 \leq i \leq m$. Hence, $\mathfrak{R}_G \cong K_{mn-n,m(n)}$ and the result follows. □

Corollary 4.9. *If $m = 3$ and $n \neq 2$, $\gamma(\mathfrak{R}_{N\times D_6}) = \lceil \frac{(n-1)(3n-2)}{2} \rceil$.*

Proposition 4.10. *Let G be a non-nilpotent group of order pq, where p and q are primes with $p \mid q-1$. Then, $\mathfrak{R}_G$ is isomorphic to the $(q+1)-$partite graph $K_{q-1,q(p-1)}$. In particular, $\gamma(\mathfrak{R}_G) = \gamma(K_{q-1,q(p-1)})$.*

Proof. Note that if $x \in G \setminus \{e\}$ then $|\langle x \rangle| = p$ or q. In either case we have $nil_G(x) = \langle x \rangle$. It is now not difficult to see that the nilpotent graph of G is isomorphic to the complete $q+1$-partite graph $K_{q-1,p-1,\dots,p-1}$. Thus the result follows. □

Lemma 4.11. *If the order of the centralizer of a non-trivial element of a group is pq, where p and q are primes, then the centralizer is abelian.*

Proof. If $p = q$, then we are done. So suppose $p \neq q$. Let $C_G(x)$ be the centralizer of a non-trivial element x of a group G. If x is of order pq, then we are done. So, without any lost of generality, we can assume that x is of order p. Since q is a prime dividing $|C_G(x)|$, $C_G(x)$ has an element y of order q. Then, x and y commutes, which implies that xy is an element of order pq and thus $C_G(x)$ is abelian. Hence, the result. □

Remark 4.12. *Let G be a finite group and p a prime such that $p \mid |G|$. If x is a $p-$element of G then $x \in P$, where P is a Sylow $p-$subgroup of G.*

Proposition 4.13. *Let G be a non-nilpotent group of order p^2q, where p and q are primes and $p < q$. Then,*

$$\mathfrak{R}_G \cong \begin{cases} K_{3,2,2,2,2}, & \text{if } p = 2, q = 3, |Z^*(G)| = 1, \\ K_{q-1,q(p^2-1)}, & \text{if } p \neq 2 \text{ or } q \neq 3,\ |Z^*(G)| = 1, \\ K_{pq-p,q(p^2-p)}, & \text{if } |Z^*(G)| = p. \end{cases}$$

In particular,

$$\gamma(\mathfrak{R}_G) = \begin{cases} \gamma(K_{3,2,2,2,2}), & \text{if } p = 2, q = 3, |Z^*(G)| = 1, \\ \gamma(K_{q-1,q(p^2-1)}), & \text{if } p \neq 2 \text{ or } q \neq 3,\ |Z^*(G)| = 1, \\ \gamma(K_{pq-p,q(p^2-p)}), & \text{if } |Z^*(G)| = p. \end{cases}$$

Proof. Here, every Sylow subgroup of G is abelian. So by [3, Lemma 3.5], G is an $\mathfrak{n}_p$ group, an $\mathfrak{n}_q$ group and $C_G(x) = nil_G(x)$ for every p-elements and q-elements x of G. Thus, by [3, Lemma 3.4], G is an $\mathfrak{n}$-group. Now, for any p-element or q-element x of $G \setminus Z^*(G)$, if $|C_G(x)| = p$ or p^2 or q, then $C_G(x)$ is abelian. If $|C_G(x)| = pq$, then by Lemma 4.11, $C_G(x)$ is abelian. Also, for any element x of $G \setminus Z^*(G)$ of order pq, $|C_G(x)| = |nil_G(x)| = pq$ and so $nil_G(x)$ is cyclic and hence nilpotent. Thus, G is an $\mathfrak{nn}$-group. Let n_p and n_q be the number of Sylow p-subgroups and Sylow q-subgroups of G respectively. Then, $n_p = 1$ or q and $n_q = 1$ or p^2. If $n_q = p^2$, then $p = 2, q = 3$ and thus $|G| = 12$. Hence, $G \cong A_4$ and $\mathfrak{R}_G \cong K_{3,2,2,2,2}$. So suppose that $n_q = 1$.

Case I: $|Z^*(G)| = 1$. Since $n_q = 1$, we have $n_p = q$ (as G is non-nilpotent). Let Q be the Sylow $q-$subgroup of G and $P_i, 1 \leq i \leq q$, be the Sylow $p-$subgroups of G. Suppose $x \in G \setminus Z^*(G)$ and $\circ(x) = pq$. Note that $|nil_G(x)| \neq p^2q$; otherwise $x \in Z^*(G)$, which is a contradiction. Thus $|nil_G(x)| = pq$. Let $y \in nil_G(x), \circ(y) = p$. Then $y \in P_i$ for some $i = 1, 2, \ldots, q$ and so $|nil_G(y)| = p^2q$. Thus $y \in Z^*(G)$, a contradiction as $|Z^*(G)| = 1$. Thus it follows that $G \setminus Z^*(G) = Q' \sqcup P'_1 \sqcup P'_2 \sqcup \cdots \sqcup P'_q$, where $Q' = Q \setminus Z^*(G), P'_i = P_i \setminus Z^*(G), i = 1, 2, \ldots, q$. Let $x \in P_i \setminus Z^*(G)$. Then $P_i \subseteq nil_G(x)$. Note that $|nil_G(x)| \neq p^2q$, otherwise $x \in Z^*(G)$. Thus $|nil_G(x)| = p^2$. Let $x \in Q \setminus Z^*(G)$. With the same argument as previous, we see that $|nil_G(x)| \neq pq$. Thus $|nil_G(x)| = q$. It follows that the distinct nilpotentizers of G are $Q, P_i, 1 \leq i \leq q$. Thus in view of Proposition 4.1, $\mathfrak{N}_G \cong K_{q-1,q(p^2-1)}$.

Case II: $|Z^*(G)| = p$. Since $n_q = 1$, we have $n_p = q$ (as G is non-nilpotent). Let Q be the Sylow $q-$subgroup of G and $P_i, 1 \leq i \leq q$ be the Sylow $p-$subgroups of G. Note that $P_i \cap P_j = Z^*(G)$, for $i \neq j$. Also $P_i \cap QZ^*(G) = Z^*(G)$, for all $i = 1, 2, \ldots, q$. Thus $G \setminus Z^*(G) = Q' \sqcup P'_1 \sqcup P'_2 \sqcup \cdots \sqcup P'_q$, where $Q' = QZ^*(G) \setminus Z^*(G), P'_i = P_i \setminus Z^*(G), i = 1, 2, \ldots, q$. Now, for $x \in QZ^*(G) \setminus Z^*(G)$, we have $nil_G(x) = QZ^*(G)$ and for $x \in P_i$, $nil_G(x) = P_i$. Therefore the distinct nilpotenttizers are $QZ^*(G), P_i, 1 \leq i \leq q$. Thus in view of Proposition 4.1, $\mathfrak{N}_G \cong K_{pq-p,q(p^2-p)}$.

Case III: $|Z^*(G)| = q$. For any p-element x of $G \setminus Z^*(G)$, $|nil_G(x)| = p^2q$ and so $x \in Z^*(G)$, a contradiction.

Case IV: $|Z^*(G)| = pq$. Again, for any p-element x of $G \setminus Z^*(G)$, $|nil_G(x)| = p^2q$ and so $x \in Z^*(G)$, a contradiction.

Hence, the result follows. □

Proposition 4.14. *Let G be a non-nilpotent group of order pq^2, where p and q are primes and $p < q$. Then,*

$$\mathfrak{N}_G \cong \begin{cases} K_{q^2-1,q^2(p-1)}, & \text{if } |Z^*(G)| = 1, \\ K_{q^2-q,q(pq-q)}, & \text{if } |Z^*(G)| = q. \end{cases}$$

In particular

$$\gamma(\mathfrak{N}_G) = \begin{cases} \gamma(K_{q^2-1,q^2(p-1)}), & \text{if } |Z^*(G)| = 1, \\ \gamma(K_{q^2-q,q(pq-q)}), & \text{if } |Z^*(G)| = q. \end{cases}$$

Proof. Following the same argument as in the previous proposition, we get that G is an $\mathfrak{nn}$-group. If n_p and n_q are the number of Sylow p-subgroups and Sylow q-subgroups of G respectively, then $n_q = 1$ or p. Since $p < q$, we have $n_q = 1$ and thus $n_p = q$ or

q^2.

Case I: $|Z^*(G)| = 1$. Let Q be the Sylow $q-$subgroup of G and $P_i, 1 \leq i \leq n, n = q$ or q^2, be the Sylow $p-$subgroups of G. Suppose $x \in G \setminus Z^*(G)$ with $\circ(x) = pq$. Note that $|nil_G(x)| \neq pq^2$; otherwise $x \in Z^*(G)$, which is a contradiction. Thus $|nil_G(x)| = pq$. Let $y \in nil_G(x), \circ(y) = q$. Then $y \in Q$ and so $|nil_G(y)| = pq^2$. Thus $y \in Z^*(G)$, a contradiction as $|Z^*(G)| = 1$. Thus it follows that $n = q^2$ and $G \setminus Z^*(G) = Q' \sqcup P_1' \sqcup P_2' \sqcup \cdots \sqcup P_{q^2}'$, where $Q' = Q \setminus Z^*(G), P_i' = P_i \setminus Z^*(G), i = 1, 2, \ldots, q^2$. Let $x \in P_i \setminus Z^*(G)$. Then $P_i \subseteq nil_G(x)$. With the same argument as above, we see that $|nil_G(x)| \neq pq$. Note that $|nil_G(x)| \neq pq^2$, otherwise $x \in Z^*(G)$. Thus $nil_G(x) = P_i$. Let $x \in Q \setminus Z^*(G)$. Then $nil_G(x) = Q$. It follows that the distinct nilpotentizers of G are $Q, P_i, 1 \leq i \leq q^2$. Thus in view of Proposition 4.1, $\mathfrak{R}_G \cong K_{q^2-1, q^2(p-1)}$.

Case II: $|Z^*(G)| = p, pq$. If $x \in G \setminus Z^*(G)$ is any q-element, then $|nil_G(x)| = pq^2$ and so $x \in Z^*(G)$, a contradiction.

Case III: $|Z^*(G)| = q$. Let Q be the Sylow $q-$subgroup of G and $P_i, 1 \leq i \leq n, n = q$ or q^2 be the Sylow $p-$subgroups of G. Note that $P_iZ^*(G) \cap P_jZ^*(G) = Z^*(G)$, for $i \neq j$. Also $P_iZ^*(G) \cap Q = Z^*(G)$, for all $i = 1, 2, \ldots, n$. Thus $n = q$ and $G \setminus Z^*(G) = Q' \sqcup P_1' \sqcup P_2' \sqcup \cdots \sqcup P_q'$, where $Q' = Q \setminus Z^*(G), P_i' = P_iZ^*(G) \setminus Z^*(G), i = 1, 2, \ldots, q$. Now, for $x \in Q \setminus Z^*(G)$, we have $nil_G(x) = Q$ and for $x \in P_iZ^*(G) \setminus Z^*(G)$, $nil_G(x) = P_iZ^*(G)$. Therefore the distinct nilpotenttizers are $Q, P_iZ^*(G), 1 \leq i \leq q$. Thus in view of Proposition 4.1, $\mathfrak{R}_G \cong K_{q^2-q, q(pq-q)}$.

Case IV: $|Z^*(G)| = q^2$. If $x \in G \setminus Z^*(G)$ is any p-element, then $|nil_G(x)| = pq^2$ and so $x \in Z^*(G)$, a contradiction.

Hence, the result follows. □

Lemma 4.15. *A group of order p^2q, where p, q are distinct primes, $p < q$, $p > 2$ and $p \nmid q - 1$ is nilpotent.*

Proof. Let G be a group of order p^2q. Let n_p and n_q be the number of Sylow p-subgroups and Sylow q-subgroups of G respectively. Then, $n_q = 1$ or p^2. If $n_q = p^2$, then $p^2 \equiv 1 (\text{mod q})$. Thus $q \mid (p + 1)$ and so $p = 2, q = 3$, which is a contradiction. Thus, $n_q = 1$. Since $p \nmid q - 1$, we have $n_p = 1$. Thus, every Sylow subgroup of G is unique which implies that G is the direct product of its Sylow subgroups. Hence, the result. □

Lemma 4.16. *Let x be a q-element or an element of order pq of a group G, where p and q are primes, $p < q$ and $p \nmid (q - 1)$. If $|C_G(x)| = pq^2$, then $C_G(x)$ is nilpotent.*

Proof. We have $\circ(x) = q, q^2, pq$. Note that $\langle x\rangle \subseteq Z(C_G(x))$. Thus $\left|\frac{C_G(x)}{Z(C_G(x))}\right| = 1, p, q, pq$. It follows that, since $p \nmid q-1$, we have $\frac{C_G(x)}{Z(C_G(x))}$ is cyclic and so $C_G(x) = Z(C_G(x))$, that is $C_G(x)$ is abelian. □

Proposition 4.17. *Let G be a non-nilpotent group of order p^2q^2, where p, q are distinct primes, $p < q$, $p > 2$, $p \nmid q-1$ and $p \mid q+1$. Then,*

$$\mathfrak{R}_G \cong \begin{cases} K_{q^2-1,q^2(p^2-1)}, & \text{if } |Z^*(G)| = 1,\\ K_{pq^2-p,q^2(p^2-p)}, & \text{if } |Z^*(G)| = p.\end{cases}$$

In particular

$$\gamma(\mathfrak{R}_G) = \begin{cases} \gamma(K_{q^2-1,q^2(p^2-1)}), & \text{if } |Z^*(G)| = 1,\\ \gamma(K_{pq^2-p,q^2(p^2-p)}), & \text{if } |Z^*(G)| = p.\end{cases}$$

Proof. Here, every Sylow subgroup of G is abelian and so by [3, Lemma 3.5], G is an $\mathfrak{n}_p$, an $\mathfrak{n}_q$ group and $C_G(x) = nil_G(x)$ for every p-element and q-element x of G. Thus, by [3, Lemma 3.4], G is an $\mathfrak{n}$-group. Let n_p and n_q be the number of Sylow p-subgroups and Sylow q-subgroups of G respectively. Then, $n_q = 1$ or p^2. If $n_q = p^2$, then $p = 2$ and $q = 3$, which is a contradiction. Thus, $n_q = 1$ and since G is non-nilpotent, therefore $n_p = q^2$. Now, for any p-element or q-element x of $G \setminus Z^*(G)$, if $|C_G(x)| = p^2$ or q^2, then $C_G(x)$ is abelian and thus nilpotent. If $|C_G(x)| = p^2q$ or pq^2, then by Lemma 4.15 and Lemma 4.16, $C_G(x)$ is nilpotent. If $x \in G \setminus Z^*(G)$ is an element of order pq, then $|nil_G(x)| = pq$ or p^2q or pq^2. Using Lemma 4.15 and Lemma 4.16, $nil_G(x)$ is nilpotent. If $x \in G \setminus Z^*(G)$ is an element of order p^2q, then $|nil_G(x)| = p^2q$ and so $nil_G(x)$ is cyclic and thus nilpotent. Similar argument holds if $x \in G \setminus Z^*(G)$ is an element of order pq^2. Hence, G is an $\mathfrak{nn}$-group. Let $P_1, P_2, \ldots, P_{q^2}$ be the Sylow $p-$subgroups of G. Let Q be the Sylow $q-$subgroup of G.

Case I: $|Z^*(G)| = 1$. Let $x \in Q \setminus Z^*(G)$. Suppose $|nil_G(x)| = pq^2$. Let $y \in nil_G(x)$ such that $\circ(y) = p$. Then $y \in P_i$ for some i. Therefore $nil_G(x), P_i \subset nil_G(y)$ and so $|nil_G(y)| = p^2q^2$, that is $y \in Z^*(G)$, a contradiction. Thus $|nil_G(x)| = q^2$ and so $nil_G(x) = Q$. Similarly, if $x \in P_i \setminus Z^*(G)$, then $nil_G(x) = P_i$. If $i \neq j$ and $x \in P_i \cap P_j$, then $P_i, P_j \subset nil_G(x)$ and so $nil_G(x) = G$. Thus $x = e$. It follows that $G = P_1 \cup P_2 \cup \cdots \cup P_{q^2} \cup Q$ and $P_1, P_2, \ldots, P_{q^2}, Q$ are the distinct nilpotentizers of G. Thus, in view of Proposition 4.1, we have $\mathfrak{R}_G \cong K_{q^2-1,q^2(p^2-1)}$.

Case II: $|Z^*(G)| = p$. As in Case I, we see that if $x \in P_i \setminus Z^*(G)$, then $nil_G(x) = P_i$

and if $x \in QZ^*(G) \setminus Z^*(G)$, then $nil_G(x) = QZ^*(G)$. Note that $P_i \cap P_j = Z^*(G)$ for $i \neq j$ and $P_j \cap QZ^*(G) = Z^*(G)$. Thus $G = P_1 \cup P_2 \cup \cdots \cup P_{q^2} \cup QZ^*(G)$ and $P_1, P_2, \ldots, P_{q^2}, QZ^*(G)$ are the distinct nilpotentizers of G. Thus, in view of Proposition 4.1, we have $\mathfrak{R}_G \cong K_{pq^2-p,q^2(p^2-p)}$.

Case III: If $|Z^*(G)| = q$, then $|\frac{G}{Z^*(G)}| = p^2q$ and so by Lemma 4.15, $\frac{G}{Z^*(G)}$ is nilpotent. Since $Z^*(G)$ is nilpotent, we have G is nilpotent, a contradiction. If $|Z^*(G)| = pq$, then $|\frac{G}{Z^*(G)}| = pq$ and so, since $p \nmid q-1$, we have $\frac{G}{Z^*(G)}$ is cyclic and hence nilpotent. Since $Z^*(G)$ is nilpotent, we have G is nilpotent, a contradiction. Similarly $|Z^*(G)| \neq p^2q, pq^2$.

This completes the proof. □

Lemma 4.18. *Let G be a group of order pqr, where p, q and r are distinct primes and $p < q < r$. Then, the following hold:*

- (a). *The Sylow r-subgroup is normal in G.*
- (b). *G has a normal subgroup of order qr.*
- (c). *If $q \nmid r-1$, then the Sylow q-subgroup is normal in G.*

Proof. (a). Let n_q and n_r be the number of Sylow q-subgroups and Sylow r-subgroups of G respectively. Then, $n_r = 1$ or pq. Suppose $n_r = pq$. Also, $n_q = 1$ or r or pr. If $n_q \geq r$, then G contains at least $r(q-1)$ elements of order q. Thus, number of r-elements and q-elements is $pq(r-1) + r(q-1) = pqr + r(q-1) - pq > pqr$, a contradiction. Thus, $n_q = 1$. Let Q be the normal Sylow q-subgroup of G. The quotient group G/Q has order pr which implies that G/Q has a normal Sylow r-subgroup N/Q. Then, by Correspondence theorem, N is a normal subgroup of G of order qr. Again, N has a normal Sylow r-subgroup, say M. Since, M is a normal Sylow r-subgroup of N and N is normal in G, it follows that M is normal in G, a contradiction to the fact that $n_r = pq$. Thus, (a) holds.
(b). Let H and K be Sylow $q-$subgroup and Sylow $r-$subgroup of G, respectively. Since K is normal in G, therefore HK is a subgroup of G. Also, $H \cap K$ is trivial. Thus, $|HK| = qr$. Now $I_G(HK) = p$ and since p is the smallest prime dividing $|G|$, therefore HK is normal in G. Hence, the result.
(c). Let H and K be as in (b). Since $q \nmid r-1$, the subgroup HK is cyclic, which implies that HK is abelian. Since H is a subgroup of HK, we have H is a normal subgroup of G. Thus, (c) holds. □

Proposition 4.19. *Let G be a non-nilpotent group of order pqr, where p, q, r are distinct primes, $p<q<r$ and $q \nmid r-1$. Then,*

$$\mathfrak{N}_G \cong \begin{cases} K_{qr-1,qr(p-1)}, & \text{if } |Z^*(G)|=1,\\ K_{qr-q,r(pq-q)}, & \text{if } |Z^*(G)|=q,\\ K_{qr-r,q(pr-r)}, & \text{if } |Z^*(G)|=r.\end{cases}$$

In particular

$$\gamma(\mathfrak{N}_G) = \begin{cases} \gamma(K_{qr-1,qr(p-1)}), & \text{if } |Z^*(G)|=1,\\ \gamma(K_{qr-q,r(pq-q)}), & \text{if } |Z^*(G)|=q,\\ \gamma(K_{qr-r,q(pr-r)}), & \text{if } |Z^*(G)|=r.\end{cases}$$

Proof. Here, every Sylow subgroup of G is abelian. So, following the same argument as in Proposition 4.13, we get that G is an $\mathfrak{nn}$-group. By Lemma 4.18, G has a normal Sylow q-subgroup and a normal Sylow r-subgroup and thus a normal subgroup of order qr. Since $q \nmid r-1$, this normal subgroup is abelian. Let n_p be the number of Sylow p-subgroups of G. Then, $n_p = r$ or q or qr.

Case I: $|Z^*(G)| = 1$. Let $P_1, P_2, \ldots, P_n$, be the Sylow $p-$subgroups of G. Let Q and R be the Sylow $q-$subgroup and Sylow $r-$subgroup of G, respectively. Then QR is the unique abelian subgroup of G of order qr. Let $x \in P_i \setminus Z^*(G)$. Then $|nil_G(x)| = p, pq, pr$. Suppose $|nil_G(x)| = pq$. Then there exist $y \in nil_G(x)$ such that $\circ(y) = q$. Thus $y \in Q$ and so $QR \subset nil_G(y)$. Also $\langle x\rangle \subset nil_G(y)$. Thus $|nil_G(y)| = pqr$ and so $y \in Z^*(G)$, a contradiction. Therefore, $|nil_G(x)| \neq pq$. Similarly, $|nil_G(x)| \neq pr$ and so $|nil_G(x)| = p$. It is obvious now that if $x \in QR \setminus Z^*(G)$, then $nil_G(x) = QR$ and there are no elements of order pq or pr. Thus if $x \in G$, then $x \in H = P_1 \cup P_2 \cup \ldots P_n \cup QR$ and so $G = H$, it follows that $n = qr$. Also we have shown that the distinct nilpotentizers are $P_1, P_2, \ldots, P_{qr}$ and QR. Hence, in view of Proposition 4.1, we have $\mathfrak{N}_G \cong K_{qr-1,qr(p-1)}$.

Case II: $|Z^*(G)| = p$. For any r-element or q-element $x \in G \setminus Z^*(G)$, $|nil_G(x)| = pqr$ and so $x \in Z^*(G)$, a contradiction.

Case III: $|Z^*(G)| = q$. Let $P_1, P_2, \ldots, P_n$, be the Sylow $p-$subgroups of G. Let Q and R be the Sylow $q-$subgroup and Sylow $r-$subgroup of G, respectively. Then $Q = Z^*(G)$. If $x \in QR \setminus Z^*(G)$, then $nil_G(x) = QR$. If $x \in P_iQ \setminus Z^*(G)$, then $nil_G(x) = P_iQ$. Let $x \in G \setminus Z^*(G)$. If $\circ(x) = p, q, pq$, then $x \in P_iQ$ for some $i \in \{1, 2, \ldots, n\}$. If $\circ(x) = q, r, qr$, then $x \in QR$. Suppose $\circ(x) = pr$. Then $\langle x\rangle, Z^*(G) \subset nil_G(x)$ and so $|nil_G(x)| = pqr$, that is $x \in Z^*(G)$, a contradiction. For $i \neq j$, $P_iQ \cap P_jQ = Q$

and $P_iQ \cap QR = Q$. Thus $n = r$ and $G = P_1Q \cup P_2Q \cup \cdots \cup P_rQ \cup QR$. Also, we have shown that the distinct nilpotentizers are $P_1Q, P_2Q, \ldots, P_rQ$ and QR. Hence, in view of Proposition 4.1, we have $\mathfrak{R}_G \cong K_{qr-q,r(pq-q)}$.

Case IV: $|Z^*(G)| = r$. With a similar argument as in Case III, we see that $\mathfrak{R}_G \cong K_{qr-r,q(pr-r)}$.

Case V: $|Z^*(G)| = pq$ or pr or qr. In this case, for any p-element or q-element or r-element $x \in G \setminus Z^*(G)$, we have $|nil_G(x)| = pqr$ and so $x \in Z^*(G)$, a contradiction. Hence, the result. $\square$

Proposition 4.20. *Let G be a finite Frobenius group with complement H and kernel N. If H is nilpotent, then $\mathfrak{R}_G$ is isomorphic to the complete $(n+1)-$partite graph $K_{n-1,n(m-1)}$, where $n = |N|, m = |H|$. In particular, $\gamma(\mathfrak{R}_G) = \gamma(K_{n-1,n(m-1)})$.*

Proof. By [8, Proposition 4.6], G is a non-nilpotent $\mathfrak{nn}$-group with $Z^*(G) = \{e\}$, $nil_G(x) = N$ for all $x \in N \setminus \{e\}$ and $nil_G(y) = gHg^{-1}$ for all $y \in gHg^{-1} \setminus \{e\}$. Moreover, the collection $\{gHg^{-1} \setminus \{e\} \mid g \in G\} \cup \{N \setminus \{e\}\}$ forms a partition of $G \setminus \{e\}$ and H has exactly n conjugates in G. Thus, by Proposition 4.1, $\mathfrak{R}_G \cong K_{n-1,n(m-1)}$ and the result follows. $\square$

Proposition 4.21. *Let $k \geq 2$ and $G = PSL(2, 2^k)$. Then, $\mathfrak{R}_{PSL(2,2^k)}$ is isomorphic to the complete $(2^k + 1) + (2^{2k-1} + 2^{k-1}) + (2^{2k-1} - 2^{k-1})-$partite graph*

$$K_{(2^k+1)(2^k-1),(2^{2k-1}+2^{k-1})(2^k-2),(2^{2k-1}-2^{k-1})(2^k)}.$$

In Particular

$$\gamma(\mathfrak{R}_{PSL(2,2^k)}) = \gamma(K_{(2^k+1)(2^k-1),(2^{2k-1}+2^{k-1})(2^k-2),(2^{2k-1}-2^{k-1})(2^k)}).$$

Proof. It is well-known that $PSL(2, 2^k)$ is a centerless group of order $2^k(2^{2k} - 1)$. Moreover, in view of [1, Proposition 3.21], the following assertions hold for $PSL(2, 2^k)$:

1. $PSL(2, 2^k)$ has an elementary abelian 2-subgroup P of order 2^k such that the number of conjugates of P in $PSL(2, 2^k)$ is $2^k + 1$.
2. $PSL(2, 2^k)$ has a cyclic subgroup A of order $2^k - 1$ such that the number of conjugates of A in $PSL(2, 2^k)$ is $2^{k-1}(2^k + 1)$.
3. $PSL(2, 2^k)$ has a cyclic subgroup B of order $2^k + 1$ such that the number of conjugates of B in $PSL(2, 2^k)$ is $2^{k-1}(2^k - 1)$.
4. The centralizers of the non-trivial elements of $PSL(2, 2^k)$ constitute precisely the family $\{xPx^{-1}, xAx^{-1}, xBx^{-1} \mid x \in G\}$; in particular, $PSL(2, 2^k)$ is an AC-group.

Thus, using [3, Lemma 3.6], we see that $PSL(2,2^k)$ is an $\mathfrak{nn}$-group and so $\mathfrak{R}_{PSL(2,q)} \cong K_{(2^k+1)(2^k-1),(2^{2k-1}+2^{k-1})(2^k-2),(2^{2k-1}-2^{k-1})(2^k)}$. Hence, the result follows. □

Proposition 4.22. *Let $G \cong Sz(2^{2m+1})$ be the Suzuki groups over the field with 2^{2m+1} elements, $m > 0$. Let $q = 2^{2m+1}$, $r = 2^m$ and $s = |Z^*(G)|$. Then, $\mathfrak{R}_G$ is isomorphic to the complete $(q^2+1) + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}-$partite graph*

$$K_{(q^2+1)(q^2-s),\frac{q^2(q^2+1)}{2}(q-1-s),\frac{q^2(q^2+1)(q-1)}{4(q+2r+1)}(q+2r+1-s),\frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}(q-2r+1-s)}.$$

In particular,

$$\gamma(\mathfrak{R}_G) = \gamma(K_{(q^2+1)(q^2-s),\frac{q^2(q^2+1)}{2}(q-1-s),\frac{q^2(q^2+1)(q-1)}{4(q+2r+1)}(q+2r+1-s),\frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}(q-2r+1-s)}).$$

Proof. Here, G is a simple group of order $q^2(q^2+1)(q-1)$. By [14, Proposition 3.7],

(a). G has a Sylow 2-subgroup F of order q^2 and $|\{F^x : x \in G\}| = q^2+1$.
(b). G contains a cyclic subgroup A of order $q-1$ and $|\{A^x : x \in G\}| = q^2(q^2+1)/2$.
(c). G contains a cyclic subgroup B of order $q+2r+1$ and $|\{B^x : x \in G\}| = q^2(q^2+1)(q-1)/4(q+2r+1)$.
(d). G contains a cyclic subgroup C of order $q-2r+1$ and $|\{C^x : x \in G\}| = q^2(q^2+1)(q-1)/4(q-2r+1)$.
(e). Suppose x is a non-trivial element of G. Then,

$$nil_G(x) = \begin{cases} F^x, & \text{if } x \in F^x, \\ A^x, & \text{if } x \in A^x, \\ B^x, & \text{if } x \in B^x, \\ C^x, & \text{if } x \in C^x. \end{cases}$$

In particular, G is an $\mathfrak{nn}$-group and thus by Proposition 4.1,

$$\mathfrak{R}_G \cong K_{(q^2+1)(q^2-s),\frac{q^2(q^2+1)}{2}(q-1-s),\frac{q^2(q^2+1)(q-1)}{4(q+2r+1)}(q+2r+1-s),\frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}(q-2r+1-s)}$$

and the result follows. □

Proposition 4.23. *Let $G \cong \mathbb{Z}_m \rtimes \mathbb{Z}_{2n} = \langle a, x : a^m = x^{2n} = 1, xax^{-1} = a^{-1}\rangle$, where m is odd, $n = 2^t$, $t \geq 2$. Then, $\mathfrak{R}_G$ is isomorphic to the complete $(m+1)-$partite graph $K_{mn-n,m(n)}$ and $\gamma(\mathfrak{R}_G) = \gamma(K_{mn-n,m(n)})$. In particular, if $m = 3$, then $\gamma(\mathfrak{R}_G) = \lceil \frac{(n-1)(3n-2)}{2} \rceil$.*

Proof. The group G is a non-nilpotent $\mathfrak{nn}$-group with $Z^*(G) =< x^2 >$. For $1 \leq i < m$ and j even, we have $nil_G(a^i x^j) = \langle a, Z^*(G)\rangle$. For $1 \leq i \leq m$, $1 \leq j < 2n$ and j odd, we have $nil_G(a^i x^j) = Z^*(G) \cup a^i x^j Z^*(G)$. Thus, the distinct nilpotentizers of the non-hypercenter elements are $\langle a, Z^*(G)\rangle$ and $Z^*(G) \cup a^i x^j Z^*(G)$, where $1 \leq i < m$ and $1 \leq j < 2n$, j odd. By Proposition 4.1, $\mathfrak{R}_G \cong K_{mn-n,m(n)}$ and hence by Lemma 2.3, the result follows. □

Proposition 4.24. *If N is a nilpotent group of order s and $H = \mathbb{Z}_m \rtimes \mathbb{Z}_{2n} = \langle a, x : a^m = x^{2n} = 1, xax^{-1} = a^{-1}\rangle$, where m is odd, $n = 2^t$, $t \geq 2$, then $\mathfrak{R}_{N\times(\mathbb{Z}_m\rtimes\mathbb{Z}_{2n})} \cong K_{mns-ns,m(ns)}$ and $\gamma(\mathfrak{R}_{N\times(\mathbb{Z}_m\rtimes\mathbb{Z}_{2n})}) = \gamma(K_{mns-ns,m(ns)})$. In particular, if $m = 3$, then $\gamma(\mathfrak{R}_G) = \lceil \frac{(ns-1)(3ns-2)}{2} \rceil$.*

Proof. Let $G = N \times H$. Then the group G is a non-nilpotent $\mathfrak{nn}$-group with $Z^*(G) = N \times Z^*(H)$. For $1 \leq i < m$ and j even, we have $nil_G(n', a^i x^j) = N \times \langle a, Z^*(H)\rangle$. For $1 \leq i \leq m$, $1 \leq j < 2n$ and j odd, we have $nil_G(n', a^i x^j) = N \times (Z^*(H) \cup a^i x^j Z^*(H))$. Thus, the distinct nilpotentizers of the non-hypercenter elements are $N \times \langle a, Z^*(H)\rangle$ and $N \times (Z^*(H) \cup a^i x^j Z^*(H))$, where $1 \leq i < m$ and $1 \leq j < 2n$, j odd. Thus, by Proposition 4.1, $\mathfrak{R}_G \cong K_{mns-ns,m(ns)}$ and hence by Lemma 2.3, the result follows. □

Lemma 4.25. *Let G be a finite non-nilpotent group such that every Sylow subgroups of G is abelian. Then,*

$$|E(\mathfrak{R}_G)| \geq \frac{p-1}{2p}|G|^2,$$

where p is the smallest prime dividing the order of G.

Proof. As in [11], we have $v_0 = n_0/|G|^2$, where $n_0 = |\{(x, y) \in G^2 : < x, y >$ is non-nilpotent $\}|$. Clearly, $n_0 = \sum_{v\in V(\mathfrak{R}_G)} deg(v) = 2|E(\mathfrak{R}_G)|$ and thus $v_0 = \frac{2|E(\mathfrak{R}_G)|}{|G|^2}$. Since G is non-abelian, by [11, Lemma 5], we have $v_0 \geq (p-1)/p$. Hence, the result follows. □

Proposition 4.26. *Let G be a non-nilpotent group of order $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where $p_i's$ are distinct primes and $0 \leq a_i \leq 2$, $1 \leq i \leq k$. Then,*

$$\frac{p-1}{12p}n^2 - \frac{(n-3)}{2} \leq \gamma(\mathfrak{R}_G) \leq \gamma(K_{n-|Z^*(G)|}),$$

where p is the smallest prime dividing order of G.

Proof. Here, every Sylow subgroup of G is abelian. Thus, result follows easily from Lemma 2.4, Lemma 4.25 and the fact that $|Z^*(G)| \geq 1$. □

Proposition 4.27. *Let G be a non-nilpotent group of order n, where n is odd. Then,*

$$\frac{p-1}{12p}n^2 - \frac{(n-3)}{2} \leq \gamma(\mathfrak{N}_G) \leq \gamma(K_{n-|Z^*(G)|}),$$

where p is the smallest prime dividing $|G|$.

Proof. By the main theorem of [10], G is a solvable group. Thus, result follows easily from Lemma 2.4, [3, Proposition 4.11] and the fact that $|Z^*(G)| \geq 1$. □

Proposition 4.28. *Let G be a finite non-nilpotent group such that $\mathfrak{N}_G \cong \Gamma_G$, the non-commuting graph of G. Then,*

$$\frac{|G|^2 - k|G| - 6|G|}{12} + \frac{3}{2} \leq \gamma(\mathfrak{N}_G) \leq \gamma(K_{|G|-|Z^*(G)|}),$$

where k is the number of conjugacy classes of G.

Proof. By [1, Lemma 3.27], $2|E(\Gamma_G)| = |G|^2 - k|G|$, where Γ_G is the non-commuting graph of a group G. Since $\mathfrak{N}_G \cong \Gamma_G$, therefore $|E(\mathfrak{N}_G)| = (|G|^2 - k|G|)/2$. Thus, by Lemma 2.4 and the fact that $|Z^*(G)| \geq 1$, the result follows. □

5. Groups whose non-nilpotent graph is planar, toroidal, double-toroidal, triple-toroidal, quadruple-toroidal or pentuple-toroidal

In this section, we will classify all non-weakly nilpotent groups whose non-nilpotent graphs are planar, toroidal, double-toroidal, triple-toroidal, quadruple-toroidal or pentuple-toroidal.

Theorem 5.1. *Let G be a non-weakly nilpotent group. Then $\mathfrak{N}_G$ is planar if and only if $G \cong S_3$.*

Proof. Follows from Theorem 3.6 and [3, Theorem 6.1]. □

Theorem 5.2. *Let G be a non-weakly nilpotent group. Then $\mathfrak{N}_G$ is not toroidal.*

Proof. Follows from Theorem 3.6 and [14, Proposition 5.3]. □

We need the following result in the sequel.

Lemma 5.3. *Let G be a non-nilpotent solvable group such that $\gamma(\mathfrak{N}_G) = n$. Then,*

$$|G|(|G|(p-1) - 6p) \leq 6p(2n-3)$$

where p is the smallest prime dividing the order of G.

Proof. It follows from Lemma 2.4, [3, Proposition 4.11] and the fact that $|Z^*(G)| \geq 1$. □

Theorem 5.4. *Let G be a non-weakly nilpotent group. Then, the non-nilpotent graph of G is double-toroidal if and only if G is isomorphic to D_{10}.*

Theorem 5.5. *Let G be a non-weakly nilpotent group. Then, the non-nilpotent graph of G is triple-toroidal if and only if G is isomorphic to D_{12} or Q_{12}.*

Theorem 5.6. *Let G be a non-weakly nilpotent group. Then, $\mathfrak{R}_G$ is quadruple-toroidal or pentuple-toroidal if and only if $G \cong A_4$.*

Proof of Theorem 5.4, Theorem 5.5 and Theorem 5.6. By Theorem 3.6, G is a finite group. Suppose the non-nilpotent graph of G is double-toroidal, triple-toroidal, quadruple-toroidal or pentuple-toroidal. Since, $\gamma(\mathfrak{R}_G) \leq 5$, we have $\omega(\mathfrak{R}_G) \leq 11$ and by [14, Lemma 5.1], we have $|Z^*(G)| \leq \sqrt{5}+1 = 3.23$ and so $|Z^*(G)| \leq 3$. Since G is a finite non-nilpotent group, by [3, Theorem 4.2], $\omega(\mathfrak{R}_G) \geq 4$ and G is solvable. Suppose $p \geq 11$ be a prime dividing the order of G. Suppose $g \in G \setminus Z^*(G)$ such that $\circ(g) = p$. Then there exist $x \in G$ such that $< g, x >$ is not nilpotent. Let $H = \{g, g^2, ...g^{p-1}\}$ and $L = xH \cup \{x\}$. Then the induced subgraph $\mathfrak{R}_G[H \cup L]$ has a subgraph isomorphic to $K_{10,11}$, a contradiction as $\gamma(K_{10,11}) = 18 > 5$. Thus $|G| = 2^r 3^s 5^t 7^u$. If 2 is the smallest prime dividing the order of G, then by Lemma 5.3, $|G| < 17$. If 3 is the smallest prime dividing the order of G, then by Lemma 5.3, $|G| < 14$. If 5 is the smallest prime dividing the order of G, then by Lemma 5.3, $|G| < 12$. Thus G is isomorphic to S_3, D_{10}, A_4, D_{14}, Q_{12} or D_{12}. By [3, Theorem 6.1], $\mathfrak{R}_{S_3}$ is planar. The group D_{10} is a *centerless AC-group* and so by [14, Lemma 4.5], $\mathfrak{R}_G \cong \Gamma(G)$, where $\Gamma(G)$ is the non-commuting graph of G. It follows from [14, Proposition 5.5] that $\mathfrak{R}_{D_{10}}$ is double-toroidal. By [14, Remark 5.2], $\mathfrak{R}_{D_{12}} \cong \mathfrak{R}_{Q_{12}} \cong K_{4,2,2,2}$. Thus, by [9, Corollary 6.2], $\gamma(\mathfrak{R}_{D_{12}}) = \gamma(\mathfrak{R}_{Q_{12}}) = 3$. By [14, Remark 5.2], $4 \leq \gamma(\mathfrak{R}_{A_4}) \leq 5$. The non-nilpotent graph of the group D_{14} is isomorphic to $K_{6,1,1,1,1,1,1,1}$. The graph $K_{6,1,1,1,1,1,1,1}$ has 13 vertices and 63 edges and so, by Lemma 4.25, we have $\gamma(\mathfrak{R}_{D_{14}}) \geq 5$. Note that, $\mathfrak{R}_{D_{14}} \cong K_{13} - K_6$ and by [12], $K_{13} - K_6$ has no triangular embedding. Thus, $\mathfrak{R}_{D_{14}}$ has no triangular embedding. It follows from Lemma 4.25 that $\gamma(\mathfrak{R}_{D_{14}}) > 5$. This completes the proof. □

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