GENERALIZED HOMOGENEOUS DERIVATIONS ON GRADED RINGS

YASSINE AIT MOHAMED

Abstract. We introduce a notion of generalized homogeneous derivations on graded rings as a natural extension of the homogeneous derivations defined by Kanunnikov. We then define gr-generalized derivations, which preserve the degrees of homogeneous components. Several significant results originally established for prime rings are extended to the setting of gr-prime rings, and we characterize conditions under which gr-semiprime rings contain nontrivial central graded ideals. In addition, we investigate the algebraic and module-theoretic structures of these maps, establish their functorial properties, and develop categorical frameworks that describe their derivation structures in both ring and module contexts.

1. Introduction

Derivations are fundamental mappings in ring theory that capture differential-like behavior while preserving the underlying algebraic structure. They play a central role in the study of ring-theoretic properties, especially in characterizing commutativity and detecting structural invariants in algebraic systems. Since their introduction, derivations have been the subject of numerous generalizations. A major development was Brešar's introduction of generalized derivations [5], which opened a rich line of research (see, for example, [6], [7], [11], [12]). This broader framework has proved to be remarkably effective in extending many classical results that were originally established for standard derivations (cf. [4], [9], [10], [10]).

In parallel, the theory of graded rings has become a central tool in modern algebra. Graded structures naturally arise in various mathematical contexts, including group rings, polynomial rings, tensor algebras, and cohomology theories. They provide refined structural information that supports both classification and characterization results. The two strands, generalized derivations and graded structures, were partially unified by Kanunnikov in 2018 through the introduction of homogeneous derivations on graded rings [8]. Homogeneous derivations are classical derivations that, in addition, preserve the grading: they map homogeneous elements to homogeneous elements and thus simultaneously respect both the differential and graded structures of the ring.

In our earlier work [1], several classical theorems on derivations were extended to the graded setting. For instance, we established graded analogues of Posner's theorem for gr-prime rings of characteristic different from 2: if the composition of two derivations (with at least one of them homogeneous) is again a derivation, then one of them must be trivial. We also showed that if a gr-prime ring admits a nonzero homogeneous derivation that is centralizing on a nonzero graded ideal, then the ring is commutative. Moreover, we proved a graded version of Herstein's theorem: in gr-prime rings of characteristic not equal to 2, if two nonzero homogeneous derivations have Lie bracket contained in the center, then the ring is commutative. For gr-semiprime rings, we showed that homogeneous derivations satisfying suitable centralizing conditions guarantee the existence of nonzero central graded ideals. Building on these results, the present paper introduces generalized homogeneous derivations on graded rings, develops their basic properties, and studies their behavior in both ring and module contexts. Our goal is to extend the theory of generalized derivations to the graded setting in a way that preserves the rich graded structure while revealing new algebraic phenomena.

Let G denote a group with identity element e. A ring R is called G-graded if it can be decomposed as $R = \bigoplus_{\tau \in G} R_\tau$ into additive subgroups such that $R_{\tau_1} R_{\tau_2} \subset R_{\tau_1 \tau_2}$ for all $\tau_1, \tau_2 \in G$. The collection of homogeneous elements $\mathcal{H}(R) = \bigcup_{\tau \in G} R_\tau$ consists of elements $a \in R_\tau$ having degree $\deg a = \tau$. Each element $x \in R$ has a unique representation $x = \sum_{\tau \in G} x_\tau$ where $x_\tau \in R_\tau$ are the homogeneous components. The graded structure naturally extends to tensor products through

$$(R \otimes_K S)_\gamma = \bigoplus_{\tau\sigma=\gamma} R_\tau \otimes_K S_\sigma$$

for G-graded K-algebras and similarly to polynomial rings.

An ideal $\mathfrak{I} \subseteq R$ is graded when $\mathfrak{I} = \bigoplus_{\tau \in G}(\mathfrak{I} \cap R_\tau)$, where we denote $\mathfrak{I}_\tau = \mathfrak{I} \cap R_\tau$. Ring homomorphisms $\varphi : R \to S$ between G-graded rings are graded if $\varphi(R_\tau) \subseteq S_\tau$, $\mathrm{Hom}(R, S)^{gr}$ denoted the set of all graded homomorphisms. When $\mathfrak{I}$ is graded, quotient rings inherit the canonical grading structure $(R/\mathfrak{I})_\tau := \{\bar{r} \in R/\mathfrak{I} \mid r \in R_\tau\}$. A graded ring R is gr-prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$ for homogeneous elements $a, b \in \mathcal{H}(R)$, and gr-semiprime if $aRa = \{0\}$ implies $a = 0$ for $a \in \mathcal{H}(R)$.

A G-graded module over a G-graded ring R decomposes as $M = \bigoplus_{\tau \in G} M_\tau$ with the compatibility condition $R_\sigma \cdot M_\tau \subseteq M_{\sigma\tau}$. Graded homomorphisms satisfy $f(M_\tau) \subseteq N_\tau$, and tensor products exhibit the multiplicative grading

$$(M \otimes_R N)_k = \bigoplus_{\tau\sigma=k} M_\tau \otimes_R N_\sigma.$$

An additive mapping $d : R \to R$ is a derivation if it satisfies the Leibniz rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is homogeneous if $d(\mathcal{H}(R)) \subseteq \mathcal{H}(R)$ ([8]). Inner derivations have the form $d(x) = [a, x]$ for some fixed $a \in R$, where $[a, x] = ax - xa$ denotes the commutator. A generalized derivation is an additive mapping $F : R \to R$ satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is the associated derivation of F.

Organization of the paper. In §2, we define generalized homogeneous derivations on graded rings, establish their main properties, and develop a functorial framework. In §3, we introduce gr-generalized derivations, which preserve degrees of homogeneous components, and study their algebraic and Lie-theoretic structures. In §4, we study commutativity criteria for gr-prime rings. Section §4.1 examines conditions under which homogeneous derivations force commutativity via Lie brackets and Jordan products. In §4.2, we extend classical results from prime rings to gr-prime rings under the action of generalized homogeneous derivations. In §5, we identify when gr-semiprime rings contain nonzero central graded ideals by means of generalized homogeneous derivations, yielding graded analogues of Posner-type results. Finally, in §6, we extend the framework to graded modules, introduce generalized homogeneous derivations on modules, study their functorial behavior, and construct the associated category $\mathscr{M}_G^{gh}$.

Conventions. Throughout this paper, we adopt the following conventions.

- All polynomial rings $\mathbb{C}[t_1, \dots, t_n]$ are equipped with the standard $\mathbb{Z}$-grading by total degree, where $\deg(t_1^{a_1} \cdots t_n^{a_n}) = a_1 + \cdots + a_n$.
- *The $\pm$ notation:* When a condition involves the symbol $\pm$, such as

  $$F(xy) \pm xy \in Z(R) \quad \text{or} \quad F_1(x)F_2(y) \pm xy \in Z(R),$$

  we mean that at least one of the two possibilities holds. Note that the case with $-$ can always be reduced to the case with $+$ by replacing F (or F_1, F_2) by $-F$ (or $-F_1, -F_2$). Thus, the proofs typically establish the result for one sign and invoke this reduction for the other.
- *Abelian grading group:* We restrict our attention to abelian grading groups G throughout. This assumption is essential because, in general, the Lie bracket $[x, y] = xy - yx$ and the Jordan product $x \circ y = xy + yx$ do not preserve homogeneity when applied to homogeneous elements, as illustrated in the following example. The abelian condition guarantees the preservation of homogeneity, which is essential for the commutator-based techniques used in this paper.

Example 1.1. *Let $R = M_4(k)$ denote the ring of 4×4 matrices over a field k, and let $D_{10} = \langle a, b \mid a^5 = b^2 = e,\, bab = a^{-1}\rangle$ be the dihedral group of order 10. We define a D_{10}-grading on R by setting*

$$R_e := \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}, \qquad R_a := \begin{pmatrix} 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{a^2} := \begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_{a^3} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_b := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \end{pmatrix}, \qquad R_{ab} := \begin{pmatrix} 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \end{pmatrix}$$

$$R_{a^4b} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & k & 0 \end{pmatrix}, \qquad R_{a^4} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R_{a^2b} = R_{a^3b} = \{0\}.$$

One verifies by direct computation that

$$[R_b, R_{a^4}] = R_b \circ R_{a^4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ k & 0 & 0 & 0 \end{pmatrix} \not\subset \mathcal{H}(R).$$

2. Generalized homogeneous derivations

We introduce generalized homogeneous derivations on G-graded rings and establish their functorial properties.

Definition 2.1. *Let R be a ring graded by an arbitrary group G. An additive mapping $F : R \to R$ is called a* generalized homogeneous derivation *if there exists a homogeneous derivation $d : R \to R$ such that*

(i) $F(xy) = F(x)y + xd(y)$ *for all* $x, y \in R$;
(ii) $F(r) \in \mathcal{H}(R)$ *for all* $r \in \mathcal{H}(R)$.

The mapping d is called an associated homogeneous derivation of F.

We denote such a generalized homogeneous derivation by $(F, d)_h$, where the subscript 'h' emphasizes the homogeneity condition. The collection of all generalized homogeneous derivations of R is denoted by $\mathfrak{Der}_G^{gh}(R)$.

Example 2.1. *Let $R = M_n(\mathbb{C}[t])$ with $\mathbb{Z}_2$-grading where R_0 consists of matrices with polynomial entries having only even-degree monomials, and R_1 consists of matrices with polynomial entries having only odd-degree monomials. Define $d : R \to R$ by $d(A) = \frac{d}{dt}(A)$ (entrywise differentiation) and $F : R \to R$ by $F(A) = tA + d(A)$. Then $(F, d)_h$ is a generalized homogeneous derivation.*

We now highlight several key properties of generalized homogeneous derivations that follow directly from Definition 2.1.

Remark 2.1. *Let R be a G-graded ring, and let $(F_1, d_1)_h$ and $(F_2, d_2)_h$ be generalized homogeneous derivations of R. In general, the sum $F_1 + F_2$ does not define a generalized homogeneous derivation. For instance, consider the polynomial ring $\mathbb{C}[t_1, t_2, t_3]$ equipped with the standard $\mathbb{Z}$-grading. Define*

$$F_1(f) = d_1(f) = t_3 \frac{\partial f}{\partial t_1} \quad \text{and} \quad F_2(f) = d_2(f) = \frac{\partial f}{\partial t_2}.$$

Then their sum acts as

$$(F_1 + F_2)(f) = t_3 \frac{\partial f}{\partial t_1} + \frac{\partial f}{\partial t_2},$$

which does not preserve homogeneity. This example shows that the set $\mathfrak{Der}_G^{gh}(R)$ of generalized homogeneous derivations does not carry a natural additive structure.

Proposition 2.1. *Let R be a nontrivially G-graded ring. Then the following inclusions hold*

$$\mathfrak{Der}_G^{h}(R) \subsetneq \mathfrak{Der}_G^{gh}(R) \subsetneq \mathfrak{Gen}(R),$$

where $\mathfrak{Der}_G^{h}(R)$ and $\mathfrak{Gen}(R)$ denote the sets of homogeneous derivations and generalized derivations on R, respectively. Both inclusions are strict.

Proof. The inclusions follow directly from the definitions. For strictness, consider $R = \mathbb{C}[t_1, t_2]$ with the standard $\mathbb{Z}$-grading. Define

$$F(f) = t_1 f + t_1 t_2 \frac{\partial f}{\partial t_1} \quad \text{and} \quad d(f) = t_1 t_2 \frac{\partial f}{\partial t_1}.$$

Then $(F,d)_h \in \mathfrak{Der}^{gh}_G(R)$. However, $F \notin \mathfrak{Der}^h_G(R)$. For the second inclusion, define $G(f) = t_1 f + \frac{\partial f}{\partial t_2}$. Then $G \in \mathfrak{Gen}(R)$ but $G \notin \mathfrak{Der}^{gh}_G(R)$. □

In the following proposition, we establish several sufficient conditions for the existence of nonzero generalized homogeneous derivations on a graded ring R.

Proposition 2.2. *Let R be a G-graded ring. Then R admits a nonzero generalized homogeneous derivation if any of the following holds:*

1. *R has a nonzero homogeneous derivation.*
2. *$R_\sigma \cap Z(R) \neq \{0\}$ for some $\sigma \in G$.*
3. *R admits a nonzero graded endomorphism.*
4. *$C_R(R_e) = R$ and $R_e \neq \{0\}$.*

Proof. (1) If $d \neq 0$ is a homogeneous derivation, then $(d,d)_h$ is a nonzero generalized homogeneous derivation by definition.

(2) For a nonzero $a \in R_\sigma \cap Z(R)$, define $F_a(r) = ar$. Since a is central and homogeneous, F_a preserves homogeneous elements, and $(F_a, 0)_h$ satisfies the required conditions.

(3) Any nonzero graded endomorphism $\varphi : R \to R$ yields $(F,0)_h$ where $F = \varphi$, because graded endomorphisms preserve homogeneous components.

(4) For a nonzero $b \in R_e$ with $C_R(R_e) = R$, define $F_b(r) = br$. The centrality condition ensures that $(F_b, 0)_h$ is well defined and nonzero. □

Definition 2.2. *Let R be a G-graded ring and $(F,d)_h$ a generalized homogeneous derivation. A graded ideal $\mathfrak{I}$ is* gr-differential *if $d(\mathfrak{I}) \subseteq \mathfrak{I}$ and $F(\mathfrak{I}) \subseteq \mathfrak{I}$.*

Remark 2.2. *Not all graded ideals are gr-differential. For instance, in $\mathbb{R}[t_1, t_2]$ with $(F,d)_h$ defined by $F(P) = d(P) = \frac{\partial P}{\partial t_1}$, the graded ideal $\mathfrak{I} = \langle t_1 t_2 \rangle$ fails the gr-differential property since $d(t_1 t_2) = t_2 \notin \mathfrak{I}$. However, restrictions of generalized homogeneous derivations to gr-differential ideals preserve the derivation structure.*

Proposition 2.3. *Let $(F,d)_h$ be a generalized homogeneous derivation of a G-graded ring R, and let $\{\mathfrak{I}_\beta\}_{\beta \in \Lambda}$ be gr-differential ideals. Then $\bigcap_{\beta\in\Lambda} \mathfrak{I}_\beta$, $\prod_{\beta\in\Lambda} \mathfrak{I}_\beta$, $\mathfrak{I}^n$ for $n \geq 1$, and $\sum_{\beta\in\Lambda} \mathfrak{I}_\beta$ are gr-differential ideals.*

Proof. Each operation preserves both the graded ideal property and the invariance conditions $d(\mathfrak{I}) \subseteq \mathfrak{I}$ and $F(\mathfrak{I}) \subseteq \mathfrak{I}$. □

Remark 2.3. *Let $\mathfrak{I}$ be a gr-differential ideal with respect to $(F,d)_h$ in a G-graded ring R. Then $(F,d)_h$ induces a well-defined generalized homogeneous derivation $(\tilde{F}, \tilde{d})_h$ on the quotient ring $R/\mathfrak{I}$ via the natural definitions $\tilde{F}(\overline{x}) = \overline{F(x)}$ and $\tilde{d}(\overline{x}) = \overline{d(x)}$. The gr-differential property ensures independence from coset representatives, while the derivation structure transfers canonically to the quotient.*

Proposition 2.4. *Let $\{R_i\}_{i\in I}$ be a finite collection of G_i-graded rings. Then*

$$\mathfrak{Der}^{gh}_{\prod_{i\in I} G_i}\left(\prod_{i\in I} R_i\right) \cong \prod_{i\in I} \mathfrak{Der}^{gh}_{G_i}(R_i).$$

Proof. Define $\Phi : \mathfrak{Der}^{gh}_G(R) \to \prod_{i\in I} \mathfrak{Der}^{gh}_{G_i}(R_i)$ by $\Phi((F,d)_h) = ((F_1,d_1)_h, \ldots, (F_n,d_n)_h)$ where $F_i(r_i) = \pi_i(F(e_i(r_i)))$ and $d_i(r_i) = \pi_i(d(e_i(r_i)))$. Here $e_i : R_i \to R$ and $\pi_i : R \to R_i$ are the canonical embedding and projection maps. The inverse Ψ is defined by $\Psi((F_1,d_1)_h, \ldots, (F_n,d_n)_h) = (F,d)_h$ where $F(r_1, \ldots, r_n) = (F_1(r_1), \ldots, F_n(r_n))$ and $d(r_1, \ldots, r_n) = (d_1(r_1), \ldots, d_n(r_n))$. A direct verification shows $\Psi \circ \Phi = \text{id}$ and $\Phi \circ \Psi = \text{id}$. □

Proposition 2.5. *Let $\phi : R \to S$ be a surjective graded homomorphism with $\ker\phi$ a gr-differential ideal. Then ϕ induces*

$$\phi_* : \mathfrak{Der}_G^{gh}(R) \to \mathfrak{Der}_G^{gh}(S)$$

defined by $\phi_*((F_R, d_R)_h) = (F_S, d_S)_h$ where $F_S(\phi(r)) = \phi(F_R(r))$ and $d_S(\phi(r)) = \phi(d_R(r))$.

Proof. Well-definedness follows from the gr-differential property of $\ker(\phi)$. The derivation properties transfer directly via surjectivity of ϕ and the graded homomorphism properties. □

Proposition 2.6. *Let $\varphi : R \to S$ be a graded isomorphism. Then*

$$\Phi_\varphi : \mathfrak{Der}_G^{gh}(R) \to \mathfrak{Der}_G^{gh}(S), \qquad \Phi_\varphi((F_R, d_R)_h) = (\varphi \circ F_R \circ \varphi^{-1}, \varphi \circ d_R \circ \varphi^{-1})_h$$

is a bijection.

Proof. Conjugation by isomorphisms preserves derivation structures. The inverse is $\Phi_\varphi^{-1}((F_S, d_S)_h) = (\varphi^{-1} \circ F_S \circ \varphi, \varphi^{-1} \circ d_S \circ \varphi)_h$. □

Corollary 2.1. $|\mathfrak{Der}_G^{gh}(R)|$ *is invariant under graded automorphisms.*

Definition 2.3. *Let R and S be G-graded rings. Let $(F_R, d_R)_h$ and $(F_S, d_S)_h$ be generalized homogeneous derivations on R and S, respectively. A graded homomorphism $\phi : R \to S$ is called a* ghd-homomorphism *if it satisfies the compatibility conditions $\phi \circ F_R = F_S \circ \phi$ and $\phi \circ d_R = d_S \circ \phi$. These conditions ensure that the following diagrams commute:*

$$\begin{array}{ccc} R & \xrightarrow{F_R} & R \\ \downarrow\phi & & \downarrow\phi \\ S & \xrightarrow{F_S} & S \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{d_R} & R \\ \downarrow\phi & & \downarrow\phi \\ S & \xrightarrow{d_S} & S \end{array}$$

Example 2.2. *For graded rings (R, G_1) and (S, G_2) with generalized homogeneous derivations $(F_R, d_R)_h$ and $(F_S, d_S)_h$, the canonical projections $\pi_R : R \times S \to R$ and $\pi_S : R \times S \to S$ are ghd-homomorphisms with respect to $(F, d)_h$ defined by $F(r, s) = (F_R(r), F_S(s))$ and $d(r, s) = (d_R(r), d_S(s))$.*

Definition 2.4. *The category $\mathscr{G}_G^h$ has*

(i) *Objects: triples (R, F, d) where R is G-graded and $(F, d)_h$ is a generalized homogeneous derivation.*
(ii) *Morphisms: ghd-homomorphisms between the underlying rings.*

Proposition 2.7. *$\mathscr{G}_G^h$ admits finite products.*

Proof. Let $\{(R_i, F_i, d_i)\}_{i\in I}$ be a finite family of objects in $\mathscr{G}_G^h$. By Proposition 2.4, we know that $R = \prod_{i\in I} R_i$ can be equipped with a generalized homogeneous derivation $(F, d)_h$ where $F((r_i)_{i\in I}) = (F_i(r_i))_{i\in I}$ and $d((r_i)_{i\in I}) = (d_i(r_i))_{i\in I}$, making (R, F, d) an object in $\mathscr{G}_G^h$. To verify the universal property of products, let (T, F_T, d_T) be an arbitrary object in $\mathscr{G}_G^h$ and let $\{\phi_i : T \to R_i\}_{i\in I}$ be a family of morphisms in $\mathscr{G}_G^h$. We must demonstrate the existence and uniqueness of a morphism $\phi : T \to R$ such that $\pi_i \circ \phi = \phi_i$ for each $i \in I$, where $\pi_i : R \to R_i$ denotes the canonical projection. Define $\phi : T \to R$ by $\phi(t) = (\phi_i(t))_{i\in I}$ for all $t \in T$. By construction, $\pi_i \circ \phi = \phi_i$ for each $i \in I$. To verify that ϕ is a morphism in $\mathscr{G}_G^h$, we confirm its compatibility with the generalized homogeneous derivations: For any $t \in T$

$$\begin{aligned} \phi(F_T(t)) &= (\phi_i(F_T(t)))_{i\in I} \\ &= (F_i(\phi_i(t)))_{i\in I} \\ &= F((\phi_i(t))_{i\in I}) \\ &= F(\phi(t)). \end{aligned}$$

Similarly, for the derivation component

$$\begin{aligned} \phi(d_T(t)) &= (\phi_i(d_T(t)))_{i\in I} \\ &= (d_i(\phi_i(t)))_{i\in I} \\ &= d((\phi_i(t))_{i\in I}) \\ &= d(\phi(t)). \end{aligned}$$

The uniqueness of ϕ follows directly from the universal property of the categorical product. Indeed, if $\psi : T \to R$ is another morphism in $\mathscr{G}_G^h$ such that $\pi_i \circ \psi = \phi_i$ for each $i \in I$, then for any $t \in T$

$$\psi(t) = (\pi_i(\psi(t)))_{i\in I} = (\phi_i(t))_{i\in I} = \phi(t).$$

Hence, $\psi = \phi$. □

3. gr-Generalized derivations

The set $\mathfrak{Der}_G^{gh}(R)$ does not have a natural algebraic structure, since the sum of two generalized homogeneous derivations may fail to preserve homogeneity (Remark 2.1). This issue comes from the tension between the generalized derivation rule and the requirements of graded homogeneity. To overcome this, we restrict our attention to generalized homogeneous derivations that preserve the degrees of homogeneous elements.

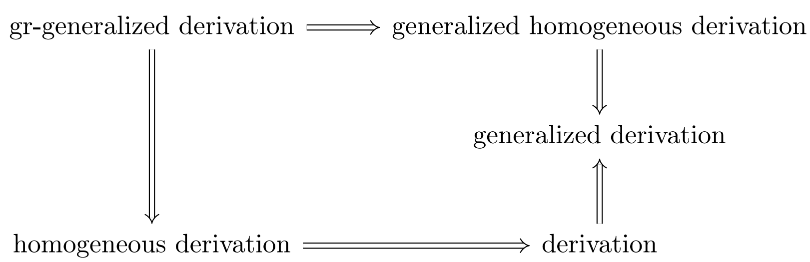
Definition 3.1. *Let R be a ring graded by an arbitrary group G. A generalized homogeneous derivation $(F, d)_h$ is called a* gr-generalized derivation *if*

$$F(R_\tau) \subseteq R_\tau \quad \text{and} \quad d(R_\tau) \subseteq R_\tau \quad \text{for all } \tau \in G.$$

The set of all such derivations is denoted by $p\mathfrak{Der}_G^{gh}(R)$.

Example 3.1. *Let $R = \mathbb{C}[t_1, t_2]$ be the polynomial ring with the standard $\mathbb{Z}$-grading. Define $F(f) = d(f) = t_1 \frac{\partial f}{\partial t_1} + t_2 \frac{\partial f}{\partial t_2}$ for all $f \in R$. Then $(F, d)_h$ is a gr-generalized derivation.*

The hierarchy of notions considered in this paper is as follows:



Remark 3.1. *Any gr-generalized derivation of R restricts to a generalized derivation on the identity component R_e.*

Example 3.2. *For $R = M_2(k)$ with $\mathbb{Z}_2$-grading where $R_0 = \{\mathrm{diag}(a, d) \mid a, d \in k\}$ and $R_1 = \{$anti-diag$(b, c) \mid b, c \in k\}$, the map $F = \lambda\cdot \mathrm{id}_R$ with $d = 0$ extends any scalar multiplication on $R_e = k\cdot I_2$ to a gr-generalized derivation.*

Conjecture 1. *Every gr-generalized derivation $F_e : R_e \to R_e$ extends to a gr-generalized derivation on R.*

Proposition 3.1. *$p\mathfrak{Der}_G^{gh}(R)$ forms a $Z(R) \cap R_e$-module under pointwise addition and scalar multiplication $(r \cdot (F, d)_h) = (rF, rd)_h$.*

Proof. The set $p\mathfrak{Der}_G^{gh}(R)$ is an additive group under pointwise addition. For the $Z(R) \cap R_e$-module structure, define scalar multiplication by $r\cdot(F, d)_h = (rF, rd)_h$ for $r \in Z(R)\cap R_e$ and $(F, d)_h \in p\mathfrak{Der}_G^{gh}(R)$. It is clear that $(rF, rd)_h \in p\mathfrak{Der}_G^{gh}(R)$, since the gr-generalized derivation property follows from the centrality of r. For degree preservation, if $x \in R_\tau$, then $F(x), d(x) \in R_\tau$. As $r \in R_e$, we have $(rF)(x) = r(F(x)) \in R_e R_\tau = R_\tau$, and similarly $(rd)(x) \in R_\tau$. □

Proposition 3.2. *$p\mathfrak{Der}_G^{gh}(R)$ admits a Lie algebra structure over $Z(R) \cap R_e$ with bracket*

$$[(F_1, d_1)_h, (F_2, d_2)_h] = (F_1 \circ F_2 - F_2 \circ F_1,\ d_1 \circ d_2 - d_2 \circ d_1)_h.$$

Proof. Well-definedness and the Lie algebra axioms follow from standard commutator properties together with degree preservation for both F_i and d_i. □

Theorem 3.1. *For gr-prime rings R, there exists a canonical decomposition*

$$p\mathfrak{Der}_G^{gh}(R) = p\mathfrak{Der}_G^{h}(R) \oplus \mathcal{C}_G(R),$$

where $\mathcal{C}_G(R) = \{F \in p\mathfrak{Der}_G^{gh}(R) \mid F$ *has zero associated derivation*$\}$.

Proof. For any $(F,d)_h \in p\mathfrak{Der}_G^{gh}(R)$, write $F = d + (F - d)$ and set $F_1 = d$ and $F_2 = F - d$. Clearly, $F_1 \in p\mathfrak{Der}_G^{h}(R)$ by definition. A straightforward calculation shows that $F_2 \in \mathcal{C}_G(R)$, since $F_2(xy) = F_2(x)y$ for all $x, y \in R$, and F_2 inherits degree preservation from F and d. To see that this decomposition is direct, it suffices to show that $p\mathfrak{Der}_G^{h}(R) \cap \mathcal{C}_G(R) = \{0\}$.

Let $H \in \mathfrak{Der}_G^{h}(R) \cap \mathcal{C}_G(R)$. The derivation property gives $H(xy) = H(x)y + xH(y)$, while the condition $H \in \mathcal{C}_G(R)$ implies $H(xy) = H(x)y$. Hence $xH(y) = 0$ for all $x, y \in R$. Thus $xRH(y) = \{0\}$ for all $x, y \in R$. For any $r \in \mathcal{H}(R) \setminus \{0\}$, we obtain $rRH(y) = \{0\}$. By [1, Proposition 2.1], it follows that $H(y) = 0$ for all $y \in R$. Therefore $H \equiv 0$. □

Proposition 3.3. *For a gr-domain R with $R[t]$ graded by $\deg(t) = e$, there exists a natural injection*

$$p\mathfrak{Der}_G^{gh}(R) \hookrightarrow p\mathfrak{Der}_G^{gh}(R[t])$$

given by $(F,d)_h \mapsto (F', d')_h$, *where* $F'(\sum r_i t^i) = \sum F(r_i)t^i$ *and* $d'(\sum r_i t^i) = \sum d(r_i)t^i$.

Proof. Let $(F,d)_h \in p\mathfrak{Der}_G^{gh}(R)$ and define $F', d' : R[t] \to R[t]$ by

$$F'\left(\sum_{i=0}^{n} r_i t^i\right) = \sum_{i=0}^{n} F(r_i)t^i, \qquad d'\left(\sum_{i=0}^{n} r_i t^i\right) = \sum_{i=0}^{n} d(r_i)t^i.$$

That $(F', d')_h \in p\mathfrak{Der}_G^{gh}(R[t])$ follows from direct computations. For homogeneity, observe that if $f(t) = \sum_{i=0}^{n} r_i t^i \in R[t]_\tau$, then $r_i \in R_\tau$, so $F(r_i), d(r_i) \in R_\tau$ by the homogeneity of $(F,d)_h$. Thus $F'(f), d'(f) \in R[t]_\tau$. For injectivity, if $(F,d)_h \neq (0,0)_h$, then there exists $r \in R$ such that either $F(r) \neq 0$ or $d(r) \neq 0$, which implies $F'(r) \neq 0$ or $d'(r) \neq 0$ when r is viewed as a constant polynomial. □

The injection in Proposition 3.3 is in general not surjective. For instance, take $R = \mathbb{C}$ with the trivial grading and endow $R[t]$ with the standard $\mathbb{Z}$-grading. Define a derivation $d : R[t] \to R[t]$ by

$$d\Big(\sum_i a_i t^i\Big) = \sum_i i\, a_i t^i,$$

so that $d(t) = t \neq 0$ and d is homogeneous of degree 0. Then $(d,d)_h \in p\mathfrak{Der}_{\mathbb{Z}}^{gh}(R[t])$, but $(d,d)_h$ cannot belong to the image of the above injection, since any element in the image satisfies $d'(t) = 0$.

Proposition 3.4. *For G-graded k-algebras R, S with $(F_R, d_R)_h \in p\mathfrak{Der}_G^{gh}(R)$ and $(F_S, d_S)_h \in p\mathfrak{Der}_G^{gh}(S)$, define*

$$F_{R\otimes S}(r \otimes s) = F_R(r) \otimes s + r \otimes F_S(s), \quad d_{R\otimes S}(r \otimes s) = d_R(r) \otimes s + r \otimes d_S(s).$$

Then $(F_{R\otimes S}, d_{R\otimes S})_h \in p\mathfrak{Der}_G^{gh}(R \otimes_k S)$.

Proof. We extend the definition of $F_{R\otimes S}$ and $d_{R\otimes S}$ to all of $R \otimes_k S$ by linearity. To verify that $(F_{R\otimes S}, d_{R\otimes S})_h$ is a generalized homogeneous derivation, we must verify that

$$F_{R\otimes S}(uv) = F_{R\otimes S}(u)v + ud_{R\otimes S}(v), \quad d_{R\otimes S}(uv) = d_{R\otimes S}(r_1r_2 \otimes s_1s_2)$$

for all $u, v \in R \otimes_K S$. By linearity, it suffices to check this identity for homogeneous tensors $u = r_1 \otimes s_1$ and $v = r_2 \otimes s_2$. We have

$$\begin{aligned}
F_{R\otimes S}((r_1 \otimes s_1)(r_2 \otimes s_2)) &= F_{R\otimes S}(r_1r_2 \otimes s_1s_2) \\
&= F_R(r_1r_2) \otimes s_1s_2 + r_1r_2 \otimes F_S(s_1s_2) \\
&= (F_R(r_1)r_2 + r_1d_R(r_2)) \otimes s_1s_2 + r_1r_2 \otimes (F_S(s_1)s_2 + s_1d_S(s_2)) \\
&= F_R(r_1)r_2 \otimes s_1s_2 + r_1d_R(r_2) \otimes s_1s_2 + r_1r_2 \otimes F_S(s_1)s_2 + r_1r_2 \otimes s_1d_S(s_2) \\
&= [F_R(r_1) \otimes s_1) + r_1 \otimes F_S(s_1)](r_2 \otimes s_2) + (r_1 \otimes s_1)[d_R(r_2) \otimes s_2 + r_2 \otimes d_S(s_2)] \\
&= F_{R\otimes S}(r_1 \otimes s_1)(r_2 \otimes s_2) + (r_1 \otimes s_1)d_{R\otimes S}(r_2 \otimes s_2)
\end{aligned}$$

For the homogeneity condition, observe that if $r \in R_\tau$ and $s \in S_\sigma$, then $r \otimes s \in (R \otimes_K S)_{\tau\sigma}$. Since $F_R(r) \in R_\tau$ and $F_S(s) \in S_\sigma$, we have $F_{R\otimes S}(r \otimes s) = F_R(r) \otimes s + r \otimes F_S(s) \in (R \otimes_K S)_{\tau\sigma}$ Additionally, we must verify that $d_{R\otimes S}$ satisfies the Leibniz rule for a homogeneous derivation. For homogeneous tensors $u = r_1 \otimes s_1$ and $v = r_2 \otimes s_2$

$$\begin{aligned} d_{R\otimes S}(uv) &= d_{R\otimes S}(r_1 r_2 \otimes s_1 s_2) \\ &= (d_R(r_1)r_2 + r_1 d_R(r_2)) \otimes s_1 s_2 + r_1 r_2 \otimes (d_S(s_1)s_2 + s_1 d_S(s_2)) \\ &= d_R(r_1)r_2 \otimes s_1 s_2 + r_1 d_R(r_2) \otimes s_1 s_2 + r_1 r_2 \otimes d_S(s_1)s_2 + r_1 r_2 \otimes s_1 d_S(s_2) \\ &= d_R(r_1) \otimes s_1(r_2 \otimes s_2) + r_1 \otimes d_S(s_1)(r_2 \otimes s_2) \\ &= d_{R\otimes S}(r_1 \otimes s_1)(r_2 \otimes s_2) + (r_1 \otimes s_1) d_{R\otimes S}(r_2 \otimes s_2) \end{aligned}$$

Moreover, $d_{R\otimes S}(r \otimes s) \in (R \otimes_K S)_{\tau\sigma}$. Hence, $(F_{R\otimes S}, d_{R\otimes S})_h \in p\mathfrak{Der}^{gh}_G(R \otimes_K S)$. □

4. **Some commutativity criteria on gr-prime rings**

4.1. *Results on homogeneous derivations.*

Proposition 4.1. *Let R be a gr-prime ring and $\mathfrak{I}$ a nonzero graded ideal of R such that*

$$[x, y] \in Z(R) \quad or \quad x \circ y \in Z(R)$$

for all $x, y \in \mathfrak{I}$. Then R is a commutative graded ring.

To prove this proposition we need the following lemma.

Lemma 4.1. *Let R be a gr-prime ring. Then the following assertions hold.*

(1) *If $\mathfrak{I}$ is a nonzero graded ideal of R and $a\mathfrak{I}b = \{0\}$ where $a \in \mathcal{H}(R)$ or $b \in \mathcal{H}(R)$, then $a = 0$ or $b = 0$.*
(2) *If d is a homogeneous derivation of R and $ad(x) = 0$ or $d(x)a = 0$ for all $x \in R$, then either $a = 0$ or $d = 0$.*

Proof. (1) Let $a = \sum_{g\in G} a_\tau \in R$ and $b \in \mathcal{H}(R) \setminus \{0\}$ such that $a\mathfrak{I}b = \{0\}$. Then, for all $r \in \mathfrak{I} \cap \mathcal{H}(R)$ we have $arb = 0$, which implies $\sum_{\tau\in G} a_\tau rb = 0$. Since every element of R has a unique homogeneous decomposition, it follows that $a_\tau rb = 0$ for all $r \in \mathfrak{I} \cap \mathcal{H}(R)$ and all $\tau \in G$. Hence $a_\tau \mathfrak{I}Rb = \{0\}$ for all $\tau \in G$. By [1, Proposition 2.1], we obtain $a_\tau \mathfrak{I} = \{0\}$ for all $\tau \in G$, i.e. $a_\tau R\mathfrak{I} = \{0\}$ for all $\tau \in G$. Thus $a_\tau = 0$ for all $\tau \in G$, and therefore $a = 0$.
(2) Suppose that $ad(x) = 0$ and $a \neq 0$. Replacing x by xy, we obtain $axd(y) = 0$ for all $x, y \in R$. In particular, $aRd(x) = \{0\}$ for all $x \in R$. Hence $aRd(r) = \{0\}$ for all $r \in \mathcal{H}(R)$. By [1, Proposition 2.1], we conclude that $d(r) = 0$ for all $r \in \mathcal{H}(R)$, and thus $d = 0$.

□

Proof of Proposition 4.1. Assume first that $[x, y] \in Z(R)$ for all $x, y \in \mathfrak{I}$. Then

$$[z, [x, y]] = 0 \tag{1}$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. Replacing y by yx in (1) and simplifying, we obtain

$$[x, y][z, x] = 0 \tag{2}$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. Substituting zy for z in (2) yields

$$[x, y]z[x, y] = 0$$

for all $x, y \in \mathfrak{I}$ and $z \in R$, which implies

$$[x, y]R[x, y] = \{0\}$$

for all $x, y \in \mathfrak{I}$. Since $\mathfrak{I}$ is a graded ideal, we also have

$$[r_1, r_2]R[r_1, r_2] = \{0\}$$

for all $r_1, r_2 \in \mathfrak{I} \cap \mathcal{H}(R)$. By gr-primeness of R, it follows that $[r_1, r_2] = 0$ for all $r_1, r_2 \in \mathfrak{I} \cap \mathcal{H}(R)$. Hence $[x, y] = 0$ for all $x, y \in \mathfrak{I}$, so $\mathfrak{I}$ is commutative. In view of [1, Proposition 2.1], R is then commutative.

Now assume that $x \circ y \in Z(R)$ for all $x, y \in \mathfrak{I}$. Then

$$[x \circ y, z] = 0 \tag{3}$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. Replacing y by yx in (3) and simplifying, we obtain

$$(x \circ y)[x, z] = 0 \tag{4}$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. Substituting sz for z in (4), we get

$$(x \circ y)s[x, z] = 0$$

for all $x, y \in \mathfrak{I}$ and $s, z \in R$. Hence

$$(x \circ y)R[x, z] = \{0\}$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. In particular,

$$(r_1 \circ r_2)R[r_1, z] = \{0\}$$

for all $r_1, r_2 \in \mathfrak{I} \cap \mathcal{H}(R)$ and $z \in R$. According to [1, Proposition 2.1], it follows that

$$r_1 \circ r_2 = 0 \quad \text{or} \quad [r_1, z] = 0$$

for all $r_1, r_2 \in \mathfrak{I} \cap \mathcal{H}(R)$ and $z \in R$. Thus

$$x \circ y = 0 \quad \text{or} \quad [x, z] = 0$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. In the latter case, $\mathfrak{I}$ is a central graded ideal, and applying [1, Proposition 2.1], we conclude that R is commutative.

We may therefore assume that $x \circ y = 0$ for all $x, y \in \mathfrak{I}$. Replacing y by yz gives

$$y[x, z] = 0$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. Since $\mathfrak{I}$ is a nonzero ideal of R, there exists $a \in \mathfrak{I} \setminus \{0\}$ such that

$$a[x, z] = 0$$

for all $x \in \mathfrak{I}$ and $z \in R$. Fix $r \in \mathfrak{I} \cap \mathcal{H}(R)$ and let d_r be the inner homogeneous derivation associated with r, i.e. $d_r(z) = [r, z]$ for $z \in R$. Then

$$ad_r(z) = 0$$

for all $z \in R$. By Lemma 4.1(2), we obtain $d_r(z) = [r, z] = 0$ for all $z \in R$. Hence $[x, z] = 0$ for all $x \in \mathfrak{I}$ and $z \in R$. In both cases, we find that $\mathfrak{I}$ is a central graded ideal of R. Therefore, R is commutative. □

The next result characterizes when compositions of homogeneous derivations force commutativity.

Theorem 4.1. *Let R be a gr-prime ring of characteristic different from 2. Suppose d_1 and d_2 are nonzero homogeneous derivations of R such that*

$$d_1 d_2(x) \in Z(R)$$

for all $x \in R$. Then R is a commutative graded ring.

Proof. By hypothesis,

$$d_1 d_2(x) \in Z(R) \tag{5}$$

for all $x \in R$. Replacing x by $[x, y]$ in (5) and expanding, we obtain

$$[d_2(x), d_1(y)] + [d_1(x), d_2(y)] \in Z(R) \tag{6}$$

for all $x, y \in R$. Putting $y = d_2(z)$ in (6) yields

$$[d_1(x), d_2^2(z)] \in Z(R)$$

for all $x, z \in R$. In particular,

$$[d_2^2(r), d_1(y)] \in Z(R)$$

for all $r \in \mathcal{H}(R)$ and $y \in R$. By [1, Lemma 2.2], it follows that either $d_2^2(r) \in Z(R)$ for all $r \in \mathcal{H}(R)$ or $d_1 = 0$. The latter is impossible by assumption, so $d_2^2(x) \in Z(R)$ for all $x \in R$. Taking $[x, z]$ instead of x, we obtain

$$2[d_2(x), d_2(z)] \in Z(R)$$

for all $x, z \in R$. Since $\operatorname{char} R \neq 2$, it follows that

$$[d_2(x), d_2(z)] \in Z(R)$$

for all $x, z \in R$. By [1, Theorem 3.5], we conclude that R is commutative. □

The following example proves that the gr-primeness hypothesis in Theorem 4.1 is not superfluous. In particular, our theorem cannot be extended to gr-semiprime rings.

Example 4.1. *Consider the ring $R = \mathbb{C}[t_1, t_2, t_3, t_4] \times M_2(\mathbb{C})$ with $\mathbb{Z} \times \mathbb{Z}_4$-grading. R is gr-semiprime. We define homogeneous derivations $d_1, d_2 : R \to R$ by*

$$d_1(f, M) = \left(t_2 t_4 \frac{\partial f}{\partial t_1}, 0\right), \qquad d_2(f, M) = \left(t_1 t_3 \frac{\partial f}{\partial t_2}, 0\right).$$

We have $d_1 d_2(x) \in Z(R)$ for all $x \in R$, thus satisfying the condition of Theorem 4.1. Nevertheless, R is noncommutative.

4.2. *Results on generalized homogeneous derivations.* In this subsection, we extend classical commutativity theorems from prime ring theory to the graded setting, giving necessary and sufficient conditions under which generalized homogeneous derivations force gr-prime rings to be commutative.

Proposition 4.2. *Let R be a gr-prime ring and $(F, d)_h$ a generalized homogeneous derivation of R. If $d \neq 0$, then $F \neq 0$.*

Proof. Assume $F = 0$. For any elements $x, y \in R$, we have $F(xy) = 0$. Since

$$F(xy) = F(x)y + xd(y),$$

it follows that $xd(y) = 0$ for all $x, y \in R$. Hence $xRd(y) = \{0\}$ for all $x, y \in R$. In particular, for some nonzero homogeneous element $r \in \mathcal{H}(R) \setminus \{0\}$, we have $rRd(y) = \{0\}$ for all $y \in R$. According to [1, Proposition 2.1], this implies $d(y) = 0$ for all $y \in R$. Thus $d = 0$, which contradicts the assumption. □

In [2], it was shown that a prime ring R with a nonzero ideal $\mathfrak{I}$ is commutative if it admits a generalized derivation F satisfying

$$F(xy) \pm xy \in Z(R) \quad \text{or} \quad F(x)F(y) \pm xy \in Z(R)$$

for all $x, y \in \mathfrak{I}$. We now extend this result to gr-prime rings in the context of generalized homogeneous derivations.

Theorem 4.2. *Let R be a gr-prime ring and $\mathfrak{I}$ a nonzero graded ideal of R. If R admits a generalized homogeneous derivation F with associated nonzero homogeneous derivation d such that*

$$F(xy) \pm xy \in Z(R)$$

for all $x, y \in \mathfrak{I}$, then R is commutative.

Proof. Consider the case

$$F(xy) - xy \in Z(R)$$

for all $x, y \in \mathfrak{I}$. Using the same reasoning as in the proof of [2, Theorem 2.1], we obtain the identity

$$[z, z_1]xyd(z) = 0$$

for all $x, y, z, z_1 \in \mathfrak{I}$, which yields $[z, z_1]xR\mathfrak{I}d(z) = \{0\}$ for all $x, z, z_1 \in \mathfrak{I}$. Since $\mathfrak{I}$ is a graded ideal of R, we have

$$[r', z_1]xRrd(r') = \{0\}$$

for all $x, z_1 \in \mathfrak{I}$ and $r, r' \in \mathfrak{I} \cap \mathcal{H}(R)$. According to [1, Proposition 2.1], either $[z_1, r']x = 0$ or $rd(r') = 0$ for all $x, z_1 \in \mathfrak{I}$ and $r, r' \in \mathfrak{I} \cap \mathcal{H}(R)$. This implies that

$$[z, z_1]\mathfrak{I} = \{0\} \quad \text{or} \quad \mathfrak{I}d(z) = \{0\}$$

for all $z, z_1 \in \mathfrak{I}$. Define

$$\mathfrak{I}_1 = \{z \in \mathfrak{I} \mid [z, z_1]\mathfrak{I} = \{0\} \text{ for all } z_1 \in \mathfrak{I}\}, \quad \mathfrak{I}_2 = \{z \in \mathfrak{I} \mid \mathfrak{I}d(z) = \{0\}\}.$$

Then $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are additive subgroups of $\mathfrak{I}$ with $\mathfrak{I} = \mathfrak{I}_1 \cup \mathfrak{I}_2$. Since a group cannot be expressed as the union of two proper subgroups, either $\mathfrak{I}_1 = \mathfrak{I}$ or $\mathfrak{I}_2 = \mathfrak{I}$. We consider these cases separately.

Case 1: If $[z_1, z]\mathfrak{I} = \{0\}$ for all $z, z_1 \in \mathfrak{I}$, then, since $\mathfrak{I}$ is an ideal, we obtain $[z_1, z]R\mathfrak{I} = \{0\}$ for all $z, z_1 \in \mathfrak{I}$. As $\mathfrak{I}$ is a nonzero graded ideal, there exists $r \in \mathfrak{I} \cap \mathcal{H}(R) \setminus \{0\}$ such that

$$[z, z_1]Rr = \{0\}$$

for all $z, z_1 \in \mathfrak{I}$. By [1, Proposition 2.1], it follows that $[z, z_1] = 0$ for all $z, z_1 \in \mathfrak{I}$. Hence $\mathfrak{I}$ is commutative. Therefore, R is commutative.
Case 2: If $\mathfrak{I}d(z) = \{0\}$ for all $z \in \mathfrak{I}$, then $\mathfrak{I}Rd(z) = \{0\}$ for all $z \in \mathfrak{I}$. In particular, $rRd(z) = \{0\}$ for some $r \in \mathfrak{I} \cap \mathcal{H}(R) \setminus \{0\}$ and all $z \in \mathfrak{I}$. Using [1, Proposition 2.1], we obtain $d(z) = 0$ for all $z \in \mathfrak{I}$. Hence d vanishes on $\mathfrak{I}$. By [1, Lemma 2.6], d is zero on R, which is a contradiction.

For the second case $F(xy) + xy \in Z(R)$ for all $x, y \in \mathfrak{I}$, the argument reduces to the first case by considering $-F$ instead of F. □

Next, we extend [2, Theorem 2.5] to gr-prime rings by considering a pair of generalized homogeneous derivations F_1 and F_2 satisfying

$$F_1(x)F_2(y) \pm xy \in Z(R)$$

for all x, y in a graded ideal $\mathfrak{I}$ of R.

Theorem 4.3. *Let R be a gr-prime ring and $\mathfrak{I}$ a nonzero graded ideal of R. If R admits two generalized homogeneous derivations F_1 and F_2 with associated nonzero homogeneous derivations d_1 and d_2, respectively, such that*

$$F_1(x)F_2(y) \pm xy \in Z(R)$$

for all $x, y \in \mathfrak{I}$, then R is commutative.

Proof. Consider the case

$$F_1(x)F_2(y) - xy \in Z(R) \tag{7}$$

for all $x, y \in \mathfrak{I}$. Substituting yz for y in (7), we obtain

$$\big(F_1(x)F_2(y) - xy\big)z + F_1(x)yd_2(z) \in Z(R) \tag{8}$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. Taking the commutator of (8) with z, we obtain

$$F_1(x)[yd_2(z), z] + [F_1(x), z]yd_2(z) = 0 \tag{9}$$

for all $x, y \in \mathfrak{I}$ et $z \in R$. Substituting $F_1(x)y$ for y in (9), we arrive at

$$[F_1(x), z]F_1(x)yd_2(z) = 0 \tag{10}$$

for all $x, y \in \mathfrak{I}$ and $z \in R$. This implies $[F_1(x), z]F_1(x)R\mathfrak{I}d_2(z) = \{0\}$ for all $x \in \mathfrak{I}$ et $z \in R$. In particular,

$$[F_1(r), r']F_1(r)R\mathfrak{I}d_2(r') = \{0\}$$

for all $r \in \mathfrak{I} \cap \mathcal{H}(R)$ and $r' \in \mathcal{H}(R)$. According to [1, Proposition 2.1], either

$$[F_1(r), r']F_1(r) = 0 \quad \text{or} \quad \mathfrak{I}d_2(r') = \{0\}$$

for all $r \in \mathfrak{I} \cap \mathcal{H}(R)$ and $r' \in \mathcal{H}(R)$. Thus, $[F_1(x), z]F_1(x) = 0$ or $\mathfrak{I}d_2(z) = \{0\}$ for all $x \in \mathfrak{I}$ and $z \in R$. Let

$$\mathfrak{J}_1 = \{z \in R \mid [F_1(x), z]F_1(x) = 0 \text{ for all } x \in \mathfrak{I}\}, \quad \mathfrak{J}_2 = \{z \in R \mid \mathfrak{I}d_2(z) = \{0\}\}.$$

Clearly, $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are additive subgroups of R whose union is R. Since a group cannot be the union of two proper subgroups, either $\mathfrak{J}_1 = R$ or $\mathfrak{J}_2 = R$.

If $\mathfrak{J}_2 = R$, then $\mathfrak{I}d_2(z) = \{0\}$ for all $z \in R$. Since $\mathfrak{I}$ is an ideal, $\mathfrak{I}Rd_2(z) = \{0\}$ for all $z \in R$. In particular, $rRd_2(z) = \{0\}$ for all $z \in R$ and some $r \in \mathfrak{I} \cap \mathcal{H}(R) \setminus \{0\}$. According to [1, Proposition 2.1], we conclude that $d_2(z) = 0$ for all $z \in R$. Hence $d_2 = 0$, which contradicts our assumption. Therefore, $[F_1(x), z]F_1(x) = 0$ for all $x \in \mathfrak{I}$ and $z \in R$. Replacing z by zz', we obtain $[F_1(x), z]z'F_1(x) = 0$ for all $x \in \mathfrak{I}$ and $z, z' \in R$, which implies $[F_1(x), z]RF_1(x) = \{0\}$ pour tous $x \in \mathfrak{I}$ et $z \in R$. In particular,

$$[F_1(r), z]RF_1(r) = \{0\}$$

for all $r \in \mathfrak{I} \cap \mathcal{H}(R)$ and $z \in R$. Invoking [1, Proposition 2.1], we conclude that either $F_1(r) = 0$ or $[F_1(r), z] = 0$ for all $z \in R$. Hence $F_1(x) = 0$ or $[F_1(x), z] = 0$ for all $x \in \mathfrak{I}$ and $z \in R$. In both cases, we obtain $[F_1(x), z] = 0$ for all $x \in \mathfrak{I}$ and $z \in R$. Replacing x by xz, we obtain

$$x[d_1(z), z] + [x, z]d_1(z) = 0 \tag{11}$$

for all $x \in \mathfrak{I}$ and $z \in R$. Substituting sx for x in (11), we arrive at $[s, z]xd_1(z) = 0$ for all $x \in \mathfrak{I}$ and $s, z \in R$, which implies $[s, z]R\mathfrak{I}d_1(z) = \{0\}$ for all $s, z \in R$. Using similar arguments as above, either $[s, z] = 0$ or $d_1(z) = 0$ for all $s, z \in R$. Since $d_1 \neq 0$, we must have $[s, z] = 0$ for all $s, z \in R$. Therefore, R is commutative.

For the second case $F_1(x)F_2(y) + xy \in Z(R)$ for all $x, y \in \mathfrak{I}$, the argument reduces to the first case by considering $-F_1$ instead of F_1. □

The following example shows that the gr-primeness hypothesis cannot be omitted from the above theorems.

Example 4.2. *Let* $R = \mathbb{C}[t_1, t_2, t_3] \times \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}$ *with* $\mathbb{Z} \times \mathbb{Z}_2$*-grading. Then* R *is not gr-prime. Let* $\mathfrak{I} = \mathbb{C}[t_1, t_2, t_3] \times \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{C} \right\}$*. Clearly,* $\mathfrak{I}$ *is a nonzero graded ideal of* R*. Consider the mappings:*

$$F_1 : \begin{array}{ccc} R & \to & R \\ (f, M) & \mapsto & \big(t_3\big(f + \frac{\partial f}{\partial t_3}\big), 0\big) \end{array}, \qquad F_2 = d_2 : \begin{array}{ccc} R & \to & R \\ (f, M) & \mapsto & \big(t_1 \frac{\partial f}{\partial t_2}, 0\big) \end{array},$$

and

$$d_1 : \begin{array}{ccc} R & \to & R \\ (f, M) & \longmapsto & \big(t_2 t_3 \frac{\partial f}{\partial t_3}, 0\big). \end{array}$$

Then $(F_1, d_1)_h$ *and* $(F_2, d_2)_h$ *are generalized homogeneous derivations on* R*. Moreover,* $F_1(xy) \pm xy \in Z(R)$ *and* $F_1(x)F_2(y) \pm xy \in Z(R)$ *for all* $x, y \in \mathfrak{I}$*. However,* R *is noncommutative.*

5. Existence Conditions for Central Graded Ideals in Gr-Semiprime Rings

In this section, we investigate the behavior of graded ideals under generalized homogeneous derivations, and we characterize when such rings necessarily contain nonzero central graded ideals.

In [3], it was shown that if a ring R admits generalized derivations F_1 and F_2 with associated nonzero derivations d_1 and d_2, respectively, such that

$$F_1(x)x \pm xF_2(x) = 0$$

for all $x \in \mathfrak{I}$, where $\mathfrak{I}$ is a nonzero ideal of R, then R contains a nonzero central ideal. We extend this result to the graded case by studying generalized homogeneous derivations F_1 and F_2 satisfying

$$F_1(x)y \pm xF_2(y) \in Z(R)$$

for all $x, y \in \mathfrak{I}$, where $\mathfrak{I}$ is a graded ideal of a gr-semiprime ring R.

Theorem 5.1. *Let* R *be a gr-semiprime ring and* $\mathfrak{I}$ *a nonzero graded ideal of* R*. Suppose that* R *admits generalized homogeneous derivations* F_1 *and* F_2 *with associated homogeneous derivations* d_1 *and* d_2*, respectively, with* $d_2(\mathfrak{I}) \neq \{0\}$*. If*

$$F_1(x)y \pm xF_2(y) \in Z(R)$$

for all $x, y \in \mathfrak{I}$*, then* R *contains a nonzero central graded ideal.*

Proof. We begin with the case

$$F_1(x)y - xF_2(y) \in Z(R) \tag{12}$$

for all $x, y \in \mathfrak{I}$. Substituting yz for y in (12), we obtain

$$\big(F_1(x)y - xF_2(y)\big)z - xyd_2(z) \in Z(R) \tag{13}$$

for all $x, y, z \in \mathfrak{I}$. Taking the commutator of (13) with z yields

$$xy[d_2(z), z] + x[y, z]d_2(z) + [x, z]yd_2(z) = 0 \tag{14}$$

for all $x, y, z \in \mathfrak{I}$. Substituting $d_2(z)x$ for x in (14), we obtain

$$[d_2(z), z]xyd_2(z) = 0 \tag{15}$$

for all $x, y, z \in \mathfrak{I}$. Substituting $d_2(z)y$ for y in (15), we obtain

$$[d_2(z), z]xd_2(z)yd_2(z) = 0 \tag{16}$$

for all $x, y, z \in \mathfrak{I}$. By subtracting (16) from (15) and using the previous identities, we get

$$[d_2(z), z]x[d_2(z), z]y[d_2(z), z] = 0$$

for all $x, y, z \in \mathfrak{I}$. This implies

$$[d_2(z), z]\mathfrak{I}[d_2(z), z]\mathfrak{I}[d_2(z), z] = \{0\}$$

for all $z \in \mathfrak{I}$. Since R is gr-semiprime, there exists a family $\mathcal{F} := \{P_i \mid i \in \Lambda\}$ of gr-prime ideals such that $\bigcap_{i\in\Lambda} P_i = \{0\}$. Therefore,

$$[d_2(z), z]\mathfrak{I}[d_2(z), z]\mathfrak{I}[d_2(z), z] \subseteq P_i$$

for all $i \in \Lambda$ and all $z \in \mathfrak{I}$. By [1, Proposition 2.1], and since $\mathfrak{I}$ is a graded ideal, we have $[d_2(z), z] \in P_i$ for all $i \in \Lambda$ and $z \in \mathfrak{I}$. Hence $[d_2(z), z] = 0$ for all $z \in \mathfrak{I}$. Thus, by [1, Theorem 4.1], R contains a nonzero central graded ideal.

For the second case

$$F_1(x)y + xF_2(y) \in Z(R)$$

for all $x, y \in \mathfrak{I}$, the argument reduces to the first case by considering $-F_2$ instead of F_2. □

From Theorem 5.1 and [1, Proposition 2.1], we obtain the following corollary.

Corollary 5.1. *Let R be a gr-prime ring and $\mathfrak{I}$ a nonzero graded ideal of R. Suppose that R admits generalized homogeneous derivations F_1 and F_2 with associated nonzero homogeneous derivations d_1 and d_2, respectively, satisfying*

$$F_1(x)y \pm xF_2(y) \in Z(R)$$

for all $x, y \in \mathfrak{I}$. Then R is commutative.

Using similar arguments with appropriate modifications, and considering the cases $F_1 = F_2$ or $F_1 = -F_2$ in Theorem 5.1, we obtain the following result. This extends the graded version of Posner's Second Theorem [1, Theorem 3.3] to generalized homogeneous derivations on gr-prime rings, providing a characterization of commutativity.

Corollary 5.2. *Let R be a gr-prime ring and $\mathfrak{I}$ a nonzero graded ideal of R. Suppose that R admits a generalized homogeneous derivation F with associated homogeneous derivation d such that*

$$[F(x), x] \in Z(R)$$

for all $x \in \mathfrak{I}$. Then R is commutative.

Corollary 5.3. *Let R be a gr-prime ring and $\mathfrak{I}$ a nonzero graded ideal of R. Suppose that R admits a generalized homogeneous derivation F with associated nonzero homogeneous derivation d such that*

$$F(x) \circ x \in Z(R)$$

for all $x \in \mathfrak{I}$. Then R is commutative.

The following example shows that the gr-semiprimeness assumption in Theorem 5.1 cannot be omitted.

Example 5.1. *Let*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

be a $\mathbb{Z}_2$-graded ring. Clearly, R is not gr-semiprime. Define generalized homogeneous derivations $(F_1, d_1)_h$ and $(F_2, d_2)_h$ on R by

$$F_1 : \begin{array}{ccc} R & \longrightarrow & R \\ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad F_2 : \begin{array}{ccc} R & \longrightarrow & R \\ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & a+2b \\ 0 & 0 \end{pmatrix} \end{array}$$

and

$$d_1\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \qquad d_2 = F_2.$$

Let

$$\mathfrak{I} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}$$

be a graded ideal of R. Even though F_1 and F_2 satisfy the conditions of Theorem 5.1, the ring R has no nonzero central graded ideal.

6. Generalized Homogeneous Derivations on Graded Modules

In this section, we systematically extend the theory to graded modules by introducing generalized homogeneous derivations on modules, establishing their functorial properties, and constructing the associated categorical framework.

Definition 6.1. *Let R be a G-graded ring and M a G-graded R-module. An additive mapping $F_M : M \to M$ is a* generalized homogeneous derivation *if there exists a homogeneous derivation $d : R \to R$ such that*

(i) *$F_M(rm) = d(r)m + rF_M(m)$ for all $r \in R$, $m \in M$;*
(ii) *$F_M(m) \in \mathcal{H}(M)$ for all $m \in \mathcal{H}(M)$.*

We denote such pairs by $(F_M, d)_{h,M}$ and let $\mathfrak{Der}_G^{gh}(R, M)$ denote the set of all generalized homogeneous derivations on M.

Example 6.1. *Let $R = \mathbb{C}[t_1, t_2]$ with the standard $\mathbb{Z}$-grading, and let $M = R^2$ with grading $M_n = \{(f_1, f_2) \mid f_i \in R_n\}$. Define*

$$F(f_1, f_2) = \left(\frac{\partial f_1}{\partial t_1}, \frac{\partial f_2}{\partial t_1}\right)$$

with associated derivation $d(f) = \frac{\partial f}{\partial t_1}$. Then $(F, d)_{h,M} \in \mathfrak{Der}_G^{gh}(R, M)$.

Definition 6.2. *A graded submodule $N \subseteq M$ is* gr-differential *with respect to $(F_M, d)_{h,M}$ if $F_M(N) \subseteq N$.*

Example 6.2. *Consider the generalized homogeneous derivation $(F_M, d)_{h,M}$ from Example 6.1. Then the graded submodule $N = \{0\} \oplus R \subseteq M$ is gr-differential with respect to $(F_M, d)_{h,M}$.*

Definition 6.3. *A generalized homogeneous derivation $(F_M, d)_{h,M}$ is* gr-generalized *if $F_M(M_\tau) \subseteq M_\tau$ and $d(R_\tau) \subseteq R_\tau$ for all $\tau \in G$.*

The set of gr-generalized derivations on M is denoted $p\mathfrak{Der}_G^{gh}(R, M)$.

Proposition 6.1. *$p\mathfrak{Der}_G^{gh}(R, M)$ forms a $Z(R) \cap R_e$-module under pointwise operations and scalar multiplication $a \cdot (F_M, d)_{h,M} = (aF_M, ad)_{h,M}$ for $a \in Z(R) \cap R_e$.*

Proof. Centrality of scalars ensures

$$(aF_M)(rm) = aF_M(rm) = a(d(r)m + rF_M(m)) = (ad)(r)m + r(aF_M)(m),$$

while degree preservation follows from $a \in R_e$ and the grading properties of F_M and d. □

Proposition 6.2. *For finite families $\{M_i\}_{i \in I}$ of graded R-modules:*

(1) *If $(F_{M_i}, d)_{h,M_i} \in p\mathfrak{Der}_G^{gh}(R, M_i)$ share the same associated derivation d, then*

$$F_{\bigoplus M_i}((m_i)_i) = (F_{M_i}(m_i))_i$$

defines a canonical gr-generalized derivation on $\bigoplus_{i \in I} M_i$.

(2) *If $(F_M, d)_{h,M}, (F_N, d)_{h,N} \in p\mathfrak{Der}_G^{gh}(R, M), p\mathfrak{Der}_G^{gh}(R, N)$ have the same associated derivation d, then*

$$F_{M \otimes N}(m \otimes n) = F_M(m) \otimes n + m \otimes F_N(n)$$

defines a canonical gr-generalized derivation on $M \otimes_R N$.

Proof. (1) For the direct sum, let $(m_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ and $r \in R$. Then

$$F_{\bigoplus M_i}(r(m_i)_{i\in I}) = (F_{M_i}(rm_i))_{i\in I} = (d(r)m_i + rF_{M_i}(m_i))_{i\in I} = d(r)(m_i)_{i\in I} + rF_{\bigoplus M_i}((m_i)_{i\in I}).$$

For degree preservation, let $(m_i)_{i\in I}$ be homogeneous of degree $\tau \in G$ in the direct sum. Then each nonzero m_i is homogeneous of degree τ, and since each F_{M_i} preserves degrees, $F_{M_i}(m_i)$ is either zero or homogeneous of degree τ. Hence $(F_{M_i}(m_i))_{i\in I}$ is homogeneous of degree τ, so $F_{\bigoplus M_i}$ preserves the grading.

(2) For $r \in R$, $m \in M_\tau$, and $n \in N_\sigma$, we have

$$\begin{aligned} F_{M\otimes N}(r(m\otimes n)) &= F_{M\otimes N}(rm\otimes n) \\ &= F_M(rm)\otimes n + rm\otimes F_N(n) \\ &= (d(r)m + rF_M(m))\otimes n + rm\otimes F_N(n) \\ &= d(r)m\otimes n + rF_M(m)\otimes n + rm\otimes F_N(n) \\ &= d(r)(m\otimes n) + r(F_M(m)\otimes n + m\otimes F_N(n)) \\ &= d(r)(m\otimes n) + rF_{M\otimes N}(m\otimes n). \end{aligned}$$

For degree preservation, if $m \in M_\tau$ and $n \in N_\sigma$, then $m\otimes n \in (M\otimes_R N)_{\tau\sigma}$. Since $F_M(m) \in M_\tau$ et $F_N(n) \in N_\sigma$, we have

$$F_{M\otimes N}(m\otimes n) = F_M(m)\otimes n + m\otimes F_N(n) \in (M\otimes_R N)_{\tau\sigma}.$$

Thus $F_{M\otimes N}$ preserves the grading. □

Definition 6.4. *A graded R-module homomorphism $\phi : M \to N$ is a* gr-generalized homomorphism *if*

$$\phi\circ F_M = F_N\circ\phi$$

for $(F_M, d)_{h,M} \in p\mathfrak{Der}^{gh}_G(R, M)$ and $(F_N, d)_{h,N} \in p\mathfrak{Der}^{gh}_G(R, N)$.

Example 6.3. *Let $\{M_i\}_{i\in I}$ be a finite family of graded R-modules with direct sum $M = \bigoplus_{i\in I} M_i$. If each M_i admits a gr-generalized derivation $(F_{M_i}, d)_{h,M_i} \in p\mathfrak{Der}^{gh}_G(R, M_i)$ with the same associated derivation d, then the canonical projection maps $\pi_j : M \to M_j$ are gr-generalized homomorphisms with respect to the gr-generalized derivations $(F_M, d)_{h,M}$ on M and $(F_{M_j}, d)_{h,M_j}$ on M_j.*

Proposition 6.3. *Let $\phi : M \to N$ be a surjective graded R-module homomorphism between G-graded modules such that* $\ker(\phi)$ *is a gr-differential submodule of M. Then there exists a well-defined $Z(R)\cap R_e$-linear map*

$$\phi_* : p\mathfrak{Der}^{gh}_G(R, M) \to p\mathfrak{Der}^{gh}_G(R, N)$$

such that for any $(F_M, d)_{h,M} \in p\mathfrak{Der}^{gh}_G(R, M)$ with $F_M(\ker\phi) \subseteq \ker\phi$, the induced map is $(F_N, d)_{h,N} = \phi_*((F_M, d)_{h,M})$.

Proof. Since ϕ is surjective, for each $n \in N$ there exists $m \in M$ with $\phi(m) = n$. Define $F_N : N \to N$ by

$$F_N(n) = \phi(F_M(m)),$$

where m is any preimage of n. To see that F_N is well defined, suppose $\phi(m_1) = \phi(m_2) = n$. Then $m_1 - m_2 \in \ker\phi$, and by hypothesis

$$F_M(m_1 - m_2) \in \ker\phi.$$

Hence $\phi(F_M(m_1)) = \phi(F_M(m_2))$. For the gr-generalized derivation property, let $r \in R$ and $n \in N$, and choose $m \in M$ with $\phi(m) = n$. Then

$$\begin{aligned} F_N(rn) &= F_N(\phi(rm)) \\ &= \phi(F_M(rm)) \\ &= \phi(d(r)m + rF_M(m)) \\ &= d(r)\phi(m) + r\phi(F_M(m)) \\ &= d(r)n + rF_N(n). \end{aligned}$$

For homogeneity, if $n \in N_\tau$, then, since ϕ is graded, we can choose $m \in M_\tau$ with $\phi(m) = n$. As F_M preserves degrees, $F_M(m) \in M_\tau$, and thus

$$F_N(n) = \phi(F_M(m)) \in \phi(M_\tau) \subseteq N_\tau.$$

The $Z(R) \cap R_e$-linearity of ϕ_* follows from the linearity of ϕ and the module structure on $p\mathfrak{Der}_G^{gh}(R, M)$. □

Corollary 6.1. *For graded isomorphisms $\phi : M \to N$, the induced map*

$$\phi_* : p\mathfrak{Der}_G^{gh}(R, M) \to p\mathfrak{Der}_G^{gh}(R, N)$$

is a $Z(R) \cap R_e$-module isomorphism, with inverse

$$\psi_*((F_N, d)_{h,N}) = (\phi^{-1} \circ F_N \circ \phi, d)_{h,M}.$$

Proof. The canonical projection $\pi : M \to M/N$ is surjective, and since $F_M(N) \subseteq N$ by hypothesis, we have $F_M(\ker \pi) \subseteq \ker \pi$. By Proposition 6.3, there exists a well-defined $Z(R) \cap R_e$-linear map

$$\pi_* : p\mathfrak{Der}_G^{gh}(R, M) \to p\mathfrak{Der}_G^{gh}(R, M/N)$$

such that $\pi_*((F_M, d)_{h,M}) = (F_{M/N}, d)_{h,M/N}$, where $F_{M/N}(\pi(m)) = \pi(F_M(m))$ for all $m \in M$. Well-definedness of $F_{M/N}$ follows from $F_M(N) \subseteq N$. The gr-generalized derivation property is checked by verifying

$$F_{M/N}(r\overline{m}) = d(r)\overline{m} + rF_{M/N}(\overline{m})$$

for all $r \in R$ and $\overline{m} \in M/N$. For homogeneity, if $\overline{m} \in (M/N)_\tau$, write $m = m_\tau + n$ with $m_\tau \in M_\tau$ and $n \in N$. Since F_M preserves degrees and $F_M(N) \subseteq N$, we get

$$F_{M/N}(\overline{m}) = \pi(F_M(m_\tau)) \in (M/N)_\tau.$$

The commutativity relation $F_{M/N} \circ \pi = \pi \circ F_M$ makes π a gr-generalized homomorphism. Uniqueness follows because any other gr-generalized derivation $F'_{M/N}$ satisfying $F'_{M/N} \circ \pi = \pi \circ F_M$ must coincide with $F_{M/N}$ on all cosets. □

Definition 6.5. *The category $\mathscr{M}_G^{gh}$ is defined as follows.*

(i) *Objects: Triples $(R, M, (F_M, d)_{h,M})$ where R is G-graded, M is a graded R-module, and $(F_M, d)_{h,M} \in p\mathfrak{Der}_G^{gh}(R, M)$.*
(ii) *Morphisms: Pairs $(\phi, \psi) : (R, M, (F_M, d)_{h,M}) \to (S, N, (F_N, e)_{h,N})$ where $\phi : R \to S$ is a graded ring homomorphism, $\psi : M \to N$ is ϕ-semilinear, and the diagrams*

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ \downarrow{\scriptstyle F_M} & & \downarrow{\scriptstyle F_N} \\ M & \xrightarrow{\psi} & N \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ R & \xrightarrow{\phi} & S \end{array}$$

commute.

Theorem 6.1. *$\mathscr{M}_G^{gh}$ is a well-defined category.*

Proof. Composition is well defined: Let

$$(\phi, \psi) : (R, M, (F_M, d)_{h,M}) \to (S, N, (F_N, e)_{h,N})$$

and

$$(\phi', \psi') : (S, N, (F_N, e)_{h,N}) \to (T, P, (F_P, f)_{h,P})$$

be morphisms in $\mathscr{M}_G^{gh}$. We must show that $(\phi' \circ \phi, \psi' \circ \psi)$ is again a morphism. First, $\psi' \circ \psi$ is $(\phi' \circ \phi)$-semilinear:

$$\begin{aligned} (\psi' \circ \psi)(rm) &= \psi'(\psi(rm)) \\ &= \psi'(\phi(r)\psi(m)) \\ &= \phi'(\phi(r))\psi'(\psi(m)) \\ &= (\phi' \circ \phi)(r)(\psi' \circ \psi)(m). \end{aligned}$$

Next, we check compatibility with the derivations and module maps:

$$(\psi' \circ \psi) \circ F_M = \psi' \circ (F_N \circ \psi) = F_P \circ (\psi' \circ \psi),$$

and similarly

$$(\phi' \circ \phi) \circ d = f \circ (\phi' \circ \phi).$$

Identity morphisms: For any object $(R, M, (F_M, d)_{h,M})$, the pair $(\mathrm{id}_R, \mathrm{id}_M)$ satisfies

$$\mathrm{id}_M(rm) = rm = \mathrm{id}_R(r)\mathrm{id}_M(m), \quad \mathrm{id}_M \circ F_M = F_M = F_M \circ \mathrm{id}_M, \quad \mathrm{id}_R \circ d = d = d \circ \mathrm{id}_R,$$

and is therefore a morphism in $\mathscr{M}_G^{gh}$.

Associativity and identity laws: These follow directly from the associativity and identity properties of function composition. □

Declarations

Conflicts of interest The authors declare no conflict of interest.
Data availability This manuscript has no associated data.

References

[1] Y. Ait Mohamed. "On graded rings with homogeneous derivations". In: *Revista de la Unión Matemática Argentina* (2024). Published online (early view): December 22, 2024. DOI: 10.33044/revuma.4934.
[2] M. Ashraf, A. Ali, and S. Ali. "Some Commutativity Theorems for Rings with Generalized Derivations". In: *Southeast Asian Bulletin of Mathematics* 31 (2007), pp. 415–421.
[3] M. Ashraf, S. Ali, and A. Ali. "On Semiprime Rings with Generalized Derivations". In: *Bol. Soc. Paran. Mat.* 28.2 (2010).
[4] H. E. Bell and M. N. Daif. "On derivations and commutativity in prime rings". In: *Acta Math. Hungar.* 66.4 (1995), pp. 337–343. DOI: 10.1007/BF01876049.
[5] M. Brešar. "On the distance of the composition of two derivations to the generalized derivations". In: *Glasgow Mathematical Journal* 33.1 (1991), pp. 89–93. DOI: 10.1017/S0017089500008077.
[6] B. Dhara, S. Kar, and K. Singh. "Action of generalized derivations with central values in prime rings". In: *Afr. Mat.* 36 (2025), p. 24. DOI: 10.1007/s13370-025-01240-w.
[7] S. Huang and B. Davvaz. "Generalized derivations of rings and Banach algebras". In: *Comm. Algebra* 41.3 (2013), pp. 1188–1194. DOI: 10.1080/00927872.2011.642043.
[8] A. L. Kanunnikov. "Graded Quotient Rings". In: *Journal of Mathematical Sciences* 233.1 (2018). DOI: 10.1007/s10958-018-3925-7.
[9] T. K. Lee. "Generalized derivations of left faithful rings". In: *Communications in Algebra* 27.8 (1999), pp. 4057–4073. DOI: 10.1080/00927879908826682.
[10] T. K. Lee and W. K. Shiue. "Linear identities and derivations". In: *Comm. Algebra* 28.7 (2000), pp. 3317–3327. DOI: 10.1080/00927870008827027.
[11] T. K. Lee and W. K. Shiue. "Identities with generalized derivations". In: *Communications in Algebra* 29.10 (2001), pp. 4437–4450. DOI: 10.1081/AGB-100106767.
[12] L. Zhao and Z. Liu. "Generalized derivations of current Lie algebras". In: *Comm. Algebra* 52.3 (2024), pp. 1234–1252. DOI: 10.1080/00927872.2024.2354423.

Sidi Mohamed Ben Abdellah University, Fez, Morocco
Email address: y.aitmohamed@yahoo.com