GRADED QUASI-BAER $*$-RING CHARACTERIZATION OF STEINBERG ALGEBRAS

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ABSTRACT. Given a graded ample, Hausdorff groupoid $\mathcal{G}$, and an involutive field K, we consider the Steinberg algebra $A_K(\mathcal{G})$. We obtain necessary and sufficient conditions on $\mathcal{G}$ under which the annihilator of any graded ideal of $A_K(\mathcal{G})$ is generated by a homogeneous projection. This property is called graded quasi-Baer $*$. We use the Steinberg algebra model to characterize graded quasi-Baer $*$ Leavitt path algebras.

1. INTRODUCTION

Steinberg algebras were independently introduced in [9] and [18] and have attracted the attention of analysts and algebraists since then. Steinberg algebras appeared in the details of many groupoid C*-algebra constructions before they were specified by name (see, e.g., [13, 16]). In addition to providing insight into the analytic theory of groupoid C*-algebras, these algebras also gave rise to interesting examples of $*$-algebras. For instance, all Leavitt path algebras, Kumjian-Pask algebras, and discrete inverse semigroup algebras can be realized as Steinberg algebras. Furthermore, Steinberg algebras have been useful for the transfer of algebraic and analytic concepts and techniques.

In [8] the authors characterized the graded ideals of Steinberg algebras over groupoids equipped with a cocycle into a discrete group such that the inverse image of the identity doesn't have too much isotropy.

The concept of Baer rings in the context of Leavitt path algebras was first investigated by Aranda-Pino and Vaš in [1]. In [14], the notions of graded Baer and graded Rickart rings were introduced and studied in the setting of Leavitt path algebras.

In this paper, we study the annihilators of graded ideals in Steinberg algebras built from graded groupoids. We show that the annihilator of any graded ideal of $A_K(\mathcal{G})$ is generated by a homogeneous projection if and only if for each open invariant subset U of the unit space $\mathcal{G}^{(0)}$, U or the interior of $\mathcal{G}^{(0)} \setminus U$ is compact. This property (i.e., the annihilator of any graded ideal is generated by a homogeneous projection) is called graded quasi-Baer $*$. In [2, 19], the authors characterized graded quasi-Baer $*$ Leavitt path algebras. We give another characterization of graded quasi-Baer $*$ Leavitt path algebras by using the Steinberg algebra model.

2. QUASI-BAER $*$ CONDITION FOR GRADED UNITAL $*$-RINGS

2.1. **Quasi-Baer $*$-rings.** For a subset X of a ring R, the right annihilator $r_R(X)$ of X in R denotes the set $\{r \in R \mid xr = 0 \text{ for all } x \in X\}$. It is straightforward to check that $r_R(X)$ is a right of R.

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A ring R is said to be a $*$-ring or an involutive ring, if it has an involution (i.e., an operation $*: R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$ for all $x, y \in R$). A $*$-ring R is said to be a *quasi-Baer* $*$*-ring* if $r_R(I)$ is generated by a projection (i.e., a self-adjoint idempotent) for any ideal I of R. This condition is left-right symmetric. If a $*$-ring is quasi-Baer $*$, the projection which generates the right annihilator of zero is an identity. Consequently, quasi-Baer $*$-rings are necessarily unital. It turns out that the projection in the definition of a quasi-Baer $*$-ring is central.

Recall that an involution $*$ on a $*$-ring R is said to be *proper* if $xx^* = 0$ implies $x = 0$ for any element $x \in R$. Also, an involution $*$ is called a *semiproper* involution if $xRx^* = 0$ implies $x = 0$. Obviously, if $*$ is a proper involution, then $*$ is a semiproper involution. The converse does not hold true (see [3, Example 10.2.9]). It was shown in [3, Lemma 10.2.10] that the involution on a quasi-Baer $*$-ring is always semiproper.

2.2. **Graded $*$-rings.** If Γ is an abelian group with identity ε, a ring R is a Γ-*graded ring* if $R = \bigoplus_{g\in\Gamma} R_g$ such that each R_g is an additive subgroup of R and $R_gR_h \subseteq R_{gh}$ for all $g, h \in \Gamma$. The elements of $R^h = \bigcup_{g\in\Gamma} R_g$ are the *homogeneous elements* of R. If $a \in R_g$ and $a \neq 0$, we say that g is the degree of a. Note that every nonzero homogeneous idempotent has degree ε. If R is an algebra over a field K, then R is a *graded algebra* if R is a graded ring and R_g is a K-vector subspace for any $g \in \Gamma$. A Γ-graded ring R with an involution $*$ is said to be a *graded* $*$*-ring* if $R_g^* \subseteq R_{g^{-1}}$ for every $g \in \Gamma$. The notion of graded $*$-rings has been studied in details in [15].

A *graded right ideal* of R is a right ideal I such that $I = \bigoplus_{g\in\Gamma} I \cap R_g$. An ideal I of R is a graded ideal if and only if I is generated by homogeneous elements. This property implies that $r_R(X)$ is a graded ideal of R for any set X of homogeneous elements of R.

2.3. **Graded quasi-Baer $*$-rings.** In [2], the definition of quasi-Baer $*$-rings is adapted to graded $*$-rings. Recall that a graded $*$-ring R is called a *graded quasi-Baer* $*$*-ring* if the right annihilator of any graded ideal of R is generated by a homogeneous projection. It is useful to note that the homogeneous projection in the definition of a graded quasi-Baer $*$-ring is central (see [2, Remark 2]).

Recall that an involution $*$ on a graded $*$-ring R is *graded proper*, if $xx^* = 0$ implies $x = 0$ for any homogeneous element $x \in R$. Also, $*$ is called *graded semiproper*, if $xRx^* = 0$ implies $x = 0$ for any homogeneous element $x \in R$. By [2, Proposition 4], the involution on a graded quasi-Baer $*$-ring R is always graded semiproper.

3. GRADED QUASI-BAER $*$ CONDITION FOR STEINBERG ALGEBRAS

3.1. **Graded groupoids.** A *groupoid* is a small category in which every morphism is invertible. It can also be viewed as a generalisation of a group which has partial binary operation. Let $\mathcal{G}$ be a groupoid. The *unit space* of $\mathcal{G}$ is the set

$$\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} \mid \gamma \in \mathcal{G}\} = \{\gamma^{-1}\gamma \mid \gamma \in \mathcal{G}\}.$$

Groupoid *source* and *range* maps $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$ are defined such that $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$. Elements of $\mathcal{G}^{(0)}$ are units in the sense that $\gamma s(\gamma) = \gamma$ and $r(\gamma)\gamma = \gamma$ for all $\gamma \in \mathcal{G}$. For each $u \in \mathcal{G}^{(0)}$, the set $\mathcal{G}_u^u = s^{-1}(u) \cap r^{-1}(u)$ is a group, called the *isotropy group* based at u. The *isotropy group bundle* of $\mathcal{G}$ is the set

$$\mathrm{Iso}(\mathcal{G}) = \bigcup_{u\in\mathcal{G}^{(0)}} \mathcal{G}_u^u = \{\gamma \in \mathcal{G} \mid s(\gamma) = r(\gamma)\}.$$

A subset $U \subseteq \mathcal{G}^{(0)}$ is *invariant* if $s(\gamma) \in U$ implies $r(\gamma) \in U$. Equivalently, U is invariant if $r(\gamma) \in U$ implies $s(\gamma) \in U$. Given such an invariant subset U, let

$$\mathcal{G}|_U = \{\gamma \in \mathcal{G} \mid s(\gamma) \in U\}.$$

Observe that $\mathcal{G}|_U$ is a subgroupoid of $\mathcal{G}$. If, in addition, U is an open subset of $\mathcal{G}^{(0)}$, then $\mathcal{G}|_U$ is open in $\mathcal{G}$.

The set of *composable pairs* of $\mathcal{G}$ is $\mathcal{G}^{(2)} = \{(\gamma, \alpha) \in \mathcal{G} \times \mathcal{G} | s(\gamma) = r(\alpha)\}$. For $U, V \subseteq \mathcal{G}$, we define

$$UV = \{\gamma\alpha \mid \gamma \in U, \alpha \in V, (\gamma, \alpha) \in \mathcal{G}^{(2)}\}.$$

A *topological groupoid* is a groupoid endowed with a topology under which the inverse map is continuous, and such that composition is continuous with respect to the relative topology on $\mathcal{G}^{(2)}$ inherited from $\mathcal{G} \times \mathcal{G}$. An *open bisection* of $\mathcal{G}$ is an open subset $U \subseteq \mathcal{G}$ such that $s|_U$ and $r|_U$ are homeomorphisms onto an open subset of $\mathcal{G}^{(0)}$. An *étale* groupoid is a topological groupoid $\mathcal{G}$ such that its range map is a local homeomorphism from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ (the source map will consequently share that property). It is easy to see that the topology of an étale groupoid admits a basis formed by open bisections. In an étale groupoid one has that $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$. If, in addition, $\mathcal{G}$ is Hausdorff, then $\mathcal{G}^{(0)}$ is also closed in $\mathcal{G}$. We say that an étale groupoid $\mathcal{G}$ is *ample* if there is a basis consisting of compact open bisections for its topology.

An ample hausdorff groupoid $\mathcal{G}$ is called *effective* if $\mathrm{Int}(\mathrm{Iso}(\mathcal{G}))$, the interior of $\mathrm{Iso}(\mathcal{G})$ in the relative topology, is equal to $\mathcal{G}^{(0)}$. We say that $\mathcal{G}$ is *strongly effective* if $\mathcal{G}|_U$ is effective for every closed invariant subset $U \subseteq \mathcal{G}^{(0)}$. If $\mathcal{G}$ is strongly effective, then it is effective because $\mathcal{G}^{(0)}$ is a closed invariant set.

Let Γ be a discrete group with identity ε, and $\mathcal{G}$ a topological groupoid. A Γ-*grading* of $\mathcal{G}$ is a continuous map $c : \mathcal{G} \to \Gamma$ such that $c(\gamma\alpha) = c(\gamma)c(\alpha)$ for all $(\gamma, \alpha) \in \mathcal{G}^{(2)}$; such a map c is called a *cocycle* on $\mathcal{G}$. We always have $\mathcal{G}^{(0)} \subseteq c^{-1}(\varepsilon)$. Observe that for $g \in \Gamma$, $\mathrm{Iso}(c^{-1}(g)) = c^{-1}(g) \cap \mathrm{Iso}(\mathcal{G})$. We write $B^{co}_g(\mathcal{G})$ for the collection of all compact open bisections of $c^{-1}(g)$ and $B^{co}_*(\mathcal{G}) = \bigcup_{g \in \Gamma} B^{co}_g(\mathcal{G})$. Throughout this paper we only consider Γ-graded ample Hausdorff groupoids.

3.2. **Steinberg algebras.** We recall the notion of the Steinberg algebra as a universal algebra generated by certain compact open subsets of an ample Hausdorff groupoid. Let $\mathcal{G}$ be a Γ-graded ample Hausdorff groupoid and R be a commutative ring with identity. The *Steinberg R-algebra* associated to $\mathcal{G}$, denoted $A_R(\mathcal{G})$, is the algebra generated by the set $\{t_B \mid B \in B^{co}_*(\mathcal{G})\}$ with coefficients in R, subject to

(i) (R_1) $t_\emptyset = 0$;
(ii) (R_2) $t_{B_1} t_{B_2} = t_{B_1 B_2}$ for all $B_1, B_2 \in B^{co}_*(\mathcal{G})$; and
(iii) (R_3) $t_{B_1} + t_{B_2} = t_{B_1 \cup B_2}$, whenever B_1 and B_2 are disjoint elements of B^{co}_g for some $g \in \Gamma$ such that $B_1 \cup B_2$ is a bisection.

The Steinberg algebra defined above is isomorphic to the following construction:

$$A_R(\mathcal{G}) = span\{1_U \mid U \text{ is a compact open bisection of } \mathcal{G}\},$$

where $1_U : \mathcal{G} \to R$ denotes the characteristic function on U (see [9, Theorem 3.10]). Equivalently, if we give R the discrete topology, then continuous functions from $\mathcal{G}$ to R are exactly locally constant functions from $\mathcal{G}$ to R, and so $A_R(\mathcal{G}) = C_c(\mathcal{G}, R)$, the space of compactly supported continuous functions from $\mathcal{G}$ to R. Addition is point-wise and multiplication is given by convolution $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$. It is useful to note that $1_U * 1_V = 1_{UV}$ for compact open bisections U and V. By [9, Lemma 3.5], every element $f \in A_R(\mathcal{G})$ can be expressed as $f = \sum_{U \in F} a_U 1_U$, where F is a finite subset of mutually disjoint elements of $B_*^{co}(\mathcal{G})$.

The family of all idempotent elements of $A_R(\mathcal{G}^{(0)})$ is a set of local units for $A_R(\mathcal{G})$. Moreover, $A_R(\mathcal{G})$ is unital if and only if $\mathcal{G}^{(0)}$ is compact. In this case, $1_{\mathcal{G}^{(0)}}$ is the identity element of $A_R(\mathcal{G})$.

If $c : \mathcal{G} \to \Gamma$ is a cocycle, then the Steinberg algebra $A_R(\mathcal{G})$ is a Γ-graded algebra with homogeneous components

$$A_R(\mathcal{G})_g = \{f \in A_R(\mathcal{G}) \mid f(\gamma) \neq 0 \Rightarrow c(\gamma) = g\}.$$

If $^- : R \to R$ is an involution on R, then the map

$$* : A_R(\mathcal{G}) \to A_R(\mathcal{G}), \qquad f \mapsto f^*, \ \textit{where } f^*(\gamma) = \overline{f(\gamma^{-1})}$$

defines an involution on $A_R(\mathcal{G})$ making it into a $*$-algebra. Observe that $(A_R(\mathcal{G})_g)^* = A_R(\mathcal{G})_{g^{-1}}$, for each $g \in \Gamma$. It follows that $A_R(\mathcal{G})$ is a Γ-graded $*$-algebra.

A function $f \in A_R(\mathcal{G})$ is a *class function* if f satisfies the following conditions:

(i) $f(x) \neq 0 \Rightarrow s(x) = r(x)$;
(ii) $s(x) = r(x) = s(z) \Rightarrow f(zxz^{-1}) = f(x)$.

By [18, Proposition 4.13] the center of $A_R(\mathcal{G})$ is the set of class functions.

The following result is a characterization of Steinberg algebras that have graded proper involutions.

Proposition 3.1. *Let R be a commutative unital $*$-ring, $\mathcal{G}$ be an ample Hausdorff groupoid, Γ be a discrete group, and $c : \mathcal{G} \to \Gamma$ be a cocycle such that $c^{-1}(\varepsilon)$ is effective. Then the following are equivalent.*

(i) *The involution on R is proper;*
(ii) *The involution on $A_R(\mathcal{G})$ is graded proper;*
(iii) *The involution on $A_R(\mathcal{G})$ is graded semiproper.*

In particular, if K is a field with involution, then the involution on $A_K(\mathcal{G})$ is graded proper.

Proof. (i)$\Rightarrow$(ii) Assume on the contrary that there exists a nonzero homogeneous element $f \in A_R(\mathcal{G})_g$ such that $ff^* = 0$. We can express f as $f = \sum_{i=1}^n r_i 1_{U_i}$, where $r_1, \ldots, r_n \in R \setminus \{0\}$ and $U_1, \ldots, U_n \in B_*^{co}(\mathcal{G})$ are mutually disjoint. Since the U_i's are disjoint and the r_i's are nonzero, we can assume each $U_i \subseteq \mathcal{G}_g$. Without loss of generality, we can assume that $r(U_1), \ldots, r(U_n)$ are mutually disjoint compact open subsets of $\mathcal{G}^{(0)}$. For this, let $W = r(U_i) \cap r(U_j) \neq \emptyset$, for some $i \neq j$. Then W is a nonempty compact open subset of $\mathcal{G}^{(0)}$. Let $x \in W$, then there exist $\alpha \in U_i$ and $\beta \in U_j$ such that $x = r(\alpha) = r(\beta)$. So $\alpha = \beta(\beta^{-1}\alpha)$, and that $\beta^{-1}\alpha \in U_j^{-1}U_i$. It is easy to see that $U_j^{-1}U_i$ does not intercept $\mathcal{G}^{(0)}$. Then $U_j^{-1}U_i \subseteq c^{-1}(\varepsilon) \setminus \mathcal{G}^{(0)}$ is a nonempty compact open bisection. Since $c^{-1}(\varepsilon)$ is effective, [4, Lemma 3.1] implies that there exists a nonempty open subset $V \subseteq W$ such that $V(U_j^{-1}U_i)V = \emptyset$. By shrinking if necessary, we can assume V is compact. Then U_jV and U_iV are mutually disjoint compact open bisection, and $(U_jV)^{-1}(U_iV) = \emptyset$. This implies $r(U_jV) \cap r(U_iV) = \emptyset$. If we define $f' = 1_V * f$, then $f' \neq 0$, $f'f'^* = 0$, and f' can be expressed as $f' = \sum_{i=1}^n r_i 1_{U_i'}$, where $r_1, \ldots, r_n \in R \setminus \{0\}$, $U_1', \ldots, U_n' \in B_*^{co}(\mathcal{G})$ are mutually disjoint, and $r(U_1'), \ldots, r(U_n') \subseteq \mathcal{G}^{(0)}$ are mutually disjoint as desired.

Next, we have

$$ff^* = \sum_{i,j=1}^{n} r_i\overline{r_j}(1_{U_i} * 1_{U_j^{-1}}) = \sum_{i,j=1}^{n} r_i\overline{r_j}1_{U_iU_j^{-1}} = 0.$$

For each i if $j \neq i$, then $U_iU_j^{-1}$ does not intercept $\mathcal{G}^{(0)}$. Thus

$$\begin{aligned} ff^*|_{\mathcal{G}^{(0)}} &= \sum_{i,j=1}^{n} r_i\overline{r_j}(1_{U_iU_j^{-1}})|_{\mathcal{G}^{(0)}} = \sum_{i,j=1}^{n} r_i\overline{r_j}1_{U_iU_j^{-1}\cap\mathcal{G}^{(0)}} \\ &= \sum_{i=1}^{n} r_i\overline{r_i}1_{U_iU_i^{-1}} = \sum_{i=1}^{n} r_i\overline{r_i}1_{r(U_i)} = 0. \end{aligned}$$

Note that any collection of characteristic functions of mutually disjoint open compact subsets of $\mathcal{G}^{(0)}$ is linearly independent. Then $r_i\overline{r_i} = 0$ for each i. Thus, (i) yields $r_1, \dots, r_n = 0$ and that $f = 0$, a contradiction. Hence the involution on $A_R(\mathcal{G})$ is graded proper.

(ii)⇒(iii) This implication holds in any $*$-ring with local units.

(iii)⇒(i) Assume that $a\overline{a} = 0$, for $a \in R$. Then for a nonempty open compact subset $U \subseteq \mathcal{G}^{(0)}$ we have $(a1_U)A_R(\mathcal{G})(a1_U)^* = (a1_U)A_R(\mathcal{G})(\overline{a}1_U) = (a\overline{a}1_U)A_R(\mathcal{G})(1_U) = 0$. Thus, (iii) yields that $a1_U = 0$. Hence $a = 0$ as desired. □

In Proposition 3.1, if one considers Γ to be trivial, the following result is obtained.

Proposition 3.2. *Let R be a $*$-ring, and $\mathcal{G}$ be an affective ample Hausdorff groupoid. Then the following are equivalent.*

- (i) *The involution on R is proper;*
- (ii) *The involution on $A_R(\mathcal{G})$ is proper;*
- (iii) *The involution on $A_R(\mathcal{G})$ is semiproper.*

In particular, if K is a field with involution, then the involution on $A_K(\mathcal{G})$ is proper.

In the next result, we characterize the graded quasi-Baer $*$ Steinberg algebras over ample Hausdorff groupoids equipped with a cocycle taking values in a discrete group.

Theorem 3.3. *Let K be a field with involution, $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space, Γ be a discrete group, and $c : \mathcal{G} \to \Gamma$ be a cocycle such that $c^{-1}(\varepsilon)$ is strongly effective. Then $A_K(\mathcal{G})$ is a graded quasi-Baer $*$-ring if and only if for every open invariant subset U of $\mathcal{G}^{(0)}$, U or $\mathrm{Int}(\mathcal{G}^{(0)} \setminus U)$ is compact.*

Proof. Let I be a graded ideal of $A_K(\mathcal{G})$. Then by [8, Theorem 5.3], there exists an open invariant subset U of $\mathcal{G}^{(0)}$ such that $I = A_K(\mathcal{G}|_U)$. First, assume that U is compact. Then by [10, Lemma 1.6], 1_U is a class function in $A_K(\mathcal{G})$, and by [18, Proposition 4.13], 1_U is in the center of $A_K(\mathcal{G})$. Thus 1_U is a central projection. We claim that $I = A_K(\mathcal{G})1_U$. Let B be a compact open bisection in $\mathcal{G}$. Then $BU = B$ if $s(B) \subseteq U$, and $BU = \emptyset$ otherwise. So $1_B * 1_U = 1_{BU} \in A_K(\mathcal{G}|_U) = I$. Since $A_K(\mathcal{G})$ is spanned by the elements of the form 1_B, with B as above, we conclude that $A_K(\mathcal{G})1_U \subseteq I$. Let us now prove that $I \subseteq A_K(\mathcal{G})1_U$. Again we may focus on characteristic functions, meaning that all we must do is show that 1_B lies in $A_K(\mathcal{G})1_U$, for all compact open bisections $B \subseteq \mathcal{G}|_U$. Given such a B, observe that $s(B) \subseteq U$, then

$$1_B = 1_{Bs(B)} = 1_{BU} = 1_B * 1_U \in A_K(\mathcal{G})1_U,$$

This prove that $I = A_K(\mathcal{G})1_U$. It is straightforward to see that $r_{A_K(\mathcal{G})}(A_K(\mathcal{G})1_U) = (1_{\mathcal{G}^{(0)}} - 1_U)A_K(\mathcal{G})$. Thus $r_{A_K(\mathcal{G})}(I)$ is generated by $1_{\mathcal{G}^{(0)}} - 1_U$.

Now, assume that $V = \operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$ is compact. Observe that V is also invariant. Indeed, $r(s^{-1}(V))$ is an open subset of $\mathcal{G}^{(0)}$, since V is an open subset of $\mathcal{G}^{(0)}$. Notice that $\mathcal{G}^{(0)} \setminus U$ is invariant, since U is invariant. Since $\mathcal{G}^{(0)} \setminus U$ is invariant, $r(s^{-1}(V)) \subseteq \mathcal{G}^{(0)} \setminus U$. Thus $r(s^{-1}(V)) \subseteq V$, and that V is invariant. Hence 1_V is a central (homogeneous) projection. We claim that the right annihilator of I is generated by 1_V. Let $1_B \in I$, for some compact open bisection $B \subseteq \mathcal{G}|_U$. Then $s(B) \subseteq U$, and so $s(B) \cap V \subseteq s(B) \cap (\mathcal{G}^{(0)} \setminus U) = \emptyset$. Thus $BV = \emptyset$, and that $1_B * 1_V = 1_{BV} = 0$. Hence $1_V A_K(\mathcal{G}) \subseteq r_{A_K(\mathcal{G})}(I)$. Now, assume that $1_B \in r_{A_K(\mathcal{G})}(I)$, for some compact open bisection $B \subseteq \mathcal{G}$. Then for each $1_W \in I$, $1_W 1_B = 1_{WB} = 0$. Since $s(W) \subseteq U$, we have $r(B) \subseteq \mathcal{G}^{(0)} \setminus U$. Since $r(B)$ is open, $r(B) \subseteq V$. Then $1_B = 1_{Br(B)} = 1_{BV} = 1_B 1_V = 1_V 1_B \in 1_V A_K(\mathcal{G})$. This show that $r_{A_K(\mathcal{G})}(I) \subseteq 1_V A_K(\mathcal{G})$. Then $r_{A_K(\mathcal{G})}(I)$ is generated by 1_V as claimed.

Conversely, assume that $A_K(\mathcal{G})$ is a graded quasi-Baer $*$-ring. Let U be an open invariant subset of $\mathcal{G}^{(0)}$. Take $I = A_K(\mathcal{G}|_U)$. By [10, Theorem 5.3] I is a graded ideal of $A_K(\mathcal{G})$. Then there exists a central homogeneous projection $p \in A_K(\mathcal{G})$ such that the right annihilator of I is generated by p. First, assume that $p = 0$. If there is a compact open set $V \subseteq \operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$, then one can show that $1_V \in r_{A_K(\mathcal{G})}(I)$, a contradiction. Thus $\operatorname{Int}(\mathcal{G}^{(0)} \setminus U = \emptyset)$ is compact open invariant. Now, assume that $p \neq 0$. We claim that U or $\operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$ is compact. First, note that [8, Lemma 1.3] implies that there exists $\{k_i\}_{i=1}^t \subseteq K \setminus \{0\}$ such that $p = \sum_{i=1}^t k_i 1_{U_i}$ where $U_i = p^{-1}(k_i)$ is a compact open bisection for each i. Since p is central, it is a class function, so 1_{U_i} is also a class function for each i. It is well-known that every homogeneous idempotent is in the zero component. Then by [8, Lemma 1.4], U_i is a compact open invariant set for each i. Also, it is clear that $U_1, U_2, \ldots, U_t$ are mutually disjoint. Assume on the contrary that U and $\operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$ are not compact. Then $U_i \neq U$ and $U_i \neq \operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$ for each i. If $U_j \cap U \neq \emptyset$ for some j, then there is a compact open subset V of $\mathcal{G}^{(0)}$ such that $V \subseteq U_j \cap U$. Since $\mathcal{G}^{(0)}$ is open in G, V is a compact open bisection in $\mathcal{G}$, so $1_V \in A_K(\mathcal{G})$. Since $s(1_V) = V \subseteq U$, $1_V \in I$. But,

$$1_V * p = \sum_{i=1}^t k_i (1_V * 1_{U_i}) = k_j 1_V + \sum_{\substack{i=1 \\ i \neq j}}^t k_i 1_{VU_i} = k_j 1_V \neq 0,$$

a contradiction. Thus $U_j \cap U = \emptyset$ and so $U_j \subsetneq \operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$ for each i. Since $\cup_{i=1}^t U_i$ is compact and $\operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$ is not compact, $\bigcup_{i=1}^t U_i \subsetneq \operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$, and that $\operatorname{Int}(\mathcal{G}^{(0)} \setminus U) \setminus \cup_{i=1}^t U_i \neq 0$. Then there is a compact open subset V of $\mathcal{G}^{(0)}$ such that $V \subseteq \operatorname{Int}(\mathcal{G}^{(0)} \setminus U) \setminus \cup_{i=1}^t U_i$. As discussed above, V is a compact open bisection in $\mathcal{G}$, so $1_V \in A_K(\mathcal{G})$. One can show that $1_V \in r_{A_K(\mathcal{G})}(I) = pA_K(\mathcal{G})$. But,

$$p * 1_V = \sum_{i=1}^t k_i (1_{U_i} * 1_V) = \sum_{i=1}^t k_i 1_{U_i V} = 0 \neq 1_V,$$

a contradiction. Therefore, U or $\operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$ must be compact. □

It should be noted that the involution on a graded quasi-Baer $*$-ring must be graded semiproper. However, we don't need any additional assumption on the field K in the previous theorem. Indeed, Proposition 3.1 guarantees the involution on $A_K(\mathcal{G})$ is graded semiproper.

As a consequence of Theorem 3.3, we have the following.

Corollary 3.4. *Let K be a field with involution, $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space, Γ be a discrete group, and $c : \mathcal{G} \to \Gamma$ be a cocycle such that $c^{-1}(\varepsilon)$ is strongly effective. If $\mathcal{G}$ is strongly effective, then $A_K(\mathcal{G})$ is a quasi-Baer $*$-ring if and only if for every open invariant subset U of $\mathcal{G}^{(0)}$, U or $\operatorname{Int}(\mathcal{G}^{(0)} \setminus U)$ is compact.*

Proof. Note that if $\mathcal{G}$ is strongly effective, then every ideal of $A_K(\mathcal{G})$ is graded. For this, let I be an ideal of $A_K(\mathcal{G})$. Then by [7, Corollary 3.7] there is an open invariant subset $U \subseteq \mathcal{G}^{(0)}$ such that $I = A_K(\mathcal{G}|_U)$. If $f = \sum_{B\in F} a_B 1_B \in I$, where F is a finite subset of mutually disjoint elements of $B^{co}_*(\mathcal{G})$. It is then clear that 1_B is also in $A_K(\mathcal{G}|_U) = I$ for each $B \in F$, so I is indeed a graded ideal.

Then we can see that $A_K(\mathcal{G})$ is quasi-Baer $*$ if and only if it is graded quasi-Baer $*$. So the result is now a direct consequence of Theorem 3.3. ☐

4. Graded quasi-Baer $*$ condition for Leavitt path algebras

In this section we explain what Proposition 3.1 and Theorem 3.3 say about a Leavitt path algebra of a directed graph. We start by gathering background needed to state Corollaries 4.1 and 4.3.

4.1. **Graph concepts.** A directed graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 , E^1 and two maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. In this setting, if the number of vertices is finite, then the number of edges is finite as well, and we call E a *finite* graph.

A vertex v for which $s^{-1}(v)$ is empty is called a *sink*, while a vertex w for which $r^{-1}(w)$ is empty is called a *source*. A vertex $v \in E^0$ such that $|s^{-1}(v)| = \infty$, is called an *infinite emitter*. If v is either a sink or an infinite emitter, then it is called a *singular vertex*. If v is not a singular vertex, it is called a *regular vertex*. The expressions $\operatorname{Sink}(E)$, $\operatorname{Source}(E)$, $\operatorname{Reg}(E)$, and $\operatorname{Inf}(E)$ will be used to denote, respectively, the sets of sinks, sources, regular vertices, and infinite emitters of E.

A finite path μ in a graph E is a sequence of edges $\mu = \mu_1 \dots \mu_k$ such that $r(\mu_i) = s(\mu_{i+1})$, $1 \le i \le k-1$. In this case, $s(\mu) := s(\mu_1)$ is the *source* of μ, $r(\mu) := r(\mu_k)$ is the *range* of μ, and k is the *length* of μ which is denoted by $|\mu|$. The set of all finite paths of a E is denoted by $\operatorname{Path}(E)$. An infinite path is an infinite sequence of edges $\mu = \mu_1\mu_k \dots$ such that $r(\mu_i) = s(\mu_{i+1})$ for every $i \in \mathbb{N}$. The set of all infinite paths of E is denoted by E^∞.

A *preorder* $\ge$ on E^0 defined by: $v \ge w$ if there is a path $\mu \in \operatorname{Path}(E)$ such that $s(\mu) = v$ and $r(\mu) = w$. If $v \in E^0$ then the *tree* of v is the set $T(v) = \{w \mid w \in E^0, v \ge w\}$. A subset H of E^0 is called *hereditary* if $v \ge w$ and $v \in H$ imply $w \in H$. A hereditary set is *saturated* if every regular vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ is finite and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. For a hereditary subset H we denote by $\overline{H}$, the saturated closure of H, i.e., the smallest hereditary and saturated subset of E^0 containing H.

Let H be a nonempty hereditary and saturated subset of E^0.

Following [10] we define

$$F_E(H) := \{\alpha \in \operatorname{Path}(E) \mid s(\alpha_1), r(\alpha_i) \in E^0 \setminus H \text{ for } i < |\alpha|, r(\alpha_{|\alpha|}) \in H\}.$$

Given a nonempty hereditary and saturated subset H of E^0, define

$$H^{tc} = \{u \in E^0 \mid H \cap T(u) = \emptyset\}.$$

The subset H^{tc} is hereditary and saturated, and $H \cap H^{tc} = \emptyset$ (see [2, Remark 13]). We note that H^{tc} corresponds to H' from [6], $E^0 - \overline{H}$ from [5], and $H^{\perp}$ from [19].

4.2. **Leavitt path algebras.** For a graph E and a field K, the Leavitt path algebra of E, denoted by $L_K(E)$, is the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{e \mid e \in E^1\}$, and $\{e^* \mid e \in E^1\}$ with the coefficients in K, subject to the relations

(V) $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$,
(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,
(E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$,
(CK1) $e^*e' = \delta_{ee'}r(e)$ for all $e, e' \in E^1$,
(CK2) $\sum_{\{e\in E^1, s(e)=v\}} ee^* = v$, for every vertex $v \in Reg(E)$.

It can be proved that $L_K(E)$ is a unital ring if and only if E^0 is finite. If $^- : K \to K$ is an involution on K, then it is straightforward to see that the map $*$ given by

$$\Big(\sum_{i=1}^{n} k_i \alpha\beta^*\Big)^* = \sum_{i=1}^{n} \overline{k_i} \beta\alpha^*$$

defines the involution on $L_K(E)$ making it into a $*$-algebra. The *canonical* grading given to a Leavitt path algebra is a $\mathbb{Z}$-grading with the n-component

$$L_K(E)_n = \Big\{ \sum_i k_i \alpha_i \beta_i^* \Big| \alpha_i, \beta_i \in \text{Path}(E), k_i \in K, \text{ and } |\alpha_i| - |\beta_i| = n \text{ for all } i \Big\}.$$

4.3. **The Steinberg algebra model of a Leavitt path algebra.** We recall the construction of a groupoid $\mathcal{G}_E$ from an arbitrary graph E, which was introduced in [16] for row-finite graphs and generalized to arbitrary graphs in [17]. We use the notation of [10]. Define

$$X := E^{\infty} \cup \{\mu \in \text{Path}(E) \mid r(\mu) \in \text{Sink}(E)\} \cup \{\mu \in \text{Path}(E) \mid r(\mu) \in \text{Inf}(E)\},$$

and

$$\mathcal{G}_E := \{(\alpha x, |\alpha| - |\beta|, \beta x) \mid \alpha, \beta \in \text{Path}(E), x \in X, r(\alpha) = r(\beta) = s(x)\}.$$

A pair of elements in $\mathcal{G}_E$ is composable if and only if it is of the form $((x, k, y), (y, l, z))$ and then the composition and inverse maps are defined such that

$$(x, k, y)(y, l, z) := (x, k + l, z) \qquad \text{and} \qquad (x, k, y)^{-1} := (y, -k, x).$$

Thus $\mathcal{G}_E^{(0)} = \{(x, 0, x) \mid x \in X\}$, which we identify with X. Next we see how $\mathcal{G}_E$ can be viewed as an ample groupoid. For $\mu \in \text{Path}(E)$ define

$$Z(\mu) := \{\mu x \mid x \in X, r(\mu) = s(x)\} \subseteq X.$$

For $\mu \in \text{Path}(E)$ and a finite $F \subseteq s^{-1}(r(\mu))$, define

$$Z(\mu \setminus F) := Z(\mu) \cap \big(\mathcal{G}_E^{(0)} \setminus \big(\bigcup_{\alpha \in F} Z(\mu\alpha)\big).$$

The sets of the form $Z(\mu \setminus F)$ are a basis of compact open sets for a Hausdorff topology on $X = \mathcal{G}_E^{(0)}$ by [20, Theorem 2.1].

For each $\mu, \upsilon \in \text{Path}(E)$ with $r(\mu) = r(\upsilon)$, and finite $F \subseteq \text{Path}(E)$ such that $r(\mu) = s(\alpha)$ for all $\alpha \in F$, define

$$Z(\mu, \upsilon) := \{(\mu x, |\mu| - |\upsilon|, \upsilon x) \mid x \in X, r(\mu) = s(x)\},$$

and then

$$Z((\mu,\upsilon)\setminus F):=Z(\mu,\upsilon)\cap\big(\mathcal{G}_E\setminus(\bigcup_{\alpha\in F}Z(\mu\alpha,\upsilon\alpha)\big).$$

The collection $Z((\mu,\upsilon)\setminus F)$ forms a basis of compact open bisections that generates a topology such that $\mathcal{G}_E$ is a Hausdorff ample groupoid.

Observe that the map $c:\mathcal{G}_E\to\mathbb{Z}$ given by $c(x,k,y)=k$ is a continuous cocyle such that

$$\mathrm{Iso}(c^{-1}(0))=c^{-1}(0)\cap\{\gamma\in\mathcal{G}_E\mid s(\gamma)=r(\gamma)\}=\mathcal{G}_E^{(0)}.$$

Thus $c^{-1}(0)$ is a principal groupoid (and hence an effective groupoid).

[12, Example 3.2] shows that the map $\pi:L_R(E)\to A_R(\mathcal{G}_E)$ such that

$$\pi(\mu\upsilon^*-\sum_{\alpha\in F}\mu\alpha\alpha^*\upsilon^*)=1_{Z((\mu,\upsilon)\setminus F)}$$

extends to a $\mathbb{Z}$-graded $*$-isomorphism where for $n\in\mathbb{Z}$

$$A_R(\mathcal{G}_E)_n:=\{f\in A_R(\mathcal{G}_E)\mid f(x,k,y)\neq 0\Rightarrow k=n\}.$$

Using these results, and Proposition 3.1, we obtain the following. It is a generalization of [2, Proposition 11] for Leavitt path algebras over commutative $*$-rings.

Corollary 4.1. *Let E be an arbitrary graph and R a commutative unital $*$-ring. Then the following are equivalent.*

(i) *The involution on R is proper;*
(ii) *The involution on $L_R(E)$ is graded proper;*
(iii) *The involution on $L_R(E)$ is graded semiproper.*

In particular, if K is a field with involution, then the involution on $L_R(E)$ is graded proper.

Following [11, Definition 3.2] for a hereditary and saturated subset H we define

$$U_H:=\{x\in\mathcal{G}_E^{(0)}\mid r(x_n)\in H\text{ for some }n\geq 0\}.$$

By [11, Lemma 3.4] U_H is an open subset of $\mathcal{G}_E^{(0)}$.

Proposition 4.2. *Let E be a finite graph, and H be a hereditary and saturated subset of E^0. Then*

(i) *U_H is compact if and only if $F_E(H)$ is finite.*
(ii) *$\mathrm{Int}(\mathcal{G}_E^{(0)}\setminus U_H)$ is compact if and only if $F_E(H^{tc})$ is finite.*

Proof. (i) Follows from [11, Proposition 2.3].

(ii) Using Part (i) by replacing H with H^{tc}, we obtain that $F_E(H^{tc})$ is finite if and only if $U_{H^{tc}}$ is compact. Thus to prove (i), we only need to show that $U_{H^{tc}}=\mathrm{Int}(\mathcal{G}_E^{(0)}\setminus U_H)$. First, we show $U_{H^{tc}}\subseteq\mathrm{Int}(\mathcal{G}_E^{(0)}\setminus U_H)$. Note that [11, Lemma 2.1] implies that

$$U_{H^{tc}}=\big(\bigcup_{v\in H^{tc}}Z(v)\big)\cup\big(\bigcup_{\alpha\in F_E(H^{tc})}Z(\alpha)\big).$$

Thus we need to show that $Z(v)\subseteq\mathrm{Int}(\mathcal{G}_E^{(0)}\setminus U_H)$ for each $v\in H^{tc}$, and $Z(\alpha)\subseteq\mathrm{Int}(\mathcal{G}_E^{(0)}\setminus U_H)$ for each $\alpha\in F_E(H^{tc})$. Fix $v\in H^{tc}$, and let $x\in Z(v)$. Then $s(x)=v\in H^{tc}$, so $r(x_i)\notin H$ for each i. This implies $x\notin U_H$, and that $x\in\mathcal{G}_E\setminus U_H$. Thus $Z(v)\subseteq\mathcal{G}_E^{(0)}\setminus U_H$, and since $Z(v)$ is open we obtain that $Z(v)\subseteq\mathrm{Int}(\mathcal{G}_E^{(0)}\setminus U_H)$.

Now, fix $\alpha \in F_E(H^{tc})$, and let $\alpha x \in Z(\beta)$. Then $s(\alpha), r(\alpha_i) \notin H^{tc}$ for $i < |\alpha|$, and $r(\alpha_{|\alpha|}) = s(x) \in H^{tc}$. Thus $r(x_i) \notin H$ for each i, and that $s(\alpha), r(\alpha_i) \notin H$ for $i < |\alpha|$. Thus $\alpha x \notin U_H$, and so $\alpha x \in \mathcal{G}_E \setminus U_H$. Thus $Z(\alpha) \subseteq \mathcal{G}_E^{(0)} \setminus U_H$, and since $Z(\alpha)$ is open, $Z(\alpha) \subseteq \mathrm{Int}(\mathcal{G}_E^{(0)} \setminus U_H)$. Hence we conclude that $U_{H^{tc}} \subseteq \mathrm{Int}(\mathcal{G}_E^{(0)} \setminus U_H)$. For the reverse inclusion, let $x \in \mathrm{Int}(\mathcal{G}_E^{(0)} \setminus U_H)$. Then $s(x) \notin H$, and for every initial subpath α of x, $\alpha \notin F_E(H)$. If $s(x) \in H^{tc}$, then $x \in Z(s(x))$, and that $x \in U_{H^{tc}}$. Otherwise, we claim that there is a subpath α of x such that $\alpha \in F_E(H^{tc})$. Towards a contradiction, suppose that $r(x_i) \notin H^{tc}$ for each i. As $x \in \mathrm{Int}(\mathcal{G}_E^{(0)} \setminus U_H)$, there is a basic open set $Z(\alpha)$ such that $x \in Z(\alpha) \subseteq \mathcal{G}_E^{(0)} \setminus U_H$. This yields that $r(\alpha) \notin H^{tc}$. Then $T(r(\alpha)) \cap H \neq \emptyset$. So there is $w \in H$ such that $v \geq w$. Then there exists $\beta \in \mathrm{Path}(E)$ such that $s(\beta) = r(\alpha)$ and $r(\beta) = w$. Thus $\alpha\beta \in Z(\alpha)$ and $\alpha\beta \in U_H$, a contradiction. Hence there is a subpath α of x such that $\alpha \in F_E(H^{tc})$. Then $x \in Z(\alpha)$, and that $x \in U_{H^{tc}}$. Therefore, $\mathrm{Int}(\mathcal{G}_E^{(0)} \setminus U_H) \subseteq U_{H^{tc}}$ as desired. □

Now we have the following result about the characterization of graded quasi-Baer $*$ Leavitt path algebras.

Corollary 4.3 ([2], Corollary 20, and [19], proposition 4.5)**.** *Let E be a finite graph and K a field with involution. Then the following are equialent.*

- (i) *The Leavitt path algebra $L_K(E)$ is a graded quasi-Baer $*$-ring;*
- (ii) *For each nonempty hereditary and saturated subset H of E^0, no cycle outside H leads to H or no cycle outside H^{tc} leads to H^{tc};*
- (iii) *For each nonempty hereditary and saturated subset H of E^0, no vertex of a cycle of E with vertices in $E^0 \setminus (H \cup H^{tc})$ emits a path to H and a path to H^{tc}.*
- (iv) *For each nonempty hereditary and saturated subset H of E^0, the saturated closure of $H^\perp \cup H^{\perp\perp}$ is E^0;*
- (v) *For each nonempty hereditary and saturated subset H of E^0, $H^\perp \vee H^{\perp\perp} = E^0$.*

Proof. (i)⇔(ii) Note that when E is a finite graph, $F_E(H)$ is finite if and only if no cycle outside H leads to H or no cycle outside H^{tc} leads to H^{tc}. Now Proposition 4.2 and Theorem 3.3 by considering the graph groupoid $\mathcal{G}_E$ yield the result.

(ii)⇔(iii) It is clear.

(iii)⇒(iv) We show that if (iii) holds and (iv) fails, we arrive to a contradiction. Assume that (iv) fails for some H and let S denote the saturated closure of $H^\perp \cup H^{\perp\perp}$. So, there is $v_0 \in E^0 \setminus S$. If v_0 were a sink, then $v_0 \notin H^\perp$ would imply that $v_0 \in H^{\perp\perp} \subseteq S$. As $v_0 \notin S$, v_0 necessarily emits some edges. Their ranges are not all in S because $v_0 \notin S$ and S is saturated, so at least one of these edges, say e_1, has its range outside of S. If $r(e_1) = v_0$, $c = e_1$ is a cycle. If $v_1 = r(e_1) \neq v_0$, one can use the same argument as the one used to show that v_0 emits edges to conclude that v_1 emits edges (if v_1 were a sink, then $v_1 \notin H^\perp$ implies that $v_1 \in H^{\perp\perp}$ and that would contradict $v_1 \notin S$). So, there is e_2 which v_1 emits. If $r(e_2) = v_0$, then e_1e_2 is a closed path with all of its vertices outside of S, so there is also a cycle c which contains v_0 with all its vertices outside of S. If $v_2 = r(e_2) \neq v_0$, we continue the process. As E^0 is finite, this process eventually ends and we arrive to a cycle c which contains v_0 and having all its vertices outside of S. The cycle c has exits since otherwise $v_0 \notin H^\perp$ would imply that all vertices of c are in $H^{\perp\perp} \subseteq S$. By the assumption that (ii) holds, the ranges of these exits either all connect to H or all connect to $H^{tc}(= H^\perp)$. In the first case, the vertices of c would be in $H^{\perp\perp}$. In the second case, the vertices of c would be in $H^\perp$. In each case, the vertices of c end up being in S which is a contradiction.

(iv)⇒(iii) We show that if (iv) holds and (iii) fails, then we arrive to a contradiction. If (iii) fails for some H, let c be a cycle whose existence the failure of (iii) guarantees. As c emits paths to H, the vertices of c are not in $H^{\perp}$. As c emits paths to $H^{\perp}$, the vertices of c are not in $H^{\perp\perp}$. Thus, the vertices of c are outside of $H^{\perp}\cup H^{\perp\perp}$. As (iv) holds, the vertices of c are in the saturated closure of $H^{\perp}\cup H^{\perp\perp}$ which implies that any path from $v_0 = s(c)$ connects to $H^{\perp}\cup H^{\perp\perp}$. Thus, for v_0 there is n such that $v_0 \in (H^{\perp}\cup H^{\perp\perp})_n$, where

$$(H^{\perp}\cup H^{\perp\perp})_0 = H^{\perp}\cup H^{\perp\perp}, \text{ and}$$

$$(H^{\perp}\cup H^{\perp\perp})_n = (H^{\perp}\cup H^{\perp\perp})_{n-1} \cup \{v \in \mathrm{Reg}(E) \mid \{r(e) \mid s(e) = v\} \subseteq (H^{\perp}\cup H^{\perp\perp})_{n-1}\}$$

This means that the range v_1 of the edge e_0 which v_0 emits in c is in $(H^{\perp}\cup H^{\perp\perp})_{n-1}$. Repeating this argument for v_1, we arrive to v_2 in c which is in $(H^{\perp}\cup H^{\perp\perp})_{n-2}$. Eventually, we arrive to v_n in c which is in $(H^{\perp}\cup H^{\perp\perp})_0 = H^{\perp}\cup H^{\perp\perp}$. Since $H^{\perp}\cup H^{\perp\perp}$ is hereditary, this implies that every vertex of c is in $H^{\perp}\cup H^{\perp\perp}$. This is a contradiction since c was chosen with vertices outside of $H^{\perp}\cup H^{\perp\perp}$.

(iv)⇔(v) It is evident since E is finite. □

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