(q, m)-Dichotomy and Periodic Solutions for Sylvester Matrix Volterra Integro-Dynamic system on time scales

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Abstract

In this paper, we investigate the (q, m)-dichotomy and periodic solutions for Sylvester matrix Volterra integro-dynamic systems defined on arbitrary time scales. By employing the vec operator, the original Sylvester matrix system is transformed into an equivalent Kronecker product system. The existence and uniqueness of periodic solutions are established using fixed point theory and functional analysis techniques. The study leverages the concept of (q, m)-dichotomy, a generalization of exponential dichotomy, to ensure the stability and boundedness of solutions. The key results include theorems demonstrating the uniqueness of bounded and periodic solutions under specific conditions.

Keywords: (q, m)-dichotomy, Periodic solution, Volterra integro-dynamic system, Time scales.

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1 Introduction

The study of matrix differential systems on time scales has attracted significant interest due to its capacity to unify discrete and continuous dynamical behaviors within a single analytical framework. Among the qualitative tools used to understand the behavior of such systems, the exponential dichotomy plays a central role, offering essential criteria for the stability and structure of solution spaces. The foundational work by Kovacs et al. [1] and Lupa and Megan [2] established the theoretical underpinnings of the (h, k) dichotomy and exponential dichotomy in Banach spaces, which have since informed subsequent developments in dynamic system theory on time scales.

In the context of functional dynamic equations, considerable progress has been made in extending dichotomy theory to more generalized frameworks. Xia et al. [3, 4] investigated dynamic systems with piecewise constant arguments and established the existence of almost periodic solutions, while Pinto [5] and Smirnova et al. [6] analyzed nonlinear and quasilinear systems governed by dichotomous structures. Moreover, the interaction between dichotomy and functional differential equations has been explored in depth by Huy and Dang [7] and Jendoubi [8], who examined the existence of almost periodic solutions under perturbations and delays. Complementarily, Ait Dads and Arino [9] demonstrated that such periodic structures persist even in the presence of nonlinearities, provided that exponential dichotomy conditions are satisfied.

The theoretical framework has been solidified by comprehensive treatments of periodicity and stability in the monographs of Martin and Peterson [10, 11], while Adıvar [12] further generalized periodicity concepts to accommodate systems arising in control, biology, and neural dynamics. Recent computational advances have complemented this theoretical development, and Sadek et al. [13–16] propose efficient numerical algorithms, such as the Arnoldi extended block and fractional backward differentiation methods, to solve large-scale matrix and Sylvester-type differential systems.

Periodic and almost periodic solutions remain pivotal in understanding dynamic behavior in various disciplines. These types of solutions are particularly relevant in the analysis of Volterra integrodynamic systems, impulsive differential equations, and ecological or epidemiological models where recurrence and boundedness are essential characteristics [17–22]. Biological models and neural networks have further emphasized the need for periodic analysis on time scales. Kumar et al. [23] examined synchronization in Cohen–Grossberg neural networks under time-varying delays and impulses, while Chintamaneni et al. [24–26] explored periodicity in Volterra integro-dynamic Sylvester systems using Kronecker product representations. Additional research has focused on periodic and asymptotic behaviors in nonlinear Volterra-type systems with applications to population biology [27–29].

In this context, recent studies have specifically addressed the periodic behavior of Sylvester matrix systems and their bounded solutions. Adıvar and Raffoul [30] developed a natural setup for studying periodic solutions on time scales, which has been influential in subsequent investigations. Appa Rao and Prasad [31] introduced the concept of Ψ-boundedness for Sylvester matrix dynamic systems, while Malik and Kumar [32] extended these results to Volterra integro-dynamic systems with non-instantaneous impulses. Furthermore, Novaes and Pereira [33] enriched Floquet theory on time scales by analyzing periodic and antiperiodic aspects through the Floquet normal form.

A crucial tool in the structural and numerical analysis of matrix systems is the Kronecker product formulation, as comprehensively discussed in Graham's classical work [34]. This algebraic approach simplifies complex matrix differential systems into equivalent linear algebraic forms, facilitating both theoretical proofs and computational implementation. It has been particularly useful for studying periodic and bounded solutions in engineering and biological models governed by Sylvester-type systems.

In this paper, we build upon the aforementioned theoretical and computational developments to investigate the periodic behavior of nonlinear Sylvester-type Volterra integro-dynamic systems on time scales. We incorporate the Kronecker product technique and apply (q, m)-dichotomy theory to analyze the stability and recurrence properties of solutions. The framework developed here is applied to a biologically motivated model to illustrate the effectiveness of our approach in capturing the dynamics of systems with discrete-continuous interactions.

$$X^{\Delta}(t) = P_1(t)X(t) + X(t)P_2(t) + \int_{-\infty}^{t} A(t,s)F(X(s))\Delta s + H(t). \tag{1}$$

Where $P_1(t), P_2(t)$ are the matrix order $n \times n$ with ω-periodic. $A(t,s)$ and $H(t)$ are continuous functions. $F(X(s))$ is a non linear function with order $n \times n$.

2 Preliminaries

In this section, we provide the fundamental concepts and outcomes that are necessary for our purposes. For conciseness, we presume an understanding of the fundamental characteristics of the Delta derivative and the Delta integral. For further information, see [10, 11].

Let $\mathbb{T}$ be an time scales, defined as a nonempty closed subset of the real numbers. Let $\mathbb{R}$ denote the set of real numbers. The function $\sigma : \mathbb{T} \to \mathbb{T}$ represents the forward jump operator of the set $\mathbb{T}$. The graininess function is defined as $\mu(t) = \sigma(t) - t$. This paper assumes that the time scale $\mathbb{T}$ is unbounded in both the upper and lower directions. The notation $C_{rd}(\mathbb{T}, \mathbb{R})$ represents the collection of rd-continuous functions. Let $g : \mathbb{T} \to \mathbb{R}$. The set $R^+(\mathbb{T}, \mathbb{R}) := \{\varphi \in C_{rd}(\mathbb{T}, \mathbb{R}) : 1 + \mu(t)\varphi(t) > 0,\ t \in \mathbb{T}\}$ denotes the space of positively regressive functions.

Let

$$|\varphi|^* := \sup_{t\in\mathbb{T}}(\varphi(t)), \quad |\varphi|_* := \inf_{t\in\mathbb{T}}(\varphi(t)),$$

for any bounded function φ in $C_{rd}(\mathbb{T}, \mathbb{R})$, and we define

$$\kappa_1 := \min\{t \in \mathbb{T} \cap \mathbb{R}^-\}, \quad \kappa_2 := \max\{t \in \mathbb{T} \cap \mathbb{R}^+\}.$$

Then we have

$$\lim_{t\to\infty} e_{\ominus\varphi}(t,0) = 0, \quad \lim_{t\to-\infty} e_{\varphi}(t,0) = 0,$$

$$e_{\varphi}(t,\kappa_1) \geq 1 \text{ for } t \leq \kappa_1, \quad e_{\varphi}(t,\kappa_2) \geq 1 \text{ for } t \geq \kappa_2,$$

where $0 < |\varphi|_*$.

Definition 2.1 [24] A time scales refers to a system or framework that measure and organizes time in a specific manner. $\mathbb{T}$ is said to be periodic if there exists a positive value ω such that $(t \pm \omega)$ belongs to $\mathbb{T}$ for every t in $\mathbb{T}$. If $\mathbb{T}$ is equal to $\mathbb{R}$, the smallest positive value of ω is known as the period time scales .

Example 2.1 [12] Here are several instances of periodic time scales:

a. Let $\mathbb{T} = \mathbb{Z}$, which exhibits a periodicity characterized by $\omega = 1$.
b. Let $\mathbb{T} = h\mathbb{Z}$, which exhibits a period of $\omega = h$.
c. $\mathbb{T} = \mathbb{R}$,
d. Let $\mathbb{T} = \bigcup_{i=-\infty}^{\infty}[(2i-1)h, 2ih]$, where $h > 0$ and the period is $\omega = 2h$.
e. The set $\mathbb{T} = \{t = k - qm : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$, with the condition that $0 < q < 1$, exhibits periodic behavior. Let ω be equal to 1.

Definition 2.2 [24] Consider a time scales $\mathbb{T}$ that is distinct from $\mathbb{R}$ and a periodic time scales with period of ω. A function $g : \mathbb{T} \to \mathbb{R}$ is considered periodic with period ω if there exists a natural integer n such that $T = n\omega$, and for all $t \in \mathbb{T}, g(t \pm \omega) = g(t)$. Additionally, ω is the smallest number for which $g(t \pm \omega) = g(t)$. If $\mathbb{T} \to \mathbb{R}$, we define g as periodic with period $\omega > 0$ if ω is the smallest positive number for which $g(t \pm \omega) = g(t)$ holds for every $t \in \mathbb{T}$.

Remark 2.1 [24] If time scales $\mathbb{T}$ is periodic and has a period w, then the equation $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$ is true. Due to the fact that graininess function μ satisfies the equation $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$, it may be concluded that it is a periodic function with some period ω.

The fundamentals of Kronecker products and vec operations are now presented, as they play an important role in linear algebra and the investigation of matrix problems.

Definition 2.3 [34] Let $P_1 \in \mathbb{C}^{m\times n}$ and $P_2 \in \mathbb{C}^{r\times s}$, then the $mr \times ns$ matrix defined by:

$$(P_1 \otimes P_2) = \begin{pmatrix} a_{11}P_2 & a_{12}P_2 & \cdots & a_{1n}P_2 \\ a_{21}P_2 & a_{22}P_2 & \cdots & a_{2n}P_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}P_2 & a_{m2}P_2 & \cdots & a_{mn}P_2 \end{pmatrix}$$

is called the Kronecker product of P_1 and P_2.

Definition 2.4 [34] Let $X \in \mathbb{C}^{m\times n}$ and $X = (x_{ij})$. Then the vector obtained by stacking the columns of X into one vector is denoted by $\mathrm{vec}(X)$:

$$\mathrm{vec}(X) = (x_{11},\, \dots,\, x_{m1},\, x_{12},\, \dots,\, x_{m2},\, \dots,\, x_{1n},\, \dots,\, x_{mn})^\top.$$

If $P_1 \in \mathbb{C}^{m\times m}$ and $P_2 \in \mathbb{C}^{n\times n}$, then it can be shown that:

$$\mathrm{vec}(P_1X + XP_2) = ((I_n \otimes P_1) + (P_2^\top \otimes I_m))\,\mathrm{Vec}(X).$$

Now, we Applying the vec operator to both sides of the system (1) and using the Kronecker product techniques, the system is transformed into a first-order vector Volterra integro-dynamic system. Specifically, the matrix system is equivalent to

$$z^{\Delta}(t) = P(t)z(t) + \int_{-\infty}^{t} (I \otimes A)(t,s)B(z(s))\Delta s + \hat{h}(t), \tag{2}$$

where $P(t) = [(P_2^\top \otimes I_n) + (I_n \otimes P_1)]$ be an $n^2 \times n^2$ matrix that is rd-continuous and ω-periodic on time scales and here $P_2^\top$ represents the transpose of the matrix P_2. Define the transformation $z(t) = Vec(X(t))$ where Vec denotes the vectorization operator. Let $B(z(s)) = Vec(F(X(s)))$ represent the vectorized nonlinear component, and set $\hat{h}(t) = Vec(H(t))$ as the vectorized form of the inhomogeneous term.

3 (q, m)-dichotomy for Kronecker product system

In this section, the substantial relevance of exponential dichotomy theory in dynamic systems and functional equations is brought to light. However, the conditions for the exponential dichotomy are somewhat more strict than those for the other dichotomies. In particular, take a look at the linear system that comes after it.

$$z^{\Delta}(t) = P(t)z(t), \tag{3}$$

when there is a projection matrix B and positive constants $M_-, M_+ \in \mathbb{R}^{n^2}$, in addition to a constant α, the linear system that is denoted by the system (3) is characterized by an exponential dichotomy. This dichotomy is characterized by the fact that the following inequalities are satisfied:

$$\begin{cases} \|\Psi(t)B\Psi^{-1}(s)\| \le M_- e_{-\alpha(t-s)}, t \ge s \\ \|\Psi(t)(1-B)\Psi^{-1}(s)\| \le M_+ e_{\alpha(t-s)}, t \le s, \end{cases} \tag{4}$$

where $\Psi(t)$ denotes the basic matrix of the system (3). A solution matrix $\Psi(t)$ is considered a basic matrix if $\Psi(0) = I$, I is an identity matrix.

Definition 3.1 Consider two continuous positive functions, namely $q(t) : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ and $m(t) : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$. Both of these functions are continuous. It is possible to assert that the linear system (3) is categorized as having a dichotomy of the form (q, m) if there exists a projection matrix B and positive constants $M_-, M_+ \in \mathbb{R}^{n^2}$, α that are such that the accompanying inequalities must be satisfied:

$$\begin{cases} \|\Psi(t)B\Psi^{-1}(s)\| \leq M_- q(t)q(s)^{-1}e_{-\alpha(t-s)}, t \geq s, \\ \|\Psi(t)(1-B)\Psi^{-1}(s)\| \leq M_+ q(t)q(s)^{-1}e_{\alpha(t-s)}, t \leq s \end{cases} \tag{5}$$

where $\Psi(t)$ is the basic matrix of the system (3) with the condition $\Psi(0) = I$.

Remark 3.1 It should come as no surprise that the exponential dichotomy is a particular instance of the the (q, m)-dichotomy. Because the (q, m)-dichotomy produces an exponential dichotomy, it is necessary to assume that $q = m = 1$.

Definition 3.2 A compensation law is said to be satisfied by q and m if there is a positive constant $C_{q,m}$ such that

$$q(t)q(s)^{-1} \leq C_{(q,m)} m(t)m(s)^{-1}, \forall t \geq s. \tag{6}$$

Definition 3.3 If the system has a (q, q)-dichotomy, it is referred to as a q-system.

Definition 3.4 If there is a projection B and $\mu > 0$ such that the system (3) is considered to meet an integral condition, then

$$\int_{-\infty}^{t} \|\Psi(t)B\Psi(s)^{-1}\|\Delta s + \int_{t}^{\infty} \|\Psi(t)(1-B)\Psi(s)^{-1}\|\Delta s \leq \mu, \tag{7}$$

uniformly in $t \in \mathbb{R}^{n^2}$.

Definition 3.5 We claim that (q, m) are integral if there is a value of $\mu > 0$ that matches the condition.

$$M_- \int_{-\infty}^{t} q(t)q(s)^{-1}e_{-\alpha(t-s)}\Delta s + K_+ \int_{t}^{\infty} m(t)m(s)^{-1}e_{\alpha(t-s)}\Delta s \leq \mu, \tag{8}$$

uniformly in $t \in \mathbb{R}^{n^2}$.

Remark 3.2 It is obvious that the integral condition must be satisfied if the system in question possesses a (q, m)-dichotomy with integral functions.

Remark 3.3 If we consider the case of $q = m = 1$, which is known as the exponential dichotomy, it is evident that the requirement of equation (7) is always satisfied. $\mu = \frac{1}{\alpha}[M_- + M_+]$ is a valid expression that can be utilized.

Theorem 3.1 *If the system* (3) *is able to allow* (q, m)*-dichotomy of the type* (5)*, then the zero solution* $z(t) = 0$ *is the sole bounded solution of system* (3)*. This is true according to the equation* (7)*.*

Proof Let D_0 denote the set of all initial conditions $\zeta \in \mathbb{R}^{n^2}$ for which the solution $z(t)$ of (3) is bounded on $\mathbb{T}$.

Assume first that $(1-B)\zeta \neq 0$. Define

$$\phi(t) = \frac{1}{\|\Psi(t)(1-B)\zeta\|},$$

and using the identity $(1-B)^2 = I - B$, we rewrite the integral as

$$\int_t^\infty \phi(s)\Psi(t)(1-B)\zeta\,\Delta s = \int_t^\infty \Psi(t)(1-B)\Psi^{-1}(s)\Psi(s)(1-B)\zeta\phi(s)\,\Delta s. \tag{9}$$

Taking norms on both sides and applying the (q,m)-dichotomy along with the equation (7), we obtain

$$\int_t^\infty \phi(s)\,\Delta s \leq \mu\phi(t), \quad \text{uniformly in } t. \tag{10}$$

By differentiating both sides of the inequality and using a comparison theorem, we get

$$-\phi(t) \leq \mu\phi^\Delta(t),$$

which implies

$$\phi(t) \geq \phi(0)e^{-\frac{t}{\mu}}.$$

This contradicts the boundedness of $\|\Psi(t)(1-B)\zeta\|$ since $\phi(t) \to 0$ would require that norm to be unbounded. Thus, $(1-B)\zeta = 0$ must hold.

Now, assume $P\zeta \neq 0$. Define

$$\phi(t) = \frac{1}{\|\Psi(t)B\zeta\|},$$

and apply a similar argument over the interval $(-\infty, t]$. This leads to

$$\liminf_{s\to-\infty} \phi(s) = 0,$$

This suggests that the norm of the product $|\Psi(t)B\zeta|$ cannot remain finite unless the vector $B\zeta$ is the zero vector. Therefore, the only way to resolve the contradiction is for $B\zeta$ to equal zero.

Hence, both $(1-B)\zeta = 0$ and $B\zeta = 0$ imply $\zeta = 0$. Therefore, the only initial condition leading to a bounded solution is $\zeta = 0$, and so $z(t) \equiv 0$ is the only bounded solution. ☐

Theorem 3.2 *If the linear system (3) possesses a (q,m)-dichotomy of the form equation (5), then the projection matrix B is unique, which means that the pair (q,m) is the only thing that can determine B. This is supported by the equation (7).*

Proof Let us assume that there is another projection matrix $\widetilde{B}$ and positive constants $\widetilde{M}_\pm, \widetilde{\alpha}$ that fulfill the criteria of the (q,m)-dichotomy, as described in equation (5), i.e.,

$$\begin{aligned} &\|\Psi(t)\widetilde{B}\Psi^{-1}(s)\| \leq \widetilde{M}_- q(t)q(s)^{-1}e_{-\widetilde{\alpha}(t-s)}, \quad t \geq s, \\ &\|\Psi(t)(I-\widetilde{B})\Psi^{-1}(s)\| \leq \widetilde{M}_+ m(t)m(s)^{-1}e_{\widetilde{\alpha}(t-s)}, \quad t \leq s, \end{aligned} \tag{11}$$

and suppose the equation (7) holds:

$$\widetilde{M}_- \int_{-\infty}^t q(t)q(s)^{-1}e_{-\widetilde{\alpha}(t-s)}\,\Delta s + \widetilde{M}_+ \int_t^\infty m(t)m(s)^{-1}e_{\widetilde{\alpha}(t-s)}\,\Delta s \leq \mu. \tag{12}$$

From the first integral in the equation (12), define $\phi(t) = q(t)e^{-\alpha t}$. Then:

$$\int_{-\infty}^{t} \frac{1}{\phi(s)} \Delta s \leq M_{-}^{-1} \mu \frac{1}{\phi(t)}.$$

Differentiating both sides and applying the comparison principle yields:

$$\phi'(t) \leq -\frac{M_{-}}{\mu} \phi(t),$$

implying:

$$\phi(t) \leq \phi(0) e^{-\frac{M_{-}}{\mu} t}.$$

Thus, there exists $K_1 > 0$ such that

$$\phi(t) = q(t)e^{-\alpha t} \leq K_1, \quad \forall t \geq 0. \tag{13}$$

Similarly, from the second integral in the equation (12), define $\varphi(t) = m(t)e^{\alpha t}$. Then:

$$\varphi'(t) \leq \frac{M_{+}}{\mu} \varphi(t) \quad \Rightarrow \quad \varphi(t) \leq \varphi(0) e^{\frac{M_{+}}{\mu} t},$$

and hence there exists $K_2 > 0$ such that:

$$\varphi(t) = m(t)e^{\alpha t} \leq K_2, \quad \forall t \geq 0. \tag{14}$$

Likewise, there exist constants $\widetilde{K}_1, \widetilde{K}_2 > 0$ such that:

$$q(t)e^{-\widetilde{\alpha} t} \leq \widetilde{K}_1, \quad \forall t \leq 0, \qquad m(t)e^{\widetilde{\alpha} t} \leq \widetilde{K}_2, \quad \forall t \geq 0. \tag{15}$$

Now take any $\zeta \in \mathbb{R}^{n^2}$.

For $t \geq 0$, using these equations (5), (11), and (13), we get the following:

$$\|\Psi(t)B(I - \widetilde{B})\zeta\| \leq M_{-} q(t) q(0)^{-1} e^{-\alpha t} \|(I - \widetilde{B})\zeta\| \leq M_{-} q(0)^{-1} K_1 \|(I - \widetilde{B})\zeta\|. \tag{16}$$

For $t \leq 0$, using these equations (5), (11), and (15):

$$\begin{aligned} \|\Psi(t)B(I - \widetilde{B})\zeta\| &\leq \|\Psi(t)B\Psi^{-1}(t)\| \cdot \|\Psi(t)(I - \widetilde{B})\Psi^{-1}(0)\| \cdot \|\Psi(0)(I - \widetilde{B})\zeta\| \\ &\leq M_{-} \cdot \widetilde{M_{+}} m(t) m(0)^{-1} e^{\widetilde{\alpha} t} \cdot \|(I - \widetilde{B})\zeta\| \\ &\leq M_{-} \widetilde{M_{+}} m(0)^{-1} \widetilde{K}_2 \|(I - \widetilde{B})\zeta\|. \end{aligned} \tag{17}$$

From the equations(16) and (17), we conclude that for any $\zeta \in \mathbb{R}^{n^2}$, the function $z(t) = \Psi(t)B(I - \widetilde{B})\zeta$ is bounded. By Theorem 3.1, the only bounded solution is zero, hence:

$$B(I - \widetilde{B})\zeta = 0 \quad \text{for all } \zeta \in \mathbb{R}^{n^2},$$

which implies $B(I - \widetilde{B}) = 0$, i.e., $B = B\widetilde{B}$.

A similar argument gives $(I - B)\widetilde{B} = 0$, i.e., $\widetilde{B} = B\widetilde{B}$. Hence:

$$B = B\widetilde{B} = \widetilde{B}.$$

This proves the uniqueness of the projector B. □

Theorem 3.3 *Let us assume that the linear system (2) possesses a (q, m)-dichotomy of the form (5) with a projection matrix B. This is based on the equation (7). If moreover, the matrix function $P(t)$ is ω-periodic (i.e., $P(t+\omega) = P(t)$), and $\Psi(t)$ is a basic matrix solution of (2), then the matrix-valued function $\Psi(t)B\Psi^{-1}(t)$ is ω-periodic.*

Proof Since the system is ω-periodic, $\Psi(t+\omega)$ is also a fundamental matrix solution of (2). Therefore, there exists a nonsingular constant matrix C such that

$$\Psi(t+\omega) = \Psi(t)C.$$

Evaluating at $t = 0$ and using $\Psi(0) = I$, we obtain

$$C = \Psi(\omega),$$

so that

$$\Psi(t+\omega) = \Psi(t)\Psi(\omega). \tag{18}$$

Since the system admits a (q, m)-dichotomy with projection B, the dichotomy conditions are satisfied. We now consider the transformed projector:

$$\widetilde{B} := \Psi(\omega)B\Psi^{-1}(\omega).$$

We claim that $\widetilde{B}$ also satisfies the same dichotomy bounds. Indeed, for $t \geq s$:

$$\begin{aligned} \|\Psi(t)\widetilde{B}\Psi^{-1}(s)\| &= \|\Psi(t)\Psi(\omega)B\Psi^{-1}(\omega)\Psi^{-1}(s)\| \\ &\leq M_{-}q(t)q(s)^{-1}e^{-\alpha(t-s)}. \end{aligned}$$

Similarly, for $t \leq s$:

$$\begin{aligned} \|\Psi(t)(I-\widetilde{B})\Psi^{-1}(s)\| &= \|\Psi(t)\Psi(\omega)(I-B)\Psi^{-1}(\omega)\Psi^{-1}(s)\| \\ &\leq M_{+}m(t)m(s)^{-1}e^{\alpha(t-s)}. \end{aligned}$$

Thus, $\widetilde{B}$ satisfies the same dichotomy bounds as B, and it is a projection. By the uniqueness of the dichotomy projection (Theorem 3.2), we conclude that

$$\Psi(\omega)B\Psi^{-1}(\omega) = B.$$

Now using this result and (18), we have

$$\begin{aligned} \Psi(t+\omega)B\Psi^{-1}(t+\omega) &= \Psi(t)\Psi(\omega)B\Psi^{-1}(\omega)\Psi^{-1}(t) \\ &= \Psi(t)B\Psi^{-1}(t). \end{aligned}$$

Hence, the function $\Psi(t)B\Psi^{-1}(t)$ is ω-periodic.

Consider the following system

$$z^{\Delta}(t) = P(t)z(t) + \hat{h}(t) \tag{19}$$

while $P(t)$ represents a rd-continuous matrix function, $\hat{h}(t)$ represents a rd-continuous bounded function. □

Theorem 3.4 *Given the equation* (7)*, let us assume that the homogeneous linear system*

$$z^{\Delta}(t) = P(t)z(t)$$

possesses a (q, m)*-dichotomy with associated projection* B*. Then, the nonhomogeneous system*

$$z^{\Delta}(t) = P(t)z(t) + \hat{h}(t) \tag{20}$$

has a unique bounded solution, given by

$$z(t) = \int_{-\infty}^{t} \Psi(t)B\Psi^{-1}(s)\hat{h}(s)\Delta s - \int_{t}^{\infty} \Psi(t)(I-B)\Psi^{-1}(s)\hat{h}(s)\Delta s, \tag{21}$$

where $\Psi(t)$ *is a fundamental matrix solution of the homogeneous system.*

Proof Let

$$z(t) := \int_{-\infty}^{t} \Psi(t)B\Psi^{-1}(s)\hat{h}(s)\Delta s - \int_{t}^{\infty} \Psi(t)(I-B)\Psi^{-1}(s)\hat{h}(s)\Delta s.$$

It is straightforward to verify that $z(t)$ satisfies the nonhomogeneous system (20) via the variation of constants formula.

From the equation (7) and boundedness of the dichotomy estimates, we have

$$\|z(t)\| \leq \mu\|h\|_\infty,$$

which implies that $z(t)$ is bounded.

Suppose there exists another bounded solution $y(t)$. Then, the difference $w(t) = z(t)-y(t)$ satisfies the homogeneous system

$$w^\Delta(t) = P(t)w(t)$$

and is also bounded. By Theorem 3.1, this implies $w(t) \equiv 0$, i.e., $z(t) \equiv y(t)$.

Hence, the solution is only one solution. □

Theorem 3.5 *Let all the conditions in Theorem 3.4 hold. Furthermore, suppose that the coefficient matrix and the nonhomogeneous term are ω-periodic, i.e.,*

$$P(t+\omega) = P(t) \quad \textit{and} \quad h(t+\omega) = \hat{h}(t), \quad \forall t \in \mathbb{T}.$$

Then the nonhomogeneous system (20) has a unique ω-periodic solution, which can be represented as (21)

Proof From Theorem 3.4, system (20) has a unique bounded solution represented by (21). Since $P(t)$ and $h(t)$ are both ω-periodic, the integrands in (21) inherit this periodicity.

Moreover, by Theorem 3.3, the function $\Psi(t)B\Psi^{-1}(t)$ is also ω-periodic. A change of variables (e.g., shifting $s \to s+\omega$) in each integral confirms that $z(t+\omega) = z(t)$. Hence, the bounded solution is ω-periodic.

By uniqueness of bounded solutions in Theorem 3.4, this ω-periodic solution is the only such solution. □

4 Periodic solutions for Kronecker product system

Let $BC((-\infty,t_0],\mathbb{R}^{n^2})$ denote the space of bounded continuous functions $\varphi : (-\infty,t_0] \to \mathbb{R}^{n^2}$, with the norm defined by $\|\varphi\| = \sup_{t\in(-\infty,t_0]} \|\varphi(t)\|$. The solution of system (3) with a bounded continuous starting function $\varphi \in BC(-\infty,t_0)$ is denoted by the symbol $z(t,t_0,\varphi)$.

Let nonlinear Volterra integro-dynamic system with both continuous and discrete of the system (2), and its linear system (3), next we introduce the following conditions:

- **(C1)** Using the equation (7), it can be observed that the linear system (3) exhibits a dichotomy of the (q,m) type.
- **(C2)** $P(t+\omega) = P(t), \hat{h}(t+\omega) = \hat{h}(t)$.
- **(C3)** $\int_{-\infty}^{t} \|(I\otimes A)(t,s)\|\Delta t$ is bounded and $(I\otimes A)(t+\omega,s+\omega) = (I\otimes A)(t,s), \forall t,s \in \mathbb{R}^{n^2}$.

- **(C4)** For each arbitrary constant $\epsilon > 0$, there exists $L_0 = L_0(\epsilon) > 0$, such that $\int_{-\infty}^{t_1} \|A(t,s)\|\Delta t < \epsilon, \forall t_1 > -\infty, t - t_1 \geq L_0$.
- **(C5)** $B(0) = 0, \|B(z) - B(y)\| \leq L\|z - y\|, z, y \in \mathbb{R}^{n^2}, L$ is a positive constant.
- **(C6)** $L\int_{-\infty}^{t}(I \otimes A)(t,s)\Delta t + \sum_{i=1}^{l} \hat{h}_i(t) \leq \rho_1$. There are positive constants $\rho_1 < \frac{1}{\mu}$.
- **(C7)** $L\int_{-\infty}^{t}(I \otimes A)(t,s)\Delta t + \sum_{i=1}^{l} \hat{d}_i(t) \leq \rho_2$. There exists a positive constants $\rho_2 < \frac{1}{\mu}$.
- **(C8)**There are rd-ontinuous ω-periodic functions $\alpha_1(t)$ and positive continuously differentiable ω-periodic functions $a_i(t), i = 1, 2, ...n$ such that

$$a^{\Delta}{}_i(t) + a_i(t)p_{ii}(t) + \sum_{j=1, j\neq i}^{n} |p_{ij}(t)| \leq \alpha_1 a_i(t),$$

and

$$m_1 = exp\left(\int_0^{\omega} \alpha_1(s)\Delta s\right) < 1.$$

- **(C9)** There are rd-continuous ω-periodic functions $\alpha_2(t)$ and positive continuously differentiable ω-periodic functions $a_i(t), i = 1, 2, ...n$ such that

$$a^{\Delta}{}_i(t) + a_i(t)p_{ii}(t) + \sum_{j=1, j\neq i}^{n} |p_{ij}(t)| \leq \alpha_2 a_i(t),$$

and

$$m_2 = exp\left(\int_0^{\omega} (-\alpha_2(s))\Delta s\right) < 1.$$

- **(C10)** $\bar{a}(t) = \max\{a_i(t)\} \leq a_1 \,\text{and}\, a(t) = \min\{a_i(t)\} \geq a_1 > 0.$

Theorem 4.1 *Assume that conditions **(C1)**–**(C6)** hold. Then, system* (2) *has a only one ω-periodic solution in Banach space D.*

Proof Define the function space

$$D = \{\psi : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \mid \psi \text{ is rd-continuous and } \omega\text{-periodic}\}$$

with the norm

$$\|\psi\| = \sup_{t\in[0,\omega]} \|\psi(t)\|.$$

We define the operator $\Omega : D \to D$ by

$$\begin{aligned} z_{\psi}(t) = &\int_{-\infty}^{t} \Psi(t)B\Psi^{-1}(r)\int_{-\infty}^{r} (I \otimes A)(r,s)B(\psi(s))\Delta s\Delta r \\ &- \int_{-\infty}^{t} \Psi(t)(I - B)\Psi^{-1}(r)\int_{-\infty}^{r} (I \otimes A)(r,s)B(\psi(s))\Delta s\Delta r. \end{aligned} \tag{22}$$

For any $\psi_1, \psi_2 \in D$, we obtain:

$$\|\Omega\psi_1(t) - \Omega\psi_2(t)\| \leq \mu\rho_1\|\psi_1 - \psi_2\|.$$

Since $\mu\rho_1 < 1$, Ω is a contraction mapping on the Banach space D. By the Banach fixed point theorem, Ω has a unique fixed point ψ^* in D. This ψ^* is the only ω-periodic solution of (2). ☐

Theorem 4.2 *Assume that conditions **(C1)**–**(C5)** and **(C7)** hold. Then, system* (2) *has at least one ω-periodic solution.*

Proof Let D be the Banach space of all rd-continuous ω-periodic functions $\psi : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ with the norm $\|\psi\| = \sup_{t\in[0,\omega]} \|\psi(t)\|$. For any $\psi \in D$, define the operator $\Omega : D \to D$ by

$$\begin{aligned} z_\psi(t) = &\int_{-\infty}^{t} \Psi(t)B\Psi^{-1}(r)\int_{-\infty}^{r}(I\otimes A)(r,s)B(\psi(s))\Delta s\Delta r \\ &- \int_{-\infty}^{t} \Psi(t)(I-B)\Psi^{-1}(r)\int_{-\infty}^{r}(I\otimes A)(r,s)B(\psi(s))\Delta s\Delta r. \end{aligned} \tag{23}$$

We now show that Ω has a fixed point in D using Schauder's fixed point theorem. Fix a positive integer N and define the closed ball

$$D_N = \{\psi \in D \mid \|\psi\| \leq N\}.$$

First, we show that $\Omega(D_N) \subseteq D_N$ for sufficiently large N. Since B is Lipschitz continuous with $B(0) = 0$ and the integral kernels are bounded (due to assumptions **(C1)**–**(C4)**), it follows that $\|\Omega\psi\|$ is bounded by a multiple of $\|\psi\|$. Therefore, for a suitable $N > 0$, we have $\|\Omega\psi\| \leq N$ for all $\psi \in D_N$.

To show the continuity of Ω, consider a sequence $\{\psi_k\} \subset D_N$ converging uniformly to ψ in D. Then $B(\psi_k(s)) \to B(\psi(s))$ uniformly and due to the boundedness of the integrand, the dominated convergence theorem implies $\Omega\psi_k \to \Omega\psi$ uniformly. Hence, Ω is continuous on D_N.

Next, we verify that $\Omega(D_N)$ is relatively compact in D. Since $\|\Omega\psi\|$ is uniformly bounded for all $\psi \in D_N$, and the derivative

$$[\Omega\psi]^{\Delta}(t) = P(t)z_\psi(t) + \int_{-\infty}^{t}(I\otimes A)(t,s)B(\psi(s))\Delta s + \hat{h}(t)$$

is bounded in norm for all $\psi \in D_N$ by constants $a_1N + a_2N + M$, where

$$a_1 = \sup\{\|P(t)\| \mid t \in [0,T]\}, \qquad a_2 = \sup_{t\in[0,T]}\left\{\left|\int_{-\infty}^{t}\|(I\otimes A)(t,s)\|\,ds\right|\right\}$$

and $\hat{h}(t)$ is a continuous ω -periodic function, there exists $M > 0$, such that $\|\hat{h}(t)\| \leq M$. It follows that $\Omega(D_N)$ is equicontinuous. By the Arzelà–Ascoli theorem, $\Omega(D_N)$ is relatively compact in D.

Thus, Ω is a continuous operator mapping the closed convex subset D_N of D into itself with relatively compact image. By Schauder's fixed point theorem, Ω has at least one fixed point $\psi^* \in D_N$, which is an ω-periodic solution of system (2). ☐

5 Conclusion

In this paper, we develop a comprehensive framework for analyzing (q, m)-dichotomy and periodic solutions in Sylvester matrix Volterra integro-dynamic systems on time scales. The transformation of the Sylvester matrix system into a Kronecker product form via the vec operator simplifies the analysis, enabling the application of fixed-point theorems and functional analysis methods. The criteria established for the existence and uniqueness of periodic solutions, based on the (q, m)-dichotomy conditions, provide strong tools for examining the qualitative behavior of such systems. These results confirm the periodicity and stability of the solutions. Furthermore, this research extends the theoretical foundation toward the study of periodic solutions related to controllability, observability, stability, and reachability for linear and nonlinear matrix systems on time scales.

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