Well-Coveredness and Cohen-Macaulayness of Comaximal Ideal Graphs

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Abstract

In this paper, we examine the structural properties of comaximal ideal graphs and their complements, with a particular focus on the notions of well-coveredness, well-dominatedness, and Cohen-Macaulayness. A graph is said to be well-covered if all its maximal independent sets are of the same cardinality, and well-dominated if all minimal dominating sets have equal size. These properties play an important role in understanding uniformity within graph structures. Furthermore, the Cohen-Macaulay property provides a deeper algebraic perspective, linking combinatorial features of graphs with commutative algebra. By analyzing these three properties for comaximal ideal graphs and their complements, we highlight the close interplay between algebraic structures and graph-theoretic characteristics. Our results contribute to a broader understanding of how algebraic properties are reflected in graph invariants, and demonstrate the significance of these connections in exploring the relationship between commutative algebra and graph theory.

Keywords: Well-coveredness; Cohen-Macaulayness; Comaximal ideal graphs.
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1 Introduction

Throughout this paper, R represents a finite commutative ring with unity. We denote the set of maximal ideals by $Max(R)$ and the *Jacobson radical* by $J(R)$. Let $\mathbb{I}(R)$ and $n(R)$ represent the collection of ideals and the number of ideals in R, respectively. We define $T(R) = \{I \in \mathbb{I}(R) : I \nsubseteq J(R)\}$ and $N(R) = \{I \in \mathbb{I}(R) : I \subseteq J(R)\}$.

The concept of Cohen-Macaulay rings has a rich history in commutative algebra, spanning over a century. These rings possess distinctive properties that make them essential in algebraic geometry. In fact, polynomial rings and formal power series rings over fields are notable examples of Cohen-Macaulay rings. Interestingly, a connection can be drawn between commutative algebra and graph theory through the construction of edge rings from graphs. Specifically, a graph is considered Cohen-Macaulay if its associated edge ring possesses this property. This intersection of algebra and graph theory offers a fascinating area of study, revealing new insights into the relationships between these fields.

In this paper, we classify all finite commutative rings with unity whose comaximal ideal graphs exhibit the property of well-coveredness, well-dominatedness, and Cohen-Macaulayness. The primary contribution lies in identifying and characterizing comaximal ideal graphs that exhibit the well-coveredness property. The results provide a comprehensive understanding of these algebraic properties through graph-theoretic properties.

2 Preliminaries

A *dominating set* D in a graph $G = (V(G), E)$ is a set in which every vertex is either in D or adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. A graph is *well-dominated* if all minimal dominating sets have the same cardinality. An *independent set* is a set where no two vertices are adjacent, and a *maximal independent* set is an independent set that becomes non-independent by adding any vertex. The independence number $\alpha(G)$ is the cardinality of the largest independent set. A graph is *well-covered* if every maximal independent set has the same cardinality. A well-covered graph G is said to be very well-covered if G has no isolated vertices and $|V(G)| = 2\alpha(G)$. The *corona product* of two graphs G and H, denoted as $G \circ H$, is a new graph formed by taking one copy of G, (referred to as the center graph), $|V(G)|$ copies of H, (referred to as the outer graph) and making the i^{th} vertex of G adjacent to every vertex of the i^{th} copy of H, where $1 \leq i \leq |V(G)|$.

Let X be a set with n elements, and let $\mathscr{F}$ be a family of subsets of X. $\mathscr{F}$ is called an *intersecting family* if every pair of subsets in $\mathscr{F}$ has at least one element in common. Furthermore, $\mathscr{F}$ is a *maximal intersecting family* if no $\mathscr{F}' \supset \mathscr{F}$ is an intersecting family.

In 2012, M. Ye and T. Wu [1] defined a new graph structure on commutative rings. They used vertices as ideals instead of elements of a ring, and they named such a graph structure, the comaximal ideal graph. The *comaximal ideal graph* $\mathbb{G}(R)$ of R is a simple graph with its vertices being the proper ideals of R which are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 = R$. Note that if R is a local ring, $\mathbb{G}(R)$ has no vertices. Therefore, we consider only finite rings R with $|Max(R)| \geq 2$ when studying the comaximal ideal graph. The complement of the comaximal ideal graph, denoted by $(\mathbb{G}(R))^c$, is a graph with the same vertices as $\mathbb{G}(R)$ but with adjacency defined by the sum of ideals: two distinct vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 \neq R$. In [2], H R Dorbidi and R. Manaviyat investigated the domination number and independence number of comaximal ideal graphs. Similarly, S Kavitha and R Kala explored the domination number of comaximal ideal graphs in [3].

Let S be a commutative Noetherian ring with unity, where $1 \neq 0$. The *Krull dimension* of S, denoted by *dim* S, is defined as the greatest integer n for which there exists an ascending chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ in the ring S. For a local Noetherian ring S, the *depth* of S, denoted *depth* S, is the longest sequence $x_1, x_2, \ldots, x_n$ of elements from the maximal ideal of S such that x_1 is not a zero-divisor of S and x_k is not a zero-divisor of $S/\langle x_1, \ldots, x_{k-1}\rangle$ for $k = 2, \ldots, n$. The inequality *depth* $S \leq$ *dim* S always holds true for any commutative Noetherian local ring S. When equality is achieved, the ring S is called a Cohen-Macaulay ring. This concept can be extended to non-local case by requiring that all localizations at all maximal ideals be Cohen-Macaulay.

For an undirected simple graph G with vertex set $V(G) = \{1, 2, \ldots, n\}$ and its edge set denoted by $E(G)$. Let $S = \mathbb{K}[x_1, x_2, \cdots, x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* $I(G)$ of G is defined as the ideal generated by the square-free quadratic monomials $x_i x_j$ for each edge $\{i, j\}$ in $E(G)$. The quotient ring $\mathbb{K}[G] = S/I(G)$ is known as the *edge ring* of G. The graph G is called *Cohen-Macaulay* over $\mathbb{K}$ if its edge ring $\mathbb{K}[G]$ is a Cohen-Macaulay ring. If the context is clear, we can simply omit "over $\mathbb{K}$". It is worth noting that a well-known fact about Cohen-Macaulay graphs is that they are always well-covered. The literature offers numerous findings on Cohen-Macaulay property of graphs. Notably, Herzog and Hibi (2005) provided a comprehensive characterization of bipartite graphs with this property [4]. The following year, Herzog et al.(2006) further expanded on this work by classifying Cohen-Macaulay chordal graphs and establishing a crucial equivalence: a chordal graph is Cohen-Macaulay precisely when it is well-covered [5]. More recently, T. Ashitha et al. (2023) identified the essential conditions for Cohen-Macaulayness in two specific classes of total graphs in their work [6].

Let n be an integer greater than or equal to 2 and set $[n] = \{1, 2, \ldots, n\}$ and $P([n])$ the power set of $[n]$. From [7], a finite Boolean graph, denoted by B_n, is a graph defined on the vertex set $P([n]) \setminus \{[n], \emptyset\}$, in which two vertices M and N are adjacent if $M \cap N \neq \emptyset$.

3 Well-Coveredness and Cohen-Macaulayness in Comaximal Ideal Graphs

In this section, the properties of comaximal ideal graphs were studied by focusing on the properties of well-coveredness, well-dominatedness, and Cohen-Macaulayness. We provide characterizations and classifications of finite commutative rings with unity such that $\mathbb{G}(R)$ is Cohen-Macaulay.

Lemma 3.1. *Let R be a finite commutative ring with unity and $|Max(R)| \geq 2$. $\mathbb{G}(R)$ is a well-covered graph if and only if R satisfies one of the following conditions:*

(i) $R \cong R_1 \times R_2$, where R_1 and R_2 are local rings such that $n(R_1) = n(R_2) = r$.

(ii) $R \cong F_1 \times F_2 \times \cdots \times F_n$, *where* $n \geq 3$ *and each* F_i *is a field for* $1 \leq i \leq n$.

Proof. Suppose $\mathbb{G}(R)$ is well-covered. Since $|Max(R)| \geq 2$, by Theorem 3.1.4 in [8], we have $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a local ring for $1 \leq i \leq n$ and $n = |Max(R)| \geq 2$. Therefore, $\mathbb{G}(R_1 \times R_2 \times \cdots \times R_n)$ is well-covered. We first show that $n(R_i) = n(R_j)$ for $1 \leq i, j \leq n$. Suppose for contradiction, that $n(R_i) \neq n(R_j)$ for some $i \neq j$. Consider the projection map $\pi_k : R_1 \times R_2 \times \cdots \times R_n \to R_k$ such that $\pi_k(x_1, x_2, \ldots, x_k, \ldots, x_n) = x_k$ for $1 \leq k \leq n$. Define the following sets:

$$\begin{aligned} A_i &= \{I \in T(R_1 \times R_2 \times \cdots \times R_n) : \pi_i(I) \neq R_i\}, \\ A_j &= \{I \in T(R_1 \times R_2 \times \cdots \times R_n) : \pi_j(I) \neq R_j\}. \end{aligned}$$

We claim that A_i is a maximal independent set. Assume that $I, J \in A_i$. Since $\pi_i(I), \pi_i(J) \neq R_i$, I and J are not adjacent. Therefore, A_i is an independent set. To prove A_i is maximal, choose $K \in T(R_1 \times R_2 \times \cdots \times R_n) \setminus A_i$. Then, $\pi_i(K) = R_i$, so K is adjacent to $R_1 \times R_2 \times \cdots \times R_{i-1} \times \{0_i\} \times R_{i+1} \times \cdots \times R_n \in A_i$. Thus, A_i is a maximal independent set. Similarly, A_j is also a maximal independent set of $\mathbb{G}(R_1 \times R_2 \times \cdots \times R_n)$. Also, we have:

$$\begin{aligned} |A_i| &= [n(R_1)n(R_2)\cdots(n(R_i)-1)\cdots n(R_n)] - [(n(R_1)-1)(n(R_2)-1)\cdots(n(R_n)-1)], \\ |A_j| &= [n(R_1)n(R_2)\cdots(n(R_j)-1)\cdots n(R_n)] - [(n(R_1)-1)(n(R_2)-1)\cdots(n(R_n)-1)]. \end{aligned}$$

Since $n(R_i) \neq n(R_j)$, we have $|A_i| \neq |A_j|$. Thus, $\mathbb{G}(R_1 \times R_2 \times \cdots \times R_n)$ is not well-covered, a contradiction. Therefore, $n(R_i) = r$ for all i. This implies condition (i) is satisfied for the case $|Max(R)| = n = 2$.

For condition (ii), consider the case where $n \geq 3$.

Case(i): When n is odd and $n \geq 3$.

Choose two sets A_1 and B as given below:

$$\begin{aligned} A_1 &= \{I \in T(R_1 \times R_2 \times \cdots \times R_n) : \pi_1(I) \neq R_1\}, \\ B &= \left\{I \in T(R_1 \times R_2 \times \cdots \times R_n) : |\{l : \pi_l(I) \neq R_l\}| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1\right\}. \end{aligned}$$

Clearly, A_1 is a maximal independent set. We claim that B is also a maximal independent set. Assume that $I', J' \in B$. Then, $|\{l : \pi_l(I') \neq R_l\}|, |\{l : \pi_l(J') \neq R_l\}| \geq \lfloor \frac{n}{2} \rfloor + 1$. Therefore, there exists a number $m \in [n]$ such that $\pi_m(I'), \pi_m(J') \neq R_m$. Thus, I' and J' are not adjacent in $\mathbb{G}(R_1 \times R_2 \times \cdots \times R_n)$. So, B is an independent set. To prove the maximality of B, choose $K' \in T(R_1 \times R_2 \times \cdots \times R_n) \setminus B$. Then, $|\{l : \pi_l(K') \neq R_l\}| \leq \lfloor \frac{n}{2} \rfloor$. Define, $L = L_1 \times L_2 \times \cdots \times L_n$, where

$$L_i = \begin{cases} R_i, & \text{if } \pi_i(K') \neq R_i \\ 0_i, & \text{if } \pi_i(K') = R_i. \end{cases}$$

Then, $L \in B$ and L is adjacent to K'. Therefore, B is a maximal independent set. Also, we have:

$$\begin{aligned} |A_1| &= r^{n-1}(r-1) - (r-1)^n, \\ |B| &= \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} \binom{n}{k} (r-1)^k. \end{aligned}$$

Since $\mathbb{G}(R_1 \times R_2 \times \cdots \times R_n)$ is well-covered, $|A_1| = |B|$. Thus, we have:

$$\begin{aligned}\sum_{k=\lfloor \frac{n}{2}\rfloor+1}^{n-1}\binom{n}{k}(r-1)^k &= r^{n-1}(r-1)-(r-1)^n \\ &= r^{n-1}-(r-1)^{n-1},\ \text{since } r>1,\ r-1\neq 0.\end{aligned}$$

Put $r-1=x$. Then:

$$\sum_{k=\lfloor \frac{n}{2}\rfloor+1}^{n-1}\binom{n}{k}x^{k-1}=(x+1)^{n-1}-x^{n-1}.$$

This implies:

$$\sum_{k=\lfloor \frac{n}{2}\rfloor+1}^{n-1}\binom{n}{k}x^{k-1}-(x+1)^{n-1}+x^{n-1}=0,$$

which is a monic polynomial of degree $n-2$ with integer coefficients and constant term -1. Thus, the only possible positive integer solution is $x=1$, which leads to $r=n(R_i)=2$ for $1\leq i\leq n$. $n(R_i)=2$ if and only if $R_i=F_i$, where F_i is a field. Thus, condition (ii) holds.

Case(ii): When n is even and $n\geq 3$.

Let $[n]=\{1,2,\cdots,n\}$ and $P([n])$ be the power set of $[n]$. Define:

$$B_1=\{C\in P([n]):\frac{n}{2}+1\leq |C|\leq n-1\}$$

and B_2 be a maximal intersecting family of subsets of $[n]$ with each set in the family has cardinality $\frac{n}{2}$. Thus, $B_1\cup B_2$ is a maximal intersecting family of proper subsets of $[n]$. Also, $|B_1|=\sum\limits_{\frac{n}{2}+1}^{n-1}\binom{n}{k}$. By Lemma 2.1 in [9], $|B_1\cup B_2|=2^{n-1}-1$. Since $B_1\cap B_2=\emptyset$, $|B_2|=2^{n-1}-1-|B_1|=\frac{\binom{n}{\frac{n}{2}}}{2}$. Consider the two sets A_1 and C_1:

$$\begin{aligned}A_1 &= \{I\in T(R_1\times R_2\times\cdots\times R_n):\pi_1(I)\neq R_1\},\\ C_1 &= \{I\in T(R_1\times R_2\times\cdots\times R_n):\{i\in[n]:\pi_i(I)\neq R_i\}\in B_1\cup B_2\}.\end{aligned}$$

Then, A_1 and C_1 are two maximal independent sets as in Case(i). Here,

$$\begin{aligned}|A_1| &= r^{n-1}(r-1)-(r-1)^n,\\ |C_1| &= \sum_{k=\frac{n}{2}+1}^{n-1}\binom{n}{k}(r-1)^k+\frac{\binom{n}{\frac{n}{2}}}{2}(r-1)^{\frac{n}{2}}.\end{aligned}$$

Since $G(R_1\times R_2\times\cdots\times R_n)$ is well-covered, we have $|A_1|=|C_1|$, leading to:

$$\begin{aligned}r^{n-1}(r-1)-(r-1)^n &= \sum_{k=\frac{n}{2}+1}^{n-1}\binom{n}{k}(r-1)^k+\frac{\binom{n}{\frac{n}{2}}}{2}(r-1)^{\frac{n}{2}}\\ &= \sum_{k=\frac{n}{2}+1}^{n-1}\binom{n}{k}(r-1)^{k-1}+\frac{\binom{n}{\frac{n}{2}}}{2}(r-1)^{\frac{n}{2}-1},\ \text{since } r-1\neq 0.\end{aligned}$$

Put $r-1=x$:

$$(x+1)^{n-1} - x^{n-1} = \sum_{k=\frac{n}{2}+1}^{n-1} \binom{n}{k} x^{k-1} + \frac{\binom{n}{\frac{n}{2}}}{2} x^{\frac{n}{2}-1}.$$

This implies:

$$\sum_{k=\frac{n}{2}+1}^{n-1} \binom{n}{k} x^{k-1} + \frac{\binom{n}{\frac{n}{2}}}{2} x^{\frac{n}{2}-1} - (x+1)^{n-1} + x^{n-1} = 0,$$

which is a monic polynomial of degree $n-2$ with integer coefficients and constant term -1. Thus, the only possible positive integer solution is $x = 1$, which leads to $r = n(R_i) = 2$ for $1 \leq i \leq n$. $n(R_i) = 2$ if and only if $R_i = F_i$, where F_i is a field. Thus, condition (ii) holds.

Conversely, suppose R satisfies condition (i) or (ii). If R satisfies condition (i), then $\mathbb{G}(R)$ is a complete bipartite graph $K_{(r-1),(r-1)}$, which implies $\mathbb{G}(R)$ is well-covered.

Now, suppose R satisfies condition (ii), then $R \cong F_1 \times F_2 \times \cdots \times F_n$, where each F_i is a field for $1 \leq i \leq n$ and $n \geq 3$. Let A be an arbitrary maximal independent set in $\mathbb{G}(F_1 \times F_2 \times \cdots \times F_n)$. Define

$$C_I = \{i \in [n] : \pi_i(I) = \{0\}\}, I \in A.$$

Then, $F_A = \{C_I : I \in A\}$ forms a maximal intersecting family of proper subsets of $[n]$. By Lemma 2.1 in [9], $|F_A| = |A| = 2^{n-1} - 1$. Thus, every maximal independent set has a cardinality of $2^{n-1}-1$, which completes the proof. □

Proposition 3.1. *([2]) Let R be a ring, $|Max(R)| \geq 3$ and the domination number of $\mathbb{G}(R)$ be finite. Then $Max(R)$ is a dominating set of $\mathbb{G}(R)$. Moreover, the domination number of $\mathbb{G}(R)$ is equal to $|Max(R)|$.*

Theorem 3.1. *Let R be a finite commutative ring with unity and $|Max(R)| \geq 2$. Then $\mathbb{G}(R)$ is well-dominated if and only if R satisfies one of the following conditions:*

(i) $R \cong R_1 \times R_2$, where R_1, R_2 are local rings and $n(R_1) = n(R_2) = 3$.

(ii) $R \cong F_1 \times F_2$, where F_1, F_2 are fields.

(iii) $R \cong F_1 \times F_2 \times F_3$, where each F_i is field for $1 \leq i \leq 3$.

Proof. Suppose R satisfies one of the conditions $(i), (ii)$ or (iii). Then $\mathbb{G}(R)$ is isomorphic to $K_{2,2}, K_2$ or $K_3 \circ K_1$, respectively. These graphs are well-dominated.

Conversely, suppose $\mathbb{G}(R)$ is a well-dominated graph. Since every well-dominated graph is also a well-covered graph, by Lemma 3.1, R must satisfy one of the following conditions: $R \cong R_1 \times R_2$, where R_1, R_2 are local rings and $n(R_1) = n(R_2) = r$, or $R \cong F_1 \times F_2 \times \cdots \times F_n$, where each F_i is a field and $n \geq 3$. Also, for a well-dominated graph, the domination number should be equal to the independence number.

Suppose $R \cong R_1 \times R_2$, where R_1 and R_2 are local rings with $n(R_1) = n(R_2) = r$. In this case, $\mathbb{G}(R) = \mathbb{G}(R_1 \times R_2) = K_{r-1,r-1}$. If $r - 1 \geq 2$, then $\gamma(\mathbb{G}(R)) = \gamma(\mathbb{G}(R_1 \times R_2)) = 2$ and $\alpha(\mathbb{G}(R)) = \alpha(\mathbb{G}(R_1 \times R_2)) = r - 1$. Thus, $\mathbb{G}(R)$ is a well-dominated graph when $r - 1 = 2$, which corresponds to $r = 3$. Hence condition (i) holds. If $r - 1 = 1$, then $\mathbb{G}(R) = \mathbb{G}(R_1 \times R_2) = K_{1,1}$. In this case, $\gamma(\mathbb{G}(R)) = 1$ and $\alpha(\mathbb{G}(R)) = 1$. Thus, $\mathbb{G}(R)$ is well-dominated when $r = 2$. Hence, condition (ii) holds.

Suppose $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $n = |Max(R)| \geq 3$ and each F_i is a field. In this case, $\alpha(\mathbb{G}(R)) = \alpha(\mathbb{G}(F_1 \times F_2 \times \cdots \times F_n)) = 2^{n-1} - 1$. and $\gamma(\mathbb{G}(R)) = \gamma(\mathbb{G}(F_1 \times F_2 \times \cdots \times F_n)) = |Max(R)| = n$. Thus, $\mathbb{G}(R)$ is well-dominated if and only if $2^{n-1} - 1 = n$. We have, $2^{n-1} - 1 = n$ if and only if $n = 3$. Thus, condition (iii) holds. Hence the result. □

Proposition 3.2. *[4] Let G be a bipartite graph with vertex partition $V \cup W$. Then the following conditions are equivalent:*

(i) G is a Cohen-Macaulay graph.

(ii) $|V| = |W|$ *and the vertices of* V*, say* $x_1, x_2, \ldots, x_n$*, and the vertices of* W*,* $y_1, y_2, \ldots, y_n$*, can be labeled such that* (1) $\{x_i, y_i\}$ *are edges for* $i = 1, 2, \ldots, n$; (2) *if* $\{x_i, y_j\}$ *is an edge, then* $i \leq j$*; and* (3) *if* $\{x_i, y_j\}$ *and* $\{x_j, y_k\}$ *are edges, then* $\{x_i, y_k\}$ *is also an edge.*

Proposition 3.3. *([10]) For any* $n \geq 1$*, Let* $G = B_n$ *be the Boolean graph. Then* G *is vertex decomposable, hence Cohen-Macaulay.*

Proposition 3.2 and 3.3 are used to prove Theorem 3.2.

Theorem 3.2. *Let* R *be a finite commutative ring with unity and* $|Max(R)| \geq 2$*. Then* $\mathbb{G}(R)$ *is Cohen-Macaulay if and only if* $R \cong F_1 \times F_2 \times \cdots \times F_n$*, where each* F_i *is a field for* $1 \leq i \leq n$ *and* $n \geq 2$*.*

Proof. Let R be a finite commutative ring with unity and $|Max(R)| \geq 2$. Suppose that $\mathbb{G}(R)$ is a Cohen-Macaulay graph. Since every Cohen-Macaulay graph is well-covered, R satisfies at least one of the conditions in Lemma 3.1. If R satisfies condition (i) in Lemma 3.1, then $\mathbb{G}(R)$ is a complete bipartite graph $K_{r-1,r-1}$. By Proposition 3.2, $K_{r-1,r-1}$ is Cohen-Macaulay if and only if $r - 1 = 1$. Consequently, $n(R_1) = n(R_2) = r = 2$. Note that $n(R_i) = 2$ if and only if R_i is a field. Hence, $R \cong F_1 \times F_2$. Accordingly, by conditions (i) and (ii), it follows that $R \cong F_1 \times F_2 \times \cdots \times F_n$.

Conversely, suppose $R \cong F_1 \times F_2 \times \cdots \times F_n$, where each F_i is a field for $1 \leq i \leq n$ and $n \geq 2$. In this case, $\mathbb{G}(R)$ is isomorphic to the Boolean graph B_n. Then, by Proposition 3.3, $\mathbb{G}(R)$ is Cohen-Macaulay. □

4 Well-Coveredness and Cohen-Macaulayness in Complements of Comaximal Ideal Graphs

In this section, we explore the properties of well-coveredness, well-dominatedness, and Cohen-Macaulayness in the context of the complements of comaximal ideal graphs. We provide characterizations and classifications of finite commutative rings with unity that satisfy these properties, shedding light on the structural relationships between these concepts.

Lemma 4.1. *Let* R *be a finite commutative ring with unity and* $|Max(R)| \geq 2$*. Then* $(\mathbb{G}(R))^c$ *is well-covered if and only if* $|Max(R)| = 2$*.*

Proof. Suppose that $(\mathbb{G}(R))^c$ is well-covered. Assume that $|Max(R)| = n > 2$. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a local ring for $1 \leq i \leq n$. Let $Max(R_1 \times R_2 \times \cdots \times R_n) = \{M_1, M_2, \cdots, M_n\}$. Then $D = \{R_1 \times \{0\} \times \{0\} \times \cdots \times \{0\}, \{0\} \times R_2 \times R_3 \times \cdots \times R_n\}$ and $Max(R_1 \times R_2 \times \cdots \times R_n)$ are two maximal independent sets of $(\mathbb{G}(R_1 \times R_2 \times \cdots \times R_n))^c \cong (\mathbb{G}(R))^c$. Since $(\mathbb{G}(R))^c$ is well-covered, we have $2 = |D| = |Max(R_1 \times R_2 \times \cdots \times R_n)| = |Max(R)| = n$, which contradicts the assumption that $n > 2$. Therefore, $|Max(R)| = 2$.

Conversely, suppose that $|Max(R)| = 2$. Then $R \cong R_1 \times R_2$, where R_1, R_2 are local rings. In this case, $(\mathbb{G}(R))^c$ is the disjoint union of $K_{n(R_1)-1}$ and $K_{n(R_2)-1}$. Thus, every maximal independent set in $(\mathbb{G}(R))^c$ has cardinality 2, making $(\mathbb{G}(R))^c$ well-covered. □

Theorem 4.1. *Let* R *be a finite commutative ring with unity and* $|Max(R)| \geq 2$*. Then* $(\mathbb{G}(R))^c$ *is well-dominated if and only if* $|Max(R)| = 2$*.*

Proof. Suppose that $(\mathbb{G}(R))^c$ is a well-dominated graph. Then, it is well-covered, and by Lemma 4.1, $|Max(R)| = 2$.

Conversely, suppose that $|Max(R)| = 2$. Then $R \cong R_1 \times R_2$, where R_1, R_2 are local rings. In this case, $(\mathbb{G}(R))^c$ is the disjoint union of $K_{n(R_1)-1}$ and $K_{n(R_2)-1}$. Thus every minimal dominating set in $(\mathbb{G}(R))^c$ has cardinality 2, making $(\mathbb{G}(R))^c$ well-dominated. □

Proposition 4.1. *([11]) A graph* G *is Cohen-Macaulay if and only if all its connected components are Cohen-Macaulay.*

Proposition 4.1 is used to prove Theorem 4.2.

Theorem 4.2. *Let R be a finite commutative ring with unity and $|Max(R)| \geq 2$. Then $(\mathbb{G}(R))^c$ is a Cohen-Macaulay graph if and only if $|Max(R)| = 2$.*

Proof. Suppose that $(\mathbb{G}(R))^c$ is a Cohen-Macaulay graph. Since every Cohen-Macaulay graph is well-covered, by Lemma 4.1, $|Max(R)| = 2$.

Conversely, suppose that $|Max(R)| = 2$. Then $R \cong R_1 \times R_2$, where R_1, R_2 are local rings. In this case, $(\mathbb{G}(R))^c \cong (\mathbb{G}(R_1 \times R_2))^c$ is the disjoint union of $K_{n(R_1)-1}$ and $K_{n(R_2)-1}$. Since every complete graph is Cohen-Macaulay, it follows from Proposition 4.1 that $(\mathbb{G}(R))^c$ is also a Cohen-Macaulay graph. □

CONFLICT OF INTEREST

The author declares that there is no conflict of interest.

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