The influence of movement of bacteria in a porous medium on the representation of chemotaxis Navier-Stokes model: Global existence

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Abstract

This paper explores the well-posedness of the chemotaxis system involving two species featuring nonlinear diffusion in response to two stimuli, along with the Navier-Stokes equation. This system describes a biological process representing two species and the influence of chemical stimuli secreted by the species, which attract the other in a porous medium. It is proven that the system possesses global-in-time weak and bounded weak solutions in the three-dimensional spatial domain. Also, it is extended for a bounded domain with a smooth boundary.

Keywords: Chemotaxis; Navier-Stokes; Weak solutions; two-species

AMS Subject Classifications: 35Q30; 35K57; 35D30; 92C17

1 Introduction

Chemotaxis is an intriguing biological phenomenon observed in various organisms, from single-celled bacteria to complex multicellular organisms. It refers to the directed movement of cells or organisms in response to chemical gradients in their environment. Essentially, chemotaxis allows organisms to sense and navigate towards attractant chemicals or away from repellents,

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facilitating vital biological processes such as finding nutrients and avoiding toxins. First, this phenomenon was modeled in [17, 18] as

$$\left.\begin{aligned} u_t &= \Delta u - \nabla\cdot(\chi(v)u\nabla v),\\ v_t &= \Delta v - v + u, \end{aligned}\right\} \tag{1.1}$$

where u refers to the density of the organism, v refers to the concentration of chemical, and χ refers to the chemotactic sensitivity function. The study of similar model equations has led to many advances in understanding how organisms navigate and interact with their environment. The existence of radially symmetric equilibrium solutions and its stability of (1.1) with the growth rate incorporate cooperation and competition effects in [24]. The global existence of classical solutions and critical blow-up of (1.1) are studied in [13]. Global-in-time solutions and their asymptotic behavior of (1.1) are studied in [37] for small initial data. Considering (1.1) with nonlinear diffusion and with a power factor in the drift term, the existence of a global-in-time weak solution and its decay properties are studied in [14, 28]. The global existence of a classical radially symmetric solution for (1.1) is established in [12].

The behavior of bacteria living in thin fluid layers near solid-air-water contact lines modeled in [35] as

$$\left.\begin{aligned} u_t + u\cdot\nabla u &= \Delta u - \nabla\cdot(\chi(c)u\nabla c),\\ v_t + u\cdot\nabla v &= \Delta v - k(v)u,\\ z_t + \tau(z\cdot\nabla)z + \nabla p &= \Delta z - u\nabla\phi, \quad div\ z = 0, \end{aligned}\right\} \tag{1.2}$$

where u refers to the density of the bacteria, v refers to the oxygen concentration, and z, p refers to the velocity field and pressure of the fluid, respectively. The system (1.2) called the Keller-Segel Navier-Stokes system (KS-NS) and called the Keller-Segel Stokes system (KSS) if $\tau = 1$ and $\tau = 0$ respectively. The global existence of a weak solution in two dimensions of (1.2) with some assumptions on bacteria density and movement established in [9]. A numerical method was found to investigate the dynamics of KS-NS in [6]. The global existence of solutions in 2D and 3D was established for (1.2) under some initial data assumptions in [22]. A local-in-time regular solution is established in [3] for KS-NS in a two-dimensional spatial domain. Also, with stronger assumptions in the chemosensitivity function, a global-in-time weak solution is established. Considering oxygen concentration in (1.2) as parabolic/hyperbolic, the local existence of a regular solution and its blow-up criterion is established in [4].

The movement of bacteria is considered as migration in a porous medium, then (1.2) becomes

$$\left.\begin{aligned} u_t + u\cdot\nabla u &= \Delta u^{1+\alpha} - \nabla\cdot(\chi(c)u\nabla c),\\ v_t + u\cdot\nabla v &= \Delta v - k(v)u,\\ z_t + \tau(z\cdot\nabla)z + \nabla p &= \Delta z - u\nabla\phi, \quad div\ z = 0. \end{aligned}\right\} \tag{1.3}$$

For $\tau = 0$ and $\alpha \in (\frac{1}{2}, 1]$ in $\mathbb{R}^2$ and $\tau = 1$ and $\alpha \in [0.8, 1]$ in $\mathbb{R}^3$ global existence of solutions established for (1.3) in [11]. Further, global-in-time solution for (1.3) in $\mathbb{R}^2$ when $\tau = 0$ and $\alpha = \frac{1}{3}$ proved in [21]. Also, the global existence of weak solution for (1.3) in three dimension for $\alpha = \frac{2}{3}$ in [5]. Considering initial data are sufficiently regular and positive, the global existence of weak solutions of (1.3) for $\alpha > \frac{1}{7}$ with $\tau = 0$ proved in [31]. Under some conditions on the initial data, the global existence of bounded weak solutions of (1.3) for any $\alpha > 0$ with $\tau = 0$ was proved in [30]. The global-in-time weak and bounded weak solutions under some assumptions on sensitivity functions and diffusive exponent were established in [8]. Similar studies are also performed for nonlinear diffusion operators in [7].

Besides the papers mentioned above, chemotaxis models also encompass the modeling of species evolution dynamics. It means that a species produces a chemical signal that either attracts or repels based on the concentration of chemicals. It is modelled in [23] as follows:

$$\left.\begin{aligned} &u_t = \Delta u - \nabla \cdot (\chi(v) u \nabla v) + \nabla \cdot (\xi(w) u \nabla w), \\ &v_t = \Delta v + \beta u - \gamma v, \\ &w_t = \Delta w + \delta u - \eta w, \end{aligned}\right\} \tag{1.4}$$

where u denotes to density of the species, v denotes the chemoattractant concentration, and w denotes the chemorepellant concentration. By assuming that $\gamma = \eta$, global solvability, blow up, and asymptotic behavior of (1.4) were studied in [29]. However, the existence of solutions of (1.4) was proved in [10] without any additional assumptions on the given data. In one-dimensional space, the global existence of classical solutions and steady states of (1.4) was proved in [20]. An attraction-repulsion chemotaxis system with logistic source was considered in [38] and established the existence of a global bounded classical solution. The global existence of a classical solution in 2D and a weak solution in 3D is established for (1.4) with the assumption that $\xi\delta - \chi\beta > 0$ in [16].

The model describing two competitive species attracted by one chemical signal is proposed in [33],

$$\left.\begin{aligned} &u_t = \Delta u - \nabla \cdot (\chi_1(v) u \nabla w) + \mu_1 u(1 - u - a_1 v), \\ &v_t = \Delta v - \nabla \cdot (\chi_2(w) v \nabla w) + \mu_2 v(1 - a_2 u - v), \\ &\tau w_t = \Delta w + \delta u + \eta v - \gamma w, \end{aligned}\right\} \tag{1.5}$$

and discussed global existence and asymptotic behavior of the system (1.5) for $\tau = 0$ with some assumptions on sensitivity functions, a_1 and a_2. The dynamics of (1.5) and decay of species were studied in [27]. The global bounded solutions of (1.5) for $\tau = 1$ and its uniqueness, asymptotic behavior for large μ_1 and μ_2 were studied in [1]. The global asymptotic stability under the small assumption on chemotactic sensitivity was established in [2] for (1.5) with weak competition. Various results about global existence and steady states of (1.5) were established in [15, 25, 26].

A Chemotaxis system involving two species and two signals with Lotka–Volterra-type kinetics can be modeled as

$$\left.\begin{aligned} &u_t = \Delta u - \nabla \cdot (\chi_1(v) u \nabla v) + \mu_1 u(1 - u - a_1 w), \\ &\tau v_t = \Delta v + \beta w - \gamma v, \\ &w_t = \Delta w - \nabla \cdot (\chi_2(z) w \nabla z) + \mu_2 w(1 - a_2 u - w), \\ &\tau z_t = \Delta z + \delta u - \eta z. \end{aligned}\right\} \tag{1.6}$$

The global classical solution, boundedness, and blow-up of (1.6) with $\tau = \mu_1 = \mu_2 = 0$ were established in [32]. The result was extended to a quasi-linear chemotaxis model of (1.6) in [39]. Under the assumption of initial conditions, the unique global classical solution for (1.6) was obtained in [19]. A global bounded classical solution for (1.6) with $\tau = 0$ in two dimensions was established in [40]. Further, a global bounded classical solution was established for (1.6) with $\tau = 1$ in $n \geq 1$ under some assumptions on the chemotactic sensitivity function. The existence of a global bounded classical solution and its rates of convergence under some assumptions, sensitivity functions were obtained in [34]. The conditions assumed in [34] are improved in [36], and similar results were established.

The novel contributions of this paper can be summarized as follows:

- Investigates a two-species system living in a fluid environment.
- Considers chemicals secreted by each species that attract the other species while repelling their own.
- Incorporated the effect of the fluid environment by modeling porous medium diffusion for both species.

The proposed model offers an advantage over earlier ones by representing bacterial movement and diffusion through the nonlinear diffusion of porous medium type for bacterial movement. This provides a realistic representation of the underlying biological processes.

The rest of the paper is organized as follows. In Section 2, we define and formulate our proposed model. In Section 3, we define the weak solution of the proposed problem (2.1) with $\tau = 1$, and using a suitable approximation problem, we deduce some uniform estimates. Further, a weak solution for (2.1) with $\tau = 1$ is established. In Section 4, we prove the existence of a bounded weak solution for (2.1) with $\tau = 0$ in $\mathbb{R}^3$ using uniform estimates derived in the previous section.

2 Mathematical Model

In this paper, we propose a model to describe the chemotaxis system involving two species featuring nonlinear diffusion in response to two secreted stimuli along with the Navier-Stokes equation in $\mathbb{R}^3 \times [0, \Lambda)$, as

$$\left.\begin{aligned}
&u_t + n \cdot \nabla u = \Delta u^{1+\alpha} - \nabla \cdot (\chi_1(v) u \nabla v) + \nabla \cdot (\xi_1(z) u \nabla z),\\
&v_t + n \cdot \nabla v = \Delta v + \beta w - \gamma v,\\
&w_t + n \cdot \nabla w = \Delta w^{1+\alpha} - \nabla \cdot (\chi_2(z) w \nabla z) + \nabla \cdot (\xi_2(v) w \nabla v),\\
&z_t + n \cdot \nabla z = \Delta z + \delta u - \eta z,\\
&n_t + \tau (n \cdot \nabla) n + \nabla p = \Delta n - (u + w) \nabla \phi.
\end{aligned}\right\} \tag{2.1}$$

Here u, w refer to the densities of two different species, v refers concentration of chemical produced by w, z refers concentration of chemical produced by u, χ_1, χ_2 refers chemoattraction sensitivity coefficient, ξ_1, ξ_2 chemorepulsion sensitivity coefficient, $\alpha, \beta, \gamma, \delta, \eta$ are positive constants.

To the best of our knowledge, there is no paper in the literature which studies the solvability of the above-proposed model. This paper aims to establish the global existence of solutions of (2.1). The novelty of the proposed model (2.1) is briefed as follows: from the biological point of view, the species moving in a fluid environment and two chemical stimuli secreted by the species, which possess attraction-repulsion chemotaxis phenomena.

3 Weak Solutions

In this section, we define the weak solution and introduce a suitable approximation problem for the proposed model (2.1) with $\tau = 1$. Then we deduce a uniform estimate using the approximation problem. Using the uniform estimate, we deduce weak solutions for (2.1) with $\tau = 1$ in $\mathbb{R}^3$ and extend the result to a bounded domain with smooth boundary. We consider the system (2.1) with $\tau = 1$ as

$$\left.\begin{aligned} &u_t + n\cdot\nabla u = \Delta u^{1+\alpha} - \nabla\cdot(\chi_1(v)u\nabla v) + \nabla\cdot(\xi_1(z)u\nabla z),\\ &v_t + n\cdot\nabla v = \Delta v + \beta w - \gamma v,\\ &w_t + n\cdot\nabla w = \Delta w^{1+\alpha} - \nabla\cdot(\chi_2(z)w\nabla z) + \nabla\cdot(\xi_2(v)w\nabla v),\\ &z_t + n\cdot\nabla z = \Delta z + \delta u - \eta z,\\ &n_t + (n\cdot\nabla)n + \nabla p = \Delta n - (u+w)\nabla\phi. \end{aligned}\right\} \tag{3.1}$$

Definition 3.1. *For $\alpha > 0$ and $\Lambda \in (0,\infty)$, a quintuple (u, v, w, z, n) is said to be a weak solution of (2.1) if*

(i) $u, v, w, z \geq 0$,

(ii) $u(1 + |x| + |\log u|),\ w(1 + |x| + |\log w|) \in L^\infty(0,\Lambda; L^1(\mathbb{R}^3))$,

(iii) For $1 \leq p \leq 1+\alpha$,

$$u,\ w \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3)) \text{ and } \nabla u^{\frac{p+\alpha}{2}},\ \nabla w^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda; L^2(\mathbb{R}^3)),$$

(iv) $v,\ z \in L^\infty(0,\Lambda; H^1(\mathbb{R}^3)) \cap L^2(0,\Lambda; H^2(\mathbb{R}^3))$, *and* $v,\ z \in L^\infty(\mathbb{R}^3 \times [0,\Lambda))$,

(v) n is a vector-valued on $\mathbb{R}^3 \times (0,\Lambda)$ and

$$n \in L^\infty(0,\Lambda; L^2(\mathbb{R}^3)), \quad \nabla n \in L^2(0,\Lambda; L^2(\mathbb{R}^3)),$$

(vi) For any $\varphi \in L^\infty(0,\Lambda; L^2(\mathbb{R}^3))$ and $\psi \in C_0^\infty(\mathbb{R}^3 \times [0,\Lambda), \mathbb{R}^3)$ with $\nabla\cdot\psi = 0$,

$$\int_0^\Lambda \int_{\mathbb{R}^3} -u\varphi_t - un\cdot\nabla\varphi + \nabla u^{1+\alpha}\cdot\nabla\varphi - u\chi_1\nabla v\cdot\nabla\varphi + u\xi_1\nabla z\cdot\nabla\varphi \; dx\; dt = \int_{\mathbb{R}^3} u_0\varphi(\cdot,0)dx,$$

$$\int_0^\Lambda \int_{\mathbb{R}^3} -v\varphi_t - vn\cdot\nabla\varphi + \nabla v\cdot\nabla\varphi - \beta w\varphi + \gamma v\varphi \; dx\; dt = \int_{\mathbb{R}^3} v_0\varphi(\cdot,0)dx,$$
$$\int_0^\Lambda \int_{\mathbb{R}^3} -w\varphi_t - wn\cdot\nabla\varphi + \nabla w^{1+\alpha}\cdot\nabla\varphi - w\chi_2\nabla z\cdot\nabla\varphi + w\xi_2\nabla v\cdot\nabla\varphi \; dxdt = \int_{\mathbb{R}^3} w_0\varphi(\cdot,0)dx,$$
$$\int_0^\Lambda \int_{\mathbb{R}^3} -z\varphi_t - zn\cdot\nabla\varphi + \nabla z\cdot\nabla\varphi - \delta u\varphi + \eta z\varphi \; dx\; dt = \int_{\mathbb{R}^3} z_0\varphi(\cdot,0)dx,$$
$$\int_0^\Lambda \int_{\mathbb{R}^3} -n\cdot\psi_t + \nabla n\cdot\nabla\psi + ((n\cdot\nabla)n)\cdot\psi + (u+w)\nabla\phi\cdot\psi \; dx\; dt = \int_{\mathbb{R}^3} n_0\cdot\psi(\cdot,0)dx.$$

The model we are addressing here is complicated as it has strong degeneracy of the diffusion terms. To reduce the complication, we consider the approximation problem of the proposed model is given as

$$\left.\begin{aligned}
&u_{\epsilon_t} + n_\epsilon\cdot\nabla u_\epsilon = \Delta(u_\epsilon+\epsilon)^{1+\alpha} - \nabla\cdot(\chi_1(v_\epsilon)u_\epsilon\nabla v_\epsilon) + \nabla\cdot(\xi_1(z_\epsilon)u_\epsilon\nabla z_\epsilon),\\
&v_{\epsilon_t} + n_\epsilon\cdot\nabla v_\epsilon = \Delta v_\epsilon + \beta w_\epsilon - \gamma v_\epsilon,\\
&w_{\epsilon_t} + n_\epsilon\cdot\nabla w_\epsilon = \Delta(w_\epsilon+\epsilon)^{1+\alpha} - \nabla\cdot(\chi_2(z_\epsilon)w_\epsilon\nabla z_\epsilon) + \nabla\cdot(\xi_2(v_\epsilon)w_\epsilon\nabla v_\epsilon),\\
&z_{\epsilon_t} + n_\epsilon\cdot\nabla z_\epsilon = \Delta z_\epsilon + \delta u_\epsilon - \eta z_\epsilon,\\
&n_{\epsilon_t} + (n_\epsilon\cdot\nabla)n_\epsilon + \nabla p_\epsilon = \Delta n_\epsilon - (u_\epsilon + w_\epsilon)\nabla\phi,
\end{aligned}\right\} \tag{3.2}$$

with smooth initial conditions

$$u_{0_\epsilon} = \phi_\epsilon * u_0,\; v_{0_\epsilon} = \phi_\epsilon * v_0,\; w_{0_\epsilon} = \phi_\epsilon * w_0,\; z_{0_\epsilon} = \phi_\epsilon * z_0 \text{ and } n_{0_\epsilon} = \phi_\epsilon * n_0, \tag{3.3}$$

where ϕ_ϵ refers usual mollifier with $\epsilon \in (0,1)$. Using standard theory of existence and regularity, (3.2) possesses local-in-time classical solution for each $\epsilon \in (0,1)$. This allows us to overcome the complications raised by degeneracy and progress in analysis. We derive some uniform estimates for the approximate problem, independent of ϵ. Using that the obtained local-in-time solution extended globally and derived a weak solution for (2.1). For the simplicity of notation, hereafter, we use variables $(u_\epsilon, v_\epsilon, w_\epsilon, z_\epsilon, n_\epsilon)$ as (u,v,w,z,n). We define the functionals as:

$$\begin{aligned}E(t) := \int_{\mathbb{R}^3} u(t)(\log u(t) + 2\langle x\rangle) + w(t)(\log w(t) + 2\langle x\rangle)\; dx + \big\|u(t)\big\|_{1+\alpha}^{1+\alpha} + \big\|w(t)\big\|_{1+\alpha}^{1+\alpha}\\
+\big\|\nabla v(t)\big\|_2^2 + \big\|\nabla z(t)\big\|_2^2 + \|n(t)\|_2^2\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}D(t) := \big\|\nabla u(t)^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u(t)^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\nabla w(t)^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla w(t)^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v(t)\big\|_2^2\\
+\big\|\Delta z(t)\big\|_2^2 + \|\nabla n(t)\|_2^2,\end{aligned} \tag{3.5}$$

where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. Now, we prove a lemma that gives an estimate for the solution of (3.2).

Lemma 3.1. *Suppose that* (u,v,w,z,n) *is a classical solution for system* (3.2) *independent of* ϵ *and* (3.3) *satisfies*

$$\left.\begin{aligned}
&u_0(1+|x|+|\log u_0|) \text{ and } w_0(1+|x|+|\log w_0|) \in L^1(\mathbb{R}^3),\\
&u_0, w_0 \in L^{1+\alpha}(\mathbb{R}^3), v_0 \text{ and } z_0 \in L^\infty(\mathbb{R}^3)\cap H^1(\mathbb{R}^3),\quad n_0 \in L^2(\mathbb{R}^3).
\end{aligned}\right\} \tag{3.6}$$

Further, assume that

$$\alpha > \frac{1}{6}, \quad \phi \in W^{2,\infty}(\mathbb{R}^3) \text{ and } \chi_i^{'}, \xi_i^{'} \in L^\infty_{loc} \text{ for } i \in \{1,2\}. \tag{3.7}$$

Then, there exists $C > 0$, *independent of* ϵ, *such that for any* $0 < t \leq \Lambda$

$$\sup_{0\leq\tau\leq t} E(\tau) + \int_0^t D(\tau)d\tau < C. \tag{3.8}$$

Proof. Integrating $(3.2)_1$ and $(3.2)_3$, we get $\|u(t)\|_1 \equiv \|u_0\|_1$ and $\|w(t)\|_1 \equiv \|w_0\|_1$. Using maximal principle for $(3.2)_2$ and $(3.2)_4$, we get $\|v\|_{L^\infty(\mathbb{R}^3\times[0,\Lambda))} \leq \|v_0\|_\infty$ and $\|z\|_{L^\infty(\mathbb{R}^3\times[0,\Lambda))} \leq \|z_0\|_\infty$. We split the proof of the lemma into three cases by range of α as: (i) $\frac{1}{6} < \alpha \leq \frac{1}{3}$, (ii) $\frac{1}{3} < \alpha \leq 1$, and (iii) $\alpha > 1$.

Case (i): $\frac{1}{6} < \alpha \leq \frac{1}{3}$

Using $\log u$ as test function in $(3.2)_1$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u \log u \, dx + \int_{\mathbb{R}^3} \nabla \log u \cdot \nabla (u+\epsilon)^{1+\alpha} \, dx = \int_{\mathbb{R}^3} \nabla u \cdot (\chi_1 \nabla v) \, dx - \int_{\mathbb{R}^3} \nabla u \cdot (\xi_1 \nabla z) \, dx. \tag{3.9}$$

Using $\nabla \log u \cdot (1+\alpha) u^\alpha \nabla u = \frac{4}{1+\alpha}\left|\nabla u^{\frac{1+\alpha}{2}}\right|^2$, the second term of LHS in above estimated as

$$\int_{\mathbb{R}^3} \nabla \log u \cdot \nabla (u+\epsilon)^{1+\alpha} \, dx \geq \frac{4}{1+\alpha}\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2. \tag{3.10}$$

Using $|\nabla u| = \frac{2}{1+\alpha} u^{\frac{1-\alpha}{2}}\left|\nabla u^{\frac{1+\alpha}{2}}\right|$ and Young's inequality, the first term of RHS in (3.9) estimated as

$$\int_{\mathbb{R}^3} \nabla u \cdot (\chi_1 \nabla v) \, dx \leq \frac{2\overline{\chi_1}}{1+\alpha}\left(\epsilon\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + C(\epsilon)\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 \, dx\right), \tag{3.11}$$

where $\overline{\chi_1} := \sup\limits_{L^\infty(\mathbb{R}^3\times[0,\Lambda))} |\chi_1(v(\cdot))|$. Using $\left|\nabla u^{1-\alpha}\right| = C u^{\frac{1-3\alpha}{2}}\left|\nabla u^{\frac{1+\alpha}{2}}\right|$ and Young's inequality, the last term of RHS in above estimated as

$$\begin{aligned}
\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2 \, dx &= \int_{\mathbb{R}^3} u^{1-\alpha}\nabla v \cdot \nabla v \, dx \\
&\leq C_0\Big(\int_{\mathbb{R}^3} |\nabla u^{1-\alpha}||\nabla v| \, dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v| \, dx\Big) \\
&= C_0\Big(\int_{\mathbb{R}^3} C u^{\frac{1-3\alpha}{2}}\left|\nabla u^{\frac{1+\alpha}{2}}\right||\nabla v| \, dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v| \, dx\Big) \\
&\leq C C_0 \int_{\mathbb{R}^3} \epsilon\left|\nabla u^{\frac{1+\alpha}{2}}\right|^2 + C(\epsilon) u^{1-3\alpha}|\nabla v|^2 \, dx \\
&\qquad + C_0 \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v| \, dx.
\end{aligned}$$

Using above in (3.11), we get

$$\int_{\mathbb{R}^3} \nabla u \cdot (\chi_1 \nabla v) \, dx \leq C_1\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + C_2\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2 \, dx + C_3 \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v| \, dx. \tag{3.12}$$

Similarly, we get

$$-\int_{\mathbb{R}^3} \nabla u \cdot (\xi_1 \nabla z) \, dx \leq C_4\left\|\nabla u^{\frac{1+\alpha}{2}}\right\|_2^2 + C_5\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla z|^2 \, dx + C_6 \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta z| \, dx. \tag{3.13}$$

Using (3.10), (3.12) and (3.13) in (3.9) and choosing sufficiently small C_1, C_4 such that $C > 0$, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\ dx + C\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 \le C_2\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\ dx + C_3\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\ dx \\ + C_5\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla z|^2\ dx + C_6\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta z|\ dx. \quad (3.14)\end{aligned}$$

Similarly, using $\log w$ as a test function in $(3.2)_3$, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} w\log w\ dx + C_0\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 \le C_7\int_{\mathbb{R}^3} w^{1-3\alpha}|\nabla z|^2\ dx + C_8\int_{\mathbb{R}^3} w^{1-\alpha}|\Delta z|\ dx \\ + C_9\int_{\mathbb{R}^3} w^{1-3\alpha}|\nabla v|^2\ dx + C_{10}\int_{\mathbb{R}^3} w^{1-\alpha}|\Delta v|\ dx. \\ (3.15)\end{aligned}$$

Using $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ as test function in $(3.2)_1$ and simple algebraic calculations in order to bound $\log u$ in (3.14), we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\langle x\rangle u\ dx \\ &\quad= \int_{\mathbb{R}^3} un\cdot\nabla\langle x\rangle dx + \int_{\mathbb{R}^3}(u+\epsilon)^{1+\alpha}\Delta\langle x\rangle\ dx + \int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot u\chi_1\nabla v\ dx - \int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot u\xi_1\nabla z\ dx \\ &\quad\le C\big(1+\|n\|_2^2+\|\nabla v\|_2^2+\|\nabla z\|_2^2\big) + \big(C(\epsilon)+\epsilon\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2\big). \quad (3.16)\end{aligned}$$

Similarly, to bound $\log w$ in (3.15), we get

$$\frac{d}{dt}\int_{\mathbb{R}^3}\langle x\rangle w\ dx \ \le\ C_1\big(1+\|n\|_2^2+\|\nabla z\|_2^2+\|\nabla v\|_2^2\big) + \big(C(\epsilon_1)+\epsilon_1\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2\big). \quad (3.17)$$

Using u^α as test function in $(3.2)_1$, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \int_{\mathbb{R}^3}\nabla u^\alpha\cdot\nabla(u+\epsilon)^{1+\alpha}\ dx = \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi_1\nabla v)\ dx - \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi_1\nabla z)\ dx. \quad (3.18)$$

Using $\big|\nabla u^{\frac{1+2\alpha}{2}}\big|^2 = \dfrac{(1+2\alpha)^2}{4}u^{\frac{2\alpha-1}{2}}\big|\nabla u\big|^2$, the second term of LHS in above estimated as

$$\int_{\mathbb{R}^3}\nabla u^\alpha\cdot\nabla(u+\epsilon)^{1+\alpha}\ dx \ \ge\ \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2. \quad (3.19)$$

Using Young's inequality, the first term of RHS in (3.18) is estimated as

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi\nabla v)\ dx \ &\le\ C\int_{\mathbb{R}^3}\big|\nabla u^{\frac{1+2\alpha}{2}}\big|(u^{\frac{1}{2}}|\nabla v|)\ dx \\ &\le\ C\Big(\epsilon\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C(\epsilon)\int_{\mathbb{R}^3} u|\nabla v|^2\ dx\Big) \\ &\le\ C\Big(\epsilon\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C'\Big(\int_{\mathbb{R}^3}|\nabla u||\nabla v|\ dx + \int_{\mathbb{R}^3} u|\Delta v|\ dx\Big)\Big) \\ &=\ C\Big(\epsilon\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C_0\int_{\mathbb{R}^3} u^{\frac{1-2\alpha}{2}}\big|\nabla u^{\frac{1+2\alpha}{2}}\big||\nabla v|dx + C'\int_{\mathbb{R}^3} u|\Delta v|dx\Big)\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\epsilon\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2+C_0\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2\\
&\qquad +C_0\int_{\mathbb{R}^3}u^{1-2\alpha}|\nabla v|^2\,dx+C'\int_{\mathbb{R}^3}u|\Delta v|\,dx\Big)\\
&\le C_1\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2+C_2\int_{\mathbb{R}^3}u^{1-2\alpha}|\nabla v|^2dx+C_3\int_{\mathbb{R}^3}u|\Delta v|dx. \qquad (3.20)
\end{aligned}
$$

Similarly, we get

$$
-\int_{\mathbb{R}^3}\nabla u^{\alpha}\cdot u(\xi_1\nabla z)\,dx\le C_4\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2+C_5\int_{\mathbb{R}^3}u^{1-2\alpha}|\nabla z|^2\,dx+C_6\int_{\mathbb{R}^3}u|\Delta z|\,dx. \quad (3.21)
$$

Using (3.19), (3.20) and (3.21) in (3.18) and choosing sufficiently small C_1, C_4 such that $C>0$, we have

$$
\begin{aligned}
\frac{d}{dt}\big\|u\big\|_{1+\alpha}^{1+\alpha}+C\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2&\le C_2\int_{\mathbb{R}^3}u^{1-2\alpha}|\nabla v|^2\,dx+C_3\int_{\mathbb{R}^3}u|\Delta v|\,dx\\
&\quad+C_5\int_{\mathbb{R}^3}u^{1-2\alpha}|\nabla z|^2\,dx+C_6\int_{\mathbb{R}^3}u|\Delta z|\,dx. \qquad (3.22)
\end{aligned}
$$

Similarly, using w^{α} as a test function in $(3.2)_3$, we get

$$
\begin{aligned}
\frac{d}{dt}\big\|w\big\|_{1+\alpha}^{1+\alpha}+C_0\big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2&\le C_7\int_{\mathbb{R}^3}w^{1-2\alpha}|\nabla z|^2\,dx+C_8\int_{\mathbb{R}^3}w|\Delta z|\,dx\\
&\quad+C_9\int_{\mathbb{R}^3}w^{1-2\alpha}|\nabla v|^2\,dx+C_{10}\int_{\mathbb{R}^3}w|\Delta v|\,dx. \qquad (3.23)
\end{aligned}
$$

Using $-\Delta v$ as test function in $(3.2)_2$, we get

$$
\frac{d}{dt}\big\|\nabla v\big\|_2^2+2\big\|\Delta v\big\|_2^2\le\int_{\mathbb{R}^3}\Delta v\cdot(n\cdot\nabla v)\,dx-\beta\int_{\mathbb{R}^3}w|\Delta v|dx+\gamma\int_{\mathbb{R}^3}v|\Delta v|dx. \qquad (3.24)
$$

The first term in RHS of the above evaluated as

$$
\begin{aligned}
\int_{\mathbb{R}^3}\Delta v\cdot(n\cdot\nabla v)\,dx&=-\int_{\mathbb{R}^3}\sum_{i,j}\partial_i v\ \partial_j v\ \partial_i n_j\,dx\\
&=\int_{\mathbb{R}^3}\sum_{i,j}v\ \partial_i\partial_j v\ \partial_i n_j\,dx\\
&\le C\ \big\|\nabla n\big\|_2\big\|\Delta v\big\|_2.
\end{aligned}
$$

Using above in (3.24), we get

$$
\frac{d}{dt}\big\|\nabla v\big\|_2^2+2\big\|\Delta v\big\|_2^2\le C\ \big\|\nabla n\big\|_2\big\|\Delta v\big\|_2-\beta\int_{\mathbb{R}^3}w|\Delta v|dx+\gamma\int_{\mathbb{R}^3}v|\Delta v|dx. \qquad (3.25)
$$

Similarly, using $-\Delta z$ as test function in $(3.2)_4$, we get

$$
\frac{d}{dt}\big\|\nabla z\big\|_2^2+2\big\|\Delta z\big\|_2^2\le C\ \big\|\nabla n\big\|_2\big\|\Delta z\big\|_2-\delta\int_{\mathbb{R}^3}u|\Delta z|dx+\eta\int_{\mathbb{R}^3}z|\Delta z|dx. \qquad (3.26)
$$

Using n as test function in $(3.2)_5$, we get

$$
\frac{d}{dt}\|n\|_2^2+2\|\nabla n\|_2^2\le C'\int_{\mathbb{R}^3}(u+w)|n|dx. \qquad (3.27)
$$

Adding (3.14)-(3.17), (3.22), (3.23), (3.25)-(3.27) and choosing sufficiently small constants such that $C_0 > 0$, we get

$$\begin{aligned}
&\frac{d}{dt}\bigg(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle) + w(\log w + 2\langle x\rangle)\ dx + \|u\|_{1+\alpha}^{1+\alpha} + \|w\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla z\|_2^2 + \|n\|_2^2 \bigg) \\
&\quad + C_0\bigg(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\nabla w^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla w^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta z\|_2^2 + \|\nabla n\|_2^2 \bigg) \\
&\le C_0^{'}\bigg(\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\ dx + \int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w^{1-3\alpha}|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w^{1-3\alpha}|\nabla v|^2\ dx \\
&\qquad + \int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2\ dx + \int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w^{1-2\alpha}|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w^{1-2\alpha}|\nabla v|^2\ dx \\
&\qquad + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\ dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\Delta z|\ dx + \int_{\mathbb{R}^3} w^{1-\alpha}|\Delta z|\ dx + \int_{\mathbb{R}^3} w^{1-\alpha}|\Delta v|\ dx \\
&\qquad + \int_{\mathbb{R}^3} u|\Delta v|\ dx + \int_{\mathbb{R}^3} u|\Delta z|\ dx + \int_{\mathbb{R}^3} w|\Delta z|\ dx + \int_{\mathbb{R}^3} w|\Delta v|\ dx \\
&\qquad + \int_{\mathbb{R}^3} v|\Delta v|\ dx + \int_{\mathbb{R}^3} z|\Delta z|\ dx + \int_{\mathbb{R}^3} (u+w)|n|\ dx \bigg).
\end{aligned} \tag{3.28}$$

As $0 \le 1 - 3\alpha < \frac{2}{3}$, using the Hölder inequality and Sobolev embedding, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla v|^2\ dx &\le \begin{cases} \int_{\mathbb{R}^3} \big(C(\epsilon_1) + \epsilon_1 u^{\frac{2}{3}}\big)|\nabla v|^2\ dx & \text{if } \frac{1}{6} < \alpha < \frac{1}{3}, \\ \|\nabla v\|_2^2 & \text{if } \alpha = \frac{1}{3}. \end{cases} \\
&\le \begin{cases} C(\epsilon_1)\|\nabla v\|_2^2 + \epsilon_1\|u_0\|^{\frac{2}{3}}\|\Delta v\|_2^2 & \text{if } \frac{1}{6} < \alpha < \frac{1}{3}, \\ \|\nabla v\|_2^2 & \text{if } \alpha = \frac{1}{3}. \end{cases}
\end{aligned} \tag{3.29}$$

Similarly, we get

$$\int_{\mathbb{R}^3} u^{1-3\alpha}|\nabla z|^2\ dx \le \begin{cases} C(\epsilon_2)\|\nabla z\|_2^2 + \epsilon_2\|u_0\|^{\frac{2}{3}}\|\Delta z\|_2^2 & \text{if } \frac{1}{6} < \alpha < \frac{1}{3}, \\ \|\nabla z\|_2^2 & \text{if } \alpha = \frac{1}{3}, \end{cases} \tag{3.30}$$

$$\int_{\mathbb{R}^3} w^{1-3\alpha}|\nabla z|^2\ dx \le \begin{cases} C(\epsilon_3)\|\nabla z\|_2^2 + \epsilon_3\|u_0\|^{\frac{2}{3}}\|\Delta z\|_2^2 & \text{if } \frac{1}{6} < \alpha < \frac{1}{3}, \\ \|\nabla z\|_2^2 & \text{if } \alpha = \frac{1}{3}, \end{cases} \tag{3.31}$$

$$\int_{\mathbb{R}^3} w^{1-3\alpha}|\nabla v|^2\ dx \le \begin{cases} C(\epsilon_4)\|\nabla v\|_2^2 + \epsilon_4\|w_0\|^{\frac{2}{3}}\|\Delta v\|_2^2 & \text{if } \frac{1}{6} < \alpha < \frac{1}{3}, \\ \|\nabla v\|_2^2 & \text{if } \alpha = \frac{1}{3}, \end{cases} \tag{3.32}$$

$$\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla v|^2\ dx \le C(\epsilon_5)\|\nabla v\|_2^2 + \epsilon_5\|u_0\|^{\frac{2}{3}}\|\Delta v\|_2^2, \tag{3.33}$$

$$\int_{\mathbb{R}^3} u^{1-2\alpha}|\nabla z|^2\ dx \le C(\epsilon_6)\|\nabla z\|_2^2 + \epsilon_6\|u_0\|^{\frac{2}{3}}\|\Delta z\|_2^2, \tag{3.34}$$

$$\int_{\mathbb{R}^3} w^{1-2\alpha}|\nabla z|^2\ dx \le C(\epsilon_7)\|\nabla z\|_2^2 + \epsilon_7\|w_0\|^{\frac{2}{3}}\|\Delta z\|_2^2, \tag{3.35}$$

$$\int_{\mathbb{R}^3} w^{1-2\alpha}|\nabla v|^2\ dx \le C(\epsilon_8)\|\nabla v\|_2^2 + \epsilon_8\|w_0\|^{\frac{2}{3}}\|\Delta v\|_2^2. \tag{3.36}$$

As $\dfrac{4}{3} \leq \dfrac{6-6\alpha}{2+3\alpha} < 2$, using the Hölder, Young and Gagliardo-Nierenberg inequality, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta v|\ dx &\leq C(\epsilon_9)\|u\|_{2-\alpha}^{2-\alpha} + \epsilon_9\|\Delta v\|_2^2 \\
&\leq C(\epsilon_9)C\|u_0\|_1^{\frac{1+4\alpha}{2+3\alpha}} \big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^{\frac{6-6\alpha}{2+3\alpha}} + \epsilon_9\|\Delta v\|_2^2 \\
&\leq C(\epsilon_9)C\bigg(C(\epsilon_{10})\|u_0\|_1^{\frac{1+4\alpha}{2+3\alpha}} + \epsilon_{10}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2\bigg) + \epsilon_9\|\Delta v\|_2^2 \\
&= C_1 + C_2\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \epsilon_9\|\Delta v\|_2^2. \qquad (3.37)
\end{aligned}$$

Similarly, we get

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\Delta z|\ dx \leq C_3 + C_4\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \epsilon_{11}\|\Delta z\|_2^2, \qquad (3.38)$$

$$\int_{\mathbb{R}^3} w^{1-\alpha}|\Delta z|\ dx \leq C_5 + C_6\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + \epsilon_{12}\|\Delta z\|_2^2, \qquad (3.39)$$

$$\int_{\mathbb{R}^3} w^{1-\alpha}|\Delta v|\ dx \leq C_7 + C_8\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + \epsilon_{13}\|\Delta v\|_2^2. \qquad (3.40)$$

As $\dfrac{3}{2} \leq \dfrac{6}{2+6\alpha} < 2$, using the Young and Gagliardo-Nierenberg inequality, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} u|\Delta v|\ dx &\leq C(\epsilon_{14})\|u\|_2^2 + \epsilon_{14}\|\Delta v\|_2^2 \\
&\leq C(\epsilon_{14})C_0\|u_0\|_1^{\frac{1+6\alpha}{2+6\alpha}} \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^{\frac{6}{2+6\alpha}} + \epsilon_{14}\|\Delta v\|_2^2 \\
&\leq C(\epsilon_{14})C_0\bigg(C(\epsilon_{15})\|u_0\|_1^{\frac{1+6\alpha}{2+3\alpha}} + \epsilon_{15}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2\bigg) + \epsilon_{14}\|\Delta v\|_2^2 \\
&= C_9 + C_{10}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \epsilon_{14}\|\Delta v\|_2^2. \qquad (3.41)
\end{aligned}$$

Similarly, we get

$$\int_{\mathbb{R}^3} u|\Delta z|\ dx \leq C_{11} + C_{12}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \epsilon_{15}\|\Delta z\|_2^2. \qquad (3.42)$$

$$\int_{\mathbb{R}^3} w|\Delta z|\ dx \leq C_{12} + C_{12}\big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2 + \epsilon_{16}\|\Delta z\|_2^2. \qquad (3.43)$$

$$\int_{\mathbb{R}^3} w|\Delta v|\ dx \leq C_{13} + C_{12}\big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2 + \epsilon_{17}\|\Delta v\|_2^2. \qquad (3.44)$$

Also, using Young's inequality, we get

$$\int_{\mathbb{R}^3} v|\Delta v|\ dx \leq C_{14}\big\|v\big\|_2^2 + \epsilon_{18}\|\Delta v\|_2^2. \qquad (3.45)$$

$$\int_{\mathbb{R}^3} z|\Delta z|\ dx \leq C_{15}\big\|z\big\|_2^2 + \epsilon_{19}\|\Delta z\|_2^2. \qquad (3.46)$$

As $1 < \frac{6}{5} < 3+6\alpha$ and $0 < \frac{2}{2+6\alpha} < 2$, using the Young, Gagliardo-Nierenberg, and Sobolev inequality, we get

$$\int_{\mathbb{R}^3} u|n|\ dx \leq C_0\|u\|_{\frac{6}{5}}\|n\|_6 \quad \leq \quad \epsilon_{20}\|u\|_{\frac{6}{5}}^2 + C(\epsilon_{20})\|\nabla n\|_2^2$$

$$
\begin{aligned}
&\leq \; C'\|u_0\|_1^{\frac{3+10\alpha}{2+6\alpha}}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^{\frac{2}{2+6\alpha}}+C(\epsilon_{20})\|\nabla n\|_2^2\\
&\leq \; C'\|u_0\|_1^{\frac{3+10\alpha}{2+6\alpha}}\big(\epsilon_{21}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2+C(\epsilon_{21})\big)+C(\epsilon_{20})\|\nabla n\|_2^2\\
&= \; C_{17}+C_{18}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2+C(\epsilon_{20})\|\nabla n\|_2^2. \qquad (3.47)
\end{aligned}
$$

Similarly, we get

$$
\int_{\mathbb{R}^3} w|n|\;dx \leq C_0\|w\|_{\frac{6}{5}}\|n\|_6 \;\;\leq\;\; C_{19}+C_{20}\|\nabla w^{\frac{1+2\alpha}{2}}\|_2^2+C(\epsilon_{21})\|\nabla n\|_2^2. \qquad (3.48)
$$

Substituting (3.29) $-$ (3.48) in (3.28) and and choosing sufficiently small constants such that $C,\overline{C}>0$, we have

$$
\begin{aligned}
&\frac{d}{dt}\bigg(\int_{\mathbb{R}^3} u(\log u+2\langle x\rangle)+w(\log w+2\langle x\rangle)\;dx+\big\|u\big\|_{1+\alpha}^{1+\alpha}+\big\|w\big\|_{1+\alpha}^{1+\alpha}+\big\|\nabla v\big\|_2^2+\big\|\nabla z\big\|_2^2+\|n\|_2^2\bigg)\\
&\quad+C\bigg(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2+\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2+\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2+\big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2+\big\|\Delta v\big\|_2^2+\big\|\Delta z\big\|_2^2+\|\nabla n\|_2^2\bigg)\\
&\leq \overline{C}\big(1+\|\nabla v\|_2^2+\|\nabla z\|_2^2\big).
\end{aligned}
$$

Integrating above with respect to t, we get (3.8).

Case (ii): $\frac{1}{3}<\alpha\leq 1$

Using $\log u$ as test function in $(3.2)_1$, we get

$$
\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\;dx+\int_{\mathbb{R}^3}\nabla\log u\cdot\nabla(u+\epsilon)^{1+\alpha}\;dx \leq \int_{\mathbb{R}^3}\nabla u\cdot(\chi_1\nabla v)\;dx-\int_{\mathbb{R}^3}\nabla u\cdot(\xi_1\nabla z)\;dx. \quad (3.49)
$$

As α does not affect (3.10) and (3.11), it holds in this case. Proceeding as similar as (3.11), we get

$$
-\int_{\mathbb{R}^3}\nabla u\cdot(\xi_1\nabla z)\;dx \leq \frac{2\overline{\xi_1}}{1+\alpha}\left(\epsilon_1\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2+C(\epsilon_1)\int_{\mathbb{R}^3}u^{1-\alpha}|\nabla z|^2\;dx\right). \qquad (3.50)
$$

Using (3.10), (3.11) and (3.50) in (3.49) and also choosing sufficiently small constants such that $C_1>0$, we have

$$
\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\;dx+C_1\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 \leq C_2\int_{\mathbb{R}^3}u^{1-\alpha}|\nabla v|^2\;dx+C_3\int_{\mathbb{R}^3}u^{1-\alpha}|\nabla z|^2\;dx. \qquad (3.51)
$$

Similarly, we get

$$
\frac{d}{dt}\int_{\mathbb{R}^3} w\log w\;dx+C_4\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 \leq C_5\int_{\mathbb{R}^3}w^{1-\alpha}|\nabla z|^2\;dx+C_6\int_{\mathbb{R}^3}w^{1-\alpha}|\nabla v|^2\;dx. \qquad (3.52)
$$

Using u^{α} as test function in $(3.2)_1$, we get

$$
\frac{1}{1+\alpha}\frac{d}{dt}\big\|u\big\|_{1+\alpha}^{1+\alpha}+\frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 \leq \int_{\mathbb{R}^3}\nabla u^{\alpha}\cdot u(\chi_1\nabla v)\;dx-\int_{\mathbb{R}^3}\nabla u^{\alpha}\cdot u(\xi_1\nabla z)\;dx. \quad (3.53)
$$

Using Young's inequality, the first term of RHS in the above is estimated as

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\nabla u^{\alpha}\cdot u(\chi_1\nabla v)\;dx\\
&\qquad\leq \; C\int_{\mathbb{R}^3}\big|\nabla u^{\frac{1+2\alpha}{2}}\big|(u^{\frac{1}{2}}|\nabla v|)\;dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left(\epsilon_2\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C(\epsilon_2)\int_{\mathbb{R}^3} u|\nabla v|^2\,dx\right)\\
&= C\epsilon_2\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + CC(\epsilon_2)\int_{\mathbb{R}^3} u|\nabla v|^2\,dx\\
&\leq C\epsilon_2\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + CC(\epsilon_2)\epsilon_3\left(\int_{\mathbb{R}^3}|\nabla u||\nabla v|\,dx + \int_{\mathbb{R}^3} u|\Delta v|\,dx\right)\\
&= C\epsilon_2\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + CC(\epsilon_2)\epsilon_3\int_{\mathbb{R}^3} u|\Delta v|\,dx + CC(\epsilon_2)\epsilon_3\int_{\mathbb{R}^3} u^{\frac{1-\alpha}{2}}\big|\nabla u^{\frac{1+\alpha}{2}}\big||\nabla v|\,dx\\
&\leq C\epsilon_2\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + CC(\epsilon_2)\epsilon_3\int_{\mathbb{R}^3} u|\Delta v|\,dx\\
&\qquad\qquad + CC(\epsilon_2)\epsilon_3\left(\epsilon_4\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C(\epsilon_4)\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx\right)\\
&= C_7\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C_8\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_9\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx + C_{10}\int_{\mathbb{R}^3} u|\Delta v|\,dx. \qquad (3.54)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}\nabla u^{\alpha}\cdot u(\xi_1\nabla z)\,dx\\
&\qquad\leq C_{11}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C_{12}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_{13}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla z|^2\,dx + C_{14}\int_{\mathbb{R}^3} u|\Delta z|\,dx. \qquad (3.55)
\end{aligned}
$$

Using (3.54) and (3.55) in (3.53) and choosing sufficiently small constants such that $C > 0$, we have

$$
\begin{aligned}
\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 &\leq C_{15}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_9\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx + C_{10}\int_{\mathbb{R}^3} u|\Delta v|\,dx\\
&\quad + C_{13}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla z|^2\,dx + C_{14}\int_{\mathbb{R}^3} u|\Delta z|\,dx. \qquad (3.56)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\frac{d}{dt}\|w\|_{1+\alpha}^{1+\alpha} + C'\big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2 &\leq C_{16}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_{17}\int_{\mathbb{R}^3} w^{1-\alpha}|\nabla z|^2\,dx + C_{18}\int_{\mathbb{R}^3} w|\Delta z|\,dx\\
&\quad + C_{19}\int_{\mathbb{R}^3} w^{1-\alpha}|\nabla v|^2\,dx + C_{20}\int_{\mathbb{R}^3} w|\Delta v|\,dx. \qquad (3.57)
\end{aligned}
$$

As α does not affect (3.16), (3.17) and (3.25)-(3.27), adding those with (3.51), (3.52), (3.56), (3.57) and choosing sufficiently small constants such that $C_0, C_0' > 0$, we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle) + w(\log w + 2\langle x\rangle)\,dx + \|u\|_{1+\alpha}^{1+\alpha} + \|w\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla z\|_2^2 + \|n\|_2^2\right)\\
&\quad + C_0\left(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2 + \|\Delta v\|_2^2 + \|\Delta z\|_2^2 + \|\nabla n\|_2^2\right)\\
&\leq C_0'\left(\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\nabla z|^2\,dx + \int_{\mathbb{R}^3} w^{1-\alpha}|\nabla z|^2\,dx + \int_{\mathbb{R}^3} w^{1-\alpha}|\nabla v|^2\,dx\right.\\
&\qquad + \int_{\mathbb{R}^3} u|\Delta v|\,dx + \int_{\mathbb{R}^3} u|\Delta z|\,dx + \int_{\mathbb{R}^3} w|\Delta z|\,dx + \int_{\mathbb{R}^3} w|\Delta v|\,dx\\
&\qquad \left. + \int_{\mathbb{R}^3} v|\Delta v|\,dx + \int_{\mathbb{R}^3} z|\Delta z|\,dx + \int_{\mathbb{R}^3}(u+w)|n|\,dx\right). \qquad (3.58)
\end{aligned}
$$

As $0 \le 1-\alpha < \frac{2}{3}$, using the Hölder inequality and Sobolev embedding, we get

$$\begin{aligned}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\,dx &\le \begin{cases} \displaystyle\int_{\mathbb{R}^3}\big(C(\epsilon_1)+\epsilon_1 u^{\frac{2}{3}}\big)|\nabla v|^2\,dx & \text{if } \frac{1}{3}<\alpha<1,\\ \|\nabla v\|_2^2 & \text{if } \alpha=1.\end{cases}\\ &\le \begin{cases} C(\epsilon_1)\|\nabla v\|_2^2+\epsilon_1\|u_0\|^{\frac{2}{3}}\|\Delta v\|_2^2 & \text{if } \frac{1}{3}<\alpha<1,\\ \|\nabla v\|_2^2 & \text{if } \alpha=1.\end{cases}\end{aligned} \tag{3.59}$$

Similarly, we get

$$\begin{aligned}\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla z|^2\,dx &\le \begin{cases} C(\epsilon_2)\|\nabla z\|_2^2+\epsilon_2\|u_0\|^{\frac{2}{3}}\|\Delta z\|_2^2 & \text{if } \frac{1}{3}<\alpha<1,\\ \|\nabla z\|_2^2 & \text{if } \alpha=1,\end{cases}\\ \int_{\mathbb{R}^3} w^{1-\alpha}|\nabla z|^2\,dx &\le \begin{cases} C(\epsilon_3)\|\nabla z\|_2^2+\epsilon_3\|w_0\|^{\frac{2}{3}}\|\Delta z\|_2^2 & \text{if } \frac{1}{3}<\alpha<1,\\ \|\nabla z\|_2^2 & \text{if } \alpha=1,\end{cases}\\ \int_{\mathbb{R}^3} w^{1-\alpha}|\nabla v|^2\,dx &\le \begin{cases} C(\epsilon_4)\|\nabla v\|_2^2+\epsilon_4\|w_0\|^{\frac{2}{3}}\|\Delta v\|_2^2 & \text{if } \frac{1}{3}<\alpha<1,\\ \|\nabla v\|_2^2 & \text{if } \alpha=1.\end{cases}\end{aligned}$$

As $0 < \frac{6}{2+6\alpha} < 2$ and $1 < \dfrac{6}{5} < 3+6\alpha$, (3.41)-(3.48) holds. Substituting (3.59), (3.41)-(3.48) and above inequalities in (3.58) and choosing sufficiently small constants such that $C_0, \overline{C} > 0$, we get

$$\begin{aligned}&\frac{d}{dt}\bigg(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle) + w(\log w + 2\langle x\rangle)\,dx + \big\|u\big\|_{1+\alpha}^{1+\alpha} + \big\|w\big\|_{1+\alpha}^{1+\alpha} + \big\|\nabla v\big\|_2^2 + \big\|\nabla z\big\|_2^2 + \|n\|_2^2\bigg)\\ &\quad + C_0\bigg(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v\big\|_2^2 + \big\|\Delta z\big\|_2^2 + \|\nabla n\|_2^2\bigg)\\ &\le \overline{C}\big(1+\|\nabla v\|_2^2+\|\nabla z\|_2^2\big).\end{aligned}$$

Integrating above with respect to t, we get (3.8).

Case (iii): $\alpha > 1$

Using $\log u$ as test function in $(3.2)_1$ and (3.10), we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + \frac{4}{1+\alpha}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 \le \int_{\mathbb{R}^3}\nabla u\cdot(\chi_1\nabla v)\,dx - \int_{\mathbb{R}^3}\nabla u\cdot(\xi_1\nabla z)\,dx. \tag{3.60}$$

Using our assumption $\chi^{'} \in L^\infty_{loc}$ and $\xi^{'} \in L^\infty_{loc}$ in RHS of above, we get

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla u\cdot(\chi_1\nabla v)\,dx &\le C_1\int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_2\int_{\mathbb{R}^3} u|\Delta v|\,dx,\\ -\int_{\mathbb{R}^3}\nabla u\cdot(\xi_1\nabla z)\,dx &\le C_3\int_{\mathbb{R}^3} u|\nabla z|^2\,dx + C_4\int_{\mathbb{R}^3} u|\Delta z|\,dx.\end{aligned}$$

Using above in (3.60), we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\,dx + \frac{4}{1+\alpha}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 \le C_1\int_{\mathbb{R}^3} u|\nabla v|^2\,dx + C_2\int_{\mathbb{R}^3} u|\Delta v|\,dx$$

$$+C_3\int_{\mathbb{R}^3} u|\nabla z|^2\ dx + C_4\int_{\mathbb{R}^3} u|\Delta z|\ dx. \quad (3.61)$$

Similarly, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} w\log w\ dx + \frac{4}{1+\alpha}\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 &\leq C_5\int_{\mathbb{R}^3} w|\nabla z|^2\ dx + C_6\int_{\mathbb{R}^3} w|\Delta z|\ dx\\ &\quad +C_7\int_{\mathbb{R}^3} w|\nabla v|^2\ dx + C_8\int_{\mathbb{R}^3} w|\Delta v|\ dx. \quad (3.62)\end{aligned}$$

Using u^α as test function in $(3.2)_1$, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\big\|u\big\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 \leq \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi_1\nabla v)\ dx - \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi_1\nabla z)\ dx. \quad (3.63)$$

The first term of RHS in the above estimated as

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi_1\nabla v)\ dx &\leq C\int_{\mathbb{R}^3}\big|\nabla u^{\frac{1+2\alpha}{2}}\big|(u^{\frac{1}{2}}|\nabla v|)\ dx\\ &\leq C_5\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C_6\int_{\mathbb{R}^3} u|\nabla v|^2\ dx. \quad (3.64)\end{aligned}$$

Similarly, we get

$$-\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi_1\nabla z)\ dx \leq C_7\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C_8\int_{\mathbb{R}^3} u|\nabla z|^2\ dx. \quad (3.65)$$

Using (3.64) and (3.65) in (3.63) and choosing sufficiently small C_5, C_7 such that $C_9 > 0$, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\big\|u\big\|_{1+\alpha}^{1+\alpha} + C_9\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 \leq C_6\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_8\int_{\mathbb{R}^3} u|\nabla z|^2\ dx. \quad (3.66)$$

Similarly, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\big\|w\big\|_{1+\alpha}^{1+\alpha} + C_{10}\big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2 \leq C_{11}\int_{\mathbb{R}^3} w|\nabla z|^2\ dx + C_{12}\int_{\mathbb{R}^3} w|\nabla v|^2\ dx. \quad (3.67)$$

As α does not affect (3.16), (3.17) and (3.25)-(3.27), adding those with (3.61), (3.62), (3.66), (3.67) and choosing sufficiently small constants such that $C_0, C_0' > 0$, we get

$$\begin{aligned}&\frac{d}{dt}\bigg(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle) + w(\log w + 2\langle x\rangle)\ dx + \big\|u\big\|_{1+\alpha}^{1+\alpha} + \big\|w\big\|_{1+\alpha}^{1+\alpha} + \big\|\nabla v\big\|_2^2 + \big\|\nabla z\big\|_2^2 + \|n\|_2^2\bigg)\\ &\quad + C_0\bigg(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v\big\|_2^2 + \big\|\Delta z\big\|_2^2 + \|\nabla n\|_2^2\bigg)\\ &\leq C_0'\bigg(\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + \int_{\mathbb{R}^3} u|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w|\nabla v|^2\ dx\\ &\qquad + \int_{\mathbb{R}^3} u|\Delta v|\ dx + \int_{\mathbb{R}^3} u|\Delta z|\ dx + \int_{\mathbb{R}^3} w|\Delta z|\ dx + \int_{\mathbb{R}^3} w|\Delta v|\ dx\\ &\qquad + \int_{\mathbb{R}^3} v|\Delta v|\ dx + \int_{\mathbb{R}^3} z|\Delta z|\ dx + \int_{\mathbb{R}^3}(u+w)|n|\ dx\bigg).\end{aligned} \quad (3.68)$$

As $\frac{1+\alpha}{2} > 1$, using Young's inequality, we get

$$\int_{\mathbb{R}^3} u|\nabla v|^2 dx \leq \epsilon_1\|\nabla v\|_2^2 + C(\epsilon_1)\int_{\mathbb{R}^3} u^{\frac{1+\alpha}{2}}|\nabla v|^2\ dx$$

$$
\begin{aligned}
&= \epsilon_1\|\nabla v\|_2^2 + C(\epsilon_1)\int_{\mathbb{R}^3} u^{\frac{1+\alpha}{2}}\nabla v\cdot\nabla v\ dx\\
&\leq \epsilon_1\|\nabla v\|_2^2 + C(\epsilon_1)C\bigg(\int_{\mathbb{R}^3}\nabla u^{\frac{1+\alpha}{2}}\cdot\nabla v\ dx + \int_{\mathbb{R}^3} u^{\frac{1+\alpha}{2}}\Delta v\ dx\bigg)\\
&\leq \epsilon_1\|\nabla v\|_2^2 + C_1\bigg(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \|\nabla v\|_2^2 + \|u\|_{1+\alpha}^{1+\alpha} + \|\Delta v\|_2^2\bigg). \qquad (3.69)
\end{aligned}
$$

As $\frac{6\alpha}{2+3\alpha} < 2$, using the Gagliardo-Nierenberg and Young inequality, the third term in RHS of the above is evaluated as

$$
\|u\|_{1+\alpha}^{1+\alpha} \leq C_1\|u_0\|_1^{\frac{2+2\alpha}{2+3\alpha}}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^{\frac{6\alpha}{2+3\alpha}} \leq C_1C(\epsilon_2) + \epsilon_2\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2.
$$

Using above in (3.69), we get

$$
\int_{\mathbb{R}^3} u|\nabla v|^2\ dx \leq C_2 + C_3\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_4\|\nabla v\|_2^2 + C_5\|\Delta v\|_2^2. \qquad (3.70)
$$

Similarly, we get

$$
\int_{\mathbb{R}^3} u|\nabla z|^2\ dx \leq C_6 + C_7\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_8\|\nabla z\|_2^2 + C_9\|\Delta z\|_2^2, \qquad (3.71)
$$

$$
\int_{\mathbb{R}^3} w|\nabla z|^2\ dx \leq C_{10} + C_{11}\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + C_{12}\|\nabla z\|_2^2 + C_{13}\|\Delta z\|_2^2, \qquad (3.72)
$$

$$
\int_{\mathbb{R}^3} w|\nabla v|^2\ dx \leq C_{14} + C_{15}\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + C_{16}\|\nabla v\|_2^2 + C_{17}\|\Delta v\|_2^2. \qquad (3.73)
$$

As $0 < \frac{6}{2+6\alpha} < 2$ and $1 < \dfrac{6}{5} < 3 + 6\alpha$, (3.41)-(3.48) holds. Substituting (3.70), (3.71) and (3.41)-(3.48) in (3.68) and choosing sufficiently small constants such that $C_0, \overline{C} > 0$, we get

$$
\begin{aligned}
&\frac{d}{dt}\bigg(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle) + w(\log w + 2\langle x\rangle)\ dx + \big\|u\big\|_{1+\alpha}^{1+\alpha} + \big\|w\big\|_{1+\alpha}^{1+\alpha} + \big\|\nabla v\big\|_2^2 + \big\|\nabla z\big\|_2^2 + \|n\|_2^2\bigg)\\
&\quad + C_0\bigg(\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla w^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v\big\|_2^2 + \big\|\Delta z\big\|_2^2 + \|\nabla n\|_2^2\bigg)\\
&\leq \overline{C}\big(1 + \|\nabla v\|_2^2 + \|\nabla z\|_2^2\big).
\end{aligned}
$$

Integrating above with respect to t, we get (3.8). ■

Lemma 3.2. *Suppose that* (u, v, w, z, n) *is a classical solution for system* (3.2) *independent of* ϵ *and* (3.3) *satisfies*

$$
u_0(1 + |x| + |\log u_0|)\ \textit{and}\ w_0(1 + |x| + |\log w_0|) \in L^1(\mathbb{R}^3),
$$

$$
u_0, w_0 \in L^{1+\alpha}(\mathbb{R}^3), v_0\ \textit{and}\ z_0 \in L^\infty(\mathbb{R}^3)\cap H^1(\mathbb{R}^3), \quad n_0 \in L^2(\mathbb{R}^3).
$$

Further, assume that

$$
\left.\begin{aligned}
&\alpha > \frac{1}{6}, \quad \phi \in W^{2,\infty}(\mathbb{R}^3)\ \textit{and}\\
&\chi_i', \xi_i' \in L^\infty_{loc}\ \textit{for}\ i \in \{1, 2\}\ \textit{with}\ \chi_i'(\cdot) \geq \chi_{i_0}\ \textit{for some constant}\ \chi_{i_0}.
\end{aligned}\right\} \qquad (3.74)
$$

Then, there exist $C > 0$, *independent of* ϵ, *such that for any* $0 < t \leq \Lambda$

$$
\sup_{0\leq\tau\leq t} E(\tau) + \int_0^t D(\tau)d\tau < C. \qquad (3.75)
$$

Proof. It is enough to prove for the case $0 < \alpha \leq \frac{1}{6}$ as the case $\alpha > \frac{1}{6}$ already established in previous lemma. Using $\log u$ as test function in $(3.2)_1$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\ dx + \frac{4}{1+\alpha}\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 \leq \int_{\mathbb{R}^3} \nabla u\cdot(\chi_1\nabla v)\ dx - \int_{\mathbb{R}^3}\nabla u\cdot(\xi_1\nabla z)\ dx. \tag{3.76}$$

Using our assumption $\chi' \in L^\infty_{loc}$ and $\chi_1'(\cdot) \geq \chi_{1_0}$, we get

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla u\cdot(\chi_1\nabla v)dx &= -\int_{\mathbb{R}^3}(\chi_1'|\nabla v|^2 + \chi_1\Delta v)u\ dx\\ &\leq -\chi_{1_0}\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_1\int_{\mathbb{R}^3} u^{\frac{1-\alpha}{2}}\big|\nabla u^{\frac{1+\alpha}{2}}\big||\nabla v|\ dx\\ &\leq -\chi_{1_0}\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_2\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_3\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx.\end{aligned}\tag{3.77}$$

Similarly, we get

$$-\int_{\mathbb{R}^3}\nabla u\cdot(\xi_1\nabla z)\ dx \leq -\xi_{1_0}\int_{\mathbb{R}^3} u|\nabla z|^2\ dx + C_4\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + C_5\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla z|^2\ dx. \tag{3.78}$$

Using (3.77) and (3.78) in (3.76) and choosing sufficiently small constants such that $C_1 > 0$, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3} u\log u\ dx + C_1\big\|\nabla u^{\frac{1+\alpha}{2}}\big\|_2^2 + \chi_{1_0}\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + \xi_{1_0}\int_{\mathbb{R}^3} u|\nabla z|^2\ dx\\ &\quad\leq C_3\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx + C_5\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla z|^2\ dx.\end{aligned}\tag{3.79}$$

Similarly, we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3} w\log w\ dx + C_1\big\|\nabla w^{\frac{1+\alpha}{2}}\big\|_2^2 + \chi_{2_0}\int_{\mathbb{R}^3} w|\nabla z|^2\ dx + \xi_{2_0}\int_{\mathbb{R}^3} w|\nabla v|^2\ dx\\ &\quad\leq C_6\int_{\mathbb{R}^3} w^{1-\alpha}|\nabla z|^2\ dx + C_7\int_{\mathbb{R}^3} w^{1-\alpha}|\nabla v|^2\ dx.\end{aligned}\tag{3.80}$$

Using u^α as test function in $(3.2)_1$, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 \leq \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi_1\nabla v)\ dx - \int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi_1\nabla z)\ dx. \tag{3.81}$$

As $\frac{1+2\alpha}{2} > 0$, using Young's inequality, the first term of RHS in the above is estimated as

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\chi_1\nabla v)\ dx &\leq \frac{2\alpha\overline{\chi_1}}{1+\alpha}\int_{\mathbb{R}^3}\big|\nabla u^{\frac{1+2\alpha}{2}}\big|(u^{\frac{1}{2}}|\nabla v|)\ dx\\ &\leq C_8\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C_9\int_{\mathbb{R}^3} u|\nabla v|^2\ dx.\end{aligned}$$

Similarly, we get

$$-\int_{\mathbb{R}^3}\nabla u^\alpha\cdot u(\xi_1\nabla z)\ dx \leq C_{10}\big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_2^2 + C_{11}\int_{\mathbb{R}^3} u|\nabla z|^2\ dx.$$

Using above two estimates in (3.81) and choosing sufficiently small C_8, C_{10} such that $C_9 > 0$, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\|u\|_{1+\alpha}^{1+\alpha} + C_{14}\|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 \ \leq\ C_9\int_{\mathbb{R}^3} u|\nabla v|^2\ dx + C_{11}\int_{\mathbb{R}^3} u|\nabla z|^2\ dx. \tag{3.82}$$

Similarly, we get

$$\frac{1}{1+\alpha}\frac{d}{dt}\|w\|_{1+\alpha}^{1+\alpha} + C_{15}\|\nabla w^{\frac{1+2\alpha}{2}}\|_2^2 \ \leq\ C_{12}\int_{\mathbb{R}^3} w|\nabla z|^2\ dx + C_{13}\int_{\mathbb{R}^3} w|\nabla v|^2\ dx. \tag{3.83}$$

As α does not affect (3.16), (3.17) and (3.25)-(3.27), adding those with (3.79), (3.80), (3.82) and (3.83) and choosing sufficiently small constants such that $C_0, C_0' > 0$ and $C_9, C_{11}, C_{12}, C_{12}$, we get

$$\begin{aligned}
&\frac{d}{dt}\bigg(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle) + w(\log w + 2\langle x\rangle)\ dx + \|u\|_{1+\alpha}^{1+\alpha} + \|w\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla z\|_2^2 + \|n\|_2^2\bigg)\\
&\qquad + C_0\bigg(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\nabla w^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla w^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta z\|_2^2 + \|\nabla n\|_2^2\\
&\qquad\qquad + \int_{\mathbb{R}^3} u|\nabla v|^2\ dx + \int_{\mathbb{R}^3} u|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w|\nabla v|^2\ dx\bigg)\\
&\leq\ C_0'\bigg(\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx + \int_{\mathbb{R}^3} u^{1-\alpha}|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w^{1-\alpha}|\nabla z|^2\ dx + \int_{\mathbb{R}^3} w^{1-\alpha}|\nabla v|^2\ dx\\
&\qquad + \int_{\mathbb{R}^3} v|\Delta v|dx + \int_{\mathbb{R}^3} z|\Delta z|dx - \int_{\mathbb{R}^3} u|\Delta z|dx - \int_{\mathbb{R}^3} w|\Delta v|dx + \int_{\mathbb{R}^3}(u+w)|n|dx\bigg).
\end{aligned}\tag{3.84}$$

Using Young's inequality and choosing sufficiently small ϵ, the first term of RHS in the above is estimated as

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla v|^2\ dx \leq \int_{\mathbb{R}^3}\big(C(\epsilon)+\epsilon u\big)|\nabla v|^2\ dx \leq C_1\|\nabla v\|_2^2. \tag{3.85}$$

Similarly, we get

$$\int_{\mathbb{R}^3} u^{1-\alpha}|\nabla z|^2\ dx \leq C_2\|\nabla z\|_2^2, \tag{3.86}$$

$$\int_{\mathbb{R}^3} w^{1-\alpha}|\nabla z|^2\ dx \leq C_3\|\nabla z\|_2^2, \tag{3.87}$$

$$\int_{\mathbb{R}^3} w^{1-\alpha}|\nabla v|^2\ dx \leq C_4\|\nabla v\|_2^2. \tag{3.88}$$

As α does not affect (3.45)-(3.47), it holds. Substituting (3.85)-(3.88) and (3.45)-(3.47) in (3.84) and choosing sufficiently small constants such that $C_0, \overline{C} > 0$, we get

$$\begin{aligned}
&\frac{d}{dt}\bigg(\int_{\mathbb{R}^3} u(\log u + 2\langle x\rangle) + w(\log w + 2\langle x\rangle)\ dx + \|u\|_{1+\alpha}^{1+\alpha} + \|w\|_{1+\alpha}^{1+\alpha} + \|\nabla v\|_2^2 + \|\nabla z\|_2^2 + \|n\|_2^2\bigg)\\
&\qquad + C_0\bigg(\|\nabla u^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla u^{\frac{1+2\alpha}{2}}\|_2^2 + \|\nabla w^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla w^{\frac{1+2\alpha}{2}}\|_2^2 + \|\Delta v\|_2^2 + \|\Delta z\|_2^2 + \|\nabla n\|_2^2\bigg)\\
&\leq \overline{C}\big(1 + \|\nabla v\|_2^2 + \|\nabla z\|_2^2\big).
\end{aligned}$$

Integrating above with respect to t, we get (3.75). Hence, the lemma holds. ■

Now, we are ready to present the main findings of this paper,(i.e.) the existence of weak solution for (2.1) in $\mathbb{R}^3$ and extend the same for a bounded domain with smooth boundary.

Theorem 3.1. *Assume that either* (3.7) *or* (3.74) *holds true and* $(u_0, v_0, w_0, z_0, n_0)$ *satisfies initial data* (3.6)*. Then for each* $\Lambda > 0$*, the system* (3.1) *possesses a weak solution* (u, v, w, z, n)*, such that*

$$\sup_{0\le t\le\Lambda}\Bigg(\int_{\mathbb{R}^3} u(t)(\log u(t) + 2\langle x\rangle) + w(t)(\log w(t) + 2\langle x\rangle)\ dx + \big\|u(t)\big\|_{1+\alpha}^{1+\alpha} + \big\|w(t)\big\|_{1+\alpha}^{1+\alpha}$$
$$+\big\|\nabla v(t)\big\|_2^2 + \big\|\nabla z(t)\big\|_2^2 + \|n(t)\|_2^2\Bigg)$$
$$+\int_0^\Lambda \Bigg(\big\|\nabla u(t)^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u(t)^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\nabla w(t)^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla w(t)^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v(t)\big\|_2^2$$
$$+\big\|\Delta z(t)\big\|_2^2 + \|\nabla n(t)\|_2^2\Bigg)\ dt < C,$$

where C *is a constant depends on initial conditions.*

Proof. As the convergence of $(u_{0_\epsilon}, v_{0_\epsilon}, w_{0_\epsilon}, z_{0_\epsilon}, n_{0_\epsilon})$ ensures the estimate obtained in Lemma 3.1 and 3.2 is uniform, the constant C in (3.8) can be chosen without dependence of ϵ. In similar, there exists positive constant C such that for $q < \infty$

$$\|u_\epsilon\|_{L^\infty((0,\Lambda)\times\mathbb{R}^3)} < C, \quad \big\|\nabla u_\epsilon^{\frac{q+\alpha}{2}}\big\|_{L^2((0,\Lambda)\times\mathbb{R}^3)} < C,$$
$$\|v_\epsilon\|_{L^\infty(0,\Lambda;W^{1,q}(\mathbb{R}^3))} < C, \quad \|v_\epsilon\|_{L^q(0,\Lambda;W^{2,q}(\mathbb{R}^3))} < C, \quad \|v_{\epsilon t}\|_{L^q(0,\Lambda;L^q(\mathbb{R}^3))} < C,$$
$$\|w_\epsilon\|_{L^\infty((0,\Lambda)\times\mathbb{R}^3)} < C, \quad \big\|\nabla w_\epsilon^{\frac{q+\alpha}{2}}\big\|_{L^2((0,\Lambda)\times\mathbb{R}^3)} < C,$$
$$\|z_\epsilon\|_{L^\infty(0,\Lambda;W^{1,q}(\mathbb{R}^3))} < C, \quad \|z_\epsilon\|_{L^q(0,\Lambda;W^{2,q}(\mathbb{R}^3))} < C, \quad \|z_{\epsilon t}\|_{L^q(0,\Lambda;L^q(\mathbb{R}^3))} < C,$$
$$\|n_\epsilon\|_{L^\infty(0,\Lambda;W^{1,q}(\mathbb{R}^3))} < C, \quad \|n_\epsilon\|_{L^q(0,\Lambda;W^{2,q}(\mathbb{R}^3))} < C, \quad \|n_{\epsilon t}\|_{L^q(0,\Lambda;L^q(\mathbb{R}^3))} < C.$$

As the above estimates are uniform, the local solution obtained can be extended to any time interval $(0, \Lambda)$ (Ref. [28], [21]). Let k be a positive number with $k \ge 2 + \alpha$. Then, $u_{\epsilon t}, u_{\epsilon t}^k, w_{\epsilon t}$ and $w_{\epsilon t}^k \in L^1(0, \Lambda; W^{-2,2}(\mathbb{R}^3))$. As $\epsilon \to 0$, using Aubin-Lions compactness lemma, we have a weak limit indeed a weak solution. ■

The above theorem can be extended for bounded domains with Neumann boundary conditions applied to u, v, w, z, and no-slip boundary conditions for n.(i.e.)

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \quad n = 0 \text{ on } \partial\Omega, \tag{3.89}$$

Consider the system (3.2) in $\Omega \times [0, \Lambda)$ where $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Next, we state and prove the next result of this paper.

Theorem 3.2. *Assume that either* (3.7) *or* (3.74) *holds true by replacing* $\mathbb{R}^3$ *with* Ω *and* $(u_0, v_0, w_0, z_0, n_0)$ *satisfies initial data* (3.6)*. Then for each* $\Lambda > 0$*, the system* (3.1) *with* (3.89) *possesses a weak solution* (u, v, w, z, n)*, such that*

$$\begin{aligned}
\sup_{0\le t\le\Lambda} \Bigg(\int_{\mathbb{R}^3} & u(t)(\log u(t) + 2\langle x\rangle) + w(t)(\log w(t) + 2\langle x\rangle)\; dx + \big\|u(t)\big\|_{1+\alpha}^{1+\alpha} + \big\|w(t)\big\|_{1+\alpha}^{1+\alpha} \\
&+ \big\|\nabla v(t)\big\|_2^2 + \big\|\nabla z(t)\big\|_2^2 + \|n(t)\|_2^2 \Bigg) \\
+ \int_0^\Lambda \Bigg(& \big\|\nabla u(t)^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla u(t)^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\nabla w(t)^{\frac{1+\alpha}{2}}\big\|_2^2 + \big\|\nabla w(t)^{\frac{1+2\alpha}{2}}\big\|_2^2 + \big\|\Delta v(t)\big\|_2^2 \\
&+ \big\|\Delta z(t)\big\|_2^2 + \|\nabla n(t)\|_2^2 \Bigg)\; dt < C,
\end{aligned}$$

where C *is a constant depends on initial conditions.*

Proof. As the proof is similar, we address the modification in the proof of the above theorem. As negative part of $\int_\Omega u \log u$ is bounded by $\|u\|_{L^1(\Omega)}$, (3.18) is not required. Also, (3.41) modified slightly for the case of bounded domains, and using Gagliardo-Nirenberg inequality we get

$$\|u\|_2^2 \le C\|u_0\|_{L^1(\Omega)}^{\frac{1+6\alpha}{2+6\alpha}} \big\|\nabla u^{\frac{1+2\alpha}{2}}\big\|_{L^2(\Omega)}^{\frac{6}{2+6\alpha}} + C\|u\|_{L^1(\Omega)}^2.$$

Following a similar procedure as in the proof of Theorem 3.1 completes the proof. ■

4 Bounded Weak Solutions

In this section, we define a bounded weak solution and introduce a suitable approximation problem for the proposed model (2.1) with $\tau = 0$. Then we prove a certain important lemma and using that, we establish the existence of bounded weak solution in $\mathbb{R}^3$. We consider the system (2.1) with $\tau = 0$ as

$$\left.\begin{aligned}
&u_t + n\cdot\nabla u = \Delta u^{1+\alpha} - \nabla\cdot(\chi_1(v)u\nabla v) + \nabla\cdot(\xi_1(z)u\nabla z), \\
&v_t + n\cdot\nabla v = \Delta v + \beta w - \gamma v, \\
&w_t + n\cdot\nabla w = \Delta w^{1+\alpha} - \nabla\cdot(\chi_2(z)w\nabla z) + \nabla\cdot(\xi_2(v)w\nabla v), \\
&z_t + n\cdot\nabla z = \Delta z + \delta u - \eta z, \\
&n_t + \nabla p = \Delta n - (u+w)\nabla\phi.
\end{aligned}\right\} \tag{4.1}$$

Definition 4.1. *For* $\alpha > 0$ *and* $\Lambda \in (0, \infty)$*, a weak solution* (u, v, w, z, n) *of* (4.1) *with* $\tau = 0$*, as defined in Definition 3.1, is said to be a bounded weak solution if*

(i) For any $p \in [1, \infty)$*,*

$$u,\ w \in L^\infty((0,\Lambda);\mathbb{R}^3) \text{ and } \nabla u^{\frac{p+\alpha}{2}}, \nabla w^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda; L^2(\mathbb{R}^3)),$$

(ii) For any $q \in [2, \infty)$*,*

$$v,\ z,\ n \in L^q(0,\Lambda; W^{2,q}(\mathbb{R}^3)) \text{ and } v_t,\ z_t,\ n_t \in L^q(0,\Lambda; L^q(\mathbb{R}^3)).$$

The approximation problem of the proposed model (2.1) given by

$$\left.\begin{aligned}
&u_{\epsilon t} + n_\epsilon \cdot \nabla u_\epsilon = \Delta(u_\epsilon+\epsilon)^{1+\alpha} - \nabla\cdot(\chi_1(v_\epsilon)u_\epsilon\nabla v_\epsilon) + \nabla\cdot(\xi_1(z_\epsilon)u_\epsilon\nabla z_\epsilon),\\
&v_{\epsilon t} + n_\epsilon \cdot \nabla v_\epsilon = \Delta v_\epsilon + \beta w_\epsilon - \gamma v_\epsilon,\\
&w_{\epsilon t} + n_\epsilon \cdot \nabla w_\epsilon = \Delta(w_\epsilon+\epsilon)^{1+\alpha} - \nabla\cdot(\chi_2(z_\epsilon)w_\epsilon\nabla z_\epsilon) + \nabla\cdot(\xi_2(v_\epsilon)w_\epsilon\nabla v_\epsilon),\\
&z_{\epsilon t} + n_\epsilon \cdot \nabla z_\epsilon = \Delta z_\epsilon + \delta u_\epsilon - \eta z_\epsilon,\\
&n_{\epsilon t} + \nabla p_\epsilon = \Delta n_\epsilon - (u_\epsilon + w_\epsilon)\nabla\phi,
\end{aligned}\right\} \tag{4.2}$$

with smooth initial conditions given by

$$u_{0_\epsilon} = \phi_\epsilon * u_0,\ v_{0_\epsilon} = \phi_\epsilon * v_0,\ w_{0_\epsilon} = \phi_\epsilon * w_0,\ z_{0_\epsilon} = \phi_\epsilon * z_0 \text{ and } n_{0_\epsilon} = \phi_\epsilon * n_0, \tag{4.3}$$

where ϕ_ϵ refers usual mollifier with $\epsilon \in (0,1)$. We derive some uniform estimates for the approximation problem and using that we derive a bounded weak solution for (2.1). Now, we prove a lemma that gives an estimate for the bounded weak solution of (4.2). For the simplicity of notation, hereafter, we use variables $(u_\epsilon, v_\epsilon, w_\epsilon, z_\epsilon, n_\epsilon)$ as (u, v, w, z, n).

Lemma 4.1. *Suppose that (u, v, w, z, n) is a classical solution for system (4.2) independent of ϵ and (4.3) satisfies*

$$u_0(1+|x|+|\log u_0|),\ w_0(1+|x|+|\log w_0|) \in L^1(\mathbb{R}^3),$$

$$u_0,\ w_0 \in L^{1+\alpha}(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3), \quad v_0,\ z_0 \in L^\infty(\mathbb{R}^3)\cap H^1(\mathbb{R}^3) \text{ and } n_0 \in L^2(\mathbb{R}^3),$$

and for any $q \in [2,\infty)$

$$v_0,\ z_0 \in W^{1,q}(\mathbb{R}^3), \text{ and } n_0 \in W^{1,q}(\mathbb{R}^3). \tag{4.4}$$

Further, assume that

$$\left.\begin{aligned}
&\alpha > \frac{1}{8}, \quad \phi \in W^{2,\infty}(\mathbb{R}^3), \quad \gamma = \eta = 0 \text{ and}\\
&\chi_i^{'}, \xi_i^{'} \in L^\infty_{loc} \text{ for } i \in \{1,2\} \text{ with } \chi_i^{'}(\cdot) \geq \chi_{i_0} \text{ for some constant } \chi_{i_0}.
\end{aligned}\right\} \tag{4.5}$$

Then, for any $t \in (0,\Lambda]$

$$u \text{ and } w \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3)), \tag{4.6}$$

$$\nabla u^{\frac{p+\alpha}{2}} \text{ and } \nabla w^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda; L^2(\mathbb{R}^3)), \qquad 1 \leq p \leq \infty, \tag{4.7}$$

$$v, z, n \in L^\infty(0,\Lambda; W^{1,q}(\mathbb{R}^3)) \cap L^q(0,\Lambda; W^{2,q}(\mathbb{R}^3)), \qquad 2 \leq q < \infty, \tag{4.8}$$

$$v_t, z_t, n_t \in L^q(0,\Lambda; L^q(\mathbb{R}^3)), \qquad 2 \leq q < \infty. \tag{4.9}$$

Proof. For $1 \leq p \leq 1+\alpha$, from Lemma 1.1, we have

$$\left.\begin{aligned}
&u \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3)),\ \nabla u^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda; L^2(\mathbb{R}^3)),\\
&w \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3)) \text{ and } \nabla w^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda; L^2(\mathbb{R}^3)).
\end{aligned}\right\} \tag{4.10}$$

To prove (4.6)-(4.8), it is enough to show above holds for $\alpha > \frac{1}{8}$. Above is not available in the case that $\frac{1}{8} < \alpha < \frac{1}{6}$. Using u^{p-1} as test function in $(4.2)_1$, we get

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + \int_{\mathbb{R}^3} \nabla u^{p-1} \cdot \nabla (u+\epsilon)^{1+\alpha}\, dx = -\int_{\mathbb{R}^3} u^{p-1}\nabla \cdot (\chi_1 u \nabla v)\, dx + \int_{\mathbb{R}^3} u^{p-1}\nabla \cdot (\xi_1 u \nabla z)\, dx. \tag{4.11}$$

From $\left|\nabla u^{\frac{p+\alpha}{2}}\right|_2^2 = \frac{(p+\alpha)^2}{4} u^{p+\alpha-2}|\nabla u|^2$, we get

$$\int_{\mathbb{R}^3} \nabla u^{p-1} \cdot \nabla (u+\epsilon)^{1+\alpha}\, dx \geq \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2}\left\|\nabla u^{\frac{p+\alpha}{2}}\right\|_2^2. \tag{4.12}$$

The first term in RHS of above estimated using $u^{p-1}\nabla \cdot (\chi_1 u \nabla v) \leq \dfrac{2\overline{\chi_1}}{(p+\alpha)}\left|\nabla u^{\frac{p+\alpha}{2}}\right| u^{\frac{p-\alpha}{2}}|\nabla v|$ and Young's inequality as

$$\begin{aligned} -\int_{\mathbb{R}^3} u^{p-1}\nabla \cdot (\chi_1 u \nabla v)\, dx &\leq \frac{2\overline{\chi_1}}{(p+\alpha)}\left(\epsilon\left\|\nabla u^{\frac{p+\alpha}{2}}\right\|_2^2 + C(\epsilon)\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2\, dx\right) \\ &\leq C_1\left\|\nabla u^{\frac{p+\alpha}{2}}\right\|_2^2 + C_2\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2\, dx. \end{aligned} \tag{4.13}$$

Similarly, we get

$$\int_{\mathbb{R}^3} u^{p-1}\nabla \cdot (\xi_1 u \nabla z)\, dx \leq C_3\left\|\nabla u^{\frac{p+\alpha}{2}}\right\|_2^2 + C_4\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla z|^2\, dx. \tag{4.14}$$

Substituting (4.12)-(4.14) in (4.11) and choosing sufficiently small C_1, C_3 such that $C_5 > 0$, we get

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + C_5\left\|\nabla u^{\frac{p+\alpha}{2}}\right\|_2^2 \leq C_2\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2\, dx + C_4\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla z|^2\, dx, \tag{4.15}$$

The first term in RHS of the above estimated using the Hölder, Sobolev, and Young inequality as

$$\begin{aligned} \int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2\, dx &\leq C_6\|u\|_p^{p-\alpha}\|\Delta v\|^2_{\frac{6p}{2p+3\alpha}} \\ &\leq C_6\left(\frac{\alpha}{p} + \frac{p-\alpha}{p}\|u\|_p^p\right)\|\Delta v\|^2_{\frac{6p}{2p+3\alpha}}. \end{aligned} \tag{4.16}$$

Similarly, we have

$$\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla z|^2\, dx \leq C_7\left(\frac{\alpha}{p} + \frac{p-\alpha}{p}\|u\|_p^p\right)\|\Delta z\|^2_{\frac{6p}{2p+3\alpha}}. \tag{4.17}$$

Substituting (4.16) and (4.17) in (4.15), we get

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u\|_p^p + C_5\left\|\nabla u^{\frac{p+\alpha}{2}}\right\|_2^2 \\ &\quad \leq C_8\left(\|\Delta v\|^2_{\frac{6p}{2p+3\alpha}} + \|\Delta z\|^2_{\frac{6p}{2p+3\alpha}}\right)\|u\|_p^p + C_9\left(\|\Delta v\|^2_{\frac{6p}{2p+3\alpha}} + \|\Delta z\|^2_{\frac{6p}{2p+3\alpha}}\right). \end{aligned}$$

Using the Gronwall inequality above, we get

$$\|u\|_p^p \;\leq\; \exp\left(\int_0^t \|\Delta v(s)\|^2_{\frac{6p}{2p+3\alpha}} + \|\Delta z(s)\|^2_{\frac{6p}{2p+3\alpha}}\, ds\right)$$

$$\times \int_0^t \left\|\Delta v(s)\right\|^2_{\frac{6p}{2p+3\alpha}} + \left\|\Delta z(s)\right\|^2_{\frac{6p}{2p+3\alpha}} \, ds + \|u_0\|_p^p.$$

Hence, u and $w \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3))$ for $p \in [1,\infty)$ holds, whenever the following holds

$$\int_0^\Lambda \left\|\Delta v(s)\right\|^2_{\frac{6p}{2p+3\alpha}} + \left\|\Delta z(s)\right\|^2_{\frac{6p}{2p+3\alpha}} \, ds < \infty, \qquad 1+\alpha < p < \infty. \tag{4.18}$$

We prove the above by considering α in two cases.

Case (i): $\alpha > \frac{1}{3}$

Using maximal regularity estimate of heat equation in $(4.2)_2$, we have

$$\int_0^\Lambda \left\|\Delta v(s)\right\|^2_{\frac{6p}{2p+3\alpha}} \, ds \le C_1\left(\left\|\nabla v_0\right\|^2_{\frac{6p}{2p+3\alpha}} + \int_0^\Lambda \left\|w(s)\right\|^2_{\frac{6p}{2p+3\alpha}} ds + \int_0^\Lambda \left\|n\cdot\nabla v\right\|^2_{\frac{6p}{2p+3\alpha}} ds\right). \tag{4.19}$$

As $\alpha > \frac{1}{3}$, for $p > 1+\alpha$, the third term in RHS of the above estimated using interpolation inequality as

$$\begin{aligned}\int_0^\Lambda \left\|w(s)\right\|^2_{\frac{6p}{2p+3\alpha}} \, ds &\le C_2 \int_0^t \|w(s)\|_1^{2-\frac{(1+2\alpha)(4p-3\alpha)}{2p(1+3\alpha)}} \|w(s)\|_{3+6\alpha}^{\frac{(1+2\alpha)(4p-3\alpha)}{2p(1+3\alpha)}} \, ds \\ &\le C_2 \int_0^\Lambda \left\|\nabla w^{\frac{1+2\alpha}{2}}(s)\right\|_2^{\frac{4}{1+3\alpha}-\frac{3\alpha}{p(1+3\alpha)}} \, ds. \end{aligned} \tag{4.20}$$

The last term in RHS of (4.19) evaluated using $n \in L^\infty(0,\Lambda; L^6(\mathbb{R}^3))$ as

$$\begin{aligned}\int_0^\Lambda \left\|n\cdot\nabla v\right\|^2_{\frac{6p}{2p+3\alpha}} ds &\le \int_0^\Lambda \|n(s)\|_6^2 \left\|\nabla v(s)\right\|^2_{\frac{6p}{p+3\alpha}} ds \\ &\le C \int_0^\Lambda \left\|\Delta v(s)\right\|^2_{\frac{2p}{p+\alpha}} ds. \end{aligned} \tag{4.21}$$

Using maximal regularity estimate for the heat equation in $(4.2)_2$, we get

$$\begin{aligned}\int_0^\Lambda \left\|\Delta v(s)\right\|^2_{\frac{2p}{p+\alpha}} \, ds &\le C_1\left(\left\|\nabla v_0\right\|^2_{\frac{2p}{p+\alpha}} + \int_0^\Lambda \left\|w(s)\right\|^2_{\frac{2p}{p+\alpha}} ds + \int_0^\Lambda \left\|n\cdot\nabla v\right\|^2_{\frac{2p}{p+\alpha}} ds\right) \\ &\le C_2 + C_3 \int_0^\Lambda \left\|\nabla v(s)\right\|^2_{\frac{6p}{2p+3\alpha}} ds + \int_0^\Lambda \left\|w(s)\right\|^2_{\frac{2p}{p+\alpha}} ds. \end{aligned} \tag{4.22}$$

The last term in RHS of the above is estimated as

$$\begin{aligned}\int_0^\Lambda \left\|w(s)\right\|^2_{\frac{2p}{p+\alpha}} ds &\le C_4 \int_0^\Lambda \|w(s)\|_1^{2-\frac{3(1+2\alpha)(p-\alpha)}{2(1+3\alpha)p}} \left\|w(s)\right\|_{3+6\alpha}^{2-\frac{3(1+2\alpha)(p-\alpha)}{2(1+3\alpha)p}} ds \\ &\le C_4 \int_0^\Lambda \left\|\nabla w^{\frac{1+2\alpha}{2}}(s)\right\|_2^{\frac{3(p-\alpha)}{p(1+3\alpha)}} ds. \end{aligned}$$

Using above in (4.22) and substituting the resulting equation in (4.21), we have

$$\int_0^\Lambda \left\|n\cdot\nabla v\right\|^2_{\frac{6p}{2p+3\alpha}} ds \le C_5 + C_6 \int_0^\Lambda \left\|\nabla v(s)\right\|^2_{\frac{6p}{2p+3\alpha}} ds + C_7 \int_0^\Lambda \left\|\nabla w^{\frac{1+2\alpha}{2}}(s)\right\|_2^{\frac{3(p-\alpha)}{p(1+3\alpha)}} ds. \tag{4.23}$$

As $\frac{6p}{2p+3\alpha} < 2$, $\int_0^\Lambda \|\nabla v(s)\|_q^2 \, ds < \infty$ for $q \in [2,6]$, $\frac{3(p-\alpha)}{p(1+3\alpha)} > 0$ and using (4.20) and (4.23), we have for $\max\{1+\alpha, 3\alpha\} < p < \infty$,

$$u \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3)) \text{ and } \nabla u^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda; L^2(\mathbb{R}^3)).$$

Proceeding as similar as above, we get for $\max\{1+\alpha, 3\alpha\} < p < \infty$,

$$w \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3)) \text{ and } \nabla w^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda; L^2(\mathbb{R}^3)).$$

Using (4.10) it can be extended for $p \in (1+\alpha, \infty)$. Also, for every $1 \leq p < \infty$, we have for all $q < \infty$

$$\begin{aligned} v_t, \nabla^2 v, z_t, \nabla^2 z, n_t, \nabla^2 n &\in L^q((0,\Lambda)\times\mathbb{R}^3) \text{ and} \\ \nabla v, \nabla w, \nabla z &\in L^\infty((0,\Lambda)\times\mathbb{R}^3). \end{aligned}$$

Using above and interpolation inequality in (4.15), we get

$$\frac{d}{dt}\|u\|_p^p \leq C_{13}\|u\|_{p-\alpha}^{p-\alpha} \leq C_{13}\|u\|_1^{\frac{\alpha}{p-1}}\|u\|_p^{\frac{p(p-\alpha-1)}{p-1}} \leq C_{13}\|u\|_p^{\frac{p(p-\alpha-1)}{p-1}}.$$

Using Gronwall's inequality above, we get for $t \leq \Lambda$

$$\|u(t)\|_p \leq (Cp^2 t)^{\frac{1}{p}} + \|u_0\|_p.$$

As $p \to \infty$, we have

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq 1 + \|u_0\|_{L^\infty(\mathbb{R}^3)}.$$

Similarly, we get

$$\|w(t)\|_{L^\infty(\mathbb{R}^3)} \leq 1 + \|w_0\|_{L^\infty(\mathbb{R}^3)}.$$

Therefore, for $1+\alpha < p < \infty$, we have u and $w \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3))$.

Case (ii): $\frac{1}{8} < \alpha \leq \frac{1}{3}$

In order to derive (4.18), it is enough to prove that for $1 \leq p < 1+4\alpha$

$$u \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3)) \text{ and } \nabla u^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda; L^2(\mathbb{R}^3)). \tag{4.24}$$

The first term in RHS of (4.15) evaluated using $\left|\nabla u^{p-\alpha}\right| = Cu^{\frac{p-3\alpha}{2}}\left|\nabla u^{\frac{p+\alpha}{2}}\right|$ and Young's inequality as

$$\begin{aligned} \int_{\mathbb{R}^3} u^{p-\alpha}|\nabla v|^2 \, dx &= \int_{\mathbb{R}^3} u^{p-\alpha}\nabla v \cdot \nabla v \, dx \\ &\leq C_1\Big(\int_{\mathbb{R}^3} |\nabla u^{p-\alpha}||\nabla v| \, dx + \int_{\mathbb{R}^3} u^{p-\alpha}|\Delta v| \, dx\Big) \\ &\leq C_1\Big(\int_{\mathbb{R}^3} Cu^{\frac{p-3\alpha}{2}}\left|\nabla u^{\frac{p+\alpha}{2}}\right||\nabla v| \, dx + \int_{\mathbb{R}^3} u^{p-\alpha}|\Delta v| \, dx\Big) \\ &\leq CC_1\int_{\mathbb{R}^3} \epsilon\left|\nabla u^{\frac{p+\alpha}{2}}\right|^2 + C(\epsilon)u^{p-3\alpha}|\nabla v|^2 \, dx + C_1\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta v| \, dx. \end{aligned}$$

Similarly, we get

$$\int_{\mathbb{R}^3} u^{p-\alpha}|\nabla z|^2 \, dx \leq C_2C_3\int_{\mathbb{R}^3} \epsilon\left|\nabla u^{\frac{p+\alpha}{2}}\right|^2 + C(\epsilon_1)u^{p-3\alpha}|\nabla z|^2 \, dx + C_3\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta z| \, dx.$$

Substituting above estimates in (4.15) and choosing sufficiently small constants such that $C_4 > 0$, we have

$$\begin{aligned}\frac{d}{dt}\|u\|_p^p + C_4\big\|\nabla u^{\frac{p+\alpha}{2}}\big\|_2^2 \;\le\;& C_5\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2\,dx + C_6\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla z|^2\,dx\\ &+ C_1\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta v|\,dx + C_3\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta z|\,dx. \qquad (4.25)\end{aligned}$$

Integrating the above with respect to t, we get

$$\begin{aligned}&\|u\|_p^p + C_4\int_0^t \big\|\nabla u^{\frac{p+\alpha}{2}}\big\|_2^2 ds\\ &\le\; C_5\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2\,dx\,ds + C_6\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla z|^2\,dx\,ds\\ &\quad +C_1\int_0^t\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta v|dx\,ds + C_3\int_0^t\int_{\mathbb{R}^3} u^{p-\alpha}|\Delta z|\,dx\,ds + \big\|u_0\big\|_p^p\\ &\le\; C_5\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2\,dx\,ds + C_6\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla z|^2\,dx\,ds\\ &\quad +C_7\int_0^t \big\|u\big\|_{p-\alpha+1}^{p-\alpha+1}\,ds + C_8\int_0^t \big\|\Delta v\big\|_{p-\alpha+1}^{p-\alpha+1} + \big\|\Delta z\big\|_{p-\alpha+1}^{p-\alpha+1}\,ds + \big\|u_0\big\|_p^p. \qquad (4.26)\end{aligned}$$

As $\frac{1}{8} < \alpha \le \frac{1}{3}$ and $1 < p < 1+4\alpha$, the first term in RHS of above evaluated using the Hölder inequality and maximal regularity estimate as

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2\,dxds \;\le\;& C\int_0^t \big\|u^{p-3\alpha}(s)\big\|_{\frac{1+\alpha}{p-3\alpha}}\big\||\nabla v(s)|^2\big\|_{\frac{1+\alpha}{1+4\alpha-p}}\,ds\\ \le\;& CC_1\int_0^t \big\|\Delta v(s)\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}\\ \le\;& CC_1\big\|\nabla v_0\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + \int_0^t \big\|w(s)\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + \big\|n\cdot\nabla v\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds,\\ &\qquad (4.27)\end{aligned}$$

for all p satisfies $1+\alpha < p < 1+4\alpha$. The third term in RHS of the above evaluated using interpolation inequality as

$$\begin{aligned}\int_0^t \big\|w(s)\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds \;\le\;& C_1\int_0^t \big\|w(s)\big\|_{1+\alpha}^{2\left(1-\frac{(1-14\alpha+3p)(\alpha+p)}{2(2\alpha+3p-1)}\right)}\big\|w(s)\big\|_{3p+3\alpha}^{\frac{(1-14\alpha+3p)(\alpha+p)}{(2\alpha+3p-1)}}ds\\ \le\;& C_2\int_0^t \left\|\nabla w^{\frac{p+\alpha}{2}}\right\|_2^{\frac{2(1-14\alpha)}{(2\alpha+3p-1)}}. \qquad (4.28)\end{aligned}$$

As $1 \le \frac{6+6\alpha}{5+14\alpha-3p} < 3$, the last term in above evaluated using maximal regularity estimate of $(4.2)_2$, $n\in L^\infty(0,\Lambda;L^6(\mathbb{R}^3))$ and interpolation inequality as

$$\begin{aligned}\int_0^t \big\|n\cdot\nabla v(s)\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}ds \;\le\;& C\int_0^t \|\nabla v\|_{\frac{6+6\alpha}{4+13\alpha-3p}}\\ \le\;& C_1\int_0^t \|\Delta v\|_{\frac{2+2\alpha}{2+5\alpha-p}}\end{aligned}$$

$$
\begin{aligned}
&\leq C_1\bigg(\big\|\nabla v_0\big\|^2_{\frac{2+2\alpha}{2+5\alpha-p}} + \big\|w(s)\big\|^2_{\frac{2+2\alpha}{2+5\alpha-p}} + \big\|n\cdot\nabla v(s)\big\|^2_{\frac{2+2\alpha}{2+5\alpha-p}}\ ds\bigg)\\
&\leq C_1\big\|\nabla v_0\big\|^2_{\frac{2+2\alpha}{2+5\alpha-p}} + \bigg\|\nabla w^{\frac{p+\alpha}{2}}\bigg\|_2^{\frac{8(1+\alpha)}{(p+\alpha)(2+5\alpha-p)}} + \big\|\nabla v(s)\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}\ ds.
\end{aligned}
$$

Using (4.28) and above in (4.27), we get

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla v|^2\ dxds \ \leq\ & C_2\|\nabla v_0\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + C_3\int_0^t \big\|\nabla v(s)\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}\\
&+C_4\int_0^t \bigg\|\nabla w^{\frac{p+\alpha}{2}}\bigg\|_2^{\frac{2(1-14\alpha)}{(2\alpha+3p-1)}} + \bigg\|\nabla w^{\frac{p+\alpha}{2}}\bigg\|_2^{\frac{8(1+\alpha)}{(p+\alpha)(2+5\alpha-p)}}\ ds. \quad (4.29)
\end{aligned}
$$

Proceeding as similar as above, we can get

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}^3} u^{p-3\alpha}|\nabla z|^2\ dx\ ds \ \leq\ & C_5\|\nabla z_0\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + C_6\int_0^t \big\|\nabla z(s)\big\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}}\\
&+C_7\int_0^t \bigg\|\nabla u^{\frac{p+\alpha}{2}}\bigg\|_2^{\frac{2(1-14\alpha)}{(2\alpha+3p-1)}} + \bigg\|\nabla u^{\frac{p+\alpha}{2}}\bigg\|_2^{\frac{8(1+\alpha)}{(p+\alpha)(2+5\alpha-p)}}\ ds. \quad (4.30)
\end{aligned}
$$

Similarly, we deduce

$$
\begin{aligned}
&\int_0^t \|u\|^{p-\alpha+1}_{p-\alpha+1}\ ds + C_8\int_0^t \|\Delta v\|^{p-\alpha+1}_{p-\alpha+1} + \|\Delta z\|^{p-\alpha+1}_{p-\alpha+1}\ ds +\\
&\qquad \leq C_8\|\nabla v_0\|^2_{p-\alpha+1} + C_9\|\nabla v_0\|^2_{\frac{6(1+p-\alpha)}{7+p-\alpha}} + C_{10}\|\nabla z_0\|^2_{\frac{6(1+p-\alpha)}{7+p-\alpha}}\\
&\qquad\quad +C_{11}\bigg\|\nabla u^{\frac{p+\alpha}{2}}\bigg\|_2^{\frac{6(p-2\alpha)}{2\alpha+3p-1}} + C_{12}\bigg\|\nabla u^{\frac{p+\alpha}{2}}\bigg\|_2^{\frac{1+5\alpha-5p}{1-3\alpha-3p}} + C_{13}\bigg\|\nabla w^{\frac{p+\alpha}{2}}\bigg\|_2^{\frac{1+5\alpha-5p}{1-3\alpha-3p}}\\
&\qquad\quad +C_{14}\|\Delta v_0\|_2^{\frac{3(p-\alpha-1)}{2(p-\alpha+1)}} + C_{15}\|\Delta z_0\|_2^{\frac{3(p-\alpha-1)}{2(p-\alpha+1)}}.
\end{aligned}
$$

Using (4.29), (4.30) and above in (4.26), for $p\in(1+\alpha,1+4\alpha)$, we get

$$
\|u\|_p^p + C_4\int_0^t \|\nabla u^{\frac{p+\alpha}{2}}\|_2^2 ds \leq C_1 + C_2\int_0^\Lambda \|\nabla v\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + \|\nabla z\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} ds,\quad 0<t<\Lambda. \quad (4.31)
$$

Similarly, we can deduce

$$
\|w\|_p^p + C_7\int_0^t \|\nabla w^{\frac{p+\alpha}{2}}\|_2^2 ds \leq C_9 + \int_0^\Lambda \|\nabla v\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} + \|\nabla z\|^2_{\frac{6+6\alpha}{5+14\alpha-3p}} ds,\quad 0<t<\Lambda. \quad (4.32)
$$

As $\int_0^\Lambda \|\nabla v(s)\|_q^2\ ds<\infty$ and $\int_0^\Lambda \|\nabla z(s)\|_q^2\ ds<\infty$ for $q\in[2,6]$, we have for $p\in\big(\frac{2+11\alpha}{3},1+4\alpha\big)$

$$
u, w \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3)) \text{ and } \nabla u^{\frac{p+\alpha}{2}}, \nabla w^{\frac{p+\alpha}{2}} \in L^2(0,\Lambda;L^2(\mathbb{R}^3)). \quad (4.33)
$$

Using (4.10), it is extended to $1+\alpha<p<1+4\alpha$. Hence, (4.24) holds. Choose $r_0=\frac{3}{2}-\frac{3\alpha}{4}$. Then from (4.33), $u\in L^\infty(0,\Lambda;L^{r_0}(\mathbb{R}^3))$ as $1\leq r_0<1+4\alpha$. Using maximal regularity estimate for $(4.2)_2$, we have

$$
\int_0^\Lambda \|\Delta v\|_{r_1} \ \leq\ C_1\int_0^\Lambda \|w\|^2_{r_1}\ ds + C_1\int_0^\Lambda \|n\cdot\nabla v\|_{r_1} ds \quad (4.34)
$$

$r_1 = \frac{6p}{2p+3\alpha}$. The first term in RHS of the above evaluated as

$$\begin{aligned}\int_0^\Lambda \|w\|_{r_1}^2 \, ds &\le C_1 \int_0^\Lambda \|w\|_{r_0}^{2-\frac{(r_0+\alpha)(6p-2pr_0-3\alpha r_0)}{2(2r_0+3\alpha)}} \|w\|_{3r_0+3\alpha}^{\frac{(r_0+\alpha)(6p-2pr_0-3\alpha r_0)}{2(2r_0+3\alpha)}} ds \\ &\le C_1 \int_0^\Lambda \left\|\nabla w^{\frac{r_0+\alpha}{2}}\right\|_2^{\frac{p(12-4r_0)-6\alpha r_0}{p(2r_0+3\alpha)}} ds \\ &= C_1 \int_0^\Lambda \left\|\nabla w^{\frac{r_0+\alpha}{2}}\right\|_2^{2-\frac{2p(3\alpha+4r_0-6)+6\alpha r_0}{p(2r_0+3\alpha)}} ds. \qquad (4.35)\end{aligned}$$

The second term in RHS of the above evaluated using (4.21) as

$$\begin{aligned}\int_0^\Lambda \|n\cdot\nabla v\|_{\frac{6p}{2p+3\alpha}}^2 ds &\le C \int_0^\Lambda \|\Delta v(s)\|_{\frac{2p}{p+\alpha}}^2 ds \\ &\le C\left(\|\nabla v_0\|_{\frac{2p}{p+\alpha}}^2 + \int_0^\Lambda \|w(s)\|_{\frac{2p}{p+\alpha}}^2 ds + \int_0^\Lambda \|n\cdot\nabla v\|_{\frac{2p}{p+\alpha}}^2 ds \right) \\ &\le C_2 + C_4 \int_0^\Lambda \|w(s)\|_{1+\alpha}^{\frac{(\alpha+1)(3\alpha(2\alpha+1)+6\alpha p+p)}{(5\alpha+2)p}} \|w(s)\|_{3+6\alpha}^{\frac{3(2\alpha+1)(\alpha(\alpha+1)+(\alpha-1)p)}{(5\alpha+2)p}} \\ &\le C_2 + C_5 \int_0^\Lambda \left\|w(s)^{\frac{1+2\alpha}{2}}\right\|_{1+\alpha}^{\frac{6(\alpha-1)(\alpha-p)}{(5\alpha+2)p}}. \qquad (4.36)\end{aligned}$$

Proceeding as similar as above and using $\int_0^\Lambda \|\nabla v(s)\|_q^2 \, ds < \infty$ and $\int_0^\Lambda \|\nabla z(s)\|_q^2 \, ds < \infty$ for $q \in [2,6]$, we have (4.18) holds for this case also. Hence, we have $u, w \in L^\infty(0,\Lambda; L^p(\mathbb{R}^3))$ for $p \in (1+\alpha, \infty)$, as (4.18) holds for $\alpha > \frac{1}{8}$. As proof is similar to above, proof for boundedness of u and w in $L^\infty-$norm is omitted. ■

Now, we state and prove another result established in this work.

Theorem 4.1. *Assume that* (4.5) *holds and* $(u_0, v_0, w_0, z_0, n_0)$ *satisfies initial data* (3.6) *and* (4.4). *Then for each* $\Lambda > 0$, *the system* (4.1) *possesses a bounded weak solution* (u, v, w, z, n), *such that*

$$\begin{aligned}&\|u\|_{L^\infty((0,\Lambda)\times\mathbb{R}^3)} + \|w\|_{L^\infty((0,\Lambda)\times\mathbb{R}^3)} + \left\|\nabla u^{\frac{p+\alpha}{2}}\right\|_{L^2((0,\Lambda)\times\mathbb{R}^3)} + \left\|\nabla w^{\frac{p+\alpha}{2}}\right\|_{L^2((0,\Lambda)\times\mathbb{R}^3)} \\ &\quad + \|v\|_{L^q(0,\Lambda;W^{2,q}(\mathbb{R}^3))} + \|z\|_{L^q(0,\Lambda;L^q(\mathbb{R}^3))} + \|n\|_{L^q(0,\Lambda;W^{2,q}(\mathbb{R}^3))} \\ &\quad + \|v_t\|_{L^q(0,\Lambda;W^{2,q}(\mathbb{R}^3))} + \|z_t\|_{L^q(0,\Lambda;L^q(\mathbb{R}^3))} + \|n_t\|_{L^q(0,\Lambda;L^q(\mathbb{R}^3))} < C, \qquad (4.37)\end{aligned}$$

where C *is a constant depends on initial conditions.*

Proof. We omit the proof of the local existence of weak solution as it is similar to the proof of Theorem 3.1. From Lemma 4.1 and Aubin-Lions compactness lemma, in fact, it is a bounded weak solution. ■

The above theorem can be extended for bounded domains with (3.89) as boundary conditions. The extension is given as a corollary below.

Corollary 4.1. *Assume that* (4.5) *holds and* $(u_0, v_0, w_0, z_0, n_0)$ *satisfies initial data* (3.6) *and* (4.4)*. Then for each* $\Lambda > 0$*, the system* (4.1) *with* (3.89) *possesses a bounded weak solution* (u, v, w, z, n)*, such that*

$$\begin{aligned}
&\|u\|_{L^\infty((0,\Lambda)\times\Omega)} + \|w\|_{L^\infty((0,\Lambda)\times\Omega)} + \left\|\nabla u^{\frac{p+\alpha}{2}}\right\|_{L^2((0,\Lambda)\times\Omega)} + \left\|\nabla w^{\frac{p+\alpha}{2}}\right\|_{L^2((0,\Lambda)\times\Omega)} \\
&\quad + \|v\|_{L^q(0,\Lambda;W^{2,q}(\Omega))} + \|z\|_{L^q(0,\Lambda;L^q(\Omega))} + \|n\|_{L^q(0,\Lambda;W^{2,q}(\Omega))} \\
&\quad + \|v_t\|_{L^q(0,\Lambda;W^{2,q}(\Omega))} + \|z_t\|_{L^q(0,\Lambda;L^q(\Omega))} + \|n_t\|_{L^q(0,\Lambda;L^q(\Omega))} < C, \qquad (4.38)
\end{aligned}$$

where C *is a constant depends on initial conditions.*

5 Conclusion

A two-species chemotaxis system in a fluid environment is modeled, where each species secretes chemicals that attract the other species while repelling their own. The effect of the fluid environment is incorporated by considering porous medium diffusion for both species. A global existence of weak and bounded weak solutions is established for the system and extended to a bounded domain. In future work, we aim to extend our study by developing a finite element framework for the proposed model in a three-dimensional spatial domain, complemented by a rigorous theoretical error analysis.

CRediT authorship contribution statement

Y. Karuppusamy: Writing-review & editing. **S. Lingeshwaran** : Writing-review & editing. **M. Jeyaraj**: Writing-review & editing. **A. S. Hendy**: Writing-review & editing. **S. Abdelaliemd**: Writing-review & editing.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Relevant data can be made available upon reasonable request.

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