On the distribution of αp^3 modulo one over special primes

S. I. Dimitrov **M. D. Lazarova**

Abstract

Let $[\,x\,]$ denote the integer part of $x \in \mathbb{R}$, and let $\|x\|$ denote the distance from x to the nearest integer. In this paper we prove that, whenever α is irrational number and β is any real number, then for any fixed $\frac{47}{48} < \gamma < 1$, there exist infinitely many prime numbers p satisfying the inequality

$$\|\alpha p^3 + \beta\| < p^{\frac{47-48\gamma}{96}+\varepsilon}$$

and such that $p = [n^{1/\gamma}]$.

Keywords: Distribution modulo one, Piatetski-Shapiro primes.

2020 Math. Subject Classification: 11J25 · 11J71 · 11L07 · 11L20

1 Introduction and statement of the result

Let α be an irrational real number, β a real number, and let $||x|| = \min_{n\in\mathbb{Z}} |x - n|$. It was shown by Vinogradov [23] that if $\theta = \frac{1}{5} - \varepsilon$, then there are infinitely many primes p such that

$$\|\alpha p + \beta\| < p^{-\theta}\,. \tag{1}$$

Subsequently, inequality (1) was sharpened several times, with the best known result to date due to Matomäki [18], who obtained $\theta = \frac{1}{3} - \varepsilon$ in the case $\beta = 0$. An analogous problem has also been studied for polynomial expressions involving higher powers of primes. Ghosh [5] was the first to establish that the inequality

$$\|\alpha p^2 + \beta\| < p^{-\theta} \tag{2}$$

has infinitely many prime solutions p, for $\theta = \frac{1}{8} - \varepsilon$. The result of Ghosh was later improved by Baker and Harman [2] with $\theta = \frac{3}{20} - \varepsilon$ and by Harman [9] with $\theta = \frac{2}{13} - \varepsilon$.

A further natural extension is to consider polynomial expressions of even higher degree. Harman [7] is credited with the inequality

$$\|\alpha p^3 + \beta\| < p^{-\theta} \tag{3}$$

which holds for infinitely many primes p, with $\theta = \frac{1}{19} - \varepsilon$. Subsequently, Harman's result was sharpened by Harman [8] with $\theta = \frac{3}{49} - \varepsilon$, by Baker and Harman [2] with $\theta = \frac{1}{12} - \varepsilon$ and by Wong [24] with $\theta = \frac{5}{56} - \varepsilon$. The study of Diophantine inequalities involving prime numbers naturally leads to questions about the distribution of primes in sparse sequences. One notable class of such sequences arises from non-integer powers of integers. In 1953, Piatetski-Shapiro [19] established that for any fixed $\frac{11}{12} < \gamma < 1$, there are infinitely many primes of the form $p = [n^{1/\gamma}]$. These primes are known as Piatetski-Shapiro primes of type γ. Later, the range for γ was refined by several authors, with the best result to date provided by Rivat and Wu [20]. Specifically, they demonstrated that for any $\frac{205}{243} < \gamma < 1$, we have the following lower bound

$$\sum_{\substack{p \le X \\ p = [n^{1/\gamma}]}} 1 \gg \frac{X^\gamma}{\log X}. \tag{4}$$

The primes of the form $[n^{1/\gamma}]$ are very much in focus nowadays and many problems are solved using them. We mention, for example, papers [16] and [17]. Recently, Dimitrov [3] considered a hybrid problem, restricting the set of primes p in (1) to Piatetski-Shapiro primes. More precisely, he proved that, for any fixed $\frac{11}{12} < \gamma < 1$, there exist infinitely many Piatetski-Shapiro primes p of type γ satisfying the inequality (1), for $\theta = \frac{12\gamma - 11}{26} - \varepsilon$. Subsequently, X. Li, J. Li and Zhang [15] generalized the result of Dimitrov [3] by solving (1) with primes $p = [n_1^{1/\gamma_1}] = [n_2^{1/\gamma_2}]$, where $\frac{23}{12} < \gamma_1 + \gamma_2 < 2$, and with $\theta = \frac{12(\gamma_1+\gamma_2)-23}{38} - \varepsilon$. Finally, Baier and Rahaman [1] managed to improve Dimitrov's result by solving (1) with primes $p = [n^{1/\gamma}]$, where $\frac{8}{9} < \gamma < 1$, and with $\theta = \frac{9\gamma - 8}{10} - \varepsilon$. Very recently, the authors [4] showed that, for any fixed $\frac{13}{14} < \gamma < 1$, there are infinitely many primes $p = [n^{1/\gamma}]$ satisfying the inequality (2), for $\theta = \frac{14\gamma - 13}{29} + \varepsilon$. Building on these results, we address inequality (3) involving Piatetski-Shapiro primes.

Theorem 1. *Let γ be fixed with $\frac{47}{48} < \gamma < 1$, and assume that α is irrational and β is real. Then there exist infinitely many Piatetski-Shapiro primes p of type γ such that*

$$\|\alpha p^3 + \beta\| < p^{\frac{47-48\gamma}{96} + \varepsilon}.$$

We remark that Theorem 1 is hardly best possible. It is likely that more sophisticated exponential sum estimates and/or sieve techniques would have allowed us to extend the range of γ. Nevertheless, we chose not to pursue such directions, and our main message is that inequality (3) can be solved with infinitely many Piatetski-Shapiro primes.

2 Notations

Let C be a sufficiently large positive constant. The letter p will always denote a prime number. We use ε to denote an arbitrarily small positive number, which may vary in different occurrences. The notation $m \sim M$ means that m ranges over the interval $(M, 2M]$. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1 m_2 \ldots m_k = n$ in natural numbers $m_1, \ldots, m_k$. As usual $\Lambda(n)$ is von Mangoldt's function. We write $[x]$ for the integer part of x, $\{x\}$ for the fractional part, and $\|x\|$ for the distance from x to the nearest integer. Moreover $e(x) = e^{2\pi i x}$ and $\psi(t) = \{t\} - 1/2$. Let γ be a real constant such that $\frac{47}{48} < \gamma < 1$. Since α is irrational, there are infinitely many different convergents a/q to its continued fraction, with

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}\,, \quad (a, q) = 1\,, \quad a \neq 0 \tag{5}$$

and q is arbitrary large. Define

$$N = q^{\frac{4}{49-48\gamma}}\,; \tag{6}$$

$$\Delta = CN^{\frac{47-48\gamma}{96}+\varepsilon}\,; \tag{7}$$

$$H = \left[q^{\frac{1}{3}}\right]; \tag{8}$$

$$M = N^{\frac{49-48\gamma}{96}}\,; \tag{9}$$

$$\vartheta = N^{\frac{23}{48}}\,; \tag{10}$$

$$\mathfrak{S}(u) = \sum_{1\le h\le u}\left|\sum_{n\le N} e(\alpha h n^3)\right|; \tag{11}$$

$$\Sigma = \sum_{p\le N}\Big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\Big)e(\alpha h p^3)\log p\,. \tag{12}$$

3 Auxiliary lemmas

Lemma 1. *For any $M \geq 2$, we have*

$$\psi(t) = -\sum_{1\le |m|\le M}\frac{e(mt)}{2\pi i m} + \mathcal{O}\left(\min\left(1, \frac{1}{M\|t\|}\right)\right),$$

where

$$\min\left(1, \frac{1}{M\|t\|}\right) = \sum_{h=-\infty}^{\infty} b_h e(ht)$$

and

$$b_h \ll \min\left(\frac{\log 2M}{M}, \frac{1}{|h|}, \frac{M}{|h|^2}\right).$$

Proof. See ([10], p. 245). □

Lemma 2. *Let Q, J, L be three positive integers, which satisfy $1 \le Q \le N\log^{-1} N$, $1 \le J \le N\log^{-1} N$, $1 \le L \le N\log^{-1} N$. Then for the sum (11), we have*

$$\mathfrak{S}^8(u) \ll u^8N^8\left(\frac{1}{Q^4}+\frac{1}{J^2}+\frac{1}{L}\right)+\frac{u^7N^8}{QJL}\sum_{q=1}^{Q}\sum_{j=1}^{J}\sum_{l=1}^{L}\left|\sum_{1\le h\le u} e(6\alpha qjlh)\right|.$$

Proof. This lemma is very similar to a result of Li and Zhang [14]. By inspecting the arguments presented in [14, pp. 203–206], the reader will easily see that the proof of Lemma 2 can be obtained in the same way. □

Lemma 3. *For any real number α and $X \ge 1$, we have*

$$\left|\sum_{n\le X} e(\alpha n)\right| \le \min\left(X, \frac{1}{2\|\alpha\|}\right).$$

Proof. See ([13], Ch. 6, Lemma 4). □

Lemma 4. *Suppose that $X, Y \ge 1$, $\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}$, $(a,q)=1$. Then*

$$\sum_{n\le X} \min\left(Y, \frac{1}{\|\alpha n\|}\right) \ll \frac{XY}{q} + (X+q)\log 2q.$$

Proof. See ([21], Lemma 1). □

Lemma 5. *Let $k \ge 3$ be an integer, and suppose that $f(x) : [N, N_1] \to \mathbb{R}$ has continuous derivatives of order up to k on $[N, N_1]$, where $1 \le N < N_1 \le 2N$. Suppose further that*

$$0 < \lambda_k \le \left|f^{(k)}(x)\right| \le A\lambda_k, \quad x \in [N, N_1].$$

Then

$$\sum_{N<n\le N_1} e(f(n)) \ll_{A,k,\varepsilon} N^{1+\varepsilon}\left(\lambda_k^{\frac{1}{k(k-1)}} + N^{-\frac{1}{k(k-1)}} + N^{-\frac{2}{k(k-1)}}\lambda_k^{-\frac{2}{k^2(k-1)}}\right).$$

Proof. See ([11], Theorem 1). □

Lemma 6. *For any complex numbers $a(n)$ we have*

$$\Big|\sum_{a<n\le b} a(n)\Big|^2 \le \left(1+\frac{b-a}{Q}\right)\sum_{|q|\le Q}\left(1-\frac{|q|}{Q}\right)\sum_{a<n,\, n+q\le b} a(n+q)\overline{a(n)}\,,$$

where Q is any positive integer.

Proof. See ([12], Lemma 8.17). □

Lemma 7. *Suppose that $\alpha \in \mathbb{R},\ a \in \mathbb{Z},\ q \in \mathbb{N},\ \left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2}\,,\ (a,q)=1$. Then*

$$\sum_{p\le N} e(\alpha p^3)\log p \ll N^{1+\varepsilon}\left(\frac{1}{q}+\frac{1}{N^{\frac{1}{2}}}+\frac{q}{N^3}\right)^{\frac{1}{16}}.$$

Proof. See ([6], Theorem 1). □

4 Proof of the theorem

4.1 Initial Steps

Our method goes back to Vaughan [21]. We take a periodic with period 1 function such that

$$F_\Delta(\theta)=\begin{cases}0 & \text{if} \quad -\frac{1}{2}\le\theta<-\Delta\,,\\ 1 & \text{if} \quad -\Delta\le\theta<\Delta\,,\\ 0 & \text{if} \quad \Delta\le\theta<\frac{1}{2}\,,\end{cases}$$

where Δ is defined by (7). Every non-trivial lower bound for the sum

$$\sum_{\substack{p\le N\\ p=[n^{1/\gamma}]}} F_\Delta(\alpha p^3+\beta)\log p$$

leads directly to Theorem 1. In order to achieve this, we define the sum

$$\Gamma = \sum_{\substack{p\le N\\ p=[n^{1/\gamma}]}} \big(F_\Delta(\alpha p^3+\beta)-2\Delta\big)\log p\,. \tag{13}$$

4.2 Estimation of Γ

Lemma 8. *Suppose that $H, N \ge 1,\ \left|\alpha-\frac{a}{q}\right| < \frac{1}{q^2}\,,\ (a,q)=1$. Then*

$$\sum_{n\le N}\min\left(1,\,\frac{1}{H\|\alpha n^3+\beta\pm\Delta\|}\right) \ll (NHq)^\varepsilon\Big(Nq^{-\frac{1}{8}}+N^{\frac{7}{8}}+NH^{-\frac{1}{8}}+N^{\frac{5}{8}}H^{-\frac{1}{8}}q^{\frac{1}{8}}\Big)\,.$$

Proof. Using Lemma 1, we write

$$\sum_{n\le N}\min\left(1,\,\frac{1}{H\|\alpha n^3+\beta\pm\Delta\|}\right)=\sum_{n\le N}\sum_{h=-\infty}^{\infty} b_h e\big(h(\alpha n^3+\beta\pm\Delta)\big)$$

$$=\sum_{n\le N}\left(b_0+\sum_{1\le|h|\le H^2} b_h e\big(h(\alpha n^3+\beta\pm\Delta)\big)+\sum_{|h|>H^2} b_h e\big(h(\alpha n^3+\beta\pm\Delta)\big)\right)$$

$$\ll(\log H)\sum_{1\le|h|\le H^2}\min\left(\frac{1}{H},\frac{1}{|h|}\right)\left|\sum_{n\le N} e(\alpha h n^3)\right|+NH\sum_{|h|>H^2}\frac{1}{h^2}+\frac{N\log H}{H}$$

$$\ll(\log H)\Psi(H)+NH^{-1}\log H\,, \tag{14}$$

where

$$\Psi(H)=\sum_{1\le h\le H^2}\min\left(\frac{1}{H},\frac{1}{h}\right)\left|\sum_{n\le N} e(\alpha h n^3)\right|. \tag{15}$$

By (11), (15) and Abel's summation formula, we obtain

$$\Psi(H)=\frac{\mathfrak{S}(H^2)}{H^2}+\int\limits_{H}^{H^2}\frac{\mathfrak{S}(u)}{u^2}\,du\ll(\log H)\max_{H\le u\le H^2}\frac{\mathfrak{S}(u)}{u}\,. \tag{16}$$

It remains to estimate $\mathfrak{S}(u)$. In Lemma 2, we set

$$Q=J=L=\left[\frac{N}{\log N}\right]\,,\qquad k=6qjl\,. \tag{17}$$

Now (11), (17), Lemma 2, Lemma 3 and Lemma 4 imply

$$\mathfrak{S}^8(u)\ll u^8N^{7+\varepsilon}+u^7N^{5+\varepsilon}\sum_{q=1}^{Q}\sum_{j=1}^{J}\sum_{l=1}^{L}\min\left(u,\,\frac{1}{\|6\alpha qjl\|}\right)$$

$$\ll u^8N^{7+\varepsilon}+u^7N^{5+\varepsilon}\sum_{k=1}^{6QJL}\sum_{q=1}^{Q}\sum_{j=1}^{J}\sum_{\substack{l=1\\6qjl=k}}^{L}\min\left(u,\,\frac{1}{\|\alpha k\|}\right)$$

$$\ll u^8N^{7+\varepsilon}+u^7N^{5+\varepsilon}\sum_{k=1}^{6QJL}\tau_3(k)\min\left(u,\,\frac{1}{\|\alpha k\|}\right)$$

$$\ll u^8N^{7+\varepsilon}+u^7N^{5+\varepsilon}\sum_{k=1}^{6QJL}\min\left(u,\,\frac{1}{\|\alpha k\|}\right)$$

$$\ll u^8N^{7+\varepsilon}+u^7N^{5+\varepsilon}\left(\frac{uQJL}{q}+QJL\log q+q\log q\right)$$

$$\ll(Nq)^{\varepsilon}\Big(u^8N^7+u^7N^8+u^8N^8q^{-1}+u^7N^5q\Big)\,.$$

Hence it follows that

$$\mathfrak{S}(u) \ll (Nq)^{\varepsilon}\Big(uN^{\frac{7}{8}} + u^{\frac{7}{8}}N + uNq^{-\frac{1}{8}} + u^{\frac{7}{8}}N^{\frac{5}{8}}q^{\frac{1}{8}}\Big) . \tag{18}$$

Bearing in mind (14), (16) and (18), we establish the statement in the lemma. □

Lemma 9. *Let* $\frac{47}{48} < \gamma < 1$. *For the sum denoted by* (12), *the following estimate holds*

$$\Sigma \ll N^{\frac{48\gamma+47}{96}+\varepsilon} .$$

Proof. From (9), (12), Lemma 1 and the simplest splitting up argument, we get

$$\Sigma \ll \big(\Sigma_1 + \Sigma_2\big)\log^2 N + N^{1/2} , \tag{19}$$

where

$$\Sigma_1 = \sum_{m\sim M_1}\frac{1}{m}\left|\sum_{n\sim N_1}\Lambda(n)e(\alpha h n^3)\Big(e\big(-mn^{\gamma}\big) - e\big(-m(n+1)^{\gamma}\big)\Big)\right| , \tag{20}$$

$$\Sigma_2 = \sum_{n\sim N_1}\min\left(1, \frac{1}{M\|n^{\gamma}\|}\right) , \tag{21}$$

$$M_1 \leq \frac{M}{2} , \quad N_1 \leq \frac{N}{2} . \tag{22}$$

Working as in [14, p. 214], and using (21) and (22), we deduce

$$\Sigma_2 \ll \Big(NM^{-1} + N^{\frac{\gamma}{2}}M^{\frac{1}{2}} + N^{1-\gamma}\Big)\log M . \tag{23}$$

In view of (9) and (23), we derive

$$\Sigma_2 \ll N^{\frac{48\gamma+47}{96}+\varepsilon} . \tag{24}$$

Next, we estimate Σ_1. By (20) and Abel's summation formula, we have

$$\Sigma_1 \ll N_1^{\gamma-1}\sum_{m\sim M_1}\max_{N_2\in[N_1,2N_1]}|\mathfrak{F}(N_1,N_2)| , \tag{25}$$

where

$$\mathfrak{F}(N_1,N_2) = \sum_{N_1<n\leq N_2}\Lambda(n)e\big(\alpha h n^3 - mn^{\gamma}\big) . \tag{26}$$

Assume that

$$N_1 \leq N^{\frac{48\gamma-1}{48\gamma}} . \tag{27}$$

Based on (9), (22), (25), (26) and (27), we conclude that

$$\Sigma_1 \ll N^{\frac{48\gamma+47}{96}+\varepsilon}\,. \tag{28}$$

Henceforth, we suppose that

$$N^{\frac{48\gamma-1}{48\gamma}} < N_1 \leq \frac{N}{2}\,. \tag{29}$$

We now estimate the sum (26). Define

$$f(d,l) = \alpha h d^3 l^3 - m d^\gamma l^\gamma\,. \tag{30}$$

By (26), (30) and Vaughan's identity (see [22]), we establish

$$\mathfrak{F}(N_1,N_2) = \Phi_1 - \Phi_2 - \Phi_3 - \Phi_4\,, \tag{31}$$

where

$$\Phi_1 = \sum_{d\leq\vartheta}\mu(d)\sum_{\frac{N_1}{d}<l\leq\frac{N_2}{d}} e(f(d,l))\log l\,, \tag{32}$$

$$\Phi_2 = \sum_{d\leq\vartheta}c(d)\sum_{\frac{N_1}{d}<l\leq\frac{N_2}{d}} e(f(d,l))\,, \tag{33}$$

$$\Phi_3 = \sum_{\vartheta<d\leq\vartheta^2}c(d)\sum_{\frac{N_1}{d}<l\leq\frac{N_2}{d}} e(f(d,l))\,, \tag{34}$$

$$\Phi_4 = \sum\sum_{\substack{N_1<dl\leq N_2\\ d>\vartheta,\, l>\vartheta}} a(d)\Lambda(l)e(f(d,l)) \tag{35}$$

and

$$|c(d)| \leq \log d, \quad |a(d)| \leq \tau_2(d)\,, \tag{36}$$

and ϑ is defined by (10). Let us first consider the sum Φ_2, as defined in (33). In view of (30), we get

$$\left|\frac{\partial^4 f(d,l)}{\partial l^4}\right| \asymp m d^4 N_1^{\gamma-4}\,. \tag{37}$$

Now (37) and Lemma 5 for $k=4$ lead to

$$\sum_{\frac{N_1}{d}<l\leq\frac{N_2}{d}} e(f(d,l)) \ll m^{\frac{1}{12}}d^{-\frac{2}{3}}N_1^{\frac{\gamma}{12}+\frac{2}{3}+\varepsilon} + N_1^{\frac{11}{12}+\varepsilon}d^{-\frac{11}{12}} + m^{-\frac{1}{24}}d^{-1}N_1^{1-\frac{\gamma}{24}+\varepsilon}\,. \tag{38}$$

From (9), (10), (33), (36) and (38), we find

$$\Phi_2 \ll \left(m^{\frac{1}{12}}\vartheta^{\frac{1}{3}}N_1^{\frac{\gamma}{12}+\frac{2}{3}} + N_1^{\frac{11}{12}}\vartheta^{\frac{1}{12}} + m^{-\frac{1}{24}}N_1^{1-\frac{\gamma}{24}}\right)N^\varepsilon\,. \tag{39}$$

To estimate Φ_1 defined by (32), we use Abel's summation formula. Then, following the same reasoning as in the estimation of Φ_2, we have

$$\Phi_1 \ll \left(m^{\frac{1}{12}}\vartheta^{\frac{1}{3}}N_1^{\frac{\gamma}{12}+\frac{2}{3}} + N_1^{\frac{11}{12}}\vartheta^{\frac{1}{12}} + m^{-\frac{1}{24}}N_1^{1-\frac{\gamma}{24}}\right)N^{\varepsilon}\,. \tag{40}$$

Our next goal is to obtain estimates for the sums defined by (34) and (35). Proceeding as in [3], we deduce that it suffices to estimate the sum

$$\Phi_4' = \sum_{D<d\le 2D} a(d) \sum_{\substack{L<l\le 2L\\ N_1<dl\le N_2}} \Lambda(l)e(f(d,l)) \tag{41}$$

under the given conditions

$$\frac{N_1}{4} \le DL \le 2N_1\,, \quad \frac{N_1^{\frac{1}{2}}}{2} \le D \le \vartheta^2\,. \tag{42}$$

Consequently, we obtain

$$\Phi_4 \ll |\Phi_4'|\log N_1 \tag{43}$$

and the resulting estimate also holds for Φ_3. Now (36), (41), (42), Cauchy's inequality and Lemma 6 with $Q \le \frac{L}{2}$ give us

$$|\Phi_4'|^2 \ll \left(\frac{LD}{Q}\sum_{1\le q\le Q}\sum_{L<l\le 2L}\left|\sum_{D_1<d\le D_2} e(g(d))\right| + \frac{(LD)^2}{Q}\right)N^{\varepsilon}\,, \tag{44}$$

where

$$D_1 = \max\left\{D, \frac{N_1}{l}, \frac{N_1}{l+q}\right\}, \quad D_2 = \min\left\{2D, \frac{N_2}{l}, \frac{N_2}{l+q}\right\} \tag{45}$$

and

$$g(d) = f(d,l+q) - f(d,l)\,. \tag{46}$$

By (30) and (46), it follows that

$$|g^{(4)}(d)| \asymp mD^{\gamma-4}qL^{\gamma-1}\,. \tag{47}$$

Now (45), (47) and Lemma 5 for $k=4$ yield

$$\sum_{D_1<d\le D_2} e(g(d)) \ll m^{\frac{1}{12}}q^{\frac{1}{12}}D^{\frac{\gamma}{12}+\frac{2}{3}+\varepsilon}L^{\frac{\gamma}{12}-\frac{1}{12}} + D^{\frac{11}{12}+\varepsilon} + m^{-\frac{1}{24}}q^{-\frac{1}{24}}D^{1-\frac{\gamma}{24}+\varepsilon}L^{\frac{1}{24}-\frac{\gamma}{24}}\,. \tag{48}$$

Set

$$Q = \min\Big([L/4]\,, [Q_0]\Big)\,, \tag{49}$$

where

$$Q_0 = m^{-1} D^{3-\gamma} L^{1-\gamma} \,. \tag{50}$$

From (9), (22), (29), (42) and (50), we have

$$Q_0 > N^{\frac{4367}{4512}} \,.$$

Bearing in mind (44), (48), (49) and (50), we derive

$$\begin{aligned}
|\Phi_4'|^2 &\ll \big(D^2L^2Q^{-1} + m^{\frac{1}{12}}Q^{\frac{1}{12}}D^{\frac{\gamma}{12}+\frac{5}{3}}L^{\frac{\gamma}{12}+\frac{23}{12}} + D^{\frac{23}{12}}L^2 + m^{-\frac{1}{24}}Q^{-\frac{1}{24}}D^{2-\frac{\gamma}{24}}L^{\frac{49}{24}-\frac{\gamma}{24}}\big)N^{\varepsilon} \\
&\ll \Big(D^2L^2L^{-1} + D^2L^2Q_0^{-1} + m^{\frac{1}{12}}Q_0^{\frac{1}{12}}D^{\frac{\gamma}{12}+\frac{5}{3}}L^{\frac{\gamma}{12}+\frac{23}{12}} + D^{\frac{23}{12}}L^2 \\
&\qquad\qquad + m^{-\frac{1}{24}}D^{2-\frac{\gamma}{24}}L^{\frac{49}{24}-\frac{\gamma}{24}}\big(L^{-\frac{1}{24}} + Q_0^{-\frac{1}{24}}\big)\Big)N^{\varepsilon} \\
&\ll \Big(D^2L + mD^{\gamma-1}L^{\gamma+1} + D^{\frac{23}{12}}L^2 + m^{-\frac{1}{24}}D^{2-\frac{\gamma}{24}}L^{2-\frac{\gamma}{24}} + D^{\frac{15}{8}}L^2\Big)N^{\varepsilon} \,.
\end{aligned} \tag{51}$$

Now (42), (43) and (51) imply

$$\Phi_4 \ll \Big(N_1^{\frac{1}{2}}\vartheta + M^{\frac{1}{2}}N_1^{\frac{\gamma}{2}} + N_1^{\frac{47}{48}} + m^{-\frac{1}{48}}N_1^{1-\frac{\gamma}{48}}\Big)N^{\varepsilon} \,. \tag{52}$$

Proceeding analogously to the estimation of Φ_4 for the sum (34), we deduce

$$\Phi_3 \ll \Big(N_1^{\frac{1}{2}}\vartheta + M^{\frac{1}{2}}N_1^{\frac{\gamma}{2}} + N_1^{\frac{47}{48}} + m^{-\frac{1}{48}}N_1^{1-\frac{\gamma}{48}}\Big)N^{\varepsilon} \,. \tag{53}$$

Combining (31), (39), (40), (52) and (53), we get

$$\mathfrak{F}(N_1, N_2) \ll \Big(N_1^{\frac{1}{2}}\vartheta + M^{\frac{1}{2}}N_1^{\frac{\gamma}{2}} + N_1^{\frac{47}{48}} + m^{\frac{1}{12}}\vartheta^{\frac{1}{3}}N_1^{\frac{\gamma}{12}+\frac{2}{3}} + N_1^{\frac{11}{12}}\vartheta^{\frac{1}{12}} + m^{-\frac{1}{48}}N_1^{1-\frac{\gamma}{48}}\Big)N^{\varepsilon} \,. \tag{54}$$

Based on (9), (10), (25), (29) and (54), we conclude

$$\Sigma_1 \ll N^{\frac{48\gamma+47}{96}+\varepsilon} \,. \tag{55}$$

Taking into account (19), (24), (28) and (55), we establish the statement in the lemma.

□

Lemma 10. *Let* $\frac{47}{48} < \gamma < 1$*. For the sum* Γ *defined by* (13) *the estimate*

$$\Gamma \ll N^{\frac{48\gamma+47}{96}+\varepsilon}$$

holds.

Proof. From (13), we write

$$\Gamma = \sum_{p\leq N} \big([-p^\gamma] - [-(p+1)^\gamma]\big)\big(F_\Delta(\alpha p^3+\beta) - 2\Delta\big)\log p = \Gamma_1 + \Gamma_2\,, \tag{56}$$

where

$$\Gamma_1 = \sum_{p\leq N} \big((p+1)^\gamma - p^\gamma\big)\big(F_\Delta(\alpha p^3+\beta) - 2\Delta\big)\log p\,, \tag{57}$$

$$\Gamma_2 = \sum_{p\leq N} \big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big)\big(F_\Delta(\alpha p^3+\beta) - 2\Delta\big)\log p\,. \tag{58}$$

Upper bound for $\mathbf{\Gamma_1}$

The function $F_\Delta(\theta) - 2\Delta$ is well known to have the expansion

$$\sum_{1\leq|h|\leq H} \frac{\sin 2\pi h\Delta}{\pi h}\, e(h\theta) + \mathcal{O}\Bigg(\min\left(1, \frac{1}{H\|\theta+\Delta\|}\right) + \min\left(1, \frac{1}{H\|\theta-\Delta\|}\right)\Bigg)\,. \tag{59}$$

Now (57), (59) and the formula

$$(p+1)^\gamma - p^\gamma = \gamma p^{\gamma-1} + \mathcal{O}\left(p^{\gamma-2}\right)$$

lead to

$$\Gamma_1 = \gamma \sum_{p\leq N} p^{\gamma-1}\log p \sum_{1\leq|h|\leq H} \frac{\sin 2\pi h\Delta}{\pi h}\, e\big(h(\alpha p^3+\beta)\big) + \mathcal{O}\big(\Omega\log N\big)\,, \tag{60}$$

where

$$\Omega = \sum_{n=1}^{N} \left(\min\left(1, \frac{1}{H\|\alpha n^3+\beta+\Delta\|}\right) + \min\left(1, \frac{1}{H\|\alpha n^3+\beta-\Delta\|}\right)\right)\,. \tag{61}$$

By (5), (6), (8), (61) and Lemma 8, we derive

$$\Omega \ll N^\varepsilon\Big(Nq^{-\frac{1}{8}} + N^{\frac{7}{8}} + NH^{-\frac{1}{8}} + N^{\frac{5}{8}}H^{-\frac{1}{8}}q^{\frac{1}{8}}\Big) \ll N^{1+\varepsilon}H^{-\frac{1}{8}} \ll N^{\frac{48\gamma+47}{96}+\varepsilon}\,. \tag{62}$$

Now (60) and (62) yield

$$\Gamma_1 \ll \sum_{h=1}^{H} \min\left(\Delta, \frac{1}{h}\right) \left|\sum_{p\leq N} p^{\gamma-1} e(\alpha h p^3)\log p\right| + N^{\frac{48\gamma+47}{96}+\varepsilon}\,. \tag{63}$$

Set

$$\mathfrak{X}(u) = \sum_{h\leq u} \left|\sum_{p\leq N} p^{\gamma-1} e(\alpha h p^3)\log p\right|\,. \tag{64}$$

By applying Abel's summation formula, we obtain

$$\sum_{h=1}^{H} \min\left(\Delta, \frac{1}{h}\right) \left| \sum_{p\le N} p^{\gamma-1} e(\alpha h p^3) \log p \right| = \frac{\mathfrak{X}(H)}{H} + \int\limits_{\Delta^{-1}}^{H} \frac{\mathfrak{X}(u)}{u^2}\, du$$

$$\ll (\log H) \max_{\Delta^{-1}\le u\le H} \frac{\mathfrak{X}(u)}{u} \,. \tag{65}$$

Using Abel's summation once more, we get

$$\sum_{p\le N} p^{\gamma-1} e(\alpha h p^3) \log p = N^{\gamma-1} G(N) + (1-\gamma) \int\limits_{2}^{N} G(y) y^{\gamma-2}\, dy \,, \tag{66}$$

where

$$G(y) = \sum_{p\le y} e(\alpha h p^3) \log p \,. \tag{67}$$

According to Dirichlet's approximation theorem, there exist integers a_h and q_h such that

$$\left|\alpha h - \frac{a_h}{q_h}\right| \le \frac{1}{q_h q^2}\,, \quad (a_h, q_h) = 1\,, \quad 1 \le q_h \le q^2 \,. \tag{68}$$

In view of (67), (68) and Lemma 7, we deduce

$$G(y) \ll y^{1+\varepsilon} \left(q_h^{-\frac{1}{16}} + y^{-\frac{1}{32}} + y^{-\frac{3}{16}} q_h^{\frac{1}{16}} \right) . \tag{69}$$

From (64), (66) and (69), we find

$$\mathfrak{X}(u) \ll N^{\gamma-1+\varepsilon} \sum_{h\le u} \left(N q_h^{-\frac{1}{16}} + N^{\frac{31}{32}} + N^{\frac{13}{16}} q_h^{\frac{1}{16}} \right) . \tag{70}$$

Combining (5), (8), (68) and following the reasoning in [3], we derive

$$q_h \in \left(\frac{q^{\frac{2}{3}}}{\log N}, q^2 \right] . \tag{71}$$

Based on (6), (70) and (71), we obtain

$$\mathfrak{X}(u) \ll u N^{\gamma+\varepsilon} q^{-\frac{1}{24}} \ll u N^{\frac{144\gamma-49}{96}+\varepsilon} . \tag{72}$$

By (6), (8), (63), (65) and (72), it follows that

$$\Gamma_1 \ll N^{\frac{48\gamma+47}{96}+\varepsilon} . \tag{73}$$

Upper bound for Γ_2

From (58), (59) and proceeding as in Γ_1, we conclude that

$$\Gamma_2 \ll \sum_{h=1}^{H} \min\left(\Delta, \frac{1}{h}\right) \left| \sum_{p\leq N} \big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) e(\alpha h p^3) \log p \right| + N^{\frac{48\gamma+47}{96}+\varepsilon} . \quad (74)$$

Define

$$S(u) = \sum_{h\leq u} \left| \sum_{p\leq N} \big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) e(\alpha h p^3) \log p \right| . \quad (75)$$

By (75) and Abel's summation formula, we deduce

$$\sum_{h=1}^{H} \min\left(\Delta, \frac{1}{h}\right) \left| \sum_{p\leq N} \big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) e(\alpha h p^3) \log p \right|$$
$$= \frac{S(H)}{H} + \int\limits_{\Delta^{-1}}^{H} \frac{S(u)}{u^2}\, du . \quad (76)$$

Now (6), (8), (74) – (76) and Lemma 9 give us

$$\Gamma_2 \ll (\log H) \max_{\Delta^{-1}\leq u\leq H} \frac{S(u)}{u} + N^{\frac{48\gamma+47}{96}+\varepsilon} \ll N^{\frac{48\gamma+47}{96}+\varepsilon} . \quad (77)$$

Summarizing (56), (73) and (77), we complete the proof of the lemma. □

4.3 The end of the proof

Taking into account (4), (7), (13) and Lemma 10, we get

$$\sum_{\substack{p\leq N \\ p=[n^{1/\gamma}]}} F_\Delta(\alpha p^3 + \beta) \log p \gg N^{\frac{48\gamma+47}{96}+\varepsilon} .$$

This completes the proof of Theorem 1.

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S. I. Dimitrov

Faculty of Applied Mathematics and Informatics
Technical University of Sofia
Blvd. St.Kliment Ohridski 8
Sofia 1000, Bulgaria
e-mail: sdimitrov@tu-sofia.bg

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str. Bl. 105, Sofia 1113, Bulgaria
e-mail: xyzstoyan@gmail.com

M. D. Lazarova

Faculty of Applied Mathematics and Informatics
Technical University of Sofia
Blvd. St.Kliment Ohridski 8
Sofia 1000, Bulgaria
e-mail: meglena.laz@tu-sofia.bg

Department of Mathematics, Infrormatics and Physics
Burgas State University Prof. Dr. Assen Zlatarov
Blvd. Prof. Yakimov 1
Burgas 8010, Bulgaria
e-mail: meglena-lazarova@uniburgas.bg