

Author's Report

Title: Diophantine approximation with mixed powers of Piatetski-Shapiro primes

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Dear Deputy Editor-in-Chief,

Thank you for the opportunity to make these formal changes, which bring greater clarity and readability to the text. The corrections mainly concern the placement of indices on $\Sigma(t)$, $U(t)$ and $\Omega(t)$.

1 – 5.

$$S_k(t) = \sum_{\substack{\lambda_0 X < p^k \leq X \\ p = [n^{1/\gamma}]}} p^{1-\gamma} e(tp^k) \log p, \quad k = 1, 2, \dots; \quad (1)$$

$$\Sigma_k(t) = \sum_{\lambda_0 X < p^k \leq X} e(tp^k) \log p, \quad k = 1, 2, \dots; \quad (2)$$

$$U_k(t) = \sum_{\lambda_0 X < n^k \leq X} e(tn^k), \quad k = 1, 2, \dots; \quad (3)$$

$$\Omega_k(t) = \sum_{\lambda_0 X < p^k \leq X} p^{1-\gamma} (\psi(-(p+1)^\gamma) - \psi(-p^\gamma)) e(tp^k) \log p, \quad k = 1, 2, \dots; \quad (4)$$

$$I_k(t) = \int_{(\lambda_0 X)^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(ty^k) dy, \quad k = 1, 2, \dots. \quad (5)$$

6.

$$\Omega_2(t) \ll X^{\frac{21-7\gamma}{29} + \delta}.$$

7.

$$\int_{-Y}^Y |\Sigma_k(t) - U_k(t)|^2 dt \ll \frac{X^{\frac{2}{k}-2} \log^2 X}{Y} + Y^2 X + X^{\frac{2}{k}-1} \exp\left(-c_1 \left(\frac{\log X}{\log \log X}\right)^{1/3}\right).$$

8.

$$S_2(t) = \gamma \Sigma_2(t) + \mathcal{O}\left(X^{\frac{21-7\gamma}{29}+\delta}\right).$$

9.

$$= \gamma \Sigma_2(t) + \Omega_2(t) + \mathcal{O}(1).$$

10.

$$J(X) = \gamma^3 \int_{|t|<\Delta} \Theta(t) I_1(\lambda_1 t) I_1(\lambda_2 t) I_2(\lambda_3 t) e(\eta t) dt.$$

11.

$$+\varepsilon \int_{|t|<\Delta} |S_1(\lambda_1 t)| |S_1(\lambda_2 t)| \left| \gamma \Sigma_2(\lambda_3 t) - \gamma I_2(\lambda_3 t) + \mathcal{O}\left(X^{\frac{21-7\gamma}{29}+\delta}\right) \right| dt$$

12.

$$\ll \varepsilon \frac{X^{\frac{3}{2}}}{e^{(\log X)^{1/6}}} + \varepsilon \int_{|t|<\Delta} |S_1(\lambda_1 t)| |S_1(\lambda_2 t)| \left| \Sigma_2(\lambda_3 t) - U_2(\lambda_3 t) \right| dt$$

13.

$$+\varepsilon \int_{|t|<\Delta} |S_1(\lambda_1 t)| |S_1(\lambda_2 t)| \left| U_2(\lambda_3 t) - I_2(\lambda_3 t) \right| dt$$

14.

$$J_1 \ll X \left(\int_{-\Delta}^{\Delta} |S_1(\lambda_1 t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{-\Delta}^{\Delta} |\Sigma_2(\lambda_3 t) - U_2(\lambda_3 t)|^2 dt \right)^{\frac{1}{2}} \ll \frac{X^{\frac{3}{2}}}{e^{(\log X)^{1/6}}}.$$

15.

$$I_2(t) - U_2(t) \ll 1 + |t|X.$$

DIOPHANTINE APPROXIMATION WITH MIXED POWERS OF PIATETSKI-SHAPIRO PRIMES

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Abstract—Let $[\cdot]$ denote the floor function. In this paper, we show that whenever η is real and the constants λ_i satisfy some necessary conditions, then for any fixed $\frac{63}{64} < \gamma < 1$ and $\theta > 0$, there exist infinitely many prime triples p_1, p_2, p_3 satisfying the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^2 + \eta| < (\max\{p_1, p_2, p_3^2\})^{\frac{63-64\gamma}{52} + \theta}$$

and such that $p_i = [n_i^{1/\gamma}]$, $i = 1, 2, 3$.

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Keywords: Diophantine approximation, Piatetski-Shapiro primes

1. Introduction and Statement of the Result

The study of Diophantine inequalities involving prime numbers constitutes a rapidly evolving field within analytic number theory. In 1967, A. Baker [1] proved that if $\lambda_1, \lambda_2, \lambda_3$ are nonzero real numbers, not all of the same sign, λ_1/λ_2 is irrational, η is real and $A > 0$, then there exist infinitely many prime triples p_1, p_2, p_3 such that

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon_1, \quad (1)$$

where $\varepsilon_1 = (\log \max p_j)^{-A}$. Subsequently, the right-hand side of (1) was improved by Ramachandra [2], Vaughan [3], Lau and Liu [4], Baker and Harman [5], and Harman [6]. The best result to date is due to Matomäki [7], with $\varepsilon_1 = (\max p_j)^{-\frac{2}{9} + \delta}$ and $\delta > 0$. In 2018, Gambini, Languasco, and Zaccagnini [8] proved the existence of infinitely many triples of primes p_1, p_2 , and p_3 such that

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^2 + \eta| < \varepsilon_2, \quad (2)$$

where $\varepsilon_2 = (\max\{p_1, p_2, p_3^2\})^{-\frac{1}{12} + \delta}$ and $\delta > 0$. Weaker results were previously obtained in [9] and [10]. Another interesting question is the study of Diophantine inequalities involving special prime numbers. Let P_l is a number with at most l prime factors. Very recently Todorova and Georgieva [11] solved inequality (2) with prime numbers p_1, p_2 , and p_3 such that $p_i + 2 = P_{l_i}$, $i = 1, 2, 3$. In 1953, Piatetski-Shapiro [12] showed that for any fixed $\frac{11}{12} < \gamma < 1$, there exist infinitely many prime numbers of the form $p = [n^{1/\gamma}]$. Such primes are called Piatetski-Shapiro primes of type γ . Subsequently, the interval for γ was sharpened many times and the best result to date has been supplied by Rivat and Wu [13] with $\frac{205}{243} < \gamma < 1$. The primes of the form $[n^{1/\gamma}]$ are very much in focus nowadays and many problems are solved using them. We mention, for example, the papers [14–16]. In 2022, the author [17] proved that for any fixed $\frac{37}{38} < \gamma < 1$, inequality (1) is solvable with infinitely many Piatetski-Shapiro prime triples p_1, p_2, p_3 of type γ . As a continuation of these studies, we solve (2) with Piatetski-Shapiro primes.

Theorem 1. *Suppose that $\lambda_1, \lambda_2, \lambda_3$ are nonzero real numbers, not all of the same sign, that λ_1/λ_2 is irrational, and that η is real. Let $\theta > 0$ and γ be fixed with $\frac{63}{64} < \gamma < 1$. Then there exist infinitely many ordered triples of Piatetski-Shapiro primes p_1, p_2 , and p_3 of type γ such that*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^2 + \eta| < (\max\{p_1, p_2, p_3^2\})^{\frac{63-64\gamma}{52} + \theta}.$$

2. Notations

The letter p will always denote a prime number. By δ we denote an arbitrarily small positive number, not the same in all appearances. As usual, $[t]$ and $\{t\}$ denote the integer part and the fractional part of t , respectively. Moreover $\psi(t) = \{t\} - \frac{1}{2}$. We write $e(t) = e^{2\pi it}$. Let γ , θ , and λ_0 be real constants such that $\frac{63}{64} < \gamma < 1$, $\theta > 0$, and $0 < \lambda_0 < 1$. Since λ_1/λ_2 is irrational, there are infinitely many different convergents a_0/q_0 to its continued fraction, with $|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}| < \frac{1}{q_0^2}$, $(a_0, q_0) = 1$, $a_0 \neq 0$ and q_0 is arbitrary large. Denote

$$X = q_0^{\frac{13}{6}}; \tag{3}$$

$$\Delta = X^{-\frac{12}{13}} \log X; \tag{4}$$

$$\varepsilon = X^{\frac{63-64\gamma}{52} + \theta}; \tag{5}$$

$$H = \frac{\log^2 X}{\varepsilon}; \tag{6}$$

$$1 \quad S_k(t) = \sum_{\substack{\lambda_0 X < p^k \leq X \\ p = [n^{1/\gamma}]}} p^{1-\gamma} e(tp^k) \log p, \quad k = 1, 2; \tag{7}$$

$$2 \quad \Sigma(t) = \sum_{\lambda_0 X < p^2 \leq X} e(tp^2) \log p; \tag{8}$$

$$3 \quad U(t) = \sum_{\lambda_0 X < n^2 \leq X} e(tn^2); \tag{9}$$

$$4 \quad \Omega(t) = \sum_{\lambda_0 X < p^2 \leq X} p^{1-\gamma} (\psi(-(p+1)^\gamma) - \psi(-p^\gamma)) e(tp^2) \log p; \tag{10}$$

$$5 \quad I_k(t) = \int_{(\lambda_0 X)^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(ty^k) dy, \quad k = 1, 2. \tag{11}$$

3. Preliminary Lemmas

Lemma 1. *Let $\varepsilon > 0$ and $k \in \mathbb{N}$. There exists a function $\theta(y)$ which is k times continuously differentiable and such that*

$$\begin{aligned} \theta(y) &= 1 && \text{for } |y| \leq 3\varepsilon/4; \\ 0 < \theta(y) &< 1 && \text{for } 3\varepsilon/4 < |y| < \varepsilon; \\ \theta(y) &= 0 && \text{for } |y| \geq \varepsilon, \end{aligned}$$

and its Fourier transform

$$\Theta(x) = \int_{-\infty}^{\infty} \theta(y) e(-xy) dy$$

satisfies the inequality

$$|\Theta(x)| \leq \min\left(\frac{7\varepsilon}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left(\frac{k}{2\pi|x|\varepsilon/8}\right)^k\right).$$

PROOF. See [18]. \square

Lemma 2. For any fixed $\frac{2426}{2817} < \gamma < 1$, we have

$$\sum_{\substack{p \leq X \\ p = [n^{1/\gamma}]} 1 \sim \frac{X^\gamma}{\log X}.$$

PROOF. See [19, Theorem 1]. \square

Lemma 3. We have

- (i) $\int_{-\Delta}^{\Delta} |S_1(t)|^2 dt \ll X \log^3 X$ and $\int_0^1 |S_1(t)|^2 dt \ll X^{2-\gamma} \log X$,
- (ii) $\int_{-\Delta}^{\Delta} |I_1(t)|^2 dt \ll X$ and $\int_{-\Delta}^{\Delta} |I_2(t)|^2 dt \ll 1$.

PROOF. For (i) see [17, Lemma 6]. For (ii) see [11, Lemma 15]. \square

Lemma 4. Let $|t| \leq \Delta$. Then the asymptotic formula

$$S_1(t) = \gamma I_1(t) + \mathcal{O}\left(\frac{X}{e^{(\log X)^{1/5}}}\right)$$

holds.

PROOF. See [17, Lemma 5]. \square

Lemma 5. Let $\frac{13}{14} < \gamma < 1$. Then

$$6 \quad \Omega(t) \ll X^{\frac{21-7\gamma}{29} + \delta}.$$

PROOF. See [20, Lemma 6]. \square

Lemma 6. Let $k \geq 1$ and $1/2X \leq Y \leq 1/2X^{1-\frac{5}{6k}+\delta}$. Then there exists a positive constant $c_1(\delta)$, which does not depend on k , such that

$$7 \quad \int_{-Y}^Y |\Sigma(t) - U(t)|^2 dt \ll \frac{X^{\frac{2}{k}-2} \log^2 X}{Y} + Y^2 X + X^{\frac{2}{k}-1} \exp\left(-c_1 \left(\frac{\log X}{\log \log X}\right)^{1/3}\right).$$

PROOF. See [8, Lemmas 1 and 2]. \square

Lemma 7. Let $\frac{11}{12} < \gamma < 1$ and $\Delta \leq |t| \leq H$. Then there exists a sequence of real numbers $X_1, X_2, \dots \rightarrow \infty$ such that

$$\min\{|S_1(\lambda_1 t)|, |S_1(\lambda_2 t)|\} \ll X_j^{\frac{37-12\gamma}{26}} \log^5 X_j, \quad j = 1, 2, \dots$$

PROOF. See [17, Lemma 7]. \square

Lemma 8. We have

$$\int_0^1 |S_2(t)|^4 dt \ll X^{2-\gamma+\delta}.$$

PROOF. See [21, (12)]. \square

4. Beginning of the Proof

Consider the sum

$$\Gamma(X) = \sum_{\substack{\lambda_0 X < p_1, p_2, p_3^2 \leq X \\ p_i = [n_i^{1/\gamma}], i=1,2,3}} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^2 + \eta) p_1^{1-\gamma} p_2^{1-\gamma} p_3^{1-\gamma} \log p_1 \log p_2 \log p_3. \quad (12)$$

Using the inverse Fourier transform for the function $\theta(x)$, we obtain

$$\begin{aligned} \Gamma(X) &= \sum_{\substack{\lambda_0 X < p_1, p_2, p_3^2 \leq X \\ p_i = [n_i^{1/\gamma}], i=1,2,3}} p_1^{1-\gamma} p_2^{1-\gamma} p_3^{1-\gamma} \log p_1 \log p_2 \log p_3 \\ &\quad \times \int_{-\infty}^{\infty} \Theta(t) e((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^2 + \eta)t) dt \\ &= \int_{-\infty}^{\infty} \Theta(t) S_1(\lambda_1 t) S_1(\lambda_2 t) S_2(\lambda_3 t) e(\eta t) dt. \end{aligned}$$

We decompose $\Gamma(X)$ as follows:

$$\Gamma(X) = \Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X), \quad (13)$$

where

$$\Gamma_1(X) = \int_{|t| < \Delta} \Theta(t) S_1(\lambda_1 t) S_1(\lambda_2 t) S_2(\lambda_3 t) e(\eta t) dt, \quad (14)$$

$$\Gamma_2(X) = \int_{\Delta \leq |t| \leq H} \Theta(t) S_1(\lambda_1 t) S_1(\lambda_2 t) S_2(\lambda_3 t) e(\eta t) dt, \quad (15)$$

$$\Gamma_3(X) = \int_{|t| > H} \Theta(t) S_1(\lambda_1 t) S_1(\lambda_1 t) S_2(\lambda_3 t) e(\eta t) dt. \quad (16)$$

We will estimate $\Gamma_1(X)$, $\Gamma_2(X)$, and $\Gamma_3(X)$, respectively, in Sections 5–7. In Section 8 we will complete the proof of Theorem 1.

5. Lower Bound of $\Gamma_1(X)$

Lemma 9. *Let $\frac{13}{14} < \gamma < 1$. Then*

$$S_2(t) = \gamma \Sigma(t) + \mathcal{O}(X^{\frac{21-7\gamma}{29} + \delta}).$$

PROOF. From (7), (8), (10), and the well-known asymptotic formula

$$(p+1)^\gamma - p^\gamma = \gamma p^{\gamma-1} + \mathcal{O}(p^{\gamma-2})$$

we write

$$\begin{aligned} S_2(t) &= \sum_{\lambda_0 X < p^2 \leq X} p^{1-\gamma} ([-p^\gamma] - [-(p+1)^\gamma]) e(tp^2) \log p \\ &= \sum_{\lambda_0 X < p^2 \leq X} p^{1-\gamma} ((p+1)^\gamma - p^\gamma) e(tp^2) \log p \\ &\quad + \sum_{\lambda_0 X < p^2 \leq X} p^{1-\gamma} (\psi(-(p+1)^\gamma) - \psi(-p^\gamma)) e(tp^2) \log p \\ &= \gamma \Sigma(t) + \Omega(t) + \mathcal{O}(1). \end{aligned} \quad (17)$$

8

9

Bearing in mind (17) and Lemma 5, we establish the statement in the lemma. \square

Put

$$10 \quad J(X) = \gamma^3 \int_{|t|<\Delta} \Theta(t) I_1(\lambda_1 t, X) I_1(\lambda_2 t, X) I_2(\lambda_3 t, X) e(\eta t) dt. \quad (18)$$

Now (4), (7), (8), (9), (11), (14), (18), Cauchy's inequality, Lemmas 1, 3, 4, and 9 imply

$$\begin{aligned} \Gamma_1(X) - J(X) &= \gamma^2 \int_{|t|<\Delta} \Theta(t) (S_1(\lambda_1 t) - \gamma I_1(\lambda_1 t)) I_1(\lambda_2 t) I_2(\lambda_3 t) e(\eta t) dt \\ &\quad + \gamma \int_{|t|<\Delta} \Theta(t) S_1(\lambda_1 t) (S_1(\lambda_2 t) - \gamma I_1(\lambda_2 t)) I_2(\lambda_3 t) e(\eta t) dt \\ &\quad + \int_{|t|<\Delta} \Theta(t) S_1(\lambda_1 t) S_1(\lambda_2 t) (S_2(\lambda_3 t) - \gamma I_2(\lambda_3 t)) e(\eta t) dt \\ &\ll \varepsilon \frac{X}{e^{(\log X)^{1/5}}} \left[\left(\int_{|t|<\Delta} |I_1(\lambda_2 t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{|t|<\Delta} |I_2(\lambda_3 t)|^2 dt \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\int_{|t|<\Delta} |S_1(\lambda_1 t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{|t|<\Delta} |I_2(\lambda_3 t)|^2 dt \right)^{\frac{1}{2}} \right] \\ 11 \quad &\quad + \varepsilon \int_{|t|<\Delta} |S_1(\lambda_1 t)| |S_1(\lambda_2 t)| |\gamma \Sigma(\lambda_3 t) - \gamma I_2(\lambda_3 t) + \mathcal{O}(X^{\frac{21-7\gamma}{29} + \delta})| dt \\ 12 \quad &\ll \varepsilon \frac{X^{\frac{3}{2}}}{e^{(\log X)^{1/6}}} + \varepsilon \int_{|t|<\Delta} |S_1(\lambda_1 t)| |S_1(\lambda_2 t)| |\Sigma(\lambda_3 t) - U(\lambda_3 t)| dt \\ 13 \quad &\quad + \varepsilon \int_{|t|<\Delta} |S_1(\lambda_1 t)| |S_1(\lambda_2 t)| |U(\lambda_3 t) - I_2(\lambda_3 t)| dt \quad (19) \\ &= \varepsilon \left(\frac{X^{\frac{3}{2}}}{e^{(\log X)^{1/6}}} + J_1 + J_2 \right), \end{aligned}$$

say. Using Cauchy's inequality and Lemmas 2, 3, and 6, we get

$$14 \quad J_1 \ll X \left(\int_{-\Delta}^{\Delta} |S_1(\lambda_1 t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{-\Delta}^{\Delta} |\Sigma(\lambda_3 t) - U(\lambda_3 t)|^2 dt \right)^{\frac{1}{2}} \ll \frac{X^{\frac{3}{2}}}{e^{(\log X)^{1/6}}}. \quad (20)$$

By Euler's summation formula, we have

$$15 \quad I_2(t) - U(t) \ll 1 + |t|X. \quad (21)$$

From (21), Cauchy's inequality, and Lemma 3, we derive

$$J_2 \ll (1 + \Delta X) \left(\int_{-\Delta}^{\Delta} |S_1(\lambda_1 t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{-\Delta}^{\Delta} |S_1(\lambda_2 t)|^2 dt \right)^{\frac{1}{2}} \ll \Delta X^2 \log^3 X. \quad (22)$$

On the other hand for the integral defined by (18), we write

$$J(X) = B(X) + \Phi, \quad (23)$$

where

$$B(X) = \gamma^3 \int_{-\infty}^{\infty} \Theta(t) I_1(\lambda_1 t) I_1(\lambda_2 t) I_2(\lambda_3 t) e(\eta t) dt.$$

and

$$\Phi \ll \int_{\Delta}^{\infty} |\Theta(t)| |I_1(\lambda_1 t) I_1(\lambda_2 t) I_2(\lambda_3 t)| dt. \quad (24)$$

Arguing as in [22, Lemma 4], we deduce that if

$$\lambda_0 < \min\left(\frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16}\right)$$

then

$$B(X) \gg \varepsilon X^{\frac{3}{2}}. \quad (25)$$

By (11) and [23, Lemma 4.2], we get

$$I_k(t) \ll X^{\frac{1}{k}-1} \min(X, |t|^{-1}). \quad (26)$$

Using (24), (26), and Lemma 1, we obtain

$$\Phi \ll \frac{\varepsilon X^{\frac{1}{2}}}{\Delta}. \quad (27)$$

Bearing in mind (4), (19), (20), (22), (23), (25), and (27), we establish

$$\Gamma_1(X) \gg \varepsilon X^{\frac{3}{2}}. \quad (28)$$

6. Upper Bound of $\Gamma_2(X)$

Put

$$\mathfrak{S}(t, X) = \min\{|S_1(\lambda_1 t)|, |S_1(\lambda_2 t)|\}. \quad (29)$$

Taking into account (15), (29), Lemma 1, and Lemma 7, we deduce

$$\begin{aligned} \Gamma_2(X_j) &\ll \varepsilon \int_{\Delta \leq |t| \leq H} \mathfrak{S}(t, X_j)^{\frac{1}{2}} |S_1(\lambda_1 t)|^{\frac{1}{2}} |S_1(\lambda_2 t)| |S_2(\lambda_3 t)| dt \\ &\quad + \varepsilon \int_{\Delta \leq |t| \leq H} \mathfrak{S}(t, X_j)^{\frac{1}{2}} |S_1(\lambda_1 t)| |S_1(\lambda_2 t)|^{\frac{1}{2}} |S_2(\lambda_3 t)| dt \\ &\ll \varepsilon X_j^{\frac{37-12\gamma}{52} + \delta} (\Psi_1 + \Psi_2), \end{aligned} \quad (30)$$

where

$$\Psi_1 = \int_{\Delta}^H |S_1(\lambda_1 t)|^{\frac{1}{2}} |S_1(\lambda_2 t)| |S_2(\lambda_3 t)| dt, \quad (31)$$

$$\Psi_2 = \int_{\Delta}^H |S_1(\lambda_1 t)| |S_1(\lambda_2 t)|^{\frac{1}{2}} |S_2(\lambda_3 t)| dt. \quad (32)$$

We estimate only Ψ_1 and the estimation of Ψ_2 proceeds in the same way. From (31) and Cauchy's inequality, we derive

$$\Psi_1 \ll \left(\int_{\Delta}^H |S_1(\lambda_2 t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{\Delta}^H |S_1(\lambda_1 t)|^2 dt \right)^{\frac{1}{4}} \left(\int_{\Delta}^H |S_2(\lambda_3 t)|^4 dt \right)^{\frac{1}{4}}. \quad (33)$$

Using Lemma 3(i), we obtain

$$\int_{\Delta}^H |S_1(\lambda_k t)|^2 dt \ll H X_j^{2-\gamma} \log X_j, \quad k = 1, 2. \quad (34)$$

By Lemma 8, we find

$$\int_{\Delta}^H |S_2(\lambda_3 t)|^4 dt \ll H X_j^{2-\gamma+\delta}. \quad (35)$$

Now (33)–(35) yield

$$\Psi_1 \ll H X_j^{2-\gamma+\delta}. \quad (36)$$

Combining (5), (6), (30), and (36), we get

$$\Gamma_2(X_j) \ll X_j^{\frac{37-12\gamma}{52}+\delta} X_j^{2-\gamma+\delta} = X_j^{\frac{141-64\gamma}{52}+\delta} \ll \frac{\varepsilon X_j^{\frac{3}{2}}}{\log X_j}. \quad (37)$$

7. Upper Bound of $\Gamma_3(X)$

By (7), (16), and Lemmas 1 and 2, it follows

$$\Gamma_3(X) \ll X^3 \int_H^{\infty} \frac{1}{t} \left(\frac{k}{2\pi t \varepsilon / 8} \right)^k dt = \frac{X^3}{k} \left(\frac{4k}{\pi \varepsilon H} \right)^k. \quad (38)$$

Choosing $k = [\log X]$ from (6) and (38), we deduce

$$\Gamma_3(X) \ll 1. \quad (39)$$

8. Proof of the Theorem

Summarizing (5), (13), (28), (37), and (39), we derive

$$\Gamma(X_j) \gg \varepsilon X_j^{\frac{3}{2}} = X_j^{\frac{141-64\gamma}{52}+\theta}.$$

The last estimation implies

$$\Gamma(X_j) \rightarrow \infty \quad \text{as } X_j \rightarrow \infty. \quad (40)$$

Bearing in mind (12) and (40) we establish Theorem 1.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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