ON THE FOURTH MOMENTS OF EXPONENTIAL SUMS WEIGHTED BY DIRICHLET CHARACTERS

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Abstract. In this paper, we investigate the fourth moments of two-term exponential sums weighted by Dirichlet characters. By using properties of quadratic residue and the third-order character ψ modulo a prime p, we first present an explicit identity for the fourth moment of the cubic exponential sums weighted by Legendre's symbol when $(3, p-1) = 3$. Subsequently, through the analyses of solutions counting for certain congruence equations, we derive some results on the fourth mixed moments associated with quartic exponential sums weighted by any Dirichlet character $\chi \bmod p$.

1. Introduction and Main Results

As usual, let $q \geq 3$ be a fixed integer. For the integers m, n, $k > h \geq 1$, we define the generalized two-term exponential sums $C(m, n, k, h, \chi; q)$ as follows:

$$C(m,n,k,h,\chi;q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^k + na^h}{q}\right), \tag{1.1}$$

where χ is a Dirichlet character modulo q, $e(y) = e^{2\pi i y}$, $i^2 = -1$.

The study of the generalized two-term exponential sums has been a hot topic in number theory, many scholars have carried out various researches on it. For example, A. Weil [13] proved that for a prime p, if χ is non-principal character modulo p or χ is principal and $p \nmid (m, n)$, then

$$|C(m,n,k,1,\chi;p)| \leq k \cdot p^{1/2}. \tag{1.2}$$

T. Cochrane and Z. Y. Zheng [4] extended (1.2) to the case of prime power moduli p^s with $s \geq 2$.

1.1. The Moments of Two Term Exponential Sums.

The preceding analyses reveal that the distribution of the generalized two-term exponential sums exhibit significant irregularity. This observation has led researchers to investigate the associated moments as a mean to characterize their average behavior. For instance, R. Duan and W. P. Zhang [5] considered the fourth moments of the cubic exponential sums and got that for any odd prime p with $(3, p-1) = 1$ and any integer n with $(n, p) = 1$, one has

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 3p^3 - 8p^2, & \text{if } \chi = \chi_2, \\ 2p^3 - 7p^2, & \text{if } \chi \neq \chi_0, \chi_2, \\ 2p^3 - 3p^2 - 3p - 1, & \text{if } \chi = \chi_0, \end{cases} \tag{1.3}$$

where χ_0 is the principal character modulo p, $\chi_2(*) = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol. In [11], the second author complemented the problem in (1.3) by showing that for any prime p

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Keywords: exponential sums; the fourth moments; characters; quadratic residue; Legendre's symbol
MR(2020) Subject Classification: 11L03, 11L05, 11L07

with $(3, p-1) = 3$ and any integer n with $(n, p) = 1$,

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3+na}{p}\right) \right|^4 = \begin{cases} 2p^3 - 5p^2 - 15p + 4dp - 1, & \text{if } \chi = \chi_0, \\ 2p^3 - 11p^2 - 2p \cdot M(\chi, \psi), & \text{if } \chi = \lambda^3, \\ 3p^3 - 12p^2 - 2p \cdot M(\chi, \psi), & \text{if } \chi = \chi_2, \\ 2p^3 - 5p^2, & \text{otherwise}, \end{cases} \tag{1.4}$$

where d is uniquely determined by $d \equiv 1 \bmod 3$ and $4p = d^2 + 27b^2$ $(d, b \in \mathbb{Z})$, λ denotes any non-real character modulo p, ψ is a third-order character modulo p, $M(\chi, \psi)$ is defined by

$$M(\chi, \psi) := \left| \sum_{a=1}^{p-1} \chi(a) \psi(a^3 - 1) \right|^2. \tag{1.5}$$

What merits our attention is that as the power of the mean value increases, the computational difficulty grows significantly, making it harder to obtain satisfactory results. For this reason, mathematicians have approached the study of generalized two-term exponential sums from a different perspective—namely, by considering their mixed mean values. For example, X. Y. Liu and W. P. Zhang [7] took the mixed moments of $C(m, n, 3, 1, \chi; p)$ into account and deduced that for any prime p with $3 \nmid (p-1)$, one has

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3+a}{p}\right) \right|^6 = p(p-1)(6p^3 - 28p^2 + 39p + 5). \tag{1.6}$$

J. Zhang and X. X. Li [15] complemented the problem in (1.6) by proving that for any prime p with $3 \mid (p-1)$, one has

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3+a}{p}\right) \right|^6 = 6p^5 + O(p^4). \tag{1.7}$$

L. Wang and Y. Y. Meng [12] paid attention to the fourth power mean of the generalized polynomial exponential sums. They drew the following conclusions:
For any odd prime p with $3 \nmid (p-1)$, one has

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a^3)+ma^3+a^2}{p}\right) \right|^4 = p(p-1)^2 \cdot (2p-3).$$

For any odd prime p with $3 \mid (p-1)$, one has

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a^3)+ma^3+a^2}{p}\right) \right|^4$$
$$= \begin{cases} p(p-1)(2p^2 - 5p + 15), & \text{if } p \equiv 7 \bmod 12, \\ p(p-1)(2p^2 - 5p + 15 - 12\sqrt{p}), & \text{if } p \equiv 1 \bmod 12 \text{ and } 3 \nmid \alpha, \\ p(p-1)(2p^2 - 5p + 15 + 4\sqrt{p}), & \text{if } p \equiv 1 \bmod 12 \text{ and } 3 \mid \alpha, \end{cases}$$

where $f(x)$ is a polynomial with integral coefficients, α is a constant defined by

$$p = \alpha^2 + \beta^2 = \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + \bar{a}}{p} \right) \right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + r\bar{a}}{p} \right) \right)^2,$$

$\left(\frac{*}{p}\right)$ denotes the Legendre's symbol and r is any quadratic non-residue modulo p. There are many other results on the generalized two-term exponential sums, which will not be listed here but can be found in the literatures (e.g. [6, 8, 9, 10, 16, 17]).

It is worth noting that for the quartic exponential sums, obtaining strong results for general moduli $q \geq 3$ remains challenging. Even in the case of prime moduli $p \geq 3$, no explicit identities exist so far. Given these difficulties, a natural approach is to study the mixed moments of the generalized quartic exponential sum

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^4 + na^t}{p}\right) \right|^4, \tag{1.8}$$

where p is an odd prime, n is any positive integer with $(n, p) = 1$, t denotes an integer satisfying $1 \leq t \leq 3$.

1.2. **Main results.**

This paper examines the fourth moment of the cubic exponential sums and the fourth mixed moments of the quartic exponential sums. By using the properties of third-order character ψ modulo p and quadratic residue, we first derive an explicit expression for $M(\chi_2, \psi)$ in (1.5), which further refines the fourth moments of $C(m, n, 3, 1, \chi_2; p)$ in the case of $(3, p-1) = 3$.

Subsequently, we focus on the mixed moments (1.8) for $t = 1, 2, 3$ respectively. The case of $t = 1$ is trivial, so our main efforts concentrate on the cases of $t = 2$ and $t = 3$. By utilizing the properties of quadratic and the third-order Dirichlet characters, along with the analyses related to the number of solutions to congruence equations, we present the exact identities for $t = 1, 2$, and the corresponding identities and asymptotic formula for $t = 3$ in different conditions. Now, we state our main results.

Theorem 1.1. *Let p be a prime with $(3, p-1) = 3$ and n be any integer with $(n, p) = 1$, then*

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ma^3 + na}{p}\right) \right|^4 = 3p^3 - 14p^2 - 4pd^2,$$

where $\chi_2(*)$ denotes the Legendre's symbol, d is uniquely determined by $d \equiv 1 \bmod 3$ and $4p = d^2 + 27b^2$ $(d, b \in \mathbb{Z})$.

It is easy to deduce from the definition of d that

$$|d| \leq 2\sqrt{p}. \tag{1.9}$$

Combining Theorem 1.1 and (1.9), we can immediately get the following corollary.

Corollary 1.2. *For the primes p with $p \equiv 1 \bmod 3$ and the integers n with $(n, p) = 1$, we have*

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ma^3 + na}{p}\right) \right|^4 = 3p^3 + O(p^2).$$

For the mixed moments of the quartic exponential sums, we have the following results.

Theorem 1.3. *For any prime p and any integer n with $(n, p) = 1$, we have*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^4 + na}{p}\right) \right|^4 = \begin{cases} p(p-1)(2p^2 - 11p + 24), & \text{if } p \equiv 1 \bmod 4, \\ p(p-1)(2p^2 - 7p + 8), & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Theorem 1.4. *For any prime p and any integer n with $(n, p) = 1$, we have*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^4 + na^2}{p}\right) \right|^4$$

$$= \begin{cases} p(p-1)^2 \cdot (4p-8), & \textit{if } p \equiv 3 \bmod 4, \\ p(p-1)(4p^2-16p-12\sqrt{p}\cdot\chi_2(n)+24), & \textit{if } p \equiv 1 \bmod 8, \\ p(p-1)(4p^2-16p+4\sqrt{p}\cdot\chi_2(n)+24), & \textit{if } p \equiv 5 \bmod 8, \end{cases}$$

where $\chi_2(*)$ *denotes the Legendre's symbol.*

Theorem 1.5. *For any prime* p *with* $(3,p-1)=1$ *and any integer* n *with* $(n,p)=1$*, we have*

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+na^3}{p}\right)\right|^4 = \begin{cases} p(p-1)(2p^2-11p+24), & \textit{if } p \equiv 5 \bmod 12, \\ p(p-1)(2p^2-7p+8), & \textit{if } p \equiv 11 \bmod 12. \end{cases}$$

Theorem 1.6. *Let* p *be a prime with* $(3,p-1)=3$*. If* 2 *is a cubic residue modulo* p*, then*

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+a^3}{p}\right)\right|^4$$

$$= \begin{cases} p(p-1)(2p^2+p+24), & \textit{if } p \equiv 1 \bmod 12, \\ p(p-1)(2p^2-7p+8+2A^2-5A), & \textit{if } p \equiv 7 \bmod 12, \end{cases}$$

where $A = \sum_{a=0}^{p-1} e\left(\frac{a^3}{p}\right)$ *is real with* $|A| \le 2\sqrt{p}$*.*

Theorem 1.7. *Let* p *be a prime with* $(3,p-1)=3$*. If* 2 *is not a cubic residue modulo* p*, then we have the asymptotic formula*

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+a^3}{p}\right)\right|^4 = 2p^4 + O(p^3).$$

Notations. Throughout this paper, we adopt the following notations: p always denotes an odd prime, (a,b) denotes the greatest common divisor of integers a and b, χ_0 represents the principal character modulo p, $\chi_2(*) = \left(\frac{*}{p}\right)$ stands for the Legendre's symbol.

2. Several Preliminary Lemmas

To complete the proofs of our main theorems, we require several auxiliary lemmas here. Throughout our arguments, we shall employ fundamental properties of the classical Gauss sum and the third-order characters modulo p. These results are standard in the literature on elementary and analytic number theory (see, for instance, [1]). Therefore, we shall not repeat them here. We begin with the following preliminary results.

Lemma 2.1. *If* p *is a prime with* $(3,p-1)=3$*, then for the quadratic character* χ_2 *and any third-order character* $\overline{\psi}$ mod p*, we have*

$$\tau(\chi_2\overline{\psi}) = \frac{\overline{\psi}(2)\cdot\tau^2(\psi)\cdot\tau(\chi_2)}{p}.$$

Proof. See G. H. Chen and W. P. Zhang [3]. □

Lemma 2.2. *Let* p *be a prime with* $(3,p-1)=3$*. For the third-order characters* ψ *and* $\overline{\psi}$ *modulo* p*, we have the identity*

$$\tau^3(\psi)+\tau^3(\overline{\psi}) = dp,$$

where $\tau(\chi) = \sum_{a=1}^{p-1}\chi(a)e(\frac{a}{p})$ *denotes the classical Gauss sum,* $4p = d^2+27b^2$ $(d,b\in\mathbb{Z})$*,* d *is uniquely determined by* $d \equiv 1 \bmod 3$*.*

Proof. See W. P. Zhang and J. Y. Hu [14] or B. C. Berndt and R. J. Evans [2]. □

Lemma 2.3. *For any prime p, we have the following identities*

$$\sum_{a=1}^{p-1}\sum_{\substack{b=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1} 1 = \begin{cases} 8p-24, & \text{if } p\equiv 1 \bmod 4,\\ 4p-8, & \text{if } p\equiv 3 \bmod 4.\end{cases}$$

Proof. If $p\equiv 3 \bmod 4$, then $a^4\equiv 1 \bmod p$ has two solutions in a reduced residue system modulo p, that is 1 and -1, and hence

$$\sum_{a=1}^{p-1}\sum_{\substack{b=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1} 1 = 2\sum_{\substack{a=1\\a^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1 - \sum_{a=1}^{p-1}\sum_{\substack{b=1\\a^4\equiv b^4\equiv 1 \bmod p}}^{p-1} 1 = 4(p-1)-4 = 4p-8.$$

If $p\equiv 1 \bmod 4$, then $a^4\equiv 1 \bmod p$ has four solutions in a reduced residue system modulo p, that is 1, -1, u, $-u$, where $u\not\equiv \pm 1 \bmod p$, and hence we have

$$\sum_{a=1}^{p-1}\sum_{\substack{b=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1} 1 = 2\sum_{\substack{a=1\\a^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1 - \sum_{a=1}^{p-1}\sum_{\substack{b=1\\a^4\equiv b^4\equiv 1 \bmod p}}^{p-1} 1 = 8(p-1)-16 = 8p-24.$$

□

Lemma 2.4. *For any prime p with $(3,p-1)=1$, we have*

$$\sum_{a=1}^{p-1}\sum_{\substack{b=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p\\(a^3-1)(b^3-1)\equiv 0 \bmod p}}^{p-1} 1 = 2p-3. \tag{2.1}$$

For any prime p with $(3,p-1)=3$, we have

$$\sum_{a=1}^{p-1}\sum_{\substack{b=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p\\(a^3-1)(b^3-1)\equiv 0 \bmod p}}^{p-1} 1 = \begin{cases} 2p+9, & \text{if } p\equiv 1 \bmod 12,\\ 2p+1, & \text{if } p\equiv 7 \bmod 12.\end{cases} \tag{2.2}$$

Proof. If $(3,p-1)=1$, then $a^3\equiv 1 \bmod p$ has only one solution in a reduced residue system modulo p, that is 1, so we have

$$\sum_{a=1}^{p-1}\sum_{\substack{b=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p\\(a^3-1)(b^3-1)\equiv 0 \bmod p}}^{p-1} 1 = 2\cdot\sum_{b=1}^{p-1} 1 - \sum_{\substack{a=1\\a^3\equiv 1 \bmod p}}^{p-1} 1\cdot \sum_{\substack{b=1\\b^3\equiv 1 \bmod p}}^{p-1} 1 = 2p-3. \tag{2.3}$$

If $(3,p-1)=3$, then $a^3\equiv 1 \bmod p$ has three distinct solutions in a reduced residue system modulo p, we denote them by 1, v, v^2. Recalling that $a^4\equiv 1 \bmod p$ has two solutions in a reduced residue system modulo p: 1 and -1 whenever $p\equiv 7 \bmod 12$, so one has

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{\substack{b=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p\\(a^3-1)(b^3-1)\equiv 0 \bmod p}}^{p-1} 1 &= \sum_{a=1}^{p-1}\sum_{\substack{b=1\\(a^2-1)(b^2-1)\equiv 0 \bmod p\\(a-1)(b-1)(a^2+a+1)(b^2+b+1)\equiv 0 \bmod p}}^{p-1} 1 = 2\cdot\sum_{b=1}^{p-1} 1 - 1 + \sum_{a=2}^{p-1}\sum_{\substack{b=2\\(a+1)(b+1)\equiv 0 \bmod p\\(a^2+a+1)(b^2+b+1)\equiv 0 \bmod p}}^{p-1} 1\\ &= 2p-3+2\sum_{\substack{b=2\\b^2+b+1\equiv 0 \bmod p}}^{p-1} 1 = 2p-3+4 = 2p+1.\end{aligned} \tag{2.4}$$

However, $a^4 \equiv 1 \bmod p$ has four solutions in a reduced residue system modulo p: 1, -1, u, $-u$ with $u \neq \pm 1$ whenever $p \equiv 1 \bmod 12$, then we have

$$\begin{aligned}
\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\(a^3-1)(b^3-1)\equiv 0 \bmod p}} 1 =& 2\sum_{b=1}^{p-1} 1 - 1 + \mathop{\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}}_{\substack{(a+1)(b+1)(a^2+1)(b^2+1)\equiv 0 \bmod p\\(a^2+a+1)(b^2+b+1)\equiv 0 \bmod p}} 1\\
=& 2p-3+2\sum_{\substack{b=2\\b^2+b+1\equiv 0 \bmod p}}^{p-1} 1 + \mathop{\sum_{a=2}^{p-2}\sum_{b=2}^{p-2}}_{\substack{(a^2+1)(b^2+1)\equiv 0 \bmod p\\(a^2+a+1)(b^2+b+1)\equiv 0 \bmod p}} 1\\
=& 2p+1+2\left(\sum_{\substack{a=2\\a^2+1\equiv 0 \bmod p}}^{p-2} 1\right)\cdot\left(\sum_{\substack{b=2\\b^2+b+1\equiv 0 \bmod p}}^{p-2} 1\right) - \mathop{\sum_{a=2}^{p-2}\sum_{b=2}^{p-2}}_{\substack{a^2+1\equiv 0 \bmod p\\b^2+1\equiv 0 \bmod p\\ab\equiv 0 \bmod p}} 1\\
=& 2p+1+8 = 2p+9. \qquad (2.5)
\end{aligned}$$

Then Lemma 2.4 follows from (2.3)–(2.5). ☐

Lemma 2.5. *Let* $\overline{\psi}$ *be a third-order character modulo* p*. If* p *is a prime with* $p \equiv 7 \bmod 12$*, then we have*

$$\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p} \overline{\psi}(a^3-1)\cdot\overline{\psi}(b^3-1) = -4\cdot\overline{\psi}(2) - \psi(2) + \frac{2\cdot\overline{\psi}(2)\tau^3(\overline{\psi})}{p}. \qquad (2.6)$$

If p *is a prime with* $p \equiv 1 \bmod 12$*, then we have*

$$\begin{aligned}
&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p} \overline{\psi}(a^3-1)\cdot\overline{\psi}(b^3-1)\\
=& 2\cdot\tau^3(\overline{\psi})\cdot[\overline{\psi}(2)+B(\overline{\psi},u)]/p - 4\cdot[\overline{\psi}(2)+B(\overline{\psi},u)] - \psi(2) - 2\cdot\overline{\psi}(2)\cdot B(\overline{\psi},u) - B^2(\overline{\psi},u),
\end{aligned} \qquad (2.7)$$

where

$$B(\overline{\psi},u) = \overline{\psi}(u-1)+\overline{\psi}(u+1),$$

$2 \le u \le p-2$ *denotes one of solutions of the congruence equation* $a^4 \equiv 1 \bmod p$*.*

Proof. If $p \equiv 7 \bmod 12$, then $a^4 \equiv 1 \bmod p$ has two solutions in a reduced residue system modulo p: 1 and -1. Moreover, by the properties of the third-order character modulo p, we have $\psi^2 = \overline{\psi}$, and hence $\psi(-1) = \overline{\psi}(-1) = 1$, $\tau(\psi)\cdot\tau(\overline{\psi}) = p$, then

$$\begin{aligned}
\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p} \overline{\psi}(a^3-1)\cdot\overline{\psi}(b^3-1) =& \mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^2-1)(b^2-1)\equiv 0 \bmod p} \overline{\psi}(a^3-1)\cdot\overline{\psi}(b^3-1)\\
=2\cdot\overline{\psi}(-2)\sum_{b=1}^{p-1}\overline{\psi}(b^3-1) - \overline{\psi}(-2)\cdot\overline{\psi}(-2) =& 2\cdot\overline{\psi}(2)\sum_{b=1}^{p-1}\overline{\psi}(b^3-1) - \psi(2). \qquad (2.8)
\end{aligned}$$

Meanwhile, by the trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q, & \text{if } \ q \mid n,\\ 0, & \text{if } \ q \nmid n,\end{cases} \qquad (2.9)$$

and the separability of Gauss sum, one has

$$\begin{aligned}
\sum_{b=1}^{p-1}\overline{\psi}(b^3-1) =& \sum_{b=1}^{p-1}(1+\psi(b)+\overline{\psi}(b))\cdot\overline{\psi}(b-1)\\
=&\sum_{b=1}^{p-1}\overline{\psi}(b-1)+\sum_{b=1}^{p-1}\psi(b)\overline{\psi}(b-1)+\sum_{b=1}^{p-1}\overline{\psi}(b)\overline{\psi}(b-1)\\
=&\frac{1}{\tau(\psi)}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\psi(c)e\left(\frac{c(b-1)}{p}\right)+\frac{1}{\tau(\psi)}\sum_{b=1}^{p-1}\psi(b)\sum_{c=1}^{p-1}\psi(c)e\left(\frac{c(b-1)}{p}\right)\\
&+\frac{1}{\tau(\psi)}\sum_{b=1}^{p-1}\overline{\psi}(b)\sum_{c=1}^{p-1}\psi(c)e\left(\frac{c(b-1)}{p}\right)\\
=&\frac{1}{\tau(\psi)}\sum_{c=1}^{p-1}\psi(c)e\left(\frac{-c}{p}\right)\sum_{b=1}^{p-1}e\left(\frac{bc}{p}\right)+\frac{1}{\tau(\psi)}\sum_{c=1}^{p-1}\psi(c)e\left(\frac{-c}{p}\right)\sum_{b=1}^{p-1}\psi(b)e\left(\frac{bc}{p}\right)\\
&+\frac{1}{\tau(\psi)}\sum_{c=1}^{p-1}\psi(c)e\left(\frac{-c}{p}\right)\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{bc}{p}\right)\\
=&-2+\frac{\tau^2(\overline{\psi})}{\tau(\psi)}=-2+\frac{\tau^3(\overline{\psi})}{p}.
\end{aligned}\tag{2.10}$$

Then (2.6) follows from (2.8) and (2.10).

On the other hand, recalling that there are four solutions of $a^4\equiv 1 \bmod p$ in the case of $p\equiv 1 \bmod 12$, that is ±1, $\pm u$ with $u\not\equiv\pm1 \bmod p$; and there are three distinct solutions of $a^3\equiv 1 \bmod p$: 1, v and v^2. Therefore,

$$\begin{aligned}
&\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\overline{\psi}(a^3-1)\cdot\overline{\psi}(b^3-1)=\sum_{\substack{a=2\\(a+1)(b+1)(a^2+1)(b^2+1)\equiv 0 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\overline{\psi}(a^3-1)\cdot\overline{\psi}(b^3-1)\\
=&2\cdot\overline{\psi}(-2)\sum_{b=2}^{p-1}\overline{\psi}(b^3-1)-\overline{\psi}(-2)\cdot\overline{\psi}(-2)+\sum_{\substack{a=2\\(a^2+1)(b^2+1)\equiv 0 \bmod p}}^{p-2}\sum_{b=2}^{p-2}\overline{\psi}(a^3-1)\overline{\psi}(b^3-1)\\
=&2\cdot\overline{\psi}(2)\cdot\left(-2+\frac{\tau^3(\overline{\psi})}{p}\right)-\psi(2)+2\sum_{\substack{a=2\\a^2+1\equiv 0 \bmod p}}^{p-2}\overline{\psi}(a^3-1)\cdot\sum_{b=2}^{p-2}\overline{\psi}(b^3-1)\\
&-\sum_{\substack{a=2\\a^2+1\equiv 0 \bmod p}}^{p-2}\overline{\psi}(a^3-1)\cdot\sum_{\substack{b=2\\b^2+1\equiv 0 \bmod p}}^{p-2}\overline{\psi}(b^3-1)\\
=&2\cdot\overline{\psi}(2)\cdot\frac{\tau^3(\overline{\psi})}{p}-4\cdot\overline{\psi}(2)-\psi(2)+2\sum_{\substack{a=2\\a^2+1\equiv 0 \bmod p}}^{p-2}\overline{\psi}(a-1)\cdot\left(\sum_{b=1}^{p-1}\overline{\psi}(b^3-1)-\overline{\psi}(-2)\right)\\
&-\sum_{\substack{a=2\\a^2+1\equiv 0 \bmod p}}^{p-2}\overline{\psi}(a-1)\cdot\sum_{\substack{b=2\\b^2+1\equiv 0 \bmod p}}^{p-2}\overline{\psi}(b-1)\\
=&2\cdot\overline{\psi}(2)\cdot\frac{\tau^3(\overline{\psi})}{p}-4\cdot\overline{\psi}(2)-\psi(2)+2\cdot\left(\overline{\psi}(u-1)+\overline{\psi}(-u-1)\right)\cdot\left(-2+\frac{\tau^3(\overline{\psi})}{p}-\overline{\psi}(2)\right)
\end{aligned}$$

$$-\Big(\overline{\psi}(u-1)+\overline{\psi}(-u-1)\Big)^2. \tag{2.11}$$

For convenience, we let

$$B(\overline{\psi},u)=\overline{\psi}(u-1)+\overline{\psi}(-u-1)=\overline{\psi}(u-1)+\overline{\psi}(u+1). \tag{2.12}$$

Then (2.7) follows from (2.11) and (2.12). □

3. Proof of Theorem 1.1

In this section, we shall give an exact identity of $M(\chi_2,\psi)$ in (1.5). Then combining $M(\chi_2,\psi)$ and (1.4), we can obtain the fourth moment of $C(m,n,3,1,\chi_2;p)$ in the case of $(3,p-1)=3$ directly.

If $(3,p-1)=3$, from the properties of quadratic residue and the third-order character ψ modulo p, we have

$$\begin{aligned}&\sum_{a=1}^{p-1}\chi_2(a)\psi(a^3-1)=\sum_{a=1}^{p-1}\chi_2(a^3)\psi(a^3-1)=\sum_{a=1}^{p-1}\big(1+\psi(a)+\overline{\psi}(a)\big)\cdot\chi_2(a)\psi(a-1)\\&=\sum_{a=1}^{p-1}\chi_2(a)\psi(a-1)+\sum_{a=1}^{p-1}\chi_2(a)\psi(a)\psi(a-1)+\sum_{a=1}^{p-1}\chi_2(a)\overline{\psi}(a)\psi(a-1).\end{aligned} \tag{3.1}$$

Since all non-principal characters modulo p are primitive, by the separability of Gauss sum, we have

$$\begin{aligned}\sum_{a=1}^{p-1}\chi_2(a)\psi(a-1)=&\frac{1}{\tau(\overline{\psi})}\sum_{a=1}^{p-1}\chi_2(a)\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{b(a-1)}{p}\right)\\=&\frac{1}{\tau(\overline{\psi})}\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{-b}{p}\right)\cdot\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ba}{p}\right)\\=&\frac{\tau(\chi_2)}{\tau(\overline{\psi})}\sum_{b=1}^{p-1}\chi_2(b)\overline{\psi}(b)e\left(\frac{-b}{p}\right)\\=&\chi_2(-1)\cdot\frac{\tau(\chi_2)\tau(\chi_2\overline{\psi})}{\tau(\overline{\psi})}.\end{aligned} \tag{3.2}$$

$$\begin{aligned}\sum_{a=1}^{p-1}\chi_2(a)\psi(a)\psi(a-1)=&\frac{1}{\tau(\overline{\psi})}\sum_{a=1}^{p-1}\chi_2(a)\psi(a)\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{b(a-1)}{p}\right)\\=&\frac{1}{\tau(\overline{\psi})}\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{-b}{p}\right)\cdot\sum_{a=1}^{p-1}\chi_2(a)\psi(a)e\left(\frac{ba}{p}\right)\\=&\frac{\tau(\chi_2\psi)}{\tau(\overline{\psi})}\sum_{b=1}^{p-1}\chi_2(b)\psi(b)e\left(\frac{-b}{p}\right)\\=&\chi_2(-1)\cdot\frac{\tau^2(\chi_2\psi)}{\tau(\overline{\psi})}.\end{aligned} \tag{3.3}$$

$$\sum_{a=1}^{p-1}\chi_2(a)\overline{\psi}(a)\psi(a-1)=\frac{1}{\tau(\overline{\psi})}\sum_{a=1}^{p-1}\chi_2(a)\overline{\psi}(a)\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{b(a-1)}{p}\right)$$

$$
\begin{aligned}
&=\frac{1}{\tau(\overline{\psi})}\sum_{b=1}^{p-1}\overline{\psi}(b)e\left(\frac{-b}{p}\right)\cdot\sum_{a=1}^{p-1}\chi_2(a)\overline{\psi}(a)e\left(\frac{ba}{p}\right)\\
&=\frac{\tau(\chi_2\overline{\psi})}{\tau(\overline{\psi})}\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{-b}{p}\right)\\
&=\chi_2(-1)\cdot\frac{\tau(\chi_2)\tau(\chi_2\overline{\psi})}{\tau(\overline{\psi})}. \qquad (3.4)
\end{aligned}
$$

Then by Lemma 2.1, Lemma 2.2, (3.1)–(3.4), and recalling that $\chi_2(-1)\cdot\tau^2(\chi_2)=|\tau(\chi_2)|^2=p$, $\psi(-1)=\overline{\psi}(-1)=1$, $\tau(\psi)\cdot\tau(\overline{\psi})=|\tau(\psi)|^2=p$, we can easily deduce

$$
\begin{aligned}
&\sum_{a=1}^{p-1}\chi_2(a)\psi(a^3-1)=\frac{2\chi_2(-1)\tau^2(\chi_2)\cdot\overline{\psi}(2)\tau^2(\psi)}{p\tau(\overline{\psi})}+\frac{\chi_2(-1)\tau^2(\chi_2)\cdot\overline{\psi}(2)\tau^4(\overline{\psi})}{p^2\cdot\tau(\overline{\psi})}\\
=&\frac{2\overline{\psi}(2)\cdot\tau^2(\psi)}{\tau(\overline{\psi})}+\frac{\overline{\psi}(2)\tau^3(\overline{\psi})}{p}=\frac{2\overline{\psi}(2)\cdot\tau^3(\psi)}{p}+\frac{\overline{\psi}(2)\tau^3(\overline{\psi})}{p}=\frac{\overline{\psi}(2)}{p}\cdot\left(2\tau^3(\psi)+\tau^3(\overline{\psi})\right)\\
=&\frac{\overline{\psi}(2)}{p}\cdot\left(\tau^3(\psi)+dp\right). \qquad (3.5)
\end{aligned}
$$

Combining the definition of $M(\chi_2,\psi)$ in (1.5) and (3.5), we have

$$
\begin{aligned}
M(\chi_2,\psi)=&\left|\sum_{a=1}^{p-1}\chi_2(a)\psi(a^3-1)\right|^2=\frac{1}{p^2}\cdot\left[(\tau^3(\psi)+dp)\cdot(\tau^3(\overline{\psi})+dp)\right]\\
=&\frac{1}{p^2}\cdot\left[p^3+dp\cdot(\tau^3(\psi)+\tau^3(\overline{\psi}))+d^2p^2\right]=\frac{1}{p^2}\cdot(p^3+2d^2p^2)=p+2d^2. \qquad (3.6)
\end{aligned}
$$

Then Theorem 1.1 follows from (1.4) and (3.6).

4. **Proofs of Theorems 1.3–1.4**

First, applying the trigonometric identity (2.9) and the orthogonality of characters modulo p, we have

$$
\begin{aligned}
&\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+na}{p}\right)\right|^4\\
=&p(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ a^4+b^4\equiv c^4+d^4 \bmod p\\ ab\equiv cd \bmod p}}^{p-1}e\left(\frac{n(a+b-c-d)}{p}\right)\\
=&p(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ a^4+b^4\equiv c^4+1 \bmod p\\ ab\equiv c \bmod p}}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{nd(a+b-c-1)}{p}\right)\\
=&p(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ a^4+b^4\equiv a^4c^4+1 \bmod p\\ b\equiv c \bmod p}}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{nd(a+b-ac-1)}{p}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=p(p-1)\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{nd(a-1)(1-b)}{p}\right)\\
&=p^2\cdot(p-1)\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p\\(a-1)(b-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1-p(p-1)\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1. \qquad (4.1)
\end{aligned}
$$

It is easy to show that

$$
\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p\\(a-1)(b-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1=\sum_{\substack{a=1\\(a-1)(b-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1=2(p-1)-1=2p-3. \qquad (4.2)
$$

Then Theorem 1.3 follows from (4.1), (4.2) and Lemma 2.3. □

Similarly, we have

$$
\begin{aligned}
&\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+na^2}{p}\right)\right|^4\\
&=p(p-1)\sum_{\substack{a=1\\a^4+b^4\equiv c^4+d^4 \bmod p\\ab\equiv cd \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{n(a^2+b^2-c^2-d^2)}{p}\right)\\
&=p(p-1)\sum_{\substack{a=1\\a^4+b^4\equiv c^4+1 \bmod p\\ab\equiv c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{nd^2(a^2+b^2-c^2-1)}{p}\right)\\
&=p(p-1)\sum_{\substack{a=1\\a^4+b^4\equiv a^4c^4+1 \bmod p\\ab\equiv ac \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{nd^2(a^2+b^2-a^2c^2-1)}{p}\right)\\
&=p(p-1)\sum_{\substack{a=1\\a^4+b^4\equiv a^4c^4+1 \bmod p\\b\equiv c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{nd^2(a^2+b^2-a^2c^2-1)}{p}\right)\\
&=p(p-1)\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{nd^2(a^2-1)(1-b^2)}{p}\right). \qquad (4.3)
\end{aligned}
$$

By the properties of Gauss sum, we have

$$
\begin{aligned}
&\sum_{d=1}^{p-1} e\left(\frac{nd^2(a^2-1)(1-b^2)}{p}\right)\\
&=\begin{cases} p-1, & \text{if } (a^2-1)(b^2-1)\equiv 0 \bmod p,\\ -1+\tau(\chi_2)\cdot\chi_2(n)\chi_2(a^2-1)\chi_2(1-b^2), & \text{if } (a^2-1)(b^2-1)\not\equiv 0 \bmod p.\end{cases} \qquad (4.4)
\end{aligned}
$$

It follows from (4.3) and (4.4) that

$$\begin{aligned}
&\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+na^2}{p}\right)\right|^4\\
=&p^2\cdot(p-1)\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^2-1)(b^2-1)\equiv 0 \bmod p}} 1-p(p-1)\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p} 1\\
&+p(p-1)\cdot\tau(\chi_2)\chi_2(n)\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^2-1)(b^2-1)\not\equiv 0 \bmod p}}\chi_2(a^2-1)\chi_2(1-b^2). \qquad (4.5)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^2-1)(b^2-1)\equiv 0 \bmod p}} 1=&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^2-1)(b^2-1)\equiv 0 \bmod p} 1=2\sum_{\substack{a=1\\ a^2\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1-\sum_{\substack{a=1\\ a^2\equiv 1 \bmod p}}^{p-1}1\cdot\sum_{\substack{b=1\\ b^2\equiv 1 \bmod p}}^{p-1}1\\
=&4(p-1)-4=4p-8. \qquad (4.6)
\end{aligned}$$

If $p\equiv 3 \bmod 4$, there are two trivial solutions of $a^4\equiv 1 \bmod p$: 1 and -1. Thus,

$$\begin{aligned}
&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^2-1)(b^2-1)\not\equiv 0 \bmod p}}\chi_2(a^2-1)\chi_2(1-b^2)\\
=&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p}\chi_2(a^2-1)\chi_2(1-b^2)-\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^2-1)(b^2-1)\equiv 0 \bmod p}}\chi_2(a^2-1)\chi_2(1-b^2)\\
=&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p}\chi_2(a^2-1)\chi_2(1-b^2)=\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^2-1)(b^2-1)\equiv 0 \bmod p}\chi_2(a^2-1)\chi_2(1-b^2)=0. \qquad (4.7)
\end{aligned}$$

If $p\equiv 1 \bmod 4$, then $\chi_2(-1)=1$, and there are four solutions of $a^4\equiv 1 \bmod p$: 1, -1, u and $-u$, where $u\not\equiv\pm 1 \bmod p$, so we have

$$\begin{aligned}
&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^2-1)(b^2-1)\not\equiv 0 \bmod p}}\chi_2(a^2-1)\chi_2(1-b^2)=\mathop{\sum_{a=2}^{p-2}\sum_{b=2}^{p-2}}_{(a^2+1)(b^2+1)\equiv 0 \bmod p}\chi_2(a^2-1)\chi_2(b^2-1)\\
=&2\sum_{\substack{a=2\\ a^2+1\equiv 0 \bmod p}}^{p-2}\chi_2(a^2-1)\cdot\sum_{b=2}^{p-2}\chi_2(b^2-1)-\sum_{\substack{a=2\\ a^2+1\equiv 0 \bmod p}}^{p-2}\chi_2(a^2-1)\cdot\sum_{\substack{b=2\\ b^2+1\equiv 0 \bmod p}}^{p-2}\chi_2(b^2-1).
\end{aligned}$$

$$(4.8)$$

Moreover, one has

$$\sum_{\substack{a=2\\ a^2+1\equiv 0 \bmod p}}^{p-2}\chi_2(a^2-1)=\chi_2(-2)\cdot\sum_{\substack{a=2\\ a^2\equiv -1 \bmod p}}^{p-2}1=2\cdot\chi_2(2)=\begin{cases}2, & \text{if } p\equiv 1 \bmod 8,\\ -2, & \text{if } p\equiv 5 \bmod 8,\end{cases} \qquad (4.9)$$

and

$$\sum_{b=2}^{p-2}\chi_2(b^2-1)=\sum_{b=1}^{p-1}\chi_2(b^2-1)=\sum_{b=1}^{p-2}\chi_2(b)\chi_2(b+2)=\frac{1}{\tau(\chi_2)}\cdot\sum_{b=1}^{p-2}\chi_2(b)\sum_{c=1}^{p-1}\chi_2(c)e\left(\frac{c(b+2)}{p}\right)$$
$$=\frac{1}{\tau(\chi_2)}\cdot\sum_{c=1}^{p-1}\chi_2(c)e\left(\frac{2c}{p}\right)\cdot\sum_{b=1}^{p-2}\chi_2(b)e\left(\frac{bc}{p}\right)=\sum_{c=1}^{p-1}e\left(\frac{2c}{p}\right)-\frac{1}{\tau(\chi_2)}\sum_{c=1}^{p-1}\chi_2(c)e\left(\frac{c}{p}\right)=-2. \tag{4.10}$$

From (4.8)–(4.10), we get

$$\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p\\(a^2-1)(b^2-1)\not\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi_2(a^2-1)\chi_2(1-b^2)=\begin{cases}-12, & \text{if } p\equiv 1 \bmod 8,\\ 4, & \text{if } p\equiv 5 \bmod 8.\end{cases} \tag{4.11}$$

Recall that

$$\tau(\chi_2)=\sqrt{p},\quad \text{if } p\equiv 1 \bmod 4. \tag{4.12}$$

It follows from (4.5), (4.6), Lemma 2.3 and (4.7) that for any prime p with $p\equiv 3 \bmod 4$,

$$\begin{aligned}\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+na^2}{p}\right)\right|^4 =&p^2\cdot(p-1)(4p-8)-p(p-1)(4p-8)\\ =&p(p-1)^2\cdot(4p-8).\end{aligned} \tag{4.13}$$

And we can gain from (4.5), (4.6), Lemma 2.3, (4.11) and (4.12) that

$$\begin{aligned}&\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+na^2}{p}\right)\right|^4\\ =&\begin{cases}p(p-1)(4p^2-16p-12\sqrt{p}\cdot\chi_2(n)+24), & \text{if } p\equiv 1 \bmod 8,\\ p(p-1)(4p^2-16p+4\sqrt{p}\cdot\chi_2(n)+24), & \text{if } p\equiv 5 \bmod 8.\end{cases}\end{aligned} \tag{4.14}$$

Then Theorem 1.4 follows from (4.13) and (4.14). □

5. Proofs of Theorems 1.5–1.7

By the trigonometric identity (2.9) and the orthogonality of characters modulo p, we have

$$\begin{aligned}&\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+na^3}{p}\right)\right|^4\\ =&p(p-1)\sum_{\substack{a=1\\a^4+b^4\equiv c^4+d^4 \bmod p\\ab\equiv cd \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{n(a^3+b^3-c^3-d^3)}{p}\right)\\ =&p(p-1)\sum_{\substack{a=1\\a^4+b^4\equiv c^4+1 \bmod p\\ab\equiv c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{nd^3(a^3+b^3-c^3-1)}{p}\right)\end{aligned}$$

$$
\begin{aligned}
=&p(p-1)\sum_{\substack{a=1\\ a^4+b^4\equiv a^4c^4+1 \bmod p\\ b\equiv c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{nd^3(a^3+b^3-a^3c^3-1)}{p}\right)\\
=&p(p-1)\sum_{\substack{a=1\\ (a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{nd^3(a^3-1)(1-b^3)}{p}\right). \qquad (5.1)
\end{aligned}
$$

5.1. **Case I.** $(3, p-1) = 1$.

In the case of $(3, p-1) = 1$, if d passes through a reduced residue system mod p, then d^3 also passes through a reduced residue system mod p. Recalling that $(n, p) = 1$, so we have

$$
\begin{aligned}
&\sum_{d=1}^{p-1} e\left(\frac{nd^3(a^3-1)(1-b^3)}{p}\right)=\sum_{d=1}^{p-1} e\left(\frac{nd(a^3-1)(1-b^3)}{p}\right)\\
=&\begin{cases} p-1, & \text{if } (a^3-1)(b^3-1)\equiv 0 \bmod p,\\ -1, & \text{if } (a^3-1)(b^3-1)\not\equiv 0 \bmod p. \end{cases} \qquad (5.2)
\end{aligned}
$$

Then from (2.1) in Lemma 2.4, Lemma 2.3, (5.1) and (5.2), one can derive that

$$
\begin{aligned}
&\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4+na^3}{p}\right)\right|^4\\
=&p^2\cdot(p-1)\sum_{\substack{a=1\\ (a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^3-1)(b^3-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1 - p(p-1)\sum_{\substack{a=1\\ (a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1\\
=&\begin{cases} p(p-1)(2p^2-11p+24), & \text{if } p\equiv 5 \bmod 12,\\ p(p-1)(2p^2-7p+8), & \text{if } p\equiv 11 \bmod 12. \end{cases} \qquad (5.3)
\end{aligned}
$$

Therefore, we complete the proof of Theorem 1.5. □

5.2. **Case II.** $(3, p-1) = 3$.

If $(a^3-1)(b^3-1)\equiv 0 \bmod p$, then we have

$$
\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)=p-1. \qquad (5.4)
$$

If $(a^3-1)(b^3-1)\not\equiv 0 \bmod p$, then by the properties of cubic residue and Gauss sum, one has

$$
\begin{aligned}
&\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)=\sum_{d=1}^{p-1}(1+\psi(d)+\overline{\psi}(d))e\left(\frac{d(a^3-1)(1-b^3)}{p}\right)\\
=&\sum_{d=1}^{p-1} e\left(\frac{d(a^3-1)(1-b^3)}{p}\right)+\sum_{d=1}^{p-1}(\psi(d)+\overline{\psi}(d))e\left(\frac{d(a^3-1)(1-b^3)}{p}\right)\\
=&-1+\tau(\psi)\cdot\overline{\psi}(a^3-1)\overline{\psi}(b^3-1)+\tau(\overline{\psi})\cdot\psi(a^3-1)\psi(b^3-1). \qquad (5.5)
\end{aligned}
$$

It is easy to show from (5.4) and (5.5) that

$$
\begin{aligned}
&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p}\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)\\
=&p\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^3-1)(b^3-1)\equiv 0 \bmod p}} 1-\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p} 1+\tau(\psi)\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^3-1)(b^3-1)\not\equiv 0 \bmod p}}\overline{\psi}(a^3-1)\overline{\psi}(b^3-1)\\
&+\tau(\overline{\psi})\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^3-1)(b^3-1)\not\equiv 0 \bmod p}}\psi(a^3-1)\psi(b^3-1)\\
=&p\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{\substack{(a^4-1)(b^4-1)\equiv 0 \bmod p\\ (a^3-1)(b^3-1)\equiv 0 \bmod p}} 1-\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p} 1+\tau(\psi)\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p}\overline{\psi}(a^3-1)\overline{\psi}(b^3-1)\\
&+\tau(\overline{\psi})\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p}\psi(a^3-1)\psi(b^3-1).
\end{aligned}\tag{5.6}
$$

5.2.1. $p\equiv 7 \bmod 12$.

In this case, by Lemma 2.3, (2.2) in Lemma 2.4, (2.6) in Lemma 2.5 and (5.6), we have

$$
\begin{aligned}
&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p}\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)\\
=&p\cdot(2p+1)-(4p-8)+\tau(\psi)\cdot\left(-4\cdot\overline{\psi}(2)-\psi(2)+\frac{2\cdot\overline{\psi}(2)\tau^3(\overline{\psi})}{p}\right)\\
&+\tau(\overline{\psi})\cdot\left(-4\cdot\psi(2)-\overline{\psi}(2)+\frac{2\cdot\psi(2)\tau^3(\psi)}{p}\right)\\
=&2p^2-3p+8-4\cdot[\overline{\psi}(2)\tau(\psi)+\psi(2)\tau(\overline{\psi})]+2\cdot[\overline{\psi}(2)\cdot\tau^2(\overline{\psi})+\psi(2)\cdot\tau^2(\psi)]\\
&-[\psi(2)\tau(\psi)+\overline{\psi}(2)\tau(\overline{\psi})].
\end{aligned}\tag{5.7}
$$

If 2 is a cubic residue mod p, then $\psi(2)=\overline{\psi}(2)=1$. From (5.7) we get

$$
\begin{aligned}
&\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p}\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)\\
=&2p^2-3p+8-5\cdot[\tau(\psi)+\tau(\overline{\psi})]+2\cdot[\tau(\psi)+\tau(\overline{\psi})]^2-4p\\
=&2p^2-7p+8-5A+2A^2,
\end{aligned}\tag{5.8}
$$

where $A=\tau(\psi)+\tau(\overline{\psi})=\sum_{a=0}^{p-1} e\left(\frac{a^3}{p}\right)$ is real with $|A|\le 2\sqrt{p}$.

If 2 is not a cubic residue modulo p, then from (5.7) and $|\tau(\psi)|=\left|\tau(\overline{\psi})\right|=\sqrt{p}$, we have

$$
\mathop{\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}}_{(a^4-1)(b^4-1)\equiv 0 \bmod p}\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)=2p^2+O(p).\tag{5.9}
$$

5.2.2. $p \equiv 1 \bmod 12$.

In this case, by Lemma 2.3, (2.2) in Lemma 2.4, (2.7) in Lemma 2.5, (5.6), and the fact $|B(\psi,u)| = \left|B(\overline{\psi},u)\right| \le 2$, we have

$$\begin{aligned}
&\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)\\
=&p\cdot(2p+9)-(8p-24)+2\cdot\tau^2(\overline{\psi})\cdot[\overline{\psi}(2)+B(\overline{\psi},u)]-4\cdot\tau(\psi)\cdot[\overline{\psi}(2)+B(\overline{\psi},u)]\\
&-\psi(2)\tau(\psi)-2\cdot\overline{\psi}(2)\tau(\psi)\cdot B(\overline{\psi},u)-\tau(\psi)\cdot B^2(\overline{\psi},u)+2\cdot\tau^2(\psi)\cdot[\psi(2)+B(\psi,u)]\\
&-4\cdot\tau(\overline{\psi})\cdot[\psi(2)+B(\psi,u)]-\overline{\psi}(2)\tau(\overline{\psi})-2\cdot\psi(2)\tau(\overline{\psi})\cdot B(\psi,u)-\tau(\overline{\psi})\cdot B^2(\psi,u)\\
=&2p^2+p+24+2\cdot[\overline{\psi}(2)\tau^2(\overline{\psi})+\psi(2)\tau^2(\psi)]-4\cdot[\overline{\psi}(2)\tau(\psi)+\psi(2)\tau(\overline{\psi})]\\
&-[\overline{\psi}(2)\tau(\overline{\psi})+\psi(2)\tau(\psi)]+2\cdot[B(\overline{\psi},u)\cdot\tau^2(\overline{\psi})+B(\psi,u)\cdot\tau^2(\psi)]\\
&-4\cdot[B(\overline{\psi},u)\tau(\psi)+B(\psi,u)\tau(\overline{\psi})]-2\cdot[\overline{\psi}(2)B(\overline{\psi},u)\cdot\tau(\psi)+\psi(2)B(\psi,u)\cdot\tau(\overline{\psi})]\\
&-[B^2(\overline{\psi},u)\cdot\tau(\psi)+B^2(\psi,u)\cdot\tau(\overline{\psi})]
\end{aligned}\tag{5.10}$$

Noting that $u^2+1\equiv 0 \bmod p$, $2\le u\le p-2$, so we have

$$\begin{cases}\psi(u)=\overline{\psi}(u^2)=\overline{\psi}(-1)=1,\\[2ex] \overline{\psi}(u)=\psi(u^2)=\psi(-1)=1.\end{cases}\tag{5.11}$$

From (2.12) we know $\overline{B(\psi,u)}=B(\overline{\psi},u)$. If 2 is a cubic residue modulo p, then

$$\begin{aligned}
|B(\psi,u)|^2=&[\psi(u-1)+\psi(u+1)]\cdot[\overline{\psi}(u-1)+\overline{\psi}(u+1)]\\
=&2+\psi(u-1)\overline{\psi}(u+1)+\psi(u+1)\overline{\psi}(u-1)\\
=&2+\psi(u-1)\cdot\psi^2(u+1)+\psi(u+1)\cdot\psi^2(u-1)\\
=&2+\psi(u^3+u^2-u-1)+\psi(u^3-u^2-u+1)\\
=&2+\psi(u^3+2u^2-u)+\psi(u^3-2u^2-u)\\
=&2+\psi(u^2+2u-1)+\psi(u^2-2u-1)\\
=&2+\psi(2u^2+2u)+\psi(2u^2-2u)\\
=&2+\psi(u+1)+\psi(u-1)\\
=&2+B(\psi,u).
\end{aligned}\tag{5.12}$$

From (5.12), we can immediately deduce

$$B(\psi,u)=B(\overline{\psi},u)=-1,\ \text{ if 2 is a cubic residue modulo } p.\tag{5.13}$$

Then by (5.10) and (5.12), we can derive that

$$\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)=2p^2+p+24.\tag{5.14}$$

If 2 is not a cubic residue modulo p, then from (5.10), $|\tau(\psi)|=\left|\tau(\overline{\psi})\right|=\sqrt{p}$ and $|B(\psi,u)|=\left|B(\overline{\psi},u)\right|\le 2$, we have

$$\sum_{\substack{a=1\\(a^4-1)(b^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{d^3(a^3-1)(1-b^3)}{p}\right)=2p^2+O(p).\tag{5.15}$$

Therefore, Theorem 1.6 follows from (5.1), (5.8) and (5.14). Theorem 1.7 follows from (5.1), (5.9) and (5.15). ☐

Acknowledgement

The authors would like to express the most sincere gratitude to Professor Wenguang Zhai for his valuable advice and constant encouragement. This work is partially supported by National Natural Science Foundations of China (Grant Nos. 12471009, 12301006) and partially supported by Beijing Natural Science Foundation (Grant No. 1242003).

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