Order Structure of the Space of Partial Integral Operators

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Abstract. It is known that the space of kernel operators forms a band in the Dedekind complete vector lattice of order bounded operators, acting in ideal spaces of measurable functions. In this paper we investigate the order structure of the space of partial integral operators. We prove that the space of all absolute partial integral operators forms a band in the Dedekind complete vector lattice of order bounded operators acting in order dense ideal spaces of measurable functions.

Keywords: Kernel operator, Partial integral operator, Order bounded operator, Vector lattice.

1. Introduction

The theory of partial integral operators has numerous applications in many areas of mathematics (see [1, 2, 3]). Various properties of these operators are studied in [4, 5, 7, 8]. It is well known that kernel operators forms a band in Dedekind complete vector lattice of order bounded operators, acting in ideal spaces of measurable functions (see for example [9, 10]). The aim of this work is to investigate the order structure of the space of partial integral operators, acting in order dense ideal spaces of measurable functions. In Theorem 3.7 we prove that the space of all absolute partial integral operators forms a band in the Dedekind complete vector lattice of order bounded operators acting in order dense ideal spaces of measurable functions.

We use the standard notation and terminology of Aliprantis and Burkinshaw [11] for the theory of vector lattices (see also Meyer-Nieberg [9], Zaanen [10]). Throughout the text we assume that all vector spaces are defined over the field of reals and all vector lattices are Archimedean. We let $:=$ denote the assignment by definition, while $\mathbb{N}$ and $\mathbb{R}$ symbolize the naturals and the reals.

2. Preliminaries

In this section we give some necessary preliminaries for further discussion. Let (Ω, Σ, μ) and $(S, \mathcal{F}, m)$ be measure spaces with σ-finite complete measures μ and m respectively, and $(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$ be their product. We will denote by $\mathcal{L}^0(\mu) := \mathcal{L}^0(\Omega, \Sigma, \mu)$ the set of all real-valued μ-a.e. finite Σ-measurable functions on Ω, and $L^0(\mu) := L^0(\Omega, \Sigma, \mu)$ denotes the collection of all equivalence classes of functions from $\mathcal{L}^0(\mu)$. As usual, functions differing on a set of measure zero are called equivalent. Since μ is σ-finite measure, in view of [9, Lemma 2.6] $L^0(\mu)$ is a super Dedekind complete vector lattice under the μ-a.e. pointwise algebraic and lattice operations. In particular, for an arbitrary $f \in L^0(\mu)$ the symbol $f \geqslant 0$ means that $f(\omega) \geqslant 0$ for μ-almost every $\omega \in \Omega$. We identify an arbitrary equivalence class $\tilde{f} \in L^0(\mu)$ containing a function $f \in \mathcal{L}^0(\mu)$ with f. Recall that a linear subspace E is called *an ideal space in* $L^0(\mu)$ if for any $f \in L^0(\mu)$ and $g \in E$ the inequality $|f| \leqslant |g|$ implies $f \in E$. A linear subspace E is called *an order dense* if for any $0 < f \in L^0(\mu)$ there exists an element $g \in E$ such that $0 < g \leqslant f$.

Further E and F will denote arbitrary order dense ideal spaces in $L^0(\mu \otimes m) := L^0(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$, $E_+ := \{f \in E : \ f \geqslant 0\}$. A linear operator $T : E \to F$ is called *positive* (in symbol $T \geqslant 0$) if $Tf \in F_+$ for all $f \in E_+$, and *regular* if it can be written as a difference of two positive operators. The set $L^r(E, F)$ of all regular operators from E to F is a Dedekind complete vector lattice. The symbol $L(E, F)_+$ denotes the set of all positive operators from E to F.

Let $(f_n)_{n=1}^{\infty}$ be a sequence in E and $f \in E$. The symbol $0 \leqslant f_n \uparrow f$ means that $0 \leqslant f_n(\omega, t) \leqslant f_{n+1}(\omega, t)$ and $f(\omega, t) = \lim_n f(\omega, t) = \sup_n f_n(\omega, t)$ for

$\mu\otimes m$-almost every $(\omega, t) \in \Omega\times S$ and for all $n \in \mathbb{N}$. Similarly, the symbol $f_n \downarrow f$ means that $f_n(\omega, t) \geqslant f_{n+1}(\omega, t)$ and $f(\omega, t) = \lim_n f(\omega, t) = \inf_n f_n(\omega, t)$ for $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$ and for all $n \in \mathbb{N}$.

Definition 2.1. A linear operator $T : E \to F$ is said to be *partial integral*, if there exists a measurable function $k \in \mathcal{L}^0(\Omega\times S\times S, \Sigma\otimes\mathcal{F}\otimes\mathcal{F}, \mu\otimes m\otimes m)$ such that

$$Tf(\omega, t) = \int\limits_S k(\omega, t, s)f(\omega, s)dm(s)$$

holds for all $f \in E$ and $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$. The function k is called the *kernel* of T.

Definition 2.2. A linear operator $T : E \to F$ is said to be $L^\infty(\Omega, \Sigma, \mu)$-*homogeneous*, if $T(hf) = hTf$ for all $f \in E$ and $h \in L^\infty(\Omega, \Sigma, \mu)$. Clearly, every partial integral operator is $L^\infty(\mu)$-homogeneous.

When is a positive $L^\infty(\Omega, \Sigma, \mu)$-homogeneous operator from E to F partial integral? The answer is given in the following result.

Theorem 2.3. *Let (Ω, Σ, μ), $(S, \mathcal{F}, m)$ be measure spaces with σ-finite measures μ and m respectively. Let E, F be order dense ideal spaces in $L^0(\Omega\times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$ and $T : E \to F$ be a positive $L^\infty(\Omega, \Sigma, \mu)$-homogeneous operator. Then the following are equivalent:*

(1) *There exists a measurable function $k \in \mathcal{L}^0(\Omega \times S \times S, \Sigma \otimes \mathcal{F} \otimes \mathcal{F})$ such that $k \geqslant 0$ $\mu \otimes m \otimes m$-a.e. and*

$$Tf(\omega, t) = \int_S k(\omega, t, s)f(\omega, s)\ dm(s) \tag{1}$$

holds for all $f \in E$ and $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$.

(2) *For any sequence $(f_n)_{n=1}^\infty \subset E$ such that $0 \leqslant f_n \leqslant f \in E$ for all $n \in \mathbb{N}$, $f_n \to 0$ in measure $\mu \otimes m$ on every subset of finite measure, and for any set $C \in \Sigma \otimes \mathcal{F}$ such that $\chi_C Tf \in L^1(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$*

$$\int_\Omega \chi_C(\omega, t)Tf_n(\omega, t)\ d\mu(\omega) \to 0$$

holds for m-almost every $t \in S$.

◁ The proof given in [12, Theorem 1]. ▷

3. Main Results

In this section we investigate the order structure of the space of partial integral operators. As in section 2, E and F will denote arbitrary order dense ideal spaces in $L^0(\mu \otimes m) := L^0(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$.

Lemma 3.1. *Let T be a partial integral operator from E into F with kernel $k \in \mathcal{L}^0(\Omega \times S \times S, \Sigma \otimes \mathcal{F} \otimes \mathcal{F})$. Then the following holds:*

(i) T is positive if and only if $k \geqslant 0$ holds $\mu \otimes m \otimes m$-a.e. on $\Omega \times S \times S$.

(ii) $T = 0$ if and only if $k = 0$ holds $\mu \otimes m \otimes m$-a.e. on $\Omega \times S \times S$.

$\lhd$ (i) It is evident that if $k(\omega, t, s) \geqslant 0$ holds $\mu \otimes m \otimes m$-almost everywhere on $\Omega \times S \times S$ then T is positive. For the converse, assume that T is positive. By [10, Theorem 86.2] there exists a sequence $(C_n)_{n=1}^\infty \subset \Sigma \otimes \mathcal{F}$ such that $C_1 \subset C_2 \subset \ldots$, $\bigcup_{n=1}^\infty C_n = \Omega \times S$, $\mu \otimes m(C_n) < \infty$ and $\chi_{C_n} \in E$ for all $n \in \mathbb{N}$. For an arbitrary $n \in \mathbb{N}$ denote by T_n the operator from E to F defined by

$$T_n f(\omega, t) = \int_S \chi_{C_n}(\omega, t) k(\omega, t, s) f(\omega, s)\ dm(s)$$

for all $f \in E$ and for $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$. Then T_n is positive partial integral operator from E to F with kernel $\chi_{C_n} k$ and $1_{\Omega \times S} \in E$, where $1_{\Omega \times S}$ is the equivalence class of the function identically equal to one everywhere on $\Omega \times S$. Since $\lim_n \chi_{C_n}(\omega, t) k(\omega, t, s) = k(\omega, t, s)$ for all $(\omega, t, s) \in \Omega \times S \times S$ it is enough to show that $\chi_{C_n} k \geqslant 0$ holds $\mu \otimes m \otimes m$-almost everywhere for all $n \in \mathbb{N}$.

Thus, without loss of generality we can assume that $\mu \otimes m(\Omega \times S) < \infty$ and $L^\infty(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m) \subset E$. From $1_{\Omega \times S} \in E$ it follows that the function $h := \int_S |k(\omega, t, s)|\ dm(s)$ is finite $\mu \otimes m$-almost everywhere on $\Omega \times S$. So, $0 \leqslant h \in L^0(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$. Denote by U_n the following set

$$U_n := \{(\omega, t) \in \Omega \times S : h(\omega, t) \leqslant n\}$$

for all $n \in \mathbb{N}$. Then $\chi_{U_n} h \in L^1(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$ for all $n \in \mathbb{N}$ and $\chi_{U_n} \uparrow 1_{\Omega \times S}$ holds. For an arbitrary $n_0 \in \mathbb{N}$ define the operator $T_{n_0} : L^\infty(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m) \to L^0(S, \mathcal{F}, m)$ by formula

$$(T_{n_0} f)(t) := \int_{\Omega \times S} \chi_{U_{n_0}}(\omega, t) k(\omega, t, s) f(\omega, s)\ d\mu \otimes m(\omega, s)$$

for all $f \in L^\infty(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m) \subset E$ and for m-almost every $t \in S$. Then T_{n_0} is a positive kernel operator with kernel $\chi_{U_{n_0}} k$ and by [10, Theorem 91.1] $\chi_{U_{n_0}} k \geqslant 0$ $\mu \otimes m \otimes m$-almost everywhere on $\Omega \times S \times S$. Thus, $\chi_{U_n} k \geqslant 0$ $\mu \otimes m \otimes m$-almost everywhere on $\Omega \times S \times S$ for all $n \in \mathbb{N}$. Since $\chi_{U_n} \uparrow 1_{\Omega \times S}$ holds, we have $\chi_{U_n} k \to k$ $\mu \otimes m \otimes m$-almost everywhere on $\Omega \times S \times S$. So,$k \geqslant 0$ $\mu \otimes m \otimes m$-almost everywhere on $\Omega \times S \times S$, as desired.

(ii) Follows by applying the result in part (i) to T and $-T$. $\rhd$

Lemma 3.2. *Let T, S be leaner operators from E into F such that $0 \leqslant T \leqslant S$. If S is an $L^\infty(\Omega, \Sigma, \mu)$-homogeneous operator then T is also an $L^\infty(\Omega, \Sigma, \mu)$-homogeneous operator.*

$\lhd$ Let $0 \leqslant T \leqslant S$ with $L^\infty(\Omega, \Sigma, \mu)$-homogeneous S. First we show that $T\chi_A = \chi_A T$ holds for all $A \in \Sigma$, where χ_A is the characteristic function of $A \in \Sigma$. Fix an arbitrary $A \in \Sigma$. It follows from the inequalities $0 \leqslant T \leqslant S$ and $L^\infty(\Omega, \Sigma, \mu)$-homogeneity of S that $0 \leqslant \chi_{A^c} T \chi_A \leqslant \chi_{A^c} S \chi_A = S \chi_{A^c} \chi_A = 0$, where $\chi_{A^c} = 1 - \chi_A = \chi_{\Omega \setminus A}$. Hence, $\chi_{A^c} T \chi_A = 0$, and so we have $T\chi_A = \chi_A T \chi_A + \chi_{A^c} T \chi_A = \chi_A T \chi_A$. Thus,

$$T\chi_A = \chi_A T \chi_A \tag{2}$$

holds for all $A \in \Sigma$. It follows from (2) that $T\chi_{A^c} = \chi_{A^c} T \chi_{A^c}$ holds for all $A \in \Sigma$. That is, $T(1-\chi_A) = (1-\chi_A)T(1-\chi_A) = T(1-\chi_A) - \chi_A T + \chi_A T \chi_A$. Thus, $\chi_A T = \chi_A T \chi_A$ and in view of (2), we deduce that

$$T\chi_A = \chi_A T \tag{3}$$

holds for all $A \in \Sigma$. It follows from (3) that $Tf_n = f_n T$ holds for every step function f_n of the form $f_n = \sum_{i=1}^{l(n)} \alpha_i \chi_{A_i}$, where $A_i \in \Sigma$, $\alpha_i \in \mathbb{R}$, $n, l(n) \in \mathbb{N}$. By [11, Theorem 2.8] or [6, Theorem I.6.3] for every function $f \in L^\infty(\Omega, \Sigma, \mu)$ there exists a sequence of step functions $(f_n)_{n=1}^\infty$ such that f_n converges uniformly to f. Since every positive operator is uniformly continuous, we have $Tf = \lim_n Tf_n = \lim_n f_n T = fT$ for all $f \in L^\infty(\Omega, \Sigma, \mu)$ $\rhd$

Lemma 3.3. *Let $0 \leqslant T \leqslant U$ be positive operators from E into F. If U is a partial integral operator then T is also a partial integral operator.*

$\lhd$ Let $0 \leqslant T \leqslant U$ be positive operators from E into F with U partial integral. Since U is $L^\infty(\Omega, \Sigma, \mu)$-homogeneous, it follows from Lemma 3.2 that T is $L^\infty(\Omega, \Sigma, \mu)$-homogeneous. Therefore, it suffices to show that assertion (2) of Theorem 2.3 holds. In order to do this, let $(f_n)_{n=1}^\infty$ be a sequence in E such that $0 \leqslant f_n \leqslant f \in E$ for all $n \in \mathbb{N}$, $f_n \to 0$ in measure $\mu \otimes m$ on every subset of finite measure, and let $C \in \Sigma \otimes \mathcal{F}$ satisfy $\chi_C Tf \in L^1(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$. We must show that

$$\int_\Omega \chi_C(\omega, t) Tf_n(\omega, t)\ d\mu(\omega) \to 0$$

holds for m-almost every $t \in S$. Since $Uf \geqslant 0$, by [6, Corollary IV.3.1] there exists an increasing sequence $(C_i)_{i=1}^\infty \subset \Sigma \otimes \mathcal{F}$ such that, $\chi_{C_i} Uf \in L^1(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$ for all $i \in \mathbb{N}$ and $0 \leqslant \chi_{C_i} Uf \uparrow Uf$. For an arbitrary $n \in \mathbb{N}$ we have $0 \leqslant \chi_C Tf_n - \chi_{C_i}\chi_C Tf_n = (1 - \chi_{C_i})\chi_C Tf_n \leqslant (1 - \chi_{C_i})Uf \downarrow 0$. Thus, $0 \leqslant \chi_{C_i}\chi_C Tf_n \uparrow_i \chi_C Tf_n$ holds uniformly over all $n \in \mathbb{N}$, and so by Dominated Converges Theorem for an arbitrary $\varepsilon > 0$ there exists $i(\varepsilon) \in \mathbb{N}$ such that

$$0 \leqslant \int_\Omega \chi_C(\omega, t) Tf_n(\omega, t)\ d\mu(\omega) - \int_\Omega \chi_{C_i}(\omega, t)\chi_C(\omega, t) Tf_n(\omega, t)\ d\mu(\omega) \leqslant \varepsilon$$

for all $i \geqslant i(\varepsilon)$ and for all $n \in \mathbb{N}$. Consequently, applying Theorem 2.3 to the partial integral operator U and taking into account that $\chi_{C_{i(\varepsilon)}} Uf \in L^1(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$, we get

$$0 \leqslant \int_\Omega \chi_C(\omega, t) Tf_n(\omega, t)\ d\mu(\omega) =$$

$$\int_\Omega \chi_C(\omega, t) Tf_n(\omega, t)\ d\mu(\omega) - \int_\Omega \chi_{C_{i(\varepsilon)}}(\omega, t)\chi_C(\omega, t) Tf_n(\omega, t)\ d\mu(\omega) +$$

$$\int_\Omega \chi_{C_{i(\varepsilon)}}(\omega, t)\chi_C(\omega, t) Tf_n(\omega, t)\ d\mu(\omega) \leqslant \varepsilon + \int_\Omega \chi_{C_{i(\varepsilon)}}(\omega, t) Uf_n(\omega, t)\ d\mu(\omega) \to \varepsilon$$

as $n \to \infty$ for m-almost every $t \in S$. Thus, $\int_\Omega \chi_C(\omega, t) Tf_n(\omega, t)\ d\mu(\omega) \to 0$ holds for m-almost every $t \in S$. $\rhd$

Lemma 3.4. *Let (Ω, Σ, μ), $(S, \mathcal{F}, m)$ be measure spaces with σ-finite measures μ and m respectively. Let E, F be order dense ideals in $L^0(\Omega \times$*

$S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$ and k be a positive $\Sigma \otimes \mathcal{F} \otimes \mathcal{F}$-measurable function with values in $\mathbb{R} \cup \{+\infty\}$ such that

$$\int_S k(\omega, t, s) f(\omega, s)\ dm(s) < \infty$$

holds for $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$ an for all $f \in E$. Then $k < \infty$ $\mu \otimes m \otimes m$-a.e., that is, $k \in \mathcal{L}^0(\Omega \times S \times S, \Sigma \otimes \mathcal{F} \otimes \mathcal{F}, \mu \otimes m \otimes m)$.

$\lhd$ Put $P := \{(\omega, t, s) \in \Omega \times S \times S : k(\omega, t, s) = \infty\}$. Assume first that $1 \in E$, where $1(\omega, s) = 1$ for all $(\omega, s) \in \Omega \times S$. Then $\int_S k(\omega, t, s)\ dm(s) < \infty$ holds for $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$. Consequently, $k(\omega, t, s) < \infty$ holds for m-almost every $s \in S$ and for $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$. In other words, the section $P_{\omega,t} := \{s \in S : (\omega, t, s) \in P\}$ has measure zero for $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$. Thus, by Fubini's Theorem we have

$$\mu \otimes m \otimes m(P) = \int_{\Omega \times S} m(P_{\omega,t})\ d\mu \otimes m(\omega, t) = 0,$$

and so $k < \infty$ $\mu \otimes m \otimes m$-a.e.

Now consider the general case. By [10, Theorem 86.2] there exists a sequence $(C_n)_{n=1}^{\infty} \subset \Sigma \otimes \mathcal{F}$ such that $C_1 \subset C_2 \subset \ldots$, $\bigcup_{n=1}^{\infty} C_n = \Omega \times S$, $\mu \otimes m(C_n) < \infty$ and $\chi_{C_n} \in E$ for all $n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. Then we have

$$\int_S \chi_{C_n}(\omega, t) k(\omega, t, s) f(\omega, s)\ dm(s) < \infty$$

for $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$ and all $f \in L^\infty(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$. Consequently, since $L^\infty(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$ is an order dense ideal, $\chi_{C_n} k < \infty$ $\mu \otimes m \otimes m$-a.e. for all $n \in \mathbb{N}$. In other words, $k < \infty$ $\mu \otimes m \otimes m$-a.e. on $C_n \times S$ for all $n \in \mathbb{N}$. Thus, $k < \infty$ $\mu \otimes m \otimes m$-a.e. $\rhd$

Definition 3.5. A partial integral operator $T : E \to F$ with kernel $k \in \mathcal{L}^0(\Omega \times S \times S, \Sigma \otimes \mathcal{F} \otimes \mathcal{F}, \mu \otimes m \otimes m)$ is called *an absolute partial integral operator* if the operator $[T]$ defined by

$$[T] f(\omega, t) = \int_S |k(\omega, t, s)| f(\omega, s) dm(s)$$

for $\mu \otimes m$-almost every $(\omega, t) \in \Omega \times S$, maps E into F.

Denote by $L_{pk}(E,F)$ the space of all absolute partial integral operators from E into F.

Lemma 3.6. *If $T,U \in L_{pk}(E,F)$ are absolute partial integral operators from E into F with kernels k_T, k_U respectively, then $T \vee U$ is a absolute partial integral operator with kernel equal to the pointwise supremum of k_T and k_U $\mu \otimes m \otimes m$-a.e. In particular, $L_{pk}(E,F)$ is a vector sublattice of $L^r(E,F)$.*

$\lhd$ Let $T,U \in L_{pk}(E,F)$ with kernels k_T, k_U respectively. Then $T+U$ is a partial integral operator with kernel k_T+k_U. It follows from $|k_T+k_U| \leqslant |k_T|+|k_U|$ that $T+U$ is absolute, and so $L_{pk}(E,F)$ is a vector subspace of $L^r(E,F)$. In view of $T \vee U = \frac{1}{2}\big(T+U-|T-U|\big)$, it is sufficient to show that $|T|$ is a partial integral operator from E to F and $|T| = [T]$, where

$$[T]f(\omega,t) = \int\limits_S |k_T(\omega,t,s)| f(\omega,s)dm(s)$$

for all $f \in E$ and for $\mu \otimes m$-almost every $(\omega,t) \in \Omega \times S$. Since $[T] \geqslant T$ and $[T] \geqslant -T$, we have $[T] \geqslant |T|$. Consequently, it follows from Lemma 3.3 that $|T|$ is a partial integral operator from E to F. It remains to shaw that $[T] \leqslant |T|$. Let k_1 be the kernel of $|T|$. Since $|T| \geqslant T$ and $|T| \geqslant -T$, it follows from Lemma 3.1 that $k_1 \geqslant k_T$ and $k_1 \geqslant -k_T$, and so $k_1 \geqslant |k_T|$. Therefore, by Lemma 3.1 again $|T| \geqslant [T]$, which implies $|T| = [T]$ holds, as desired. $\rhd$

Recall that a vector lattice X is called *super Dedekind complete*, if whenever an arbitrary subset $D \subset X$ has a supremum, then there exists at most a countable subset C of D with $\sup C = \sup D$. The following theorem is the main result of this paper.

Theorem 3.7. *Let (Ω,Σ,μ), $(S,\mathcal{F},m)$ be measure spaces with σ-finite measures μ and m respectively. Let E, F be order dense ideal spaces in $L^0(\Omega \times S, \Sigma \otimes \mathcal{F}, \mu \otimes m)$. Then the space $L_{pk}(E,F)$ of all absolute partial integral operators from E into F is a band in $L^r(E,F)$ and is super Dedekind complete. Moreover, if $T,U \in L_{pk}(E,F)$ with kernels k_T, k_U respectively, then $k_T \vee k_U$ is the kernel of $T \vee U$.*

$\lhd$ It follows from Lemmas 3.3 and 3.6 that $L_{pk}(E,F)$ is an ideal in

$L^r(E,F)$. If $T, U \in L_{pk}(E,F)$ with kernels k_T, k_U respectively, then by Lemma 3.6 $k_T \vee k_U$ is the kernel of $T \vee U$.

Consider an increasing net $(T_\alpha)_{\alpha\in A} \subset L_{pk}(E,F)$ such that $0 \leqslant T_\alpha \uparrow T$ for some $T \in L^r(E,F)$. It is sufficient to show that $T \in L_{pk}(E,F)$. Let k_α be the kernel of T_α for all $\alpha \in A$. Then $(k_\alpha)_{\alpha\in A}$ is increasing net in $L^0(\Omega\times S\times S, \Sigma\otimes\mathcal{F}\otimes\mathcal{F}, \mu\otimes m\otimes m)_+$. Since $\mu\otimes m\otimes m$ is σ-finite measure, by [9, Lemma 2.6.1] $L^0(\Omega\times S\times S, \Sigma\otimes\mathcal{F}\otimes\mathcal{F}, \mu\otimes m\otimes m)$ is super Dedekind complete.

Let $P(\Omega\times S\times S) := P(\Omega\times S\times S, \Sigma\otimes\mathcal{F}\otimes\mathcal{F}, \mu\otimes m\otimes m)$ be the set of all $\Sigma\otimes\mathcal{F}\otimes\mathcal{F}$-measurable functions k on $\Omega\times S\times S$ assuming non-negative values in $\mathbb{R}\cup\{+\infty\}$. Functions differing only on a $\mu\otimes m\otimes m$-null set are identified. The set $P(\Omega, \times S\times S)$ is partial ordered as usual, i.e., $k_1 \leqslant k_2$ means that $k_1(\omega,t,s) \leqslant k_2(\omega,t,s)$ holds $\mu\otimes m\otimes m$-almost everywhere on $\Omega\times S\times S$. Then $P(\Omega\times S\times S)$ is a lattice.

It follows from [10, Lemma 94.4] that there exist an increasing sequence $(\alpha_n)_{n=1}^\infty \subset A$ and a function $0 \leqslant k \in P(\Omega\times S\times S)$ such that $k = \sup_n k_{\alpha_n} = \sup_{\alpha\in A} k_\alpha$ holds in $P(\Omega\times S\times S)$. So, we have $0 \leqslant k_{\alpha_n} \uparrow k$ and

$$(T_{\alpha_n} f)(\omega,t) = \int_S k_{\alpha_n}(\omega,t,s) f(\omega,s)\ dm(s) \leqslant Tf(\omega,t)$$

holds for all $f \in E$, $n \in \mathbb{N}$ and for $\mu\otimes m$-almost every $(\omega,t) \in \Omega\times S$. Consequently, by Levi's theorem there exists

$$\lim_n \int_S k_n(\omega,t,s) f(\omega,s)\ dm(s) = \int_S k(\omega,t,s) f(\omega,s)\ dm(s)$$

for all $0 \leqslant f \in E$ and for $\mu\otimes m$-almost every $(\omega,t) \in \Omega\times S$. If we define $T_k : E \to F$ by

$$T_k f(\omega,t) = \int_S k(\omega,t,s) f(\omega,s)\ dm(s),$$

then by Lemma 3.4 $k < \infty$ $\mu\otimes m\otimes m$-a.e., and so $0 \leqslant T_k \in L_{pk}(E,F)$. Since $T_k f = \sup_n T_{\alpha_n} f$ holds for all $f \in E$ and $T_{\alpha_n} \leqslant T$ for all $n \in \mathbb{N}$, we have $T_k \leqslant T$ in $L^r(E,F)$. On the other hand, since $k \geqslant k_\alpha$ $\mu\otimes m\otimes m$-a.e. for all $\alpha \in A$ it follows from Lemma 3.1 that $T_k \geqslant T_\alpha$ for all

$\alpha \in A$. Consequently, $T_k \geqslant T$, which implies $T_k = T$. Therefore, $T \in L_{pk}(E, F)$. Super Dedekind completeness of $L_{pk}(E, F)$ follows from equalities $T = \sup_{\alpha \in A} = \sup_{n \in \mathbb{N}} T_{\alpha_n}$. $\rhd$

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References

1. J.M. Appel, A.S. Kalitvin, P.P. Zabrejko. Partial integral operators and integro–differential equations. New York, Marcel Dekker, 2000. 560 p.
2. J.M. Appel, I.A. Eletskikh, A.S. Kalitvin. A note on the Fredholm property of partial integral equations of Romanovskij type. Journal of Integral Equations and Applications, 2004, vol. 16, no. 1, pp. 25–32.
3. V.I. Romanovsky. Sur une classe d'equations int'egrales lin'eaires. Acta Math, 1932, vol. 59, pp. 99–208.
4. A.D. Arziev, K.K. Kudaybergenov, P.R. Orinbaev, A.K. Tangirbergen. Partial Integral Operators on Banach–Kantorovich Spaces. Math Notes, 2023, vol. 114, pp. 15–29.
5. Yu.Kh. Eshkabilov, R.R. Kucharov. Partial integral operators of Fredholm type on Kaplansky–Hilbert module over L_0. Vladikavkaz Mathematical Journal, 2021, vol. 23, no. 3, pp. 80–90.
6. L.V. Kantorovich, G.P. Akilov. Funkcionalvnyy analiz [Functional Analysis], Moscow, Nauka, 1984. (in Russian).
7. K.K. Kudaybergenov, A.D. Arziev. The spectrum of an element in a Banach–Kantorovich algebra over a ring of measurable functions, Advances in Operator Theory, 2022, vol. 7, no. 1, pp. 2–15.
8. K.K. Kudaybergenov, A.D. Arziev, P.R. Orinbaev, A.K. Tanirbergen. The Mercer‘s theorem for partial integral operators. Journal of Mathematical Sciences, 2023, vol. 271, no. 6, pp. 749–761.
9. P. Meyer-Nieberg. Banach Lattices, Springer, Berlin etc. (1991).
10. A.C. Zaanen. Riesz Spaces II. Amsterdam-New York-Oxford, North-Holland publishing company. 1983. 720 p.
11. C.D. Aliprantis, O. Burkinshaw. Positive Operators. Dordrecht, Springer. 2006. 376 p.

12. P.R. Orinbaev, B.B. Tasoev. On Partial Integral Representation of Linear Positive Operators, Vladikavkaz Math. J., 2025, vol. 27, no. 1, pp. 101-111 (in Russian).