An investigation on Lie ideals in rp-algebra using d-derivation

Rinki Paul* and Paltu Sarkar†
Department of Mathematics, University of North Bengal, West Bengal-734013, India.

Abstract

In this investigation, $\mathcal{A}$ represents an associative algebra over a ring $\mathcal{R}$ with unity (not necessarily commutative), $\mathcal{U}$ a Lie ideal of $\mathcal{A}$ and $\mathcal{D}_d$ a d-derivation on $\mathcal{A}$. An additive mapping $\mathcal{D}_d$ from $\mathcal{A}$ to itself is said to be a d-derivation, if there exists a ring derivation d on $\mathcal{R}$ such that

$$\mathcal{D}_d(xy) = \mathcal{D}_d(x)y + x\mathcal{D}_d(y) \qquad \text{and} \qquad \mathcal{D}_d(rx) = d(r)x + r\mathcal{D}_d(x)$$

for all $r \in \mathcal{R}$ and $x, y \in \mathcal{A}$. An rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ is an associative algebra which contains no divisors of zero and for any $r \in \mathcal{R}$ and $p \in \mathcal{A}$, $r\mathcal{A}p = \{0\}$ implies either $r = 0$ or $p = 0$. By admitting d-derivation $\mathcal{D}_d$ on $\mathcal{A}$, we present the necessary and sufficient identities for commutativity of Lie ideals in rp-algebra. We also examine the relationship among the centre $\mathcal{Z}(\mathcal{A})$ and the centralizers $C_{\mathcal{A}}(\mathcal{D}_d(\mathcal{U}))$ and $C_{\mathcal{A}}(\mathcal{U})$ in rp-algebra. Moreover, we explore the structures of Lie ideals in rp-algebra by restricting d-derivation $\mathcal{D}_d$ to $\mathcal{U}$, which acts as a homomorphism (anti-homomorphism, Jordan homomorphism) on $\mathcal{U}$. We also address the commuting and co-commuting concepts in this Lie structure formation. In the end, we conclude our investigation by discussing Posner and Herstein's results in rp-algebra related to Lie ideals.

keywords: Algebra, d-derivation, rp-algebra, Lie ideal

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1. INTRODUCTION

Let $\mathcal{R}$ be a ring with unity (not necessarily commutative) and $\mathcal{A}$ be an associative algebra over a ring $\mathcal{R}$ with centre $\mathcal{Z}(\mathcal{A})$. A $\mathcal{R}$-submodule $\mathcal{U}$ of an associative algebra $\mathcal{A}$ over a ring $\mathcal{R}$ is said to be a Lie ideal (Lie subalgebra) if $[\mathcal{U}, \mathcal{A}] \subseteq \mathcal{U}$ (respectively $[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{U}$). A Lie ideal $\mathcal{U}$ of $\mathcal{A}$ is said to be a square closed Lie ideal if $u^2 \in \mathcal{U}$ for all $u \in \mathcal{U}$. A Lie ideal $\mathcal{U}$ is said to be non-central on $\mathcal{A}$ if $\mathcal{U} \nsubseteq \mathcal{Z}(\mathcal{A})$.

An additive mapping $\mathscr{D}$ from a ring $\mathcal{R}$ into itself is said to be a derivation if $\mathscr{D}(ab) = \mathscr{D}(a)b + a\mathscr{D}(b)$ for all $a, b \in \mathcal{R}$. Brešar [1] introduced a broader class, known as generalized ring derivation, and later, Hvala [2] carried out an algebraic analysis of this notion in prime rings. A generalized derivation is an additive mapping $\mathscr{F}$ from $\mathcal{R}$ into $\mathcal{R}$ if there exists a derivation $\mathscr{D}$ on $\mathcal{R}$ such that $\mathscr{F}(xy) = \mathscr{F}(x)y + x\mathscr{D}(y)$ for all $x, y \in \mathcal{R}$. A multiplicative derivation of $\mathcal{R}$ is a mapping $\mathscr{D}$ from $\mathcal{R}$ into itself which is not necessarily additive but satisfies $\mathscr{D}(ab) = \mathscr{D}(a)b + a\mathscr{D}(b)$ for all $a, b \in \mathcal{R}$. This multiplicative notion was further extended to a multiplicative (generalized)-derivation in such a way that the map $\mathscr{F}$, not necessarily additive, satisfies $\mathscr{F}(xy) = \mathscr{F}(x)y + x\mathscr{D}(y)$ for all $x, y \in \mathcal{R}$, where $\mathscr{D}$ is any map (need not be a derivation or even additive). For further references, see [3–7].

The symbols $[a, b]$ and $a \circ b$ stand for commutator and anti-commutator respectively of elements $a, b \in \mathcal{R}$. Let $\mathcal{S}$ be a nonempty subset of a ring $\mathcal{R}$. A mapping $\mathscr{F} : \mathcal{R} \to \mathcal{R}$ is said to be commuting (centralizing) on $\mathcal{S}$ if $[\mathscr{F}(x), x] = 0$ $\big($respectively $[\mathscr{F}(x), x] \in \mathcal{Z}(\mathcal{R})\big)$ for all $x \in \mathcal{S}$. Two mappings $\mathscr{F} : \mathcal{R} \to \mathcal{R}$ and $\mathscr{G} : \mathcal{R} \to \mathcal{R}$ are said to be co-commmuting (co-centralizing) on $\mathcal{S}$ if $\mathscr{F}(x)x - x\mathscr{G}(x) = 0$ $\big($respectively $\mathscr{F}(x)x - x\mathscr{G}(x) \in \mathcal{Z}(\mathcal{R})\big)$ for all $x \in \mathcal{S}$. Moreover, an additive map $\mathscr{F} : \mathcal{R} \to \mathcal{R}$ acts as a homomorphism (anti-homomorphism, Jordan homomorphism) on a subset $\mathcal{S}$ of $\mathcal{R}$ if $\mathscr{F}(xy) = \mathscr{F}(x)\mathscr{F}(y)$ $\big($respectively $\mathscr{F}(xy) = \mathscr{F}(y)\mathscr{F}(x)$, $\mathscr{F}(x \circ y) = \mathscr{F}(x) \circ \mathscr{F}(y)\big)$ for all $x, y \in \mathcal{S}$. The study of commuting and centralizing maps on prime rings was initiated by Posner [8], and this line of work has since been extended by many authors in diverse ways.

In derivation theory, Posner and Herstein's poineer results are very significant and a key factor for the study of non-commutative algebraic frames. Around 1950, Herstein first discovered the theory on Jordan and Lie ideals in simple, associative rings. He [9] also developed some theory on Lie ideals on rings of operators on a Hilbert space. In

*Electronic address: rinkipaul2k@gmail.com;
†Electronic address: pal2_math_796@nbu.ac.in(corresponding_author)

1973, Awtar [10] first generalized the results of Posner in prime rings with the help of Lie and Jordan ideals. After that, Bergen et al. [11] discussed the connection between Lie ideal and derivation on the same structure, and finally measured the size of $\mathscr{D}(\mathcal{U})$, where $\mathcal{U}$ is a Lie ideal. They also highlighted the question of when $\mathscr{D}^2(\mathcal{U}) = \{0\}$ holds for a non-central Lie ideal $\mathcal{U}$ in a prime ring $\mathcal{R}$. Later on, Martindale and Miers [12] worked on Lie ideals in semiprime rings with involution in 1986. Lanski and Montgomery [13] discussed the Lie structures of prime rings of characteristics 2. They proved that if $\mathcal{U}$ is a commutative Lie ideal of a prime ring $\mathcal{R}$, then $\mathcal{U} \subset \mathcal{Z}(\mathcal{R})$ unless $char(\mathcal{R}) = 2$ and $\mathcal{R}\mathcal{Z}^{-1}$ is a simple ring 4-dimentional over its centre. Brešar proved in [14] that if $\mathscr{D}$ and $\mathscr{H}$ be two non-zero co-centralizing derivations on a prime ring $\mathcal{R}$, then $\mathcal{R}$ must be commutative. Later, Lee and Wong [15] studied similar conditions for non-central Lie ideal $\mathcal{L}$ of a prime ring $\mathcal{R}$ and proved that if $\mathscr{D}(x)x - x\mathscr{H}(x) \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{L}$, then either $\mathscr{D} = O = \mathscr{H}$ or $\mathcal{R}$ satisfies $\mathcal{S}_4$. In [16], Lanski extended the Posner's result to the Lie ideal and proved that for a non-commutative Lie ideal $\mathcal{L}$ of a prime ring $\mathcal{R}$, if $[\mathscr{D}(x), x]_k = 0$ for all $x \in \mathcal{L}$ and $k > 0$ fixed, then $char(\mathcal{R}) = 2$ and $\mathcal{R} \subseteq \mathcal{M}_2(\mathcal{F})$ for a field $\mathcal{F}$.

After that, Carini [17] demonstrated that if $\mathcal{L}$ is a non-central Lie ideal of a prime ring $\mathcal{R}$ with $char(\mathcal{R}) \neq 2$ and $\mathscr{D}$ is a non-zero derivation of $\mathcal{R}$ such that $[\mathscr{D}(u), u]^n \in \mathcal{Z}(\mathcal{R})$ for all $u \in \mathcal{L}$ then $\mathcal{R}$ satisfies the standard identity of degree 4. Specifically, if $\mathscr{D}$ satisfies $[\mathscr{D}(u), u]^n = 0$ for all $u \in \mathcal{L}$ then $\mathcal{R}$ is commutative. Following this development, Filippis [18] studied the same context, considering $\mathscr{D}$ as a generalized derivation. Thereafter, Wang [19] investigated a similar scenario in which the derivation $\mathscr{D}$ is replaced with a non-trivial automorphism σ of $\mathcal{R}$. He proved that if $\mathcal{L}$ be a non-central Lie ideal of a prime ring $\mathcal{R}$ with centre $\mathcal{Z}(\mathcal{R})$ and $char(\mathcal{R})$ either greater than n or 0, then for a non-trivial automorphism σ of $\mathcal{R}$ such that $[\sigma(u), u]^n \in \mathcal{Z}(\mathcal{R})$ for all $u \in \mathcal{L}$, $\mathcal{R}$ satisfies $\mathcal{S}_4$. Filippis and Huang [20] explored the action of skew derivations on Lie ideals and they showed in prime ring $\mathcal{R}$ with $char(\mathcal{R}) \neq 2$ and 3 that if $\mathcal{L}$ be a non-central Lie ideal of $\mathcal{R}$, and $(\mathscr{D}, \sigma)$ is a skew derivation of $\mathcal{R}$ such that $[\mathscr{D}(x), x]^n = 0$ for fixed positive integer n and for all $x \in \mathcal{L}$, then $\mathcal{R}$ satisfies $\mathcal{S}_4$. Later, Asma et al. [21] proved that if $\mathcal{L}$ is a non-central Lie ideal of $\mathcal{R}$ such that $u^2 \in \mathcal{L}$ for all $u \in \mathcal{L}$ (that means, $\mathcal{L}$ is a square closed non-central Lie ideal) and $\mathscr{D}$ acts as a homomorphism or anti-homomorphism on $\mathcal{L}$, then $\mathscr{D} = O$. Meanwhile, Wang and You [22] omitted the assumption $u^2 \in \mathcal{L}$ and examined the same results. Golbasi and Kaya [23] further showed that if the generalized derivation $\mathscr{G}$ of a prime ring $\mathcal{R}$ acts as a homomorphism or anti-homomorphism on a square closed Lie ideal $\mathcal{L}$, then either $\mathscr{D} = O$ or $\mathcal{L}$ is central in $\mathcal{R}$. More detailed literature on Lie ideals can be found in the references [24–28] for further study.

From the well-established and extensive literature on Lie ideals in some known algebraic structures, we try to present an investigation on the structure of Lie ideals in rp-algebra by using d-derivation $\mathcal{D}_d$. This work is a continuation of our previous work [29]. In this article, we introduced the concept of d-derivation on an associative algebra over a ring and a prime structure, namely, rp-algebra. We also derived several differential identites related to commutativity for rp-algebra. Some important investigating identities are as follows:

1. $\mathcal{D}_d(xy) - xy \in \mathcal{Z}(\mathcal{A})$,
2. $\mathcal{D}_d([x, y]) + \mathcal{D}_d(x)\mathcal{D}_d(y) \in \mathcal{Z}(\mathcal{A})$,
3. $[\mathcal{D}_d(x), \mathcal{D}_d(y) + y] \in \mathcal{Z}(\mathcal{A})$,
4. $\mathcal{D}_d(x \circ y) + \mathcal{D}_d(x)\mathcal{D}_d(y) \in \mathcal{Z}(\mathcal{A})$,
5. $\mathcal{D}_d([\mathcal{D}_d(x), xy]) + x^2\mathcal{D}_d(y) \in \mathcal{Z}(\mathcal{A})$,

for all $x, y \in \mathcal{A}$.

In this paper, our main intention is to focus on determining the necessary and sufficient identities in connection with commutativity of the Lie ideals in rp-algebra. We delightedly establish the relationship between the centre $\mathcal{Z}(\mathcal{A})$ and the centralizers of $\mathcal{U}$ and $\mathcal{D}_d(\mathcal{U})$. We also construct certain identities by adopting the concepts of commuting and centralizing mappings. Furthermore, we categorize the Lie ideals in rp-algebra by restricting d-derivation $\mathcal{D}_d$ to $\mathcal{U}$ and considering its behavior as a homomorphism, anti-homomorphism and Jordan homomorphism. Finally, we end our study by analyzing Posner and Herstein's results in rp-algebra using Lie ideals.

2. PRELIMINARIES

This section outlines some fundamental facts related to d-derivation and rp-algebra, which we proposed in [29].

Definition 2.1 *Let $\mathcal{A}$ be an associative algebra over a ring $\mathcal{R}$. An additive map $\mathcal{D}_d : \mathcal{A} \to \mathcal{A}$ is a d-derivation if there exists a non-zero ring derivation $d : \mathcal{R} \to \mathcal{R}$ such that*

$$\mathcal{D}_d(ra) = d(r)a + r\mathcal{D}_d(a)$$
$$\text{and} \qquad \mathcal{D}_d(a_1a_2) = \mathcal{D}_d(a_1)a_2 + a_1\mathcal{D}_d(a_2)$$

for all $a_1, a_2, a \in \mathcal{A}$ and $r \in \mathcal{R}$.

Definition 2.2 *An associative algebra $\mathcal{A}$ over a ring $\mathcal{R}$ is an rp-algebra if $\mathcal{A}$ does not contain divisors of zero and for any $r \in \mathcal{R}$ and $p \in \mathcal{A}$, $r\mathcal{A}p = \{0\}$ implies either $r = 0$ or $p = 0$.*

Lemma 2.3 *Let $\mathcal{A}$ be an rp-algebra over a ring $\mathcal{R}$. For any $r(\neq 0) \in \mathcal{R}$ and $a, b \in \mathcal{A}$, the following holds:*

(i) $ra = 0 \implies a = 0.$

(ii) $ra \in \mathcal{Z}(\mathcal{A}) \implies a \in \mathcal{Z}(\mathcal{A}).$

(iii) *If $b\ (\neq 0) \in \mathcal{Z}(\mathcal{A})$ and $ab \in \mathcal{Z}(\mathcal{A})$ $\big($or $ba \in \mathcal{Z}(\mathcal{A})\big)$ then $a \in \mathcal{Z}(\mathcal{A})$.*

Theorem 2.4 *Let $\mathcal{D}_d$ be a d-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$. If $[\mathcal{D}_d(x), x] = 0$ for all $x \in \mathcal{A}$, then $\mathcal{A}$ is commutative.*

Lemma 2.5 *Let $\mathcal{A}$ be an rp-algebra over a ring $\mathcal{R}$ and $\mathcal{D}_d$ be a d-derivation on $\mathcal{A}$. If $x \circ y = 0$ for all $x, y \in \mathcal{A}$, then $\mathcal{A}$ is commutative.*

Theorem 2.6 *Let $\mathcal{D}_d$ be a d-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$. If $x \circ y \in \mathcal{Z}(\mathcal{A})$ for all $x, y \in \mathcal{A}$, then $\mathcal{A}$ is commutative.*

Theorem 2.7 *An rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ admitting a d-derivation $\mathcal{D}_d$ is commutative if $\mathcal{D}_d(x) \circ x \in \mathcal{Z}(\mathcal{A})$ for all $x \in \mathcal{A}$.*

Theorem 2.8 *Let an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char$(\mathcal{R}) \neq n$ admits a non-zero d-derivation $\mathcal{D}_d$ such that d is commuting on $\mathcal{R}$. If $\mathcal{D}_d(x^n) \in \mathcal{Z}(\mathcal{A})$ for all $x \in \mathcal{A}$, then $\mathcal{A}$ is commutative.*

Theorem 2.9 *Let $\mathcal{D}_{d_1}$ and $\mathcal{D}_{d_2}$ be respectively d_1-derivation and d_2-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char$(\mathcal{R}) \neq 2$ such that $\mathcal{D}_{d_1}\mathcal{D}_{d_2}$ is a d_1d_2-derivation on $\mathcal{A}$. Then either $\mathcal{D}_{d_1} = O$ or $\mathcal{D}_{d_2} = O$.*

Throughout the paper, we use $\mathcal{A}$ as an *rp*-algebra over a ring with unity $\mathcal{R}$ (not necessarily commutative), $\mathcal{Z}(\mathcal{A})$ as the centre of $\mathcal{A}$ and $\mathcal{Q} = \{\, r \in \mathcal{R} : d(r) \neq 0 \,\}$. By $C_{\mathcal{A}}(\mathcal{U})$, we shall mean the centralizer of $\mathcal{U}$, defined by $C_{\mathcal{A}}(\mathcal{U}) = \{\, x \in \mathcal{A} : xu = ux \text{ for all } u \in \mathcal{U} \,\}$.

3. *d*-DERIVATION ON LIE IDEALS OF *rp*-ALGEBRAS

We start our investigation with the following results, which will be referred to throughout our study.

Lemma 3.1 *Let $\mathcal{U}$ be a Lie ideal of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$. For some $r \in \mathcal{R}$ and $a \in \mathcal{A}$, if $r\mathcal{U}a = \{0\}$, then either $r = 0$ or $a = 0$.*

Proof. Suppose $r\mathcal{U}a = \{0\}$ for some $r \in \mathcal{R}$ and $a \in \mathcal{A}$. Pre-multiplying both sides by b, we get

$$r(b\mathcal{U}a) = \{0\} \quad \text{for all } b \in \mathcal{A} \implies r\mathcal{A}\mathcal{U}a = \{0\}.$$

Since $\mathcal{A}$ is an *rp*-algebra and $\mathcal{U} \neq \{0\}$, it easily follows that either $r = 0$ or $a = 0$.

Lemma 3.2 *For a Lie ideal $\mathcal{U}$ of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$, if $\mathcal{D}_d(\mathcal{U}) = \{0\}$ then*

$$\mathcal{D}_d(\mathcal{A}) \subseteq C_{\mathcal{A}}(\mathcal{U}).$$

Proof. Suppose $\mathcal{D}_d(\mathcal{U}) = \{0\}$, which means

$$\mathcal{D}_d(u) = 0 \quad \text{for all } u \in \mathcal{U}. \tag{3.1}$$

Now, replacing u by $[u, a]$ for all $a \in \mathcal{A}$, we have $\mathcal{D}_d([u, a]) = 0$. It implies that

$$[\mathcal{D}_d(u), a] + [u, \mathcal{D}_d(a)] = 0 \implies [u, \mathcal{D}_d(a)] = 0 \qquad \text{(using (3.1))}.$$

Therefore, $\mathcal{D}_d(a) \in C_{\mathcal{A}}(\mathcal{U})$ for all $a \in \mathcal{A}$ and hence, $\mathcal{D}_d(\mathcal{A}) \subseteq C_{\mathcal{A}}(\mathcal{U})$.

Lemma 3.3 *If $\mathcal{U}$ is a non-central Lie ideal of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$, then*

$$C_{\mathcal{A}}([\mathcal{U}, \mathcal{A}]) = C_{\mathcal{A}}(\mathcal{U}).$$

Proof. Let $x \in C_{\mathcal{A}}([\mathcal{U},\mathcal{A}])$. It implies that

$$[x,[u,a]] = 0 \quad \text{for all } u \in \mathcal{U} \text{ and } a \in \mathcal{A}. \tag{3.2}$$

Now, putting a by au, we get $[x,[u,au]] = 0$. Therefore,

$$[x,[u,a]u] = 0, \quad \text{which implies} \ \ [u,a][x,u] = 0 \qquad (\text{using } (3.2)).$$

Since $\mathcal{U} \nsubseteq \mathcal{Z}(\mathcal{A})$ and $\mathcal{A}$ is an *rp*-algebra, it follows that $[x,u] = 0$ for all $u \in \mathcal{U}$. Hence $x \in C_{\mathcal{A}}(\mathcal{U})$, which implies $C_{\mathcal{A}}([\mathcal{U},\mathcal{A}]) \subseteq C_{\mathcal{A}}(\mathcal{U})$.

Conversely, let $y \in C_{\mathcal{A}}(\mathcal{U})$, that is, $[y,u] = 0$ for all $u \in \mathcal{U}$. Replacing u by $[u,a]$ for all $a \in \mathcal{A}$, we obtain $[y,[u,a]] = 0$ and hence, $y \in C_{\mathcal{A}}([\mathcal{U},\mathcal{A}])$. Thus, $C_{\mathcal{A}}(\mathcal{U}) \subseteq C_{\mathcal{A}}([\mathcal{U},\mathcal{A}])$, and the equality follows.

Lemma 3.4 *In an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char$(\mathcal{R}) \neq 2$, if $\mathcal{U}$ is a non-central Lie ideal of $\mathcal{A}$ such that*

$$[\mathcal{U},\mathcal{U}] \subseteq C_{\mathcal{A}}(\mathcal{U}),$$

then $\mathcal{U}$ is commutative.

Proof. For all $u \in \mathcal{U}$ and $a \in \mathcal{A}$, we have $[u,a] \in \mathcal{U}$. Let us consider $x = [u,[u,a]] \in [\mathcal{U},\mathcal{U}] \subseteq C_{\mathcal{A}}(\mathcal{U})$. Then

$$ux = u[u,[u,a]] = [u,[u,ua]] \in C_{\mathcal{A}}(\mathcal{U}).$$

Therefore $[ux,v] = 0$ for all $v \in \mathcal{U}$. It implies that

$$[u,v]x + u[x,v] = 0 \implies [u,v]x = 0.$$

Case 1: If $x \neq 0$ then $[u,v] = 0$ for all $u,v \in \mathcal{U}$ and hence, $\mathcal{U}$ is commutative.

Case 2: If $x = 0$, that is, $[u,[u,a]] = 0$ for all $u \in \mathcal{U}$. Replacing u by $u+v$ for all $v \in \mathcal{U}$, we get

$$[u+v,[u+v,a]] = 0 \implies [u,[v,a]] + [v,[u,a]] = 0. \tag{3.3}$$

Again, using the Jacobi identity, we can write $[u,[v,a]] + [v,[a,u]] + [a,[u,v]] = 0$,

$$\text{that means,} \ \ [u,[v,a]] - [v,[u,a]] + [a,[u,v]] = 0. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$[a,[u,v]] = -2[u,[v,a]] \in C_{\mathcal{A}}(\mathcal{U}). \tag{3.5}$$

Take $c = [u,v]$. Then $[a,c] \in C_{\mathcal{A}}(\mathcal{U})$, which implies $[u,[a,c]] = 0$. Replacing $[a,c]$ by $v[a,c]$ for all $v \in \mathcal{U}$, we obtain

$$[u,v[a,c]] = 0 \ \text{ implies } \ [u,v][a,c] = 0.$$

Thus it follows that either $[u,v] = 0$ or $[a,c] = 0$. If $[u,v] = 0$ for all $u,v \in \mathcal{U}$, then $\mathcal{U}$ is commutative.
But, if $[a,c] = 0$, then by (3.5) we have $2[u,[v,a]] = 0$. Applying the d-derivation $\mathcal{D}_d$ and expanding, we get

$$2\big([\mathcal{D}_d(u),[v,a]] + [u,[\mathcal{D}_d(v),a]] + [u,[v,\mathcal{D}_d(a)]]\big) = 0. \tag{3.6}$$

Finally, substituting a by ra, where $r \in \mathcal{Q}$ and using *char*$(\mathcal{R}) \neq 2$, we obtain

$$2d(r)[u,[v,a]] = 0 \implies [u,[v,a]] = 0.$$

It implies that $u \in C_{\mathcal{A}}([\mathcal{U},\mathcal{A}])$. So, by Lemma 3.3, we obtain $u \in C_{\mathcal{A}}(\mathcal{U})$ for all $u \in \mathcal{U}$, that means, $\mathcal{U} \subseteq C_{\mathcal{A}}(\mathcal{U})$. Thus for all cases, $\mathcal{U}$ is commutative.

4. MAIN RESULTS

In the main results, we first discuss the behavior of the image of a Lie ideal $\mathcal{U}$ under $\mathcal{D}_d$.

Theorem 4.1 *If a Lie ideal $\mathcal{U}$ of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ be contained in the image $\mathcal{D}_d(\mathcal{U})$, then $\mathcal{D}_d(\mathcal{U})$ is a Lie ideal of $\mathcal{A}$.*

Proof. First, we claim that $\mathcal{D}_d(\mathcal{U})$ is an $\mathcal{R}$-submodule of $\mathcal{A}$. Consider $r \in \mathcal{R}$ and $\mathcal{D}_d(u), \mathcal{D}_d(v) \in \mathcal{D}_d(\mathcal{U})$, where $u, v \in \mathcal{U}$. Then

$$\mathcal{D}_d(u) + \mathcal{D}_d(v) = \mathcal{D}_d(u+v) \in \mathcal{D}_d(\mathcal{U}) \qquad \text{and} \qquad \mathcal{D}_d(ru) = d(r)u + r\mathcal{D}_d(u) \in \mathcal{D}_d(\mathcal{U}).$$

Since $ru, d(r)u \in \mathcal{U} \subseteq \mathcal{D}_d(\mathcal{U})$; $r\mathcal{D}_d(u) \in \mathcal{D}_d(\mathcal{U})$ and hence, $\mathcal{D}_d(\mathcal{U})$ is an $\mathcal{R}$-submodule of $\mathcal{A}$.
To prove $\mathcal{D}_d(\mathcal{U})$ as a Lie ideal, we have to show that $[\mathcal{D}_d(u), a] \in \mathcal{D}_d(\mathcal{U})$ for all $a \in \mathcal{A}$. For Lie ideal, $[u, a] \in \mathcal{U}$. Then

$$\mathcal{D}_d([u, a]) \in \mathcal{D}_d(\mathcal{U}), \quad \text{which gives} \quad [\mathcal{D}_d(u), a] + [u, \mathcal{D}_d(a)] \in \mathcal{D}_d(\mathcal{U}).$$

But, $[u, \mathcal{D}_d(a)] \in \mathcal{D}_d(\mathcal{U})$. Hence, we conclude that

$$[\mathcal{D}_d(u), a] \in \mathcal{D}_d(\mathcal{U}).$$

Therefore, $\mathcal{D}_d(\mathcal{U})$ is a Lie ideal of $\mathcal{A}$.

Theorem 4.2 *If $\mathcal{U}$ is a Lie ideal of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char$(\mathcal{R}) \neq 2$ which admits a d-derivation $\mathcal{D}_d$ such that*

$$\mathcal{D}_d{}^2(\mathcal{U}) = \{0\},$$

then $\mathcal{U} \subseteq \mathcal{Z}(\mathcal{A})$.

Proof. Suppose $\mathcal{D}_d{}^2(\mathcal{U}) = \{0\}$, that is,

$$\mathcal{D}_d{}^2(u) = 0 \quad \text{for all } u \in \mathcal{U}. \tag{4.1}$$

Now, replacing u by $[u, a]$ for all $a \in \mathcal{A}$, we get $\mathcal{D}_d{}^2([u, a]) = 0$, which implies

$$\mathcal{D}_d\big([\mathcal{D}_d(u), a] + [u, \mathcal{D}_d(a)]\big) = 0.$$

Again expanding,

$$[\mathcal{D}_d{}^2(u), a] + 2[\mathcal{D}_d(u), \mathcal{D}_d(a)] + [u, \mathcal{D}_d{}^2(a)] = 0.$$

Using (4.1), this reduces to

$$2[\mathcal{D}_d(u), \mathcal{D}_d(a)] + [u, \mathcal{D}_d{}^2(a)] = 0. \tag{4.2}$$

Again, setting a to au, we have

$$2[\mathcal{D}_d(u), \mathcal{D}_d(au)] + [u, \mathcal{D}_d{}^2(au)] = 0$$

$$\implies 2[\mathcal{D}_d(u), \mathcal{D}_d(a)]u + 2\mathcal{D}_d(a)[\mathcal{D}_d(u), u] + 2[\mathcal{D}_d(u), a]\mathcal{D}_d(u) + [u, \mathcal{D}_d{}^2(a)]u + 2[u, \mathcal{D}_d(a)]\mathcal{D}_d(u)$$

$$+2\mathcal{D}_d(a)[u, \mathcal{D}_d(u)] + [u, a]\mathcal{D}_d{}^2(u) + a[u, \mathcal{D}_d{}^2(u)] = 0.$$

Using (4.1) and (4.2), we get

$$2[\mathcal{D}_d(u), a]\mathcal{D}_d(u) + 2[u, \mathcal{D}_d(a)]\mathcal{D}_d(u) = 0. \tag{4.3}$$

Replacing a by ra, where $r \in \mathcal{Q}$, we obtain

$$2[\mathcal{D}_d(u), ra]\mathcal{D}_d(u) + 2[u, \mathcal{D}_d(ra)]\mathcal{D}_d(u) = 0$$

$$\implies 2r[\mathcal{D}_d(u), a]\mathcal{D}_d(u) + 2d(r)[u, a]\mathcal{D}_d(u) + 2r[u, \mathcal{D}_d(a)]\mathcal{D}_d(u) = 0.$$

Using (4.3), it follows that

$$2d(r)[u, a]\mathcal{D}_d(u) = 0, \quad \text{that means,} \quad [u, a]\mathcal{D}_d(u) = 0 \quad (\text{since } char(\mathcal{R}) \neq 2). \tag{4.4}$$

Setting u by $u + v$ for all $v \in \mathcal{U}$, we have

$$[u, a]\mathcal{D}_d(u) + [u, a]\mathcal{D}_d(v) + [v, a]\mathcal{D}_d(u) + [v, a]\mathcal{D}_d(v) = 0.$$

Applying equation (4.4), we deduce

$$[u, a]\mathcal{D}_d(v) + [v, a]\mathcal{D}_d(u) = 0. \tag{4.5}$$

Finally, substituting v by $r_1 v$, where $r_1 \in \mathcal{Q}$, we obtain

$$[u, a]\mathcal{D}_d(r_1 v) + [r_1 v, a]\mathcal{D}_d(u) = 0.$$

Expanding,

$$d(r_1)[u, a]v + r_1[u, a]\mathcal{D}_d(v) + r_1[v, a]\mathcal{D}_d(u) = 0.$$

Lastly, for $d(r_1) \neq 0$ and using (4.5), we get $[u, a]v = 0$ for all $v \in \mathcal{U}$. Since $\mathcal{U}$ is a non-zero Lie ideal of $\mathcal{A}$, it follows that $[u, a] = 0$ for all $u \in \mathcal{U}$ and $a \in \mathcal{A}$. Hence, $\mathcal{U} \subseteq \mathcal{Z}(\mathcal{A})$.

Corollary 4.3 *If $\mathcal{U}$ is a Lie ideal of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char$(\mathcal{R}) \neq 2$, which admits a d-derivation $\mathcal{D}_d$ such that $\mathcal{D}_d(u) \neq 0$ for all $u(\neq 0) \in \mathcal{U}$, but ${\mathcal{D}_d}^2(\mathcal{U}) = \{0\}$ then $\mathcal{U}$ is central on $\mathcal{A}$.*

Theorem 4.4 *Let $\mathcal{D}_d$ be a d-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$. If all elements of a Lie ideal $\mathcal{U}$ of $\mathcal{A}$ are idempotent, then $\mathcal{U}$ is central on $\mathcal{A}$.*

Proof. For all $u \in \mathcal{U}$, we have $\mathcal{D}_d(u^2) = \mathcal{D}_d(u)$. Replacing u by $[u, a]$ for all $a \in \mathcal{A}$, we obtain $\mathcal{D}_d([u, a]^2) = \mathcal{D}_d([u, a])$. Expanding the left-hand side, we get

$$\mathcal{D}_d([u, a])[u, a] + [u, a]\mathcal{D}_d([u, a]) = \mathcal{D}_d([u, a])$$

$$\implies [\mathcal{D}_d(u), a][u, a] + [u, \mathcal{D}_d(a)][u, a] + [u, a][\mathcal{D}_d(u), a] + [u, a][u, \mathcal{D}_d(a)] = [\mathcal{D}_d(u), a] + [u, \mathcal{D}_d(a)].$$

Now, pre-multiplying both sides by $[u, a]$, we get

$$[u, a][\mathcal{D}_d(u), a][u, a] + [u, a][u, \mathcal{D}_d(a)][u, a] + [u, a]^2[\mathcal{D}_d(u), a] + [u, a]^2[u, \mathcal{D}_d(a)] = [u, a][\mathcal{D}_d(u), a] + [u, a][u, \mathcal{D}_d(a)]$$

$$\implies [u, a][\mathcal{D}_d(u), a][u, a] + [u, a][u, \mathcal{D}_d(a)][u, a] + [u, a][\mathcal{D}_d(u), a] + [u, a][u, \mathcal{D}_d(a)] = [u, a][\mathcal{D}_d(u), a] + [u, a][u, \mathcal{D}_d(a)].$$

This implies that

$$[u, a][\mathcal{D}_d(u), a][u, a] + [u, a][u, \mathcal{D}_d(a)][u, a] = 0,$$

which can be written as

$$[u, a]\, \mathcal{D}_d([u, a])\, [u, a] = 0.$$

Therefore, if $[u, a] = 0$, then we are done. But if $\mathcal{D}_d([u, a]) = 0$, then substituting u by ru, where $r \in \mathcal{Q}$, we get

$$d(r)[u, a] + r\mathcal{D}_d([u, a]) = 0 \implies d(r)[u, a] = 0,$$

which gives $[u, a] = 0$ for all $u \in \mathcal{U}$ and $a \in \mathcal{A}$. Hence $\mathcal{U} \subseteq \mathcal{Z}(\mathcal{A})$, that means, $\mathcal{U}$ is central on $\mathcal{A}$.

Theorem 4.5 *Let $\mathcal{D}_d$ be a d-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ such that ring derivation d is not commuting on $\mathcal{R}$. Then a non-central Lie ideal $\mathcal{U}$ of $\mathcal{A}$ is commutative if and only if*

$$[\mathcal{D}_d(u), u] = 0 \quad \textit{for all } u \in \mathcal{U}.$$

Proof. Let $[\mathcal{D}_d(u), u] = 0$ for all $u \in \mathcal{U}$. Now, substitute u by $u + v$ for all $v \in \mathcal{U}$, we obtain $[\mathcal{D}_d(u+v), u+v] = 0$, which gives

$$[\mathcal{D}_d(u), v] + [\mathcal{D}_d(v), u] = 0. \tag{4.6}$$

Again, replacing u by ru for any $r \in \mathcal{Q}$, we have

$$[\mathcal{D}_d(ru), v] + [\mathcal{D}_d(v), ru] = 0$$

$$\implies d(r)[u, v] + r\big([\mathcal{D}_d(u), v] + [\mathcal{D}_d(v), u]\big) = 0.$$

Using (4.6), we get $d(r)[u, v] = 0$. This implies that $[u, v] = 0$ for all $u, v \in \mathcal{U}$. Hence $\mathcal{U}$ is commutative.

Conversely, assume that $\mathcal{U}$ is commutative, that means, $[u, v] = 0$ for all $u, v \in \mathcal{U}$. Therefore,

$$\mathcal{D}_d([u, v]) = 0, \quad \text{which implies} \quad [\mathcal{D}_d(u), v] + [u, \mathcal{D}_d(v)] = 0. \tag{4.7}$$

Now, we replace u by $[u, a]$ for all $a \in \mathcal{A}$,

$$[\mathcal{D}_d([u, a]), v] + [[u, a], \mathcal{D}_d(v)] = 0$$

$$\implies [[\mathcal{D}_d(u), a], v] + [[u, \mathcal{D}_d(a)], v] + [[u, a], \mathcal{D}_d(v)] = 0$$

$$\implies [[\mathcal{D}_d(u), a], v] + [[u, a], \mathcal{D}_d(v)] = 0. \tag{4.8}$$

Again, substitute a by $a\mathcal{D}_d(u)$, we obtain

$$[[\mathcal{D}_d(u), a\mathcal{D}_d(u)], v] + [[u, a\mathcal{D}_d(u)], \mathcal{D}_d(v)] = 0$$

$$\implies [[\mathcal{D}_d(u), a]\mathcal{D}_d(u), v] + [a[\mathcal{D}_d(u), \mathcal{D}_d(u)], v] + [\,[u, a]\mathcal{D}_d(u), \mathcal{D}_d(v)] + [a[u, \mathcal{D}_d(u)], \mathcal{D}_d(v)] = 0$$

$$\implies [\mathcal{D}_d(u), a][\mathcal{D}_d(u), v] + [[\mathcal{D}_d(u), a], v]\mathcal{D}_d(u) + [u, a][\mathcal{D}_d(u), \mathcal{D}_d(v)] + [[u, a], \mathcal{D}_d(v)]\mathcal{D}_d(u)$$

$$+ a[[u, \mathcal{D}_d(u)], \mathcal{D}_d(v)] + [a, \mathcal{D}_d(v)][u, \mathcal{D}_d(u)] = 0$$

$$\implies [\mathcal{D}_d(u), a][\mathcal{D}_d(u), v] + [u, a][\mathcal{D}_d(u), \mathcal{D}_d(v)] + a[[u, \mathcal{D}_d(u)], \mathcal{D}_d(v)] + [a, \mathcal{D}_d(v)][u, \mathcal{D}_d(u)] = 0.$$

Also, replacing u by ru for any $r \in \mathcal{Q}$,

$$[\mathcal{D}_d(ru), a][\mathcal{D}_d(ru), v] + [ru, a][\mathcal{D}_d(ru), \mathcal{D}_d(v)] + a[[ru, \mathcal{D}_d(ru)], \mathcal{D}_d(v)] + [a, \mathcal{D}_d(v)][ru, \mathcal{D}_d(ru)] = 0.$$

This simplifies to

$$d(r)r[u, a][\mathcal{D}_d(u), v] + rd(r)[u, a][u, \mathcal{D}_d(v)] = 0.$$

Finally, substituting v by u, we get

$$d(r)r[u, a][\mathcal{D}_d(u), u] + rd(r)[u, a][u, \mathcal{D}_d(u)] = 0$$

$$\implies d(r)r[u, a][\mathcal{D}_d(u), u] - rd(r)[u, a][\mathcal{D}_d(u), u] = 0$$

$$\implies [d(r), r][u, a][\mathcal{D}_d(u), u] = 0.$$

Since $\mathcal{U} \not\subseteq \mathcal{Z}(\mathcal{A})$ and $[d(r), r] \neq 0$, we conclude that $[\mathcal{D}_d(u), u] = 0$ for all $u \in \mathcal{U}$.

Theorem 4.6 *Let $\mathcal{D}_d$ be a d-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char($\mathcal{R}$) $\neq 2$. Then a non-central Lie ideal $\mathcal{U}$ of $\mathcal{A}$ is commutative if and only if*

$$[\mathcal{D}_d(u), u] \in C_{\mathcal{A}}(\mathcal{U}) \quad \textit{for all } u \in \mathcal{U}.$$

Proof. Suppose $[\mathcal{D}_d(u), u] \in C_{\mathcal{A}}(\mathcal{U})$ for all $u \in \mathcal{U}$. Replacing u by $u+v$ for all $v \in \mathcal{U}$, we obtain

$$[\mathcal{D}_d(u+v), u+v] \in C_{\mathcal{A}}(\mathcal{U})$$

$$\implies [\mathcal{D}_d(u), v] + [\mathcal{D}_d(v), u] \in C_{\mathcal{A}}(\mathcal{U}). \tag{4.9}$$

Substitute u by ru for any $r \in \mathcal{Q}$, we have

$$[\mathcal{D}_d(ru), v] + [\mathcal{D}_d(v), ru] \in C_{\mathcal{A}}(\mathcal{U})$$

$$\implies d(r)[u,v] + r\big([\mathcal{D}_d(u), v] + [\mathcal{D}_d(v), u]\big) \in C_{\mathcal{A}}(\mathcal{U}).$$

Using (4.9), it follows that $d(r)[u,v] \in C_{\mathcal{A}}(\mathcal{U})$ and consequently, $[u,v] \in C_{\mathcal{A}}(\mathcal{U})$ for all $u, v \in \mathcal{U}$, that means, $[\mathcal{U},\mathcal{U}] \subseteq C_{\mathcal{A}}(\mathcal{U})$. Hence by Lemma 3.4, we conclude that $\mathcal{U}$ is commutative.

Conversely, let $\mathcal{U}$ be commutative, that is, $[u,v] = 0$ for all $u, v \in \mathcal{U}$. Therefore,

$$[[u, \mathcal{D}_d(u)], v] = 0 \quad \text{for all } u, v \in \mathcal{U}$$

and as a result, we obtain $[u, \mathcal{D}_d(u)] \in C_{\mathcal{A}}(\mathcal{U})$.

Theorem 4.7 *In an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ that admits a d-derivation $\mathcal{D}_d$, if $\mathcal{U}$ is a Lie ideal of $\mathcal{A}$ such that $\mathcal{D}_d(\mathcal{U})$ is commutative, then $[\mathcal{U},\mathcal{U}]$ is central on $\mathcal{A}$.*

Proof. Consider $[\mathcal{D}_d(u), \mathcal{D}_d(v)] = 0$ for all $u, v \in \mathcal{U}$. Replacing u by $[u,a]$ for all $a \in \mathcal{A}$, we have

$$[\mathcal{D}_d([u,a]), \mathcal{D}_d(v)] = 0, \quad \text{which implies} \quad [\mathcal{D}_d(ua - au), \mathcal{D}_d(v)] = 0.$$

This gives

$$[\mathcal{D}_d(ua), \mathcal{D}_d(v)] - [\mathcal{D}_d(au), \mathcal{D}_d(v)] = 0.$$

Expanding, we get

$$[\mathcal{D}_d(u), \mathcal{D}_d(v)]a + \mathcal{D}_d(u)[a, \mathcal{D}_d(v)] + [u, \mathcal{D}_d(v)]\mathcal{D}_d(a) + u[\mathcal{D}_d(a), \mathcal{D}_d(v)] - [\mathcal{D}_d(a), \mathcal{D}_d(v)]u$$

$$-\mathcal{D}_d(a)[u, \mathcal{D}_d(v)] - [a, \mathcal{D}_d(v)]\mathcal{D}_d(u) - a[\mathcal{D}_d(u), \mathcal{D}_d(v)] = 0.$$

After simplification,

$$[\mathcal{D}_d(u), [a, \mathcal{D}_d(v)]] + [[u, \mathcal{D}_d(v)], \mathcal{D}_d(a)] + [u, [\mathcal{D}_d(a), \mathcal{D}_d(v)]] = 0. \tag{4.10}$$

Now, putting $u = ru$, where $r \in \mathcal{Q}$, we get

$$d(r)[u, [a, \mathcal{D}_d(v)]] + r\big([\mathcal{D}_d(u), [a, \mathcal{D}_d(v)]] + [[u, \mathcal{D}_d(v)], \mathcal{D}_d(a)] + [u, [\mathcal{D}_d(a), \mathcal{D}_d(v)]]\big) = 0.$$

Using (4.10) and $d(r) \neq 0$, we obtain

$$[u, [a, \mathcal{D}_d(v)]] = 0. \tag{4.11}$$

Again, setting a by $r_1 a$ for any $r_1 \in \mathcal{Q}$ in (4.10), we get

$$d(r_1)\big([[u, \mathcal{D}_d(v)], a] + [u, [a, \mathcal{D}_d(v)]]\big) + r_1\big([\mathcal{D}_d(u), [a, \mathcal{D}_d(v)]] + [[u, \mathcal{D}_d(v)], \mathcal{D}_d(a)] + [u, [\mathcal{D}_d(a), \mathcal{D}_d(v)]]\big) = 0.$$

Using (4.10), this reduces to

$$d(r_1)\big([[u, \mathcal{D}_d(v)], a] + [u, [a, \mathcal{D}_d(v)]]\big) = 0 \implies [[u, \mathcal{D}_d(v)], a] + [u, [a, \mathcal{D}_d(v)]] = 0.$$

Also, using (4.11), we get

$$[[u, \mathcal{D}_d(v)], a] = 0.$$

Finally, substituting $v = r_2 v$ for any $r_2 \in \mathcal{Q}$, we find

$$d(r_2)[[u, v], a] + r_2[[u, \mathcal{D}_d(v)], a] = 0, \quad \text{which reduces} \quad [[u, v], a] = 0.$$

This implies that $[u, v] \in \mathcal{Z}(\mathcal{A})$ for all $u, v \in \mathcal{U}$, that means, $[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{Z}(\mathcal{A})$. Hence $[\mathcal{U}, \mathcal{U}]$ is central on $\mathcal{A}$.

From the above theorem, we immediately conclude an important remark.

Remark 4.8 *In an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ which admits a d-derivation $\mathcal{D}_d$, if $\mathcal{U}$ is a Lie ideal of $\mathcal{A}$ such that $\mathcal{D}_d(\mathcal{U})$ is commutative then $\mathcal{U}$ may not be central. But there exists a Lie ideal of the form $[\mathcal{U}, \mathcal{U}]$ which is central.*

Theorem 4.9 *Let $\mathcal{D}_d$ be a d-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char($\mathcal{R}$) $\neq 2$. Then a non-central Lie ideal $\mathcal{U}$ of $\mathcal{A}$ is commutative if and only if*

$$[\mathcal{D}_d(\mathcal{U}), \mathcal{D}_d(\mathcal{U})] \subseteq C_{\mathcal{A}}(\mathcal{U}).$$

Proof. Suppose

$$[\mathcal{D}_d(u), \mathcal{D}_d(v)] \in C_{\mathcal{A}}(\mathcal{U}) \quad \text{for all } u, v \in \mathcal{U}. \tag{4.12}$$

Replacing u by ru for any $r \in \mathcal{Q}$, we obtain

$$[\mathcal{D}_d(ru), \mathcal{D}_d(v)] \in C_{\mathcal{A}}(\mathcal{U})$$

$$\implies d(r)[u, \mathcal{D}_d(v)] + r[\mathcal{D}_d(u), \mathcal{D}_d(v)] \in C_{\mathcal{A}}(\mathcal{U}).$$

Using (4.12) and $d(r) \neq 0$, we get $[u, \mathcal{D}_d(v)] \in C_{\mathcal{A}}(\mathcal{U})$. Again, substituting v by $r_1 v$, where $r_1 \in \mathcal{Q}$,

$$d(r_1)[u, v] + r_1[u, \mathcal{D}_d(v)] \in C_{\mathcal{A}}(\mathcal{U}).$$

It implies $[u, v] \in C_{\mathcal{A}}(\mathcal{U})$ for all $u, v \in \mathcal{U}$, that means, $[\mathcal{U}, \mathcal{U}] \subseteq C_{\mathcal{A}}(\mathcal{U})$. So, Lemma 3.4 implicates that $\mathcal{U}$ is commutative.

Conversely, suppose $\mathcal{U}$ is commutative. Then Theorem 4.5 implies $[\mathcal{D}_d(u), u] = 0$ for all $u \in \mathcal{U}$. Putting u by $u + v$ and expanding, we obtain $[\mathcal{D}_d(u), v] + [\mathcal{D}_d(v), u] = 0$. Now, applying $\mathcal{D}_d$ on both sides, we get

$$[{\mathcal{D}_d}^2(u), v] + [\mathcal{D}_d(u), \mathcal{D}_d(v)] + [\mathcal{D}_d(v), \mathcal{D}_d(u)] + [{\mathcal{D}_d}^2(v), u] = 0,$$

which reduces to

$$[{\mathcal{D}_d}^2(u), v] + [{\mathcal{D}_d}^2(v), u] = 0. \tag{4.13}$$

Replacing u by $r_2 u$, where $r_2 \in \mathcal{Q}$, we obtain

$$[{\mathcal{D}_d}^2(r_2 u), v] + [{\mathcal{D}_d}^2(v), r_2 u] = 0$$

$$\implies d^2(r_2)[u, v] + 2d(r_2)[\mathcal{D}_d(u), v] + r_2([{\mathcal{D}_d}^2(u), v] + [{\mathcal{D}_d}^2(v), u]) = 0$$

$$\implies 2d(r_2)[\mathcal{D}_d(u), v] = 0 \quad \text{(using (4.13))}.$$

Again, using char($\mathcal{R}$) $\neq 2$, we have $[\mathcal{D}_d(u), v] = 0$. Finally, setting v by $[v, \mathcal{D}_d(v)]$, we find

$$[\mathcal{D}_d(u), [v, \mathcal{D}_d(v)]] = 0$$

$$\implies [\mathcal{D}_d(u), v\mathcal{D}_d(v)] - [\mathcal{D}_d(u), \mathcal{D}_d(v)v] = 0$$

$$\implies v[\mathcal{D}_d(u), \mathcal{D}_d(v)] + [\mathcal{D}_d(u), v]\mathcal{D}_d(v) - \mathcal{D}_d(v)[\mathcal{D}_d(u), v] - [\mathcal{D}_d(u), \mathcal{D}_d(v)]v = 0$$

$$\implies [v, [\mathcal{D}_d(u), \mathcal{D}_d(v)]] = 0 \quad \text{for all } u, v \in \mathcal{U}.$$

This implicates that $[\mathcal{D}_d(u), \mathcal{D}_d(v)] \in C_{\mathcal{A}}(\mathcal{U})$ for all $u, v \in \mathcal{U}$ and hence, $[\mathcal{D}_d(\mathcal{U}), \mathcal{D}_d(\mathcal{U})] \subseteq C_{\mathcal{A}}(\mathcal{U})$.

Theorem 4.10 *Let $\mathcal{U}$ be a Lie ideal of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ which admits a d-derivation $\mathcal{D}_d$. Then*

(i) *$\mathcal{U} \subseteq \mathcal{Z}(\mathcal{A})$ if and only if $\mathcal{D}_d(\mathcal{U}) \subseteq \mathcal{Z}(\mathcal{A})$.*

(ii) *$\mathcal{U} \subseteq C_{\mathcal{A}}(\mathcal{U})$ if and only if $\mathcal{D}_d(\mathcal{U}) \subseteq C_{\mathcal{A}}(\mathcal{U})$, where $\mathcal{U}$ is non-central and d is not commuting on $\mathcal{R}$.*

Proof. **(i)** Suppose $\mathcal{U} \subseteq \mathcal{Z}(\mathcal{A})$, that is, $[u, a] = 0$ for all $u \in \mathcal{U}$ and $a \in \mathcal{A}$. Therefore $\mathcal{D}_d([u, a]) = 0$, which implies

$$[\mathcal{D}_d(u), a] + [u, \mathcal{D}_d(a)] = 0, \quad \text{that means,} \quad [\mathcal{D}_d(u), a] = 0 \quad \text{for all } a \in \mathcal{A}.$$

It concludes that $\mathcal{D}_d(u) \in \mathcal{Z}(\mathcal{A})$ for all $u \in \mathcal{U}$ and consequently, $\mathcal{D}_d(\mathcal{U}) \subseteq \mathcal{Z}(\mathcal{A})$.

Conversely, suppose $\mathcal{D}_d(\mathcal{U}) \subseteq \mathcal{Z}(\mathcal{A})$, that means, $\mathcal{D}_d(u) \in \mathcal{Z}(\mathcal{A})$ for all $u \in \mathcal{U}$. This implies that

$$[\mathcal{D}_d(u), a] = 0 \quad \text{for all } a \in \mathcal{A}. \tag{4.14}$$

Replacing u by ru for any $r \in \mathcal{Q}$, we get $[\mathcal{D}_d(ru), a] = 0$, which gives

$$d(r)[u, a] + r[\mathcal{D}_d(u), a] = 0 \implies d(r)[u, a] = 0 \quad \text{(using (4.14))}.$$

Hence $[u, a] = 0$ for all $a \in \mathcal{A}$, that means, $u \in \mathcal{Z}(\mathcal{A})$ for all $u \in \mathcal{U}$. Thus, we conclude that $\mathcal{U} \subseteq \mathcal{Z}(\mathcal{A})$.

(ii) Suppose $\mathcal{D}_d(\mathcal{U}) \subseteq C_{\mathcal{A}}(\mathcal{U})$, that is, $\mathcal{D}_d(u) \in C_{\mathcal{A}}(\mathcal{U})$ for all $u \in \mathcal{U}$. Replacing u by ru for any $r \in \mathcal{Q}$, we have

$$d(r)u + r\mathcal{D}_d(u) \in C_{\mathcal{A}}(\mathcal{U}) \implies d(r)u \in C_{\mathcal{A}}(\mathcal{U})$$

(since $C_{\mathcal{A}}(\mathcal{U})$ is a subalgebra of $\mathcal{A}$). Therefore, $u \in C_{\mathcal{A}}(\mathcal{U})$ for all $u \in \mathcal{U}$, which shows that $\mathcal{U} \subseteq C_{\mathcal{A}}(\mathcal{U})$.

Conversely, let $\mathcal{U} \subseteq C_{\mathcal{A}}(\mathcal{U})$, that is, $[u, v] = 0$ for all $u, v \in \mathcal{U}$. Now, by using Theorem 4.5, we obtain $[\mathcal{D}_d(u), u] = 0$ for all $u \in \mathcal{U}$. Hence $\mathcal{D}_d(\mathcal{U}) \subseteq C_{\mathcal{A}}(\mathcal{U})$.

Lemma 4.11 *In an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$, a square-closed Lie ideal $\mathcal{U}$ of $\mathcal{A}$ is commutative if*

$$u \circ v = 0 \quad \textit{for all } u, v \in \mathcal{U}.$$

Proof. Suppose $\mathcal{U}$ be a square-closed Lie ideal of $\mathcal{A}$, that means,

$$u^2 \in \mathcal{U} \quad \text{and} \quad uv + vu \in \mathcal{U} \quad \text{for all } u, v \in \mathcal{U}.$$

If $u \circ v = 0$ for all $u, v \in \mathcal{U}$, then we can write $[u, v] + 2vu = 0$. Replacing v by $[v, \mathcal{D}_d(v)]$, we find

$$[u, [v, \mathcal{D}_d(v)]] + 2[v, \mathcal{D}_d(v)]u = 0. \tag{4.15}$$

Again, by putting $\mathcal{D}_d(v) = v\mathcal{D}_d(v)$, we obtain

$$[u, [v, v\mathcal{D}_d(v)]] + 2[v, v\mathcal{D}_d(v)]u = 0$$

$$\implies [u, v[v, \mathcal{D}_d(v)]] + 2v[v, \mathcal{D}_d(v)]u = 0$$

$$\implies [u, v][v, \mathcal{D}_d(v)] + v\big([u, [v, \mathcal{D}_d(v)]] + 2[v, \mathcal{D}_d(v)]u\big) = 0.$$

By (4.15), it follows that $[u, v][v, \mathcal{D}_d(v)] = 0$. If $[u, v] = 0$, then the result holds. On the other hand, if $[v, \mathcal{D}_d(v)] = 0$ for all $v \in \mathcal{U}$, then Theorem 4.5 concludes that $\mathcal{U}$ is commutative.

Now, we restrict the d-derivation $\mathcal{D}_d$ to a Lie ideal $\mathcal{U}$ and hypothesize it in different forms, such as homomorphism, anti-homomorphism, and Jordan homomorphism to evaluate differential identities in connection with the commutativity of $\mathcal{U}$.

Theorem 4.12 *Let $\mathcal{D}_d$ be a derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char$(\mathcal{R}) \neq 2$. If $\mathcal{D}_d$ acts as a homomorphism (or anti-homomorphism) on a square-closed Lie ideal $\mathcal{U}$ of $\mathcal{A}$, then $\mathcal{U} \subseteq \mathcal{Z}(\mathcal{A})$, that means, $\mathcal{U}$ is central on $\mathcal{A}$.*

Proof. Suppose $\mathcal{D}_d$ acts as a homomorphism on $\mathcal{U}$. In a square-closed Lie ideal $\mathcal{U}$ of $\mathcal{A}$, we have $u^2 \in \mathcal{U}$ for all $u \in \mathcal{U}$. Therefore,

$$uv + vu = (u+v)^2 - u^2 - v^2 \in \mathcal{U} \quad \text{for all } u, v \in \mathcal{U}.$$

So, $2uv \in \mathcal{U}$ for all $u, v \in \mathcal{U}$. Let $[u,a], [v,b] \in \mathcal{U}$ for all $u, v \in \mathcal{U}$ and $a, b \in \mathcal{A}$. By the homomorphism property of $\mathcal{D}_d$, we have

$$\mathcal{D}_d(2[u,a][v,b]) = 2\mathcal{D}_d([u,a])\mathcal{D}_d([v,b]). \tag{4.16}$$

Again, using the derivation property, we obtain

$$\mathcal{D}_d(2[u,a][v,b]) = 2\big(\mathcal{D}_d([u,a])[v,b] + [u,a]\mathcal{D}_d([v,b])\big). \tag{4.17}$$

Combining (4.16) and (4.17), we get

$$2\mathcal{D}_d([u,a])\mathcal{D}_d([v,b]) - 2\big(\mathcal{D}_d([u,a])[v,b] + [u,a]\mathcal{D}_d([v,b])\big) = 0. \tag{4.18}$$

Replacing u by $r_1 u$, where $r_1 \in \mathcal{Q}$, we have

$$2\mathcal{D}_d([r_1u,a])\mathcal{D}_d([v,b]) - 2\big(\mathcal{D}_d([r_1u,a])[v,b] + [r_1u,a]\mathcal{D}_d([v,b])\big) = 0.$$

This implies that

$$2d(r_1)[u,a]\mathcal{D}_d([v,b]) + 2r_1\mathcal{D}_d([u,a])\mathcal{D}_d([v,b]) - 2d(r_1)[u,a][v,b] - 2r_1\mathcal{D}_d([u,a])[v,b] - 2r_1[u,a]\mathcal{D}_d([v,b]) = 0.$$

Using (4.18) and $char(\mathcal{R}) \neq 2$, we get

$$[u,a]\mathcal{D}_d([v,b]) - [u,a][v,b] = 0. \tag{4.19}$$

Finally, setting v by $r_2 v$ for any $r_2 \in \mathcal{Q}$, we obtain

$$[u,a]\mathcal{D}_d([r_2v,b]) - [u,a][r_2v,b] = 0$$

$$\Longrightarrow \; d(r_2)[u,a][v,b] + r_2\big([u,a]\mathcal{D}_d([v,b]) - [u,a][v,b]\big) = 0.$$

Using (4.19), this reduce to

$$d(r_2)[u,a][v,b] = 0, \quad \text{which implies} \quad [u,a][v,b] = 0 \quad \text{for all } u, v \in \mathcal{U} \text{ and } a, b \in \mathcal{A}.$$

This suggest that $\mathcal{U} \subseteq \mathcal{Z}(\mathcal{A})$.

For the other case of $\mathcal{D}_d$, proof follows easily.

Theorem 4.13 *If $\mathcal{D}_d$ acts as a Jordan homomorphism on a square closed Lie ideal $\mathcal{U}$ of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$, then $\mathcal{U}$ is commutative.*

Proof. Since $\mathcal{D}_d$ acts as a Jordan homomorphism on the square closed Lie ideal $\mathcal{U}$, we have

$$\mathcal{D}_d(u \circ v) = \mathcal{D}_d(u) \circ \mathcal{D}_d(v) \quad \text{for all } u, v \in \mathcal{U},$$

that means,

$$\mathcal{D}_d(uv + vu) = \mathcal{D}_d(u)\mathcal{D}_d(v) + \mathcal{D}_d(v)\mathcal{D}_d(u).$$

On the other hand,

$$\mathcal{D}_d(uv + vu) = \mathcal{D}_d(u)v + u\mathcal{D}_d(v) + \mathcal{D}_d(v)u + v\mathcal{D}_d(u).$$

It implies that

$$\mathcal{D}_d(u)v + u\mathcal{D}_d(v) + \mathcal{D}_d(v)u + v\mathcal{D}_d(u) - \mathcal{D}_d(u)\mathcal{D}_d(v) - \mathcal{D}_d(v)\mathcal{D}_d(u) = 0.$$

Replacing u by r_1u, where $r_1 \in \mathcal{Q}$ and then rearranging, we have

$$uv - u\mathcal{D}_d(v) + vu - \mathcal{D}_d(v)u = 0.$$

Substituting v by r_2v for any $r_2 \in \mathcal{Q}$, it reduces to

$$uv + vu = 0, \quad \text{that means,} \quad u \circ v = 0 \quad \text{for all } u, v \in \mathcal{U}.$$

Thus, Lemma 4.11 implicates that $\mathcal{U}$ is commutative.

Now we allow the concepts of co-commuting and co-centralizing for two distinct d-derivations, say d_1-derivation $\mathcal{D}_{d_1}$ and d_2-derivation $\mathcal{D}_{d_2}$ on $\mathcal{A}$ and then derive the following outcomes.

Theorem 4.14 *In an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$, let $\mathcal{U}$ be a square closed Lie ideal of $\mathcal{A}$ and $\mathcal{D}_{d_1}$, $\mathcal{D}_{d_2}$ be respectively d_1-derivation and d_2-derivation on $\mathcal{A}$. Then the following holds:*

(i) *If $\mathcal{D}_{d_1}$ and $\mathcal{D}_{d_2}$ are co-commuting on $\mathcal{U}$, then d_1 and d_2 must be co-commuting on $\mathcal{R}$.*

(ii) *If $\mathcal{D}_{d_1}$ and $\mathcal{D}_{d_2}$ are co-centralizing on $\mathcal{U}$, but d_1 and d_2 are not co-commuting on $\mathcal{R}$, then $\mathcal{U}$ is commutative.*

Proof. **(i)** Since $\mathcal{D}_{d_1}$ and $\mathcal{D}_{d_2}$ are co-commuting on $\mathcal{U}$,

$$\mathcal{D}_{d_1}(u)u - u\,\mathcal{D}_{d_2}(u) = 0 \quad \text{for all } u \in \mathcal{U}. \tag{4.20}$$

Replacing u by ru for any $r \in \mathcal{R}$, we obtain

$$\mathcal{D}_{d_1}(ru)ru - ru\mathcal{D}_{d_2}(ru) = 0$$

$$\Longrightarrow \; d_1(r)ru^2 - rd_2(r)u^2 + r^2\big(\mathcal{D}_{d_1}(u)u - u\mathcal{D}_{d_2}(u)\big) = 0.$$

Using (4.20), this gives

$$(d_1(r)r - rd_2(r))u^2 = 0.$$

Since $u^2 \neq 0$, it follows that

$$d_1(r)r - rd_2(r) = 0 \quad \text{for all } r \in \mathcal{R}.$$

This reflects that d_1 and d_2 are co-commuting on $\mathcal{R}$.

(ii) If $\mathcal{D}_{d_1}$ and $\mathcal{D}_{d_2}$ are co-centralizing on $\mathcal{U}$, then

$$\mathcal{D}_{d_1}(u)u - u\,\mathcal{D}_{d_2}(u) \in \mathcal{Z}(\mathcal{A}) \quad \text{for all } u \in \mathcal{U}. \tag{4.21}$$

Replacing u by ru for any $r \in \mathcal{R}$,

$$d_1(r)ru^2 - rd_2(r)u^2 + r^2\big(\mathcal{D}_{d_1}(u)u - u\mathcal{D}_{d_2}(u)\big) \in \mathcal{Z}(\mathcal{A}).$$

Using (4.21), this reduces to

$$(d_1(r)r - rd_2(r))u^2 \in \mathcal{Z}(\mathcal{A}).$$

Therefore for all $a \in \mathcal{A}$,

$$[(d_1(r)r - rd_2(r))u^2, a] = 0$$

$$\Longrightarrow \; (d_1(r)r - rd_2(r))[u^2, a] = 0$$

$$\Longrightarrow \; (d_1(r)r - rd_2(r))\big([u, a]u + u[u, a]\big) = 0$$

$$\Longrightarrow \; (d_1(r)r - rd_2(r))([u, a] \circ u) = 0.$$

So, either $d_1(r)r - rd_2(r) = 0$ or $[u, a] \circ u = 0$. But, d_1 and d_2 are not co-commuting on $\mathcal{R}$. It follows that

$$[u, a] \circ u = 0 \quad \text{for all } u \in \mathcal{U} \text{ and } a \in \mathcal{A}.$$

Hence, using Lemma 4.11, we implicates that $\mathcal{U}$ is commutative.

Theorem 4.15 *Let $\mathcal{D}_d$ be a d-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ with char$(\mathcal{R}) \neq 2$. For a Lie ideal $\mathcal{U}$ of $\mathcal{A}$ such that $\mathcal{U} \cap \mathcal{Z}(\mathcal{A}) = \{0\}$,*

$$C_{\mathcal{A}}(\mathcal{D}_d(\mathcal{U})) = \mathcal{Z}(\mathcal{A}) = C_{\mathcal{A}}(\mathcal{U}).$$

Proof. Let $x \in C_{\mathcal{A}}(\mathcal{D}_d(\mathcal{U}))$. Then $[\mathcal{D}_d(u), x] = 0$ for all $u \in \mathcal{U}$. Replacing u by $ua - au$ for all $a \in \mathcal{A}$, we obtain

$$[\mathcal{D}_d(ua - au), x] = 0.$$

Expanding and rearranging, we get

$$[\mathcal{D}_d(u)a, x] + [u\mathcal{D}_d(a), x] - [\mathcal{D}_d(a)u, x] - [a\mathcal{D}_d(u), x] = 0$$

$$\implies [\mathcal{D}_d(u), x]a + \mathcal{D}_d(u)[a, x] + u[\mathcal{D}_d(a), x] + [u, x]\mathcal{D}_d(a) - [\mathcal{D}_d(a), x]u$$

$$-\mathcal{D}_d(a)[u, x] - a[\mathcal{D}_d(u), x] - [a, x]\mathcal{D}_d(u) = 0$$

$$\implies \mathcal{D}_d(u)[a, x] + u[\mathcal{D}_d(a), x] + [u, x]\mathcal{D}_d(a) - [\mathcal{D}_d(a), x]u - \mathcal{D}_d(a)[u, x] - [a, x]\mathcal{D}_d(u) = 0$$

$$\implies [\mathcal{D}_d(u), [a, x]] + [u, [\mathcal{D}_d(a), x]] + [[u, x], \mathcal{D}_d(a)] = 0.$$

Again, putting $u = ru$, where $r \in \mathcal{Q}$, we have

$$[\mathcal{D}_d(ru), [a, x]] + [ru, [\mathcal{D}_d(a), x]] + [[ru, x], \mathcal{D}_d(a)] = 0$$

$$\implies d(r)[u, [a, x]] + r([\mathcal{D}_d(u), [a, x]] + [u, [\mathcal{D}_d(a), x]] + [[u, x], \mathcal{D}_d(a)]) = 0$$

$$\implies [u, [a, x]] = 0 \quad \text{for all } u \in \mathcal{U} \quad (\text{using } d(r) \neq 0). \tag{4.22}$$

Hence $[x, a] \in C_{\mathcal{A}}(\mathcal{U})$. Now, replacing a by $\mathcal{D}_d(a)a$ in (4.22), we obtain

$$[u, [\mathcal{D}_d(a)a, x]] = 0$$

$$\implies [u, [\mathcal{D}_d(a), x]a] + [u, \mathcal{D}_d(a)[a, x]] = 0$$

$$\implies [u, [\mathcal{D}_d(a), x]]a + [\mathcal{D}_d(a), x][u, a] + [u, \mathcal{D}_d(a)][a, x] + \mathcal{D}_d(a)[u, [a, x]] = 0. \tag{4.23}$$

Finally, setting a by $r_1 a$, where $r_1 \in \mathcal{Q}$, we have

$$[u, [\mathcal{D}_d(r_1a), x]]r_1a + [\mathcal{D}_d(r_1a), x][u, r_1a] + [u, \mathcal{D}_d(r_1a)][r_1a, x] + \mathcal{D}_d(r_1a)[u, [r_1a, x]] = 0.$$

Expanding,

$$d(r_1)r_1[u, [a, x]]a + d(r_1)r_1[a, x][u, a] + d(r_1)r_1[u, a][a, x] + d(r_1)r_1a[u, [a, x]]$$

$$+ {r_1}^2\big([u, [\mathcal{D}_d(a), x]]a + [\mathcal{D}_d(a), x][u, a] + [u, \mathcal{D}_d(a)][a, x] + \mathcal{D}_d(a)[u, [a, x]]\big) = 0$$

$$\implies d(r_1)r_1([a, x][u, a] + [u, a][a, x]) = 0 \quad (\text{using (4.23)}).$$

Since $[x, a] \in C_{\mathcal{A}}(\mathcal{U})$ and $char(\mathcal{R}) \neq 2$, it reduces to $[a, x][u, a] = 0$. Therefore, either $[a, x] = 0$ or $[u, a] = 0$. As $\mathcal{U}$ is non-central, $[u, a] \neq 0$. Hence $[a, x] = 0$ for all $a \in \mathcal{A}$, that means, $x \in \mathcal{Z}(\mathcal{A})$. Therefore $C_{\mathcal{A}}(\mathcal{D}_d(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{A})$. On the other hand, it is obvious that $\mathcal{Z}(\mathcal{A}) \subseteq C_{\mathcal{A}}(\mathcal{D}_d(\mathcal{U}))$, which concludes

$$C_{\mathcal{A}}(\mathcal{D}_d(\mathcal{U})) = \mathcal{Z}(\mathcal{A}). \tag{4.24}$$

Again, let $x \in C_{\mathcal{A}}(\mathcal{U})$. Then $[x, u] = 0$ for all $u \in \mathcal{U}$. Applying $\mathcal{D}_d$ on both sides and expanding, we obtain

$$[\mathcal{D}_d(x), u] + [x, \mathcal{D}_d(u)] = 0. \tag{4.25}$$

Now, replacing u by $ua - au$ for all $a \in \mathcal{A}$, we get

$$[\mathcal{D}_d(x), ua - au] + [x, \mathcal{D}_d(ua - au)] = 0$$

$$\Longrightarrow [\mathcal{D}_d(x), ua] - [\mathcal{D}_d(x), au] + [x, \mathcal{D}_d(u)a] + [x, u\mathcal{D}_d(a)] - [x, \mathcal{D}_d(a)u] - [x, a\mathcal{D}_d(u)] = 0$$

$$\Longrightarrow [\mathcal{D}_d(x), u]a + u[\mathcal{D}_d(x), a] - a[\mathcal{D}_d(x), u] - [\mathcal{D}_d(x), a]u + \mathcal{D}_d(u)[x, a] + [x, \mathcal{D}_d(u)]a + u[x, \mathcal{D}_d(a)]$$

$$+[x, u]\mathcal{D}_d(a) - [x, \mathcal{D}_d(a)]u - \mathcal{D}_d(a)[x, u] - a[x, \mathcal{D}_d(u)] - [x, a]\mathcal{D}_d(u) = 0$$

$$\Longrightarrow [u, [\mathcal{D}_d(x), a]] + [\mathcal{D}_d(u), [x, a]] + [u, [x, \mathcal{D}_d(a)]] = 0 \qquad \text{(using (4.25))}. \tag{4.26}$$

Putting x by rx for any $r \in \mathcal{Q}$, we have

$$[u, [\mathcal{D}_d(rx), a]] + [\mathcal{D}_d(u), [rx, a]] + [u, [rx, \mathcal{D}_d(a)]] = 0$$

$$\Longrightarrow d(r)[u, [x, a]] + r\big([u, [\mathcal{D}_d(x), a]] + [\mathcal{D}_d(u), [x, a]] + [u, [x, \mathcal{D}_d(a)]]\big) = 0.$$

Using (4.26) and $d(r) \neq 0$, we obtain

$$[u, [x, a]] = 0 \ \text{ for all } u \in \mathcal{U}. \tag{4.27}$$

Thus $[x, a] \in C_{\mathcal{A}}(\mathcal{U})$. Lastly, replacing a by $\mathcal{D}_d(a)a$ in (4.27), we conclude that $x \in \mathcal{Z}(\mathcal{A})$ (same as above). Therefore, $C_{\mathcal{A}}(\mathcal{U}) = \mathcal{Z}(\mathcal{A})$. Using (4.24), finally we conclude that

$$C_{\mathcal{A}}(\mathcal{U}) = \mathcal{Z}(\mathcal{A}) = C_{\mathcal{A}}(\mathcal{D}_d(\mathcal{U})).$$

In the next result, we extend the Posner's second theorem [8] for Lie ideals and discuss it in rp-algebra $\mathcal{A}$ in the following manner.

Theorem 4.16 ([8]) *Let d_1 and d_2 be two ring derivations of a prime ring $\mathcal{R}$ with char$(\mathcal{R}) \neq 2$ such that the product d_1d_2 is also a derivation of $\mathcal{R}$. Then either $d_1 = O$ or $d_2 = O$.*

Theorem 4.17 *Let $\mathcal{D}_{d_1}$ and $\mathcal{D}_{d_2}$ be respectively d_1-derivation and d_2-derivation on an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$ and $\mathcal{U}$ be a Lie ideal of $\mathcal{A}$ such that $\mathcal{U} \cap \mathcal{Z}(\mathcal{A}) = \{0\}$. If*

$$\mathcal{D}_{d_1}\mathcal{D}_{d_2}(\mathcal{U}) = \{0\},$$

then $\mathcal{D}_{d_2} = O$ on $\mathcal{U}$.

Proof. Suppose $\mathcal{D}_{d_1}\mathcal{D}_{d_2}(\mathcal{U}) = \{0\}$, that means,

$$\mathcal{D}_{d_1}\mathcal{D}_{d_2}(u) = 0 \quad \text{ for all } u \in \mathcal{U}. \tag{4.28}$$

Replacing u by $[u, a]$ for all $a \in \mathcal{A}$, we obtain

$$\mathcal{D}_{d_1}([\mathcal{D}_{d_2}(u), a] + [u, \mathcal{D}_{d_2}(a)]) = 0$$

$$\Longrightarrow [\mathcal{D}_{d_1}\mathcal{D}_{d_2}(u), a] + [\mathcal{D}_{d_2}(u), \mathcal{D}_{d_1}(a)] + [\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(a)] + [u, \mathcal{D}_{d_1}\mathcal{D}_{d_2}(a)] = 0$$

$$\Longrightarrow [\mathcal{D}_{d_2}(u), \mathcal{D}_{d_1}(a)] + [\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(a)] + [u, \mathcal{D}_{d_1}\mathcal{D}_{d_2}(a)] = 0 \quad \text{(using (4.28))}. \tag{4.29}$$

Now, we set $\mathcal{D}_{d_1}(a) = \mathcal{D}_{d_1}(a)u$ and $\mathcal{D}_{d_2}(a) = \mathcal{D}_{d_2}(a)u$ and substituting in (4.29), we obtain

$$[\mathcal{D}_{d_2}(u), \mathcal{D}_{d_1}(a)u] + [\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(a)u] + [u, \mathcal{D}_{d_1}(\mathcal{D}_{d_2}(a)u)] = 0$$

$$\implies [\mathcal{D}_{d_2}(u), \mathcal{D}_{d_1}(a)]u + \mathcal{D}_{d_1}(a)[\mathcal{D}_{d_2}(u), u] + [\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(a)]u + \mathcal{D}_{d_2}(a)[\mathcal{D}_{d_1}(u), u] + [u, \mathcal{D}_{d_1}\mathcal{D}_{d_2}(a)]u$$

$$+[u, \mathcal{D}_{d_2}(a)]\mathcal{D}_{d_1}(u) + \mathcal{D}_{d_2}(a)[u, \mathcal{D}_{d_1}(u)] = 0.$$

Using (4.29) and rearranging, we get

$$\mathcal{D}_{d_1}(a)[\mathcal{D}_{d_2}(u), u] + [u, \mathcal{D}_{d_2}(a)]\mathcal{D}_{d_1}(u) = 0. \tag{4.30}$$

Again, substitute u by ru for any $r \in \mathcal{Q}$,

$$d_2(r)r\mathcal{D}_{d_1}(a)[u, u] + r^2\mathcal{D}_{d_1}(a)[\mathcal{D}_{d_2}(u), u] + rd_1(r)[u, \mathcal{D}_{d_2}(a)]u + r^2[u, \mathcal{D}_{d_2}(a)]\mathcal{D}_{d_1}(u) = 0$$

$$\implies r^2\big(\mathcal{D}_{d_1}(a)[\mathcal{D}_{d_2}(u), u] + [u, \mathcal{D}_{d_2}(a)]\mathcal{D}_{d_1}(u)\big) + rd_1(r)[u, \mathcal{D}_{d_2}(a)]u = 0$$

$$\implies rd_1(r)[u, \mathcal{D}_{d_2}(a)]u = 0 \quad \text{(using (4.30))}$$

$$\implies d_1(r)[u, \mathcal{D}_{d_2}(a)]u = 0 \quad \text{(since } r \in \mathcal{Q},\ r \neq 0).$$

Therefore $[u, \mathcal{D}_{d_2}(a)]u = 0$ for all $u \in \mathcal{U}$. It implies that

$$[u, \mathcal{D}_{d_2}(a)] = 0 \quad \text{for all } a \in \mathcal{A}. \tag{4.31}$$

Finally, setting a by ua for all $u \in \mathcal{U}$, we obtain $[u, \mathcal{D}_{d_2}(ua)] = 0$, which gives

$$[u, \mathcal{D}_{d_2}(u)a + u\mathcal{D}_{d_2}(a)] = 0$$

$$\implies [u, \mathcal{D}_{d_2}(u)]a + \mathcal{D}_{d_2}(u)[u, a] + u[u, \mathcal{D}_{d_2}(a)] = 0$$

$$\implies \mathcal{D}_{d_2}(u)[u, a] = 0 \quad \text{for all } a \in \mathcal{A} \quad \text{(using (4.31))}.$$

But for Lie ideal $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{Z}(\mathcal{A}) = \{0\}$, we finally conclude that $\mathcal{D}_{d_2}(u) = 0$ for all $u \in \mathcal{U}$. Therefore $\mathcal{D}_{d_2} = O$ on $\mathcal{U}$.

At the end, we conclude our research work with the Herstein's theorem [9] relating to Lie ideals using d_1 and d_2-derivations in rp-algebra accordingly:

Theorem 4.18 *Let $\mathcal{U}$ be a non-central Lie ideal of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$. For d_1-derivation $\mathcal{D}_{d_1}$ and d_2-derivation $\mathcal{D}_{d_2}$ on $\mathcal{A}$, if*

$$[\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(v)] = 0 \quad \textit{for all } u, v \in \mathcal{U},$$

then either $\mathcal{D}_{d_1}$ or $\mathcal{D}_{d_2}$ is commuting on $\mathcal{U}$.

Proof. Suppose $[\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(v)] = 0$ for all $u, v \in \mathcal{U}$. Replacing u by $[u, a]$ for all $a \in \mathcal{A}$ and then expanding, we obtain

$$[\mathcal{D}_{d_1}(ua), \mathcal{D}_{d_2}(v)] - [\mathcal{D}_{d_1}(au), \mathcal{D}_{d_2}(v)] = 0$$

$$\implies [\mathcal{D}_{d_1}(u)a, \mathcal{D}_{d_2}(v)] + [u\mathcal{D}_{d_1}(a), \mathcal{D}_{d_2}(v)] - [\mathcal{D}_{d_1}(a)u, \mathcal{D}_{d_2}(v)] - [a\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(v)] = 0$$

$$\implies \mathcal{D}_{d_1}(u)[a, \mathcal{D}_{d_2}(v)] + u[\mathcal{D}_{d_1}(a), \mathcal{D}_{d_2}(v)] + [u, \mathcal{D}_{d_2}(v)]\mathcal{D}_{d_1}(a) - [\mathcal{D}_{d_1}(a), \mathcal{D}_{d_2}(v)]u$$

$$-\mathcal{D}_{d_1}(a)[u, \mathcal{D}_{d_2}(v)] - [a, \mathcal{D}_{d_2}(v)]\mathcal{D}_{d_1}(u) = 0.$$

After simplification,

$$[\mathcal{D}_{d_1}(u), [a, \mathcal{D}_{d_2}(v)]] + [u, [\mathcal{D}_{d_1}(a), \mathcal{D}_{d_2}(v)]] + [[u, \mathcal{D}_{d_2}(v)], \mathcal{D}_{d_1}(a)] = 0. \tag{4.28}$$

Again, setting u by ru for $r \in \mathcal{Q}$, we get

$$[\mathcal{D}_{d_1}(ru), [a, \mathcal{D}_{d_2}(v)]] + [ru, [\mathcal{D}_{d_1}(a), \mathcal{D}_{d_2}(v)]] + [[ru, \mathcal{D}_{d_2}(v)], \mathcal{D}_{d_1}(a)] = 0.$$

Expanding,

$$d_1(r)[u, [a, \mathcal{D}_{d_2}(v)]] + r\big([\mathcal{D}_{d_1}(u), [a, \mathcal{D}_{d_2}(v)]] + [u, [\mathcal{D}_{d_1}(a), \mathcal{D}_{d_2}(v)]] + [[u, \mathcal{D}_{d_2}(v)], \mathcal{D}_{d_1}(a)]\big) = 0$$

$$\implies d_1(r)[u, [a, \mathcal{D}_{d_2}(v)]] = 0, \text{ which implies } [u, [a, \mathcal{D}_{d_2}(v)]] = 0. \tag{4.29}$$

Finally, substitute $a = a\mathcal{D}_{d_1}(u)$, we get

$$[u, [a\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(v)]] = 0$$

$$\implies [u, [a, \mathcal{D}_{d_2}(v)]\mathcal{D}_{d_1}(u)] + [u, a[\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(v)]] = 0$$

$$\implies [a, \mathcal{D}_{d_2}(v)][u, \mathcal{D}_{d_1}(u)] + [u, [a, \mathcal{D}_{d_2}(v)]]\mathcal{D}_{d_1}(u) = 0$$

$$\implies [a, \mathcal{D}_{d_2}(v)][u, \mathcal{D}_{d_1}(u)] = 0 \qquad \text{(using (4.35))}.$$

In rp-algebra, it follows that, either $[a, \mathcal{D}_{d_2}(v)] = 0$ or $[u, \mathcal{D}_{d_1}(u)] = 0$.
Case 1: If $[u, \mathcal{D}_{d_1}(u)] = 0$ for all $u \in \mathcal{U}$, then Theorem 4.5 concludes that $\mathcal{D}_{d_1}$ is commuting on $\mathcal{U}$.
Case 2: Otherwise in $[a, \mathcal{D}_{d_2}(v)]$, replacing a by va, we have $[va, \mathcal{D}_{d_2}(v)] = 0$, which gives

$$v[a, \mathcal{D}_{d_2}(v)] + [v, \mathcal{D}_{d_2}(v)]a = 0$$

$$\implies [v, \mathcal{D}_{d_2}(v)]a = 0 \quad \text{for all } a \in \mathcal{A}.$$

Thus, it follows that $[v, \mathcal{D}_{d_2}(v)] = 0$ for all $v \in \mathcal{U}$. This leads to the conclusion that $\mathcal{D}_{d_2}$ is commuting on $\mathcal{U}$.

Corollary 4.19 *Let $\mathcal{U}$ be a non-central Lie ideal of an rp-algebra $\mathcal{A}$ over a ring $\mathcal{R}$. For d_1-derivation $\mathcal{D}_{d_1}$ and d_2-derivation $\mathcal{D}_{d_2}$ on $\mathcal{A}$, if*

$$[\mathcal{D}_{d_1}(u), \mathcal{D}_{d_2}(v)] = 0 \quad \textit{for all } u, v \in \mathcal{U},$$

then $\mathcal{U}$ is commutative.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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