WONG-ZAKAI APPROXIMATIONS FOR NON-AUTONOMOUS STOCHASTIC PARABOLIC EQUATIONS INVOLVING A SUBELLIPTIC OPERATOR DRIVEN BY NONLINEAR NOISE ON UNBOUNDED DOMAINS

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Abstract. In this paper, we study the long-term behavior of a class of stochastic parabolic equations involving a subelliptic operator on some unbounded domains perturbed by nonlinear noise. Employing the Wong-Zakai approximation on the noise term combined with the uniform estimates on the tails of solutions, we first show that the Wong-Zakai approximation equation generates a continuous random dynamical system, and then establish the existence and uniqueness of tempered pullback attractors for the Wong-Zakai approximation equation. Moreover, in the cases of additive noise and multiplicative linear noise, we prove the convergence of these attractors to those of the original equation driven by white noise when the Wong-Zakai approximation parameter vanishes. Some difficulties need to be overcome due to the fact that the operator is strongly degenerate and the domain is unbounded.

1. Introduction

In this paper, we consider the initial value problem for a class of stochastic degenerate parabolic equations that has the following form:

$$\begin{cases} du + (-\Delta_\lambda u + \gamma u)dt = (f(t,x,u) + g(t,x))dt + h(t,x,u) \circ dW(t), \\ u(\tau, x) = u_\tau(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^N, t > \tau$, γ is constant, $\tau > 0$ is the initial time of the system, f, h are given functions, $g \in L^2_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^N))$ and $W = W(t,\omega)$ is a real-valued one-dimensional independent two-sided Wiener process on a probability space to be specified later, and the symbol $\circ$ is understood in the sense of Stratonovich integration. Here, Δ_λ is the subelliptic operator (or strongly degenerate operator) of the form

$$\Delta_\lambda u := \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\lambda_j^2(x)\frac{\partial u}{\partial x_j}\right), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N.$$

This operator was introduced by Franchi and Lanconelli in [15], recently, by adding some assumptions on the functions λ_i [25], this operator is now known as Δ_λ-Laplace operator (see Subsection 2.1 below). The operator Δ_λ belongs to the class of degenerate elliptic operators which has received considerable attention over the years. For some elementary properties, typical examples and recent results on the elliptic problems involving Δ_λ-Laplace operator, we refer to the papers [2, 34, 35, 38, 39] and a recent survey paper [26].

We now review some previous results on the long-time behavior of equation of type (1.1). In the deterministic case, i.e., when $h(t,x,u) \equiv 0$, the long-time

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behavior of system (1.1) has been studied by several authors. For example, for $\gamma \equiv 0$ and $g \equiv 0$, the authors in [28] studied in the subcritical growth case, and they proved the existence of solutions and characterized their long-time behavior. They established the existence and finite fractal dimension of the global attractor of the generated semigroup and the convergence of solutions to an equilibrium solution when time tends to infinity. Later, in [31] the authors have considered the case of critical growth nonlinearity and obtained the existence of global attractors. Recently, in [40] Quyet et. al. extended the results in [28], where $g \not\equiv 0$ and a new class of nonlinearities is considered, and they also obtained the existence of a global attractor. Note that, the results mentioned above are for bounded domains. In the case of unbounded domains, the long time behavior of system (1.1) is not well understood. Up to the best of our knowledge, there are few results in this direction, for instance, the existence of a global attractor was proved in [1, 43] in various cases of nonlinearities, however, they only established the results for the degenerate operator has form $P_{\alpha,\beta}u = \Delta_x u + \Delta_y u + |x|^{2\alpha}|y|^{2\beta}\Delta_z u$, with $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, which is a special case of Δ_λ-Laplace operator. For more results on this topic, we refer the readers to the papers [27, 42, 37] and references therein.

In the stochastic case, i.e., when $h \not\equiv 0$. In the case of the classical Δ-Laplace operator, there are many results on the existence of solutions as well as the long-time behavior of solutions of some various classes of parabolic equations are obtained, both in the case of additive noise and multiplicative linear noise, see e.g. [6, 17] and references therein (see also [7, 11, 12, 14] where the diffusion term h has a very special structure and is in the context of the Navier-Stokes equations). In the case of degenerate parabolic equations, it seems very few results on the case of strongly degenerate operator Δ_λ, very recently, in [16] the authors considered the regularity of Wong-Zakai approximations of the non-autonomous stochastic degenerate parabolic equations with X-elliptic operators on bounded domains, where the existence of the pullback random attractors with a general diffusion term. Note that the X-elliptic operator was explicitly introduced in [30], based on some ideas applied in [15] and this operator contains many degenerate elliptic operators, such as sub-Laplacians on homogeneous Carnot groups, Grushin operator, and the strongly degenerate Δ_λ as mentioned above. For the long term behavior of various stochastic degenerate parabolic equations, we refer the reader to [10, 13, 29, 33, 36] and references cited therein.

In the non-autonomous cases, it is well-known in the literatures, to treat the noise term when the noise $h(t, x, u) \circ dW(t) \equiv h(x) \circ dW(t)$ or $h(t, x, u) \circ dW(t) \equiv u \circ dW(t)$, i.e., the noise is additive or multiplicative, respectively, we may use the change of variables technique to transform the original equations into the new equations which have only random coefficients, and exploit the methods as in the deterministic cases to obtain the existence of global random attractors or pullback random attractors, see e.g. [6, 13, 44, 49]. Namely, by standard techniques we first prove the stochastic differential equation generates a random dynamical system, and then show the existence of a family of compact sets, which is pullback absorbing for the solution process of the original system, see e.g., [3, 11, 12]. However, in general case, i.e., when the noise has form $h(t, x, u) \circ dW(t)$ where $h(t, x, u)$ is a nonlinear function with respect to u, it seems very difficulty to obtain the informations on the existence of random attractors because we do not have a suitable the change of variables to treat the diffusion term $h(t, x, u)$. Therefore, it seems that there are very few works in the literature dealing with the existence of random attractors for stochastic partial differential equations with general nonlinear noise. In order to deal with the nonlinear diffusion term, the author in [20] used the Wong-Zakai approximations given by a stationary process via the Wiener shift to study the existence of random attractors for the two-dimensional stochastic Navier-Stokes equations with a general Lipschitz nonlinearity. We notice that the Wong-Zakai approximations established by Wong and Zakai in 1965 (see [54, 55]), and have been used to study the solutions and dynamics of some stochastic equations, see e.g. [21, 22, 32, 47, 52, 53]. It is worth mentioning that we may use the concept of weak pullback mean random attractors was introduced in [45] to deal with the Lipschitz diffusion term. For more results on this direction, we refer the reader to [18, 19, 23, 46] and references therein.

Motivated by the above works, in this work we investigate the long-term behavior of system (1.1) driven by a general nonlinear noise on unbounded domains. Our problem here is not straightforward from the literature mentioned above, and when dealing with system (1.1) there are some essential difficulties that we must overcome. One of the main difficulty of this paper lies in the non-compactness of Sobolev embedding $H^1_\lambda(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ and so the asymptotic compactness of solutions cannot be obtained by the standard method. In the case of deterministic equations, this difficulty can be overcome by the energy equation approach, introduced by Ball in [5] and successfully applied to deterministic equations [8, 9] as well as stochastic equations [6, 24, 51]. In our case, we overcome this difficulty by using the method of uniform estimates on the tails of solutions [48] (see also [4, 41]). Precisely, for every $\epsilon > 0$, we show that there exists a large open ball $\mathcal{O}_k \subset \mathbb{R}^N$ with center at origin and radius $k > 0$ such that the solutions are uniformly less than ϵ in $L^2(\mathbb{R}^N \setminus \mathcal{O}_k)$ when time is sufficiently large. Since $\mathcal{O}_k$ is bounded and the embedding $H^1_\lambda(\mathbb{R}^N) \hookrightarrow L^2(\mathcal{O}_k)$ is compact in $L^2(\mathbb{R}^N)$, by the uniform estimates, we can prove that the solutions are compact in $L^2(\mathcal{O}_k)$. Consequently, the solutions are covered by a finite number of open balls in $L^2(\mathcal{O}_k)$ with radii less than ϵ. This along with the uniform tail-estimates implies that the solutions are covered by a finite of open balls in $L^2(\mathbb{R}^N)$ with radii less than ϵ, and hence the associated cocycle is asymptotically compact in $L^2(\mathbb{R}^N)$, (for more details see Lemma 3.5 in Subsection 3.2 below). Another difficulty that occurs when establishing the uniform estimates on the tails of solutions come from the degeneracy of the operator Δ_λ. In this case, to obtain the uniform of the solutions we cannot use the usual test function as in the case of the Laplace operator, instead, we need to choose a test function which is suitable with the structure degeneracy of Δ_λ operator, and employ more delicate computations to obtain the uniform estimates (see Lemma 3.4 in Subsection 3.2). Compared with the equations with standard Laplace operator, the uniform estimates on the tails of solutions are much more involved because of the degeneracy of the Δ_λ-Laplace operator. Our results obtained here are interesting and new for the strongly degenerate operator, even in the deterministic case on unbounded domains.

The paper is organized as follows. In Section 2, we recall some basic notations, definitions and results on strongly degenerate operator, on the theory of random dynamical systems as well as Wong-Zakai approximations. In Section 3, we prove the existence and uniqueness of random attractors to problem (1.1) driven by Wong-Zakai approximations. In the last Sections 4 and 5, we prove the convergence of solutions and attractors of approximate equations when the step size of approximations approaches zero for linear multipicative noise and additive noise, respectively.

2. PRELIMINARY RESULTS

2.1. The Δ_λ-Laplace operator.

In this subsection, we recall the definition and properties of the Δ_λ-Laplace operator as well as Sobolev spaces (see [25]).

Let $N \geq 2$, we consider the following operator

$$\Delta_\lambda u := \sum_{j=1}^{N} \partial_{x_j}(\lambda_j^2(x)\partial_{x_j} u),$$

where $\partial_{x_j} = \frac{\partial}{\partial x_j}$, $j = 1, \dots, N$. Here the functions $\lambda_j : \mathbb{R}^n \to \mathbb{R}$ are continuous, strictly positive and of class C^1 outside the coordinate hyperplanes, i.e., $\lambda_j > 0, j = 1, \dots, N$ in $\mathbb{R}^N \setminus \prod$, where $\prod = \{(x_1, \dots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0\}$. As in [25] we assume that λ_j satisfy the following properties:

(1) $\lambda_1(x) \equiv 1,\ \lambda_j(x) = \lambda_j(x_1, \dots, x_{j-1}), j = 2, \dots, N$;

(2) For every $x \in \mathbb{R}^N$, $\lambda_j(x) = \lambda_j(x^*), j = 1, \dots, N$, where
$$x^* = (|x_1|, \dots, |x_N|) \text{ if } x = (x_1, \dots, x_N);$$

(3) There exists a constant $\rho \geq 0$ such that
$$0 \leq x_k \partial_{x_k} \lambda_i(x) \leq \rho \lambda_j(x) \quad \forall k \in \{1, \dots, j-1\}, j = 2, \dots, N,$$
and for every $x \in \mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_j \geq 0\ \forall j = 1, \dots, N\}$;

(4) There exists a group of dilations $\{\delta_t\}_{t>0}$
$$\delta_t : \mathbb{R}^N \to \mathbb{R}^N, \delta_t(x) = \delta_t(x_1, \dots, x_N) = (t^{\epsilon_1} x_1, \dots, t^{\epsilon_N} x_N),$$
where $1 \leq \epsilon_1 \leq \epsilon_2 \leq \dots \leq \epsilon_N$, such that λ_j is δ_t-homogeneous of degree $\epsilon_j - 1$, i.e.,
$$\lambda_j(\delta_t(x)) = t^{\epsilon_j - 1}\lambda(x), \quad \forall x \in \mathbb{R}^N, t > 0, j = 1, \dots, N.$$
This implies that the operator Δ_λ is δ_t-homogeneous of degree two, i.e.,
$$\Delta_\lambda(u(\delta_t(x))) = t^2(\Delta_\lambda u)(\delta_t(x)), \quad \forall u \in C^\infty(\mathbb{R}^N).$$

We denote by Q the homogeneous dimension of $\mathbb{R}^N$ with respect to the group of dilations $\{\delta_t\}_{t>0}$, i.e.,

$$Q := \epsilon_1 + \dots + \epsilon_N.$$

For $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$ and $p \geq 1$, we denote by $\overset{\circ}{W}{}^{1,p}_\lambda(\mathcal{O})$ the completion of $C_0^\infty(\mathcal{O})$ in the norm

$$\|u\|_{\overset{\circ}{W}{}^{1,p}_\lambda} = \left(\int_{\mathcal{O}} |\nabla_\lambda u|^p dx\right)^{\frac{1}{p}},$$

where $\nabla_\lambda u = (\lambda_1 \partial_{x_1} u, \lambda_2 \partial_{x_2} u, \dots, \lambda_N \partial_{x_N} u)$. When $p = 2$, we denote $H^1_\lambda(\mathcal{O}) = \overset{\circ}{W}{}^{1,2}_\lambda(\mathcal{O})$, then the following useful embedding was established in [25].

Lemma 2.1. *The embedding $H^1_\lambda(\mathcal{O}) \hookrightarrow L^{2^*_\lambda}(\mathcal{O})$, where $2^*_\lambda = \frac{2Q}{Q-2}$, is continuous. Moreover, the embedding $H^1_\lambda(\mathcal{O}) \hookrightarrow L^\gamma(\mathcal{O})$ is compact for every $\gamma \in [1, 2^*_\lambda)$.*

We consider the operator $-\Delta_\lambda : H^1_\lambda(\mathcal{O}) \to L^2(\mathcal{O})$, and set $A = -\Delta_\lambda$, then by Lemma 2.1, A is a linear, positive, self-adjoint operator with compact inverse. Consequently, there exists an orthonormal basis of $L^2(\mathcal{O})$ consisting of eigenfunctions $\varphi_j \in H^1_\lambda(\mathcal{O}),\ j = 1, 2, \dots$ of the operator A with eigenvalues

$$0 < \mu_1 \leq \mu_2 \leq \cdots \text{ and } \mu_j \to +\infty \text{ as } j \to +\infty.$$

We also denote the Sobolev space

$$H^1_\lambda(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R}, u \in L^2(\mathbb{R}^N), |\nabla_\lambda u| \in L^2(\mathbb{R}^N)\}$$

equipped with the norm

$$\|u\|^2_{H^1_\lambda(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla_\lambda u|^2 + |u|^2) dx,$$

then $H^1_\lambda(\mathbb{R}^N)$ is a Hilbert space, and by [1, Lemma 2.1] (see also [34, 35]) we have the following embedding

$$H^1_\lambda(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$$

is continuous for $p \in [2, 2^*_\lambda]$.

The dual spaces of $H^1_\lambda(\mathcal{O})$ and $H^1_\lambda(\mathbb{R}^N)$ are denoted by $H^{-1}_\lambda(\mathcal{O})$ and $H^{-1}_\lambda(\mathbb{R}^N)$, respectively.

We define $W^{2,p}_\lambda(\mathbb{R}^N)$ as the space of all functions u such that

$$u \in L^p(\mathbb{R}^N),\ \lambda_i(x)\frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N),\ \lambda_i(x)\frac{\partial}{\partial x_i}\left(\lambda_j(x)\frac{\partial u}{\partial x_j}\right) \in L^p(\mathbb{R}^N),\ i,j = 1,\dots,N,$$

with the norm

$$\|u\|_{W^{2,p}_\lambda} = \left(\int_{\mathbb{R}^N} \left[|u|^p + |\nabla_\lambda u|^p + \sum_{i,j=1}^N \left|\lambda_i(x)\frac{\partial}{\partial x_i}(\lambda_j(x)\frac{\partial u}{\partial x_j})\right|^p\right] dx\right)^{\frac{1}{p}}.$$

We see that $W^{2,p}_\lambda(\mathbb{R}^N)$ is a Banach space and when $p = 2$ the space $W^{2,2}_\lambda(\mathbb{R}^N)$ becomes a Hilbert space with the following inner product

$$(u,v)_{W^{2,2}_\lambda} = (u,v)_{L^2} + \sum_{i=1}^N (\lambda_i \frac{\partial u}{\partial x_i}, \lambda_i \frac{\partial v}{\partial x_i})_{L^2} + \sum_{i,j=1}^N \left(\lambda_i \frac{\partial}{\partial x_i}(\lambda_j \frac{\partial u}{\partial x_j}), \lambda_i \frac{\partial}{\partial x_i}(\lambda_j \frac{\partial v}{\partial x_j})\right)_{L^2}.$$

For simplicity, throughout this paper we will write $\|\cdot\|_{L^2} = \|\cdot\|$.

2.2. **Random dynamical systems.** In this subsection, we recall some basic concepts on the theory of non-autonomous random attractors for random dynamical systems, for more details, we refer the readers to [3, 49, 50].

Let $(X, \|\cdot\|_X)$ be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 2.1. *$(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_s\theta_t$ for all $s, t \in \mathbb{R}$, and $\theta_t(\mathbb{P}) = \mathbb{P}$ for all $t \in \mathbb{R}$.*

Definition 2.2. *A random dynamical system (RDS for short) is a pair (θ, Φ) consists of a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and a cocycle mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$, which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X))$-measurable and satisfies the following properties:*

(i) $\Phi(0, \tau, \omega, \cdot)$ is the identity of X;

(ii) $\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, x))$ for all $\tau \in \mathbb{R}$, $t, s \in \mathbb{R}^+, x \in X$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Moreover, Φ is said to be continuous if $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t \in \mathbb{R}^+$.

Definition 2.3. *A mapping $\psi : \mathbb{R} \times \mathbb{R} \times \Omega \to X$ is called a complete orbit of Φ if for every $t \in \mathbb{R}^+, \tau, s \in \mathbb{R}$ and $\omega \in \Omega$, the map ψ satisfies the following condition*

$$\Phi(t, \tau + s, \theta_s\omega, \psi(s, \tau, \omega)) = \psi(t + s, \tau, \omega).$$

In addition, if there exists $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ such that $\psi(t, \tau, \omega)$ belongs to $D(\tau + t, \theta_t\omega)$ for every $t, \tau \in \mathbb{R}$, then ψ is called a $\mathcal{D}$-complete orbit of Φ.

Definition 2.4. *A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $T = T(D, \tau, \omega) > 0$ such that*

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{for all } t \geq T.$$

In addition, if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the set $K(\tau,\omega)$ is a closed nonempty subset of X and K is measurable in ω with respect to $\mathcal{F}$, then K is called a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.5. *The continuous cocycle Φ is called $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)_{n=1}^{\infty}$ is precompact in X whenever $t_n \to +\infty$ and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.*

Definition 2.6. *A family $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback random attractor for Φ if the following conditions are satisfied, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$:*

(i) $\mathcal{A}$ is measurable in ω with respect to $\mathcal{F}$ and $\mathcal{A}(\tau,\omega)$ is compact in X;
(ii) $\mathcal{A}$ is invariant, that is $\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\theta_t\omega)$, for all $t \geq 0$;
(iii) $\mathcal{A}$ attracts every set in $\mathcal{D}$, that is, for every $D \in \mathcal{D}$,

$$\lim_{t\to+\infty} \operatorname{dist}(\Phi(t,\tau-t,\theta_{-t}\omega, D(\tau-t,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0,$$

where dist is the Hausdorff semi-distance given by

$$\operatorname{dist}(E,F) = \sup_{x\in E}\inf_{y\in F} \|x-y\|_X \qquad \text{for any } E,F \subset X.$$

In addition, if there exists $T > 0$ such that

$$\mathcal{A}(\tau+T,\omega) = \mathcal{A}(\tau,\omega), \qquad \forall \tau \in \mathbb{R}, \forall \omega \in \Omega,$$

then we say $\mathcal{A}$ is periodic with period T.

By the above definitions, we now state the following result for the existence of $\mathcal{D}$-pullback random attractor for non-autonomous random dynamical systems, see e.g. [49].

Proposition 2.1. *Let $\mathcal{D}$ be an inclusion closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $\mathbb{R}$ and over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$. If Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$ and Φ is $\mathcal{D}$-pullback asymptotically compact in X, then Φ has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}$ in $\mathcal{D}$ which is given by*

$$\begin{aligned}\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega) &= \bigcup_{D\in\mathcal{D}} \Omega(D,\tau,\omega)\\ &= \{\psi(0,\tau,\omega) : \psi \text{ is a } \mathcal{D} - \text{complete orbit of } \Phi\},\end{aligned}$$

where $\Omega(K)$ and $\Omega(D)$ are the omega-limit sets of K and D, respectively. In addition, if Φ and K are T-periodic then $\mathcal{A}$ is also T-periodic.

We next recall some results concerned with the upper semicontinuity of non-autonomous pullback random attractors from [50]. Let I be an interval such that $\delta_0 \in I$ and for each $\delta \in I$, Φ_δ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Assume that for every $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, \delta_n \in I$ such that $\delta_n \to \delta_0$, and $x_n \to x$ for $x_n, x \in X$,

$$\lim_{n\to\infty} \Phi_{\delta_n}(t,\tau,\omega,x_n) = \Phi_{\delta_0}(t,\tau,\omega,x). \tag{2.1}$$

For each $\delta \in I$, suppose that $\mathcal{D}_\delta$ be a collection of some families of subsets of X, and for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $R_{\delta_0}(\tau,\omega) > 0$ such that

$$D = \{D(\tau,\omega) = \{x \in X : \|x\|_X \leq R_{\delta_0}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_{\delta_0}. \tag{2.2}$$

Next, for $\delta \in I$ is given and let $\mathcal{A}_\delta \in \mathcal{D}$ and $K_\delta \in \mathcal{D}$ be a $\mathcal{D}_\delta$-pullback random attractor and a $\mathcal{D}_\delta$-pullback absorbing set of Φ, respectively, such that for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\limsup_{\delta \to \delta_0} \|K_\delta(\tau, \omega)\|_X \le R_{\delta_0}(\tau, \omega), \tag{2.3}$$

where $R_{\delta_0}(\tau, \omega)$ is given as in (2.2).

Finally, we assume that for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{x_n\}_{n=1}^\infty \text{ is precompact in } X \text{ whenever } \delta_n \to \delta_0 \text{ and } x_n \in \mathcal{A}_{\delta_n}(\tau, \omega). \tag{2.4}$$

Proposition 2.2 ([50]). *Suppose that* (2.1) *and* (2.3)–(2.4) *hold. Then for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *the non-autonomous pullback random attractor is upper semicontinuous, that is,*

$$\mathrm{dist}(\mathcal{A}_\delta(\tau, \omega), \mathcal{A}_{\delta_0}(\tau, \omega)) \to 0 \qquad \text{as } \delta \to \delta_0.$$

2.3. **Wong-Zakai approximations.**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classical Wiener probability space, where

$$\Omega = C_0(\mathbb{R}, \mathbb{R}) = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

with the open compact topology, $\mathcal{F}$ is its Borel σ-algebra and $\mathbb{P}$ is the Wiener measure. The Brownian motion has the form $W(t, \omega) = \omega(t)$, and we consider the Wiener shift θ_t defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

Then from [3], we known that the probability measure $\mathbb{P}$ is an ergodic invariant measure for θ_t and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ becomes a metric dynamical system. And there exists a $\{\theta_t\}_{t \in \mathbb{R}}$-invariant subset $\Omega \subseteq \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$, we have

$$\frac{\omega(t)}{t} \to 0 \qquad \text{as } t \to \pm\infty. \tag{2.5}$$

In what follows, we will write Ω as the space $\tilde{\Omega}$. For each $\delta \in \mathbb{R}$, we denote $\mathcal{W}_\delta : \Omega \to \mathbb{R}$ is the random variable defined by

$$\mathcal{W}_\delta(\omega) = \frac{1}{\delta} \omega(\delta), \qquad \forall \omega \in \Omega,$$

then we have

$$\mathcal{W}_\delta(\theta_t \omega) = \frac{1}{\delta} \theta_t(\delta) = \frac{\omega(t + \delta) - \omega(t)}{\delta}, \tag{2.6}$$

and

$$\int_0^t \mathcal{W}_\delta(\theta_s \omega) ds = \int_t^{t+\rho} \frac{\omega(s)}{s} ds + \int_\rho^0 \frac{\omega(s)}{s} ds. \tag{2.7}$$

By properties of Brownian motions, we known that $\mathcal{W}_\delta(\theta_t \omega)$ is a stationary stochastic process with a normal distribution and is unbounded in t for almost all $\omega \in \Omega$. Hence, $\mathcal{W}_\delta(\theta_t \omega)$ can be viewed as an approximation of white noise in the following sense

$$\lim_{\delta \to 0} \sup_{t \in [0, T]} \left| \int_0^t \mathcal{W}_\delta(\theta_s \omega) ds - W(t, \omega) \right| = 0 \qquad \text{a.s. for each } T > 0.$$

Moreover, by (2.6) and (2.7) and the continuity of ω, we obtain the uniform convergence of $\mathcal{W}_\delta$ on any finite interval, i.e., for $\tau \in \mathbb{R}, \omega \in \Omega$ and $T > 0$, then for every $\epsilon > 0$, there exists $\delta_0 = \delta_0(\epsilon, \tau, \omega, T) > 0$ such that for all $0 < |t| < \delta_0$ and $t \in [\tau, \tau + T]$,

$$\left| \int_0^t \mathcal{W}_\delta(\theta_s \omega) ds - W(t, \omega) \right| < \epsilon. \tag{2.8}$$

Note that the continuity of $\omega(t)$ on $[\tau,\tau+T]$ implies that there exists $c=c(\tau,\omega,T)>0$ such that

$$|\omega(t)|\le c \quad \forall t\in[\tau,\tau+T]. \tag{2.9}$$

From this and by (2.8), we have there exist $\delta_1=\delta_1(\tau,\omega,T)>0$ and $c=c(\tau,\omega,T)>0$ such that for all $0<|\delta|<\delta_1$ and $t\in[\tau,\tau+T]$,

$$\left|\int_0^t \mathcal{W}_\delta(\theta_s\omega)ds\right| \le \left|\int_0^t \mathcal{W}_\delta(\theta_s\omega)ds - W(t,\omega)\right| + |W(t,\omega)| \le c. \tag{2.10}$$

3. Random dynamical systems for stochastic degenerate parabolic equations by Wong-Zakai approximations

To prove problem (1.1) generates a random dynamical system, we first define a continuous cocycle for random degenerate parabolic systems driven by approximate white noise (called Wong-Zakai approximations), and then show the existence of pullback random attractors. To do this, we need to the following assumptions on the nonlinearity f and the nonlinear diffusion term h.

- **Assumptions on f:** We assume the nonlinearity $f:\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

$$f(t,x,s)s \le -c_1|s|^p + f_1(t,x), \quad \forall t,s\in\mathbb{R}, x\in\mathbb{R}^N, \tag{3.1}$$

$$|f(t,x,s)| \le c_2|s|^{p-1} + f_2(t,x), \quad \forall t,s\in\mathbb{R}, x\in\mathbb{R}^N, \tag{3.2}$$

$$\frac{\partial}{\partial s}f(t,x,s) \le -c_3|s|^{p-2} + f_3(t,x), \quad \forall t,s\in\mathbb{R}, x\in\mathbb{R}^N, \tag{3.3}$$

$$\left|\frac{\partial f}{\partial s}(t,x,s)\right| \le f_4(t,x)(1+|s|^{p-2}), \tag{3.4}$$

where $p>2$ and c_1,c_2,c_3 are positive constants, $f_1\in L^1_{\text{loc}}(\mathbb{R};L^1(\mathbb{R}^N)), f_2\in L^{p_1}_{\text{loc}}(\mathbb{R};L^{p_1}(\mathbb{R}^N))$ with $\frac{1}{p_1}+\frac{1}{p}=1$, $f_3\in L^\infty_{\text{loc}}(\mathbb{R};L^\infty(\mathbb{R}^N))$, and $f_4\in L^\infty(\mathbb{R};L^\infty(\mathbb{R}^N))$.

- **Assumptions on h:** The diffusion function $h:\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

$$|h(t,x,s)| \le h_1(t,x)|s|^{q-1} + h_2(t,x), \quad \forall t,s\in\mathbb{R}, x\in\mathbb{R}^N, \tag{3.5}$$

$$\left|\frac{\partial}{\partial s}h(t,x,s)\right| \le h_3(t,x)|s|^{q-2} + h_4(t,x), \quad \forall t,s\in\mathbb{R}, x\in\mathbb{R}^N, \tag{3.6}$$

where $2\le q<p$ and $h_1\in L^{\frac{p}{p-q}}_{\text{loc}}(\mathbb{R};L^{\frac{p}{p-q}}(\mathbb{R}^N)), h_2\in L^{p_1}_{\text{loc}}(\mathbb{R};L^{p_1}(\mathbb{R}^N))$ with $\frac{1}{p_1}+\frac{1}{p}=1$, and $h_3,h_4\in L^\infty_{\text{loc}}(\mathbb{R};L^\infty(\mathbb{R}^N))$.

Remark 3.1. *The assumption* (3.1) *is a dissipativity condition (sometimes called coercivity), while condition* (3.2) *is a growth condition on f in s, which may include subcritical, critical, or even supercritical cases. The assumptions* (3.5) *and* (3.6) *ensure that the nonlinearity in the diffusion is weaker than in the reaction term. Together, conditions* (3.3) *and* (3.4) *are crucial for establishing the energy estimates and a priori bounds as well as proving global existence and uniqueness of solutions to our problem.*

3.1. Continuous cocycles.

Given $\tau,\delta\in\mathbb{R}$ with $\delta\neq 0$. We consider the following Wong-Zakai approximation of the non-autonomous stochastic degenerate parabolic equations defined for $x\in\mathbb{R}^N$ and $t>\tau$

$$\begin{cases} u_t-\Delta_\lambda u+\gamma u = f(t,x,u)+g(t,x)+h(t,x,u)\mathcal{W}_\delta(\theta_t\omega), \\ u(\tau,x)=u_\tau(x), \quad x\in\mathbb{R}^N, \end{cases} \tag{3.7}$$

where $g \in L^2_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^N))$. We now prove the existence and uniqueness of solutions of equations (3.7) in $L^2(\mathbb{R}^N)$. To do this, we first introduce the definition of weak solutions for the equation.

Definition 3.1. *Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $u_\tau \in L^2(\mathbb{R}^N)$. A function $u(\cdot, \tau, \omega, u_\tau) \in C([\tau, \infty), L^2(\mathbb{R}^N)) \cap L^2_{\text{loc}}(\tau, \infty; H^1_\lambda(\mathbb{R}^N) \cap L^{p_1}_{\text{loc}}(\tau, \infty; L^{p_1}(\mathbb{R}^N))$ and*

$$\frac{du}{dt} \in L^2_{\text{loc}}(\tau, \infty; H^{-1}_\lambda(\mathbb{R}^N)) \cap L^{p_1}_{\text{loc}}(\tau, T; L^{p_1}(\mathbb{R}^N))$$

is called a weak solution of (3.7) if $u(\tau, \tau, \omega, u_\tau) = u_\tau$ and for every $\xi \in H^1_\lambda(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \cap L^{p_1}(\mathbb{R}^N)$,

$$\begin{aligned}&\frac{d}{dt}(u, \xi) + \int_{\mathbb{R}^N} \nabla_\lambda u \cdot \nabla_\lambda \xi dx + \gamma(u, \xi)\\ &= \int_{\mathbb{R}^N} f(t, x, u)\xi dx + (g(t, \cdot), \xi) + \mathcal{W}_\delta(\theta_t \omega)(h(t, \cdot, u), \xi)\end{aligned}$$

in the sense of distribution on $[\tau, \infty)$.

Next, for each $k = 1, 2, \dots$, we denote by

$$\mathcal{O}_k = B_1(0, k^{\epsilon_1}) \times B_2(0, k^{\epsilon_2}) \times \cdots \times B_N(0, k^{\epsilon_N})$$

and consider the following equations defined in $\mathcal{O}_k$

$$\begin{cases} \dfrac{\partial u_k}{\partial t} - \Delta_\lambda u_k + \gamma u_k = f(t, x, u_k) + g(t, x) + h(t, x, u_k)\mathcal{W}_\delta(\theta_t \omega), \ t > \tau, x \in \mathcal{O}_k, \\ u_k(t, x) = 0, \quad t > \tau, x \in \partial\mathcal{O}_k, \\ u_k(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_k. \end{cases} \tag{3.8}$$

Since system (3.8) is deterministic with random coefficients defined on bounded domains $\mathcal{O}_k$, thus for every $\tau \in \mathbb{R}, \omega$ and $u_\tau \in L^2(\mathcal{O}_k)$ are given, we can use Galerkin method as in [40] to prove the well-posedness in $L^2(\mathcal{O}_k)$ of (3.8). Moreover, the solutions u_k of (3.8) are in $(\mathcal{F}, \mathcal{B}(L^2(\mathcal{O}_k)))$-measurable with respect to $\omega \in \Omega$. We now show that the solution u_k of (3.8) tends to the corresponding solution of (3.7) as $k \to \infty$.

Lemma 3.1. *Let (3.1)–(3.6) hold. Then for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $u_\tau \in L^2(\mathbb{R}^N)$, problem (3.7) has a unique solution*

$$u(\cdot, \tau, \omega, u_\tau) \in C([\tau, \infty), L^2(\mathbb{R}^N)) \cap L^2_{\text{loc}}(\tau, \infty; H^1_\lambda(\mathbb{R}^N)).$$

This solution is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^N)))$-measurable in ω and continuous in initial data u_τ in $L^2(\mathbb{R}^N)$.

Proof. The proof is divided three steps.

Step 1: Uniform estimates on the solutions u_k:

Multiplying u_k on both sides of the first equation in (3.8) we obtain for $t > \tau$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_k\|^2 + \|\nabla_\lambda u_k\|^2 + \gamma\|u_k\|^2 = \int_{\mathcal{O}_k} f(t, x, u_k)u_k dx + \int_{\mathcal{O}_k} g(t, x)u_k dx\\ + \mathcal{W}_\delta(\theta_t \omega)\int_{\mathcal{O}_k} h(t, x, u_k)u_k dx.\end{aligned} \tag{3.9}$$

By (3.1), we can have that

$$\int_{\mathcal{O}_k} f(t, x, u_k)u_k dx \le -c_1 \int_{\mathcal{O}_k} |u_k|^p dx + \int_{\mathcal{O}_k} f_1(t, x) dx, \tag{3.10}$$

and by (3.5), use Hölder's and Young's inequalities, we obtain

$$\mathcal{W}_\delta(\theta_t \omega)\int_{\mathcal{O}_k} h(t, x, u_k)u_k dx \le |\mathcal{W}_\delta(\theta_t \omega)| \int_{\mathcal{O}_k} (h_1(t, x)|u_k|^q + h_2(t, x)|u_k|) dx$$

$$
\begin{aligned}
&\leq \left(\int_{\mathcal{O}_k} |u_k|^{q\cdot p/q} dx\right)^{q/p} \left(\int_{\mathcal{O}_k} |h_1(x,t)\mathcal{W}_\delta(\theta_t\omega)|^{p/(p-q)} dx\right)^{(p-q)/q} \\
&\quad + \left(\int_{\mathcal{O}_k} |u_k|^p dx\right)^{1/p} \left(\int_{\mathcal{O}_k} |h_2(x,t)\mathcal{W}_\delta(\theta_t\omega)|^{p_1} dx\right)^{1/p_1} \\
&\leq \frac{c_1}{4}\int_{\mathcal{O}_k} |u_k|^p dx + c\int_{\mathcal{O}_k} |h_1(x,t)\mathcal{W}_\delta(\theta_t\omega)|^{p/(p-q)} dx \\
&\quad + \frac{c_1}{4}\int_{\mathcal{O}_k} |u_k|^p dx + c\int_{\mathcal{O}_k} |h_2(x,t)\mathcal{W}_\delta(\theta_t\omega)|^{p_1} dx \\
&\leq \frac{c_1}{2}\int_{\mathcal{O}_k} |u_k|^p dx + c_1|\mathcal{W}_\delta(\theta_t\omega)|^{\frac{p}{p-q}} \|h_1(t,x)\|^{\frac{p}{p-q}}_{L^{\frac{p}{p-q}}} \\
&\quad + c_2|\mathcal{W}_\delta(\theta_t\omega)|^{p_1}\|h_2(t,x)\|^{p_1}_{L^{p_1}}. \qquad (3.11)
\end{aligned}
$$

Using again Young's inequality for the last term, we have

$$
\int_{\mathcal{O}_k} g(t,x)u_k dx \leq \frac{\gamma}{4}\|u_k\|^2 + \frac{1}{\gamma}\|g(t)\|^2. \qquad (3.12)
$$

Thus, we could deduce from (3.9)–(3.12) that for $t > \tau$,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_k\|^2 + \|\nabla_\lambda u_k\|^2 + \frac{3}{4}\|u_k\|^2 &\leq -\frac{c_1}{2}\|u_k\|^p_{L^p} + \frac{1}{\gamma}\|g(t)\|^2 + \|f_1(t)\|_{L^1} \\
&\quad + c_1|\mathcal{W}_\delta(\theta_t\omega)|^{\frac{p}{p-q}}\|h_1(t,x)\|^{\frac{p}{p-q}}_{L^{\frac{p}{p-q}}} + c_2|\mathcal{W}_\delta(\theta_t\omega)|^{p_1}\|h_2(t,x)\|^{p_1}_{L^{p_1}},
\end{aligned}
$$

which indicates

$$
\begin{aligned}
\frac{d}{dt}\|u_k\|^2 + 2\|\nabla_\lambda u_k\|^2 + \frac{3}{2}\|u_k\|^2 + c_1\|u_k\|^p_{L^p} &\leq \frac{2}{\gamma}\|g(t)\|^2 + 2\|f_1(t)\|_{L^1} \\
&\quad + c_1|\mathcal{W}_\delta(\theta_t\omega)|^{\frac{p}{p-q}}\|h_1(t,x)\|^{\frac{p}{p-q}}_{L^{\frac{p}{p-q}}} + c_2|\mathcal{W}_\delta(\theta_t\omega)|^{p_1}\|h_2(t,x)\|^{p_1}_{L^{p_1}}. \qquad (3.13)
\end{aligned}
$$

Multiplying both sides of (3.13) by $e^{\frac{3}{2}\gamma t}$ and integrating on $[\tau, t]$ with $t \geq \tau$, we deduce for every $\omega \in \Omega$,

$$
\begin{aligned}
&\|u_k(t,\tau,\omega,u_\tau)\|^2 + 2\int_\tau^t e^{\frac{3}{2}\gamma(s-t)}\|\nabla_\lambda u_k(s,\tau,\omega,u_\tau)\|^2 ds \\
&\quad + c_1\int_\tau^t e^{\frac{3}{2}\gamma(s-t)}\|u_k(s,\tau,\omega,u_\tau)\|^p_{L^p} ds \\
&\leq e^{\frac{3}{2}\gamma(s-t)}\|u_\tau\|^2 + \int_\tau^t e^{\frac{3}{2}\gamma(s-t)}\left(\frac{2}{\gamma}\|g(s)\|^2 + 2\|h_1(t,x)\|_{L^1}\right) ds \\
&\quad + c_1\int_\tau^t e^{\frac{3}{2}\gamma(s-t)}|\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}}\|h_1(s)\|^{\frac{p}{p-q}}_{L^{\frac{p}{p-q}}} ds \\
&\quad + c_2\int_\tau^t e^{\frac{3}{2}\gamma(s-t)}|\mathcal{W}_\delta(\theta_s\omega)|^{p_1}\|h_2(s)\|^{p_1}_{L^{p_1}} ds.
\end{aligned}
$$

Hence, we obtain

$$
\{u_k\}_{k=1}^\infty \quad \text{is bounded in } L^\infty(\tau,T;L^2(\mathcal{O}_k)) \cap L^p(\tau,T;L^p(\mathcal{O}_k)) \cap L^2(\tau,T;H^1_\lambda(\mathcal{O}_k)), \qquad (3.14)
$$

and by (3.2), we infer that

$$
\int_{\mathcal{O}_k\times[\tau,T]} |f(t,x,u_k)|^{p_1} dxdt \leq c\int_{\mathcal{O}_k\times[\tau,T]} |u_k|^p dxdt + c\int_{\mathcal{O}_k\times[\tau,T]} |f_2(t,x)|^{p_1} dxdt,
$$

this implies that

$$
\{f(t,x,u_k)\}_{k=1}^\infty \quad \text{is bounded in } L^{p_1}(\tau,T;L^{p_1}(\mathcal{O}_k)). \qquad (3.15)
$$

And by (3.5) and Hölder's inequality, we also obtain

$$\begin{aligned}\int_{\mathcal{O}_k\times[\tau,T]}|h(t,x,u_k)|^{p_1}dxdt &\le c\int_{\mathcal{O}_k\times[\tau,T]}(|h_1(t,x)|^{p_1}|u_k|^{p_1(q-1)}+|h_2(t,x)|^{p_1})dxdt\\ &\le c\int_{\mathcal{O}_k\times[\tau,T]}(|h_1(t,x)|^{\frac{p}{p-q}}+|u_k|^p+|h_2(t,x)|^{p_1})dxdt,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p_1}=1$, thus we conclude that

$$\{h(t,x,u_k)\}_{k=1}^{\infty}\quad\text{is bounded in } L^{p_1}(\tau,T;L^{p_1}(\mathcal{O}_k)).\tag{3.16}$$

Therefore, from (3.14), (3.15) and (3.16) we can conclude the boundedness of derivative sequence, i.e.,

$$\left\{\frac{du_k}{dt}\right\}_{k=1}^{\infty}\quad\text{is bounded in } L^2(\tau,T;H_\lambda^{-1}(\mathcal{O}_k))+L^{p_1}(\tau,T;L^{p_1}(\mathcal{O}_k)).\tag{3.17}$$

Step 2: Existence of solutions: Let $T>0, t_0\in[\tau,\tau+T]$ and $u_k(t,\tau,\omega,u_\tau)$ is solutions of (3.8) defined in $\mathcal{O}_k$. Extend u_k to the whole space $\mathbb{R}^N$ by setting $u_k=0$ on $\mathbb{R}^N\setminus\mathcal{O}_k$ and for simplicity, we still denote this extension by u_k. Thus we can see from (3.14), (3.15), (3.16) and (3.17) that there exist functions

$$\begin{aligned}&\tilde{u}\in L^2(\mathbb{R}^N),\\ &u\in L^\infty(\tau,T;L^2(\mathbb{R}^N))\cap L^p(\tau,T;L^p(\mathbb{R}^N))\cap L^2(\tau,T;H_\lambda^1(\mathbb{R}^N)),\\ &\chi_1\in L^{p_1}(\tau,T;L^{p_1}(\mathbb{R}^N)),\end{aligned}$$

such that up to a subsequence,

$$u_k\rightharpoonup u\quad\text{weak-star in } L^\infty(\tau,T;L^2(\mathbb{R}^N)),\tag{3.18}$$

$$u_k\rightharpoonup u\quad\text{weakly in } L^p(\tau,T;L^p(\mathbb{R}^N)),\tag{3.19}$$

$$u_k\rightharpoonup u\quad\text{weakly in } L^2(\tau,T;H_\lambda^1(\mathbb{R}^N)),\tag{3.20}$$

$$Au_k\rightharpoonup Au\quad\text{weakly in } L^2(\tau,T;H_\lambda^{-1}(\mathbb{R}^N)),\tag{3.21}$$

$$f(t,x,u_k)+\mathcal{W}_\delta(\theta_t\omega)h(t,x,u_k)\rightharpoonup\chi_1\quad\text{weakly in } L^{p_1}(\tau,T;L^{p_1}(\mathbb{R}^N)),\tag{3.22}$$

$$u_k(t_0,\tau,\omega,u_\tau)\rightharpoonup\tilde{u}\quad\text{weakly in } L^2(\mathbb{R}^N).\tag{3.23}$$

Moreover, since the embedding $H_\lambda^1(\mathcal{O}_k)\hookrightarrow L^2(\mathcal{O}_k)$ is compact, we can choose a further subsequence (not relabeled) by a diagonal processes such that for each $k_0\in\mathbb{N}$,

$$u_k\to u\quad\text{strongly in } L^2(\tau,T;L^2(\mathcal{O}_{k_0})).\tag{3.24}$$

We next prove $\chi_1=f(t,\cdot,u)+h(t,\cdot,u)\mathcal{W}_\delta(\theta_t\omega)$. Indeed, from (3.24) we infer that (up to a subsequence)

$$u_k\to u\quad\text{a.e. } (t,x)\in(\tau,\tau+T)\times\mathcal{O}_k.$$

From this and by the continuity of f,h and $\mathcal{W}_\delta$ we obtain

$$f(t,x,u_k)+h(t,x,u_k)\mathcal{W}_\delta(\theta_t\omega)\to f(t,x,u)+h(t,x,u)\mathcal{W}_\delta(\theta_t\omega)\tag{3.25}$$

for a.e. $(t,x)\in(\tau,\tau+T)\times\mathcal{O}_k$. Hence, by the boundedness in (3.15) and (3.16), we obtain from (3.25) that

$$f(t,x,u_k)+h(t,x,u_k)\mathcal{W}_\delta(\theta_t\omega)\rightharpoonup f(t,x,u)+h(t,x,u)\mathcal{W}_\delta(\theta_t\omega)\tag{3.26}$$

weakly in $L^{p_1}(\tau,\tau+T;L^{p_1}(\mathcal{O}_k))$. From (3.22), (3.26) and by uniqueness of weak limit, we obtain

$$\chi_1=f(t,x,u)+h(t,x,u)\mathcal{W}_\delta(\theta_t\omega).\tag{3.27}$$

Now, for every $j \in \mathbb{N}$ and $\phi \in C_c^\infty(\tau, \tau+T)$, we have from (3.24) and (3.26) that

$$\begin{aligned}\lim_{k\to\infty}\int_\tau^{\tau+T}&(f(t,\cdot,u_k)+h(t,\cdot,u_k)\mathcal{W}_\delta(\theta_t\omega),\phi e_j)dt\\ &=\int_\tau^{\tau+T}(f(t,\cdot,u)+h(t,\cdot,u)\mathcal{W}_\delta(\theta_t\omega),\phi e_j)dt. \qquad (3.28)\end{aligned}$$

Hence, letting $k\to\infty$ in (3.8) and using (3.18)–(3.22) and (3.28), we have for every $j\in\mathbb{N}$ and $\phi\in C_c^\infty(\tau,\tau+T)$,

$$\begin{aligned}&-\int_\tau^{\tau+T}(u,e_j)\phi' dt+\int_\tau^{\tau+T}(Au,\phi e_j)_{H_\lambda^{-1},H_\lambda^1}dt\\ &=\gamma\int_\tau^{\tau+T}(u,\phi e_j)dt+\int_\tau^{\tau+T}(\chi_1,\phi e_j)_{L^{p_1},L^p}dt\\ &\quad+\int_\tau^{\tau+T}(f(t,\cdot,u)+h(t,\cdot,u)\mathcal{W}_\delta(\theta_t\omega),\phi e_j)dt+\int_\tau^{\tau+T}(g,\phi e_j)dt. \qquad (3.29)\end{aligned}$$

Moreover, $\operatorname{span}\{e_j, j\in\mathbb{N}\}$ is dense in $H_\lambda^1(\mathcal{O}_k)\cap L^2(\mathcal{O}_k)\cap L^p(\mathcal{O}_k)$, hence we can see that (3.29) is still valid when e_j is replaced by any element in $H_\lambda^1(\mathcal{O}_k)\cap L^2(\mathcal{O}_k)\cap L^p(\mathcal{O}_k)$. Thus, we obtain that for every $\xi\in H_\lambda^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)\cap L^p(\mathbb{R}^N)$,

$$\frac{d}{dt}(u,\xi)+(Au,\xi)_{H_\lambda^{-1},H_\lambda^1}+\gamma(u,\xi)=(\chi_1,\xi)_{(L^{p_1},L^p)}+(g(t),\xi) \qquad (3.30)$$

in the sense of distribution. Hence, (3.30) and (3.27) imply that

$$\frac{du}{dt}=-Au+f(t,x,u)+h(t,x,u)\mathcal{W}_\delta(\theta_t\omega)-\gamma u+g, \qquad (3.31)$$

in $L^2(\tau,\tau+T;H_\lambda^{-1}(\mathbb{R}^N))+L^{p_1}(\tau,\tau+T;L^{p_1}(\mathbb{R}^N))+L^2(\tau,\tau+T;L^2(\mathbb{R}^N))$, from this and by $u\in L^\infty(\tau,\tau+T;L^2(\mathbb{R}^N)\cap L^p(\tau,\tau+T;H_\lambda^1(\mathbb{R}^N))\cap L^2(\tau,\tau+T;L^2(\mathbb{R}^N)))$ and $\frac{du}{dt}\in L^2(\tau,\tau+T;H_\lambda^{-1}(\mathbb{R}^N))+L^{p_1}(\tau,\tau+T;L^{p_1}(\mathbb{R}^N))$ give us $u\in C([\tau,\tau+T],L^2(\mathbb{R}^N))$ and

$$\frac{1}{2}\frac{d}{dt}\|u\|^2=\left(\frac{du}{dt},u\right)_{H_\lambda^{-1}+L^{p_1}+L^2,H_\lambda^1\cap L^p\cap L^2}\quad\text{for a.e. } t\in(\tau,\tau+T). \qquad (3.32)$$

We now prove $u(\tau)=u_\tau$ and $u(\tau+T)=\tilde{u}$. Indeed, we choose $\phi\in C^1([\tau,\tau+T])$ and $\xi\in H_\lambda^1(\mathcal{O}_k)\cap L^{p_1}(\mathcal{O}_k)$. Multiplying (3.8) by $\phi\xi$ and taking the limits as previous, then we obtain from (3.18)–(3.23) and (3.31) that

$$\begin{aligned}(\tilde{u},\xi)\phi(\tau+T)-(u_\tau,\xi)\phi(\tau)&=\int_\tau^{\tau+T}(v,\xi)\phi' dt-\int_\tau^{\tau+T}(Au,\phi\xi)_{(H_\lambda^{-1},H_\lambda^1)}dt\\ &+\gamma\int_\tau^{\tau+T}(u,\phi\xi)dt+\int_\tau^{\tau+T}(\chi_1,\phi\xi)_{(L^{p_1},L^p)}dt+\int_\tau^{\tau+T}(g,\phi\xi)dt. \qquad (3.33)\end{aligned}$$

On the other hand, from (3.30), we deduce that the right-hand side of (3.33) is given by

$$(u(\tau+T),\xi)\phi(\tau+T)-(u(\tau),\xi)\phi(\tau).$$

Thus, we obtain

$$(u(\tau+T),\xi)\phi(\tau+T)-(u(\tau),\xi)\phi(\tau)=(\tilde{u},\xi)\phi(\tau+T)-(u_\tau,\xi)\phi(\tau).$$

Next, for $\psi\in C^1([\tau,\tau+T])$ such that $\psi(\tau)=1$ and $\psi(\tau+T)=0$, we first taking $\phi=\psi$ and then choosing $\phi=1-\psi$, we obtain

$$u(\tau)=u_\tau \quad\text{and}\quad u(\tau+T)=\tilde{u},$$

and this proves u is a weak solution of system (3.7).

Step 3: Uniqueness of solutions: Assume u_1 and u_2 are two solutions of system (3.7) with initial conditions $u_1(\tau,x)=u_{1,\tau}$ and $u_2(\tau,x)=u_{2,\tau}$, respectively. Let $\bar{u}=u_1-u_2$ and $\bar{u}$ satisfies

$$\frac{d\bar{u}}{dt}-\Delta_\lambda\bar{u}+\gamma\bar{u}=f_1(t)\bar{u}+\mathcal{W}_\delta(\theta_t\omega)h_1(t)\bar{u},$$

where we used the mean value theorem for functions f and h to obtain

$$f_1(t)=\int_0^t\frac{\partial f}{\partial s}(t,x,\eta u_1+(1-\eta)u_2)d\eta,$$
$$h_1(t)=\int_0^t\frac{\partial h}{\partial s}(t,x,\eta u_1+(1-\eta)u_2)d\eta.$$

Moreover, since $q>2$, by (3.3) and (3.6) we have for $t\in[\tau,T]$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\bar{u}\|^2&\le\int_{\mathbb{R}^N}f_1(t)|\bar{u}|^2dx+\mathcal{W}_\delta(\theta_t\omega)\int_{\mathbb{R}^N}h_1(t)|\bar{u}|^2\\&\le c_3\int_{\mathbb{R}^N}\left(\int_0^1|\eta u_1+(1-\eta)u_2|^{p-2}d\eta+f_3(t,x)\right)|\bar{u}|^2dx\\&\quad+|\mathcal{W}_\delta(\theta_s\omega)|\int_{\mathbb{R}^N}\left(h_3\int_0^1|\eta u_1+(1-\eta)u_2|^{q-2}d\eta+h_4\right)|\bar{u}|^2dx\\&\quad-c_3\int_{\mathbb{R}^N}\int_0^1|\eta u_1+(1-\eta)u_2|^{p-2}|\bar{u}|^2d\eta dx+\int_{\mathbb{R}^N}f_3|\bar{u}|^2dx\\&\quad+c\int_{\mathbb{R}^N}|h_3\mathcal{W}_\delta(\theta_t\omega)|^{\frac{p-2}{p-q}}|\bar{u}|^2dx+c_3\int_0^1|\eta u_1+(1-\eta)u_2|^{p-2}|\bar{u}|^2d\eta dx\\&\quad+|\mathcal{W}_\delta(\theta_t\omega)|\int_{\mathbb{R}^N}h_4|\bar{u}|^2dx\\&\le c\|\bar{u}\|^2.\end{aligned}$$

Therefore, for all $t\in[\tau,T]$, we obtain

$$\|u_1(t,\tau,\omega,u_{1,\tau})-u_2(t,\tau,\omega,u_{2,\tau})\|^2\le e^{c(t-\tau)}\|u_{1,\tau}-u_{2,\tau}\|^2,\tag{3.34}$$

this implies the uniqueness and continuous dependence of solutions on initial data in $L^2(\mathbb{R}^N)$.

In addition, by the uniqueness of solutions and by $\hat{u}=u(t_0)$ we obtain that for every $\omega\in\Omega$, the sequence of solutions $\{u_k(t_0,\tau,\omega,u_\tau)\}$ converges to $u(t_0,\tau,\omega,u_\tau)$ weakly in $L^2(\mathbb{R}^N)$ for any fixed $t_0\in[\tau,T]$ and $\omega\in\Omega$. And by $u_k(t,\tau,\omega,u_\tau)$ is measurable, we also have the measurablity of $u(t,\tau,\omega,u_\tau)$. Therefore, the lemma is proved. □

Next, we prove the asymptotic compactness of solutions.

Lemma 3.2. *Let $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $L^2(\mathbb{R}^N)$. Then there exist $u_0\in L^2(\tau,t;L^2(\mathbb{R}^N))$ and a subsequence $\{u(\cdot,\tau,\omega,u_{n_m})\}_{m=1}^\infty$ of $\{u(\cdot,\tau,\omega,u_n)\}_{n=1}^\infty$ such that*

$$u(\cdot,\tau,\omega,u_{n_m})\to u_0(s)\quad\text{in }L^2(\mathcal{O}_k)\tag{3.35}$$

as $m\to\infty$ for every $k\in\mathbb{N}$ and for almost all $s\in(\tau,t)$.

Proof. Given T be a sufficiently large time such that $t\in(\tau,T]$. It follows from (3.14) and (3.17) that, up to a subsequence and for every $k\in\mathbb{N}$,

$$u(\cdot,\tau,\omega,u_n)\to\hat{u}(\cdot)\quad\text{in }L^2(\tau,T;L^2(\mathcal{O}_k))$$

for some $\hat{u}\in L^2(\tau,T;L^2(\mathbb{R}^N))$. Then, for each k, there exists a sub-interval $I_k\subseteq[\tau,T]$ with $|I_k|=0$ and a subsequence $\{u^k_{n_m}\}\subset\{u_n\}$,

$$u(s,\tau,\omega,u^k_{n_m})\to\hat{u}(s)\quad\text{in }L^2(\mathcal{O}_k),\quad\forall s\in[\tau,T]\setminus I_k.$$

Then by a diagonal process, we can find a interval $I \subseteq [\tau, T]$ with $|I| = 0$ and a subsequence of u_n (we do not relabel) such that

$$u(s, \tau, \omega, u_n) \to \hat{u}(s) \quad \text{in } L^2(\mathcal{O}_k) \quad \forall s \in [\tau, T] \setminus I, \forall k \in \mathbb{N},$$

hence (3.35) follows. □

We next define a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau)$$

where u is a solution of (3.7) and u_τ is the initial condition which is given in $L^2(\mathbb{R}^N)$. Then by Lemma 3.1, we obatin that Φ is a continuous cocycle on $L^2(\mathbb{R}^N)$ over complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

In what follows, we will study existence of a unique $\mathcal{D}$-random attractor in $L^2(\mathbb{R}^N)$ for Φ. To do this, for a bounded nonempty subset D of $L^2(\mathbb{R}^N)$ is given, we denote by

$$\|D\| = \sup_{\xi \in D} \|\xi\|$$

is the Hausdorff semi-distance between D and the origin in $L^2(\mathbb{R}^N)$. Let

$$\mathcal{D} = \{D = D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega : D \text{ is tempered}\},$$

be the collection of all families of tempered nonempty subsets of $L^2(\mathbb{R}^N)$.

To show the existence and convergence of pullback attractors, we further need to the following conditions:

$$\int_{-\infty}^{\tau} e^{\gamma s}(\|g(s)\|^2 + \|f_1(s)\|_{L^1})ds < +\infty \quad \forall \tau \in \mathbb{R}, \tag{3.36}$$

and

$$\lim_{t\to-\infty} e^{ct} \int_{-\infty}^{0} e^{\gamma s}(\|g(s+t)\|^2 + \|f_1(s+t)\|_{L^1})ds = 0, \quad \forall c > 0, \tag{3.37}$$

where $\gamma > 0$.

3.2. **Existence of pullback random attractors.**

In what follows, we prove that Φ has a tempered pullback absorbing set in $L^2(\mathbb{R}^N)$ and is $\mathcal{D}$-pullback asymptotically compact, which implies the existence of a unique $\mathcal{D}$-random attractor for the cocycle Φ.

In addition, we assume that

$$h_1 \in L^\infty(\mathbb{R}; L^{p/(p-q)}(\mathbb{R}^N)), \; h_2 \in L^\infty(\mathbb{R}; L^{p_1}(\mathbb{R}^N)).$$

Lemma 3.3. *Let* (3.1)–(3.6), (3.36) *and* (3.37) *hold. Then* Φ *has a closed measurable* $\mathcal{D}$*-pullback absorbing set*

$$K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$$

with

$$K(\tau, \omega) = \{u \in L^2(\mathbb{R}^N) : \|u\|^2 \le R(\tau, \omega)\} \tag{3.38}$$

where

$$R(\tau, \omega) = M \int_{-\infty}^{0} e^{\alpha s}(\|g(s+\tau)\|^2 + \|f_1(s+\tau)\|_{L^1} + |\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1})ds, \tag{3.39}$$

here M *is a positive constant independent of* σ, τ, ω *and* D.

Proof. From (3.7), for all $\varphi \in L^2(\mathbb{R}^N)$, we have

$$\langle u_t, \varphi\rangle + (\nabla_\lambda u, \nabla_\lambda \varphi) + \gamma(u, \varphi) = (f(t,x,u) + g(t,x), \varphi) + (\mathcal{W}_\delta(\theta_t\omega)h(t,x,u), \varphi).$$

Taking $\varphi = u$, we obtain

$$\begin{aligned}\frac{d}{dt}\|u\|^2 + 2\|\nabla_\lambda u\|^2 + 2\gamma\|u\|^2 &= 2\int_{\mathbb{R}^N} (f(t,x,u) + g(t,x))u dx \\ &\quad + 2\mathcal{W}_\delta(\theta_t\omega)\int_{\mathbb{R}^N} h(t,x,u)u dx. \qquad (3.40)\end{aligned}$$

By Young's inequality we have

$$\int_{\mathbb{R}^N} g(t,x)u dx \le \frac{\gamma}{4}\|u\|^2 + \frac{1}{\gamma}\|g(t)\|^2, \qquad (3.41)$$

and by (3.1) we have

$$\int_{\mathbb{R}^N} f(t,x,u)u dx \le -c_1\int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} f_1(t,x) dx, \qquad (3.42)$$

and also by (3.5), we obtain

$$\begin{aligned}\mathcal{W}_\delta(\theta_t\omega)\int_{\mathbb{R}^N} h(t,x,u)u dx &\le |\mathcal{W}_\delta(\theta_t\omega)|\int_{\mathbb{R}^N} (h_1(t,x)|u|^q + h_2(t,x)|u|)dx \\ &\le \frac{c_1}{2}\int_{\mathbb{R}^N} |u|^p dx + c|\mathcal{W}_\delta(\theta_t\omega)|^{p/(p-q)}\|h_1(t)\|^{p/(p-q)}_{L^{p/(p-q)}} \\ &\quad + c|\mathcal{W}_\delta(\theta_t\omega)|^{p_1}\|h_2(t)\|^{p_1}_{L^{p_1}}. \qquad (3.43)\end{aligned}$$

From (3.40)–(3.43), we have

$$\begin{aligned}&\frac{d}{dt}\|u\|^2 + 2\|\nabla_\lambda u\|^2 + \frac{\gamma}{2}\|u\|^2 + c_1\|u\|^p_{L^p} \\ &\le -\gamma\|u\|^2 + \frac{2}{\gamma}\|g(t)\|^2 + 2\|f_1(t)\|_{L^1} + c(|\mathcal{W}_\delta(\theta_t\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_t\omega)|^{p_1}). \qquad (3.44)\end{aligned}$$

Multiply (3.44) by $e^{\gamma t}$ and integrate over $(\tau - t, \sigma)$, then for every $\omega \in \Omega$, we have

$$\begin{aligned}&\frac{d}{dt}(e^{\gamma t}\|u\|^2) + 2e^{\gamma t}\|u\|^2_{H^1_\lambda} + \frac{\gamma}{2}e^{\gamma t}\|u\|^2 + c_1\|u\|^p_{L^p} \\ &\le \frac{2}{\gamma}\|g(t)\|^2 + 2e^{\gamma t}\|f_1(t)\|_{L^1} + ce^{\gamma t}(|\mathcal{W}_\delta(\theta_t\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_t\omega)|^{p_1}).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + 2\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s, \tau - 1, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{H^1_\lambda} ds \\ &\quad + \frac{\gamma}{2}\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds \\ &\le e^{\gamma(\tau-t-\sigma)}\|u_{\tau-t}\|^2 + \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\big(\frac{2}{\gamma}\|g(s)\|^2 + 2\|f_1(s)\|_{L^1}\big)ds \\ &\quad + c\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}(|\mathcal{W}_\delta(\theta_{s-\tau}\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_{s-\tau}\omega)|^{p_1})ds \\ &\le e^{\gamma(\tau-t-\sigma)}\|u_{\tau-t}\|^2 + \int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)}\big(\frac{2}{\gamma}\|g(s+\tau)\|^2 + 2\|f_1(s+\tau)\|_{L^1}\big)ds \\ &\quad + c\int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)}(|\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1})ds. \qquad (3.45)\end{aligned}$$

Since $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and D is tempered, $\omega(t) \to 0$ as $t \to \pm\infty$, using (2.6)–(2.7) and (3.36), we deducing from (3.45) that

$$\limsup_{t\to+\infty} e^{\gamma(\tau-t-\sigma)}\|u_{\tau-t}\|^2 \le \limsup_{t\to+\infty} e^{\gamma(\tau-t-\sigma)}\|D(\tau-t,\theta_{-t}\omega)\|^2 = 0,$$

which shows that there exists $T = T(\sigma, \tau, \omega, D) > 0$ such that for all $t \ge T$,

$$e^{\gamma(\tau-t-\sigma)}\|u_{\tau-t}\|^2 \le \int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)}(|\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1})ds. \qquad (3.46)$$

It follows from (3.45) and (3.46), there exists $M > 0$ independent of σ, τ, ω and D such that

$$\begin{aligned}&\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + 2\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s, \tau-1, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{H^1_\lambda} ds \\ &\le M\int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)}(\|g(s+\tau)\|^2 + 2\|f_1(s+\tau)\|_{L^1} \\ &\qquad\qquad + |\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1})ds.\end{aligned}$$

We denote by

$$\begin{aligned}R(\tau,\omega) = M\int_{-\infty}^{\sigma-\tau} & e^{\gamma(s+\tau-\sigma)}(\|g(s+\tau)\|^2 + 2\|f_1(s+\tau)\|_{L^1} \\ &+ |\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1})ds,\end{aligned}$$

and

$$K(\tau,\omega) = \{u \in L^2(\mathbb{R}^N) : \|u\|^2 \le R(\tau,\omega)\}.$$

Then we claim that K is tempered. Indeed, let $\eta > 0$ be arbitrary positive number, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have

$$\begin{aligned}e^{\eta}\|K(\tau+t,\theta_t\omega)\|^2 &\le e^{\eta t}R(\tau+t,\theta_t\omega) \\ &= Me^{\eta t}\int_{-\infty}^{t} e^{\gamma s}(\|g(s+\tau+t)\|^2 + \|f_1(s+\tau+t)\|_{L^1})ds \\ &\quad + Me^{\eta}\int_{-\infty}^{t} e^{\gamma s}(|\mathcal{W}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_{s+t}\omega)|^{p_1})ds. \qquad (3.47)\end{aligned}$$

For the first term on the right-hand of (3.47), let $t \to -\infty$, we deduce from (3.37) that

$$\lim_{t\to-\infty} e^{\eta(\tau+t)}\int_{-\infty}^{0} e^{\gamma s}(\|g(s+\tau+t)\|^2 + \|f_1(s+\tau+t)\|_{L^1})ds = 0. \qquad (3.48)$$

Choose $\eta_1 = \min\{\eta,\gamma\}$. Then for the last term in (3.47), we have for $t \le 0$,

$$\begin{aligned}&e^{\eta t}\int_{-\infty}^{0} e^{\eta s}(|\mathcal{W}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_{s+t}\omega)|^{p_1})ds \\ &\le \int_{-\infty}^{0} e^{\eta_1(s+t)}(|\mathcal{W}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_{s+t}\omega)|^{p_1})ds \\ &\le \int_{-\infty}^{t} e^{\eta_1(s+t)}(|\mathcal{W}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_{s+t}\omega)|^{p_1})ds. \qquad (3.49)\end{aligned}$$

Recall that from (2.6)–(2.7) and $\frac{\omega(t)}{t} \to 0$ as $t \to \pm\infty$, we infer that

$$\int_{-\infty}^{0} e^{\eta_1 s}(|\mathcal{W}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_{s+t}\omega)|^{p_1})ds < +\infty,$$

thus we obtain from (3.49)

$$\lim_{t\to-\infty} e^{\eta t}\int_{-\infty}^{0} e^{\eta s}(|\mathcal{W}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}}+|\mathcal{W}_\delta(\theta_{s+t}\omega)|^{p_1})ds=0. \tag{3.50}$$

Combining (3.47), (3.48) and (3.50), we deduce that K belongs to $\mathcal{D}$.

Moreover, for each $\tau\in\mathbb{R}$ we have $R(\tau,\cdot):\Omega\to\mathbb{R}$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable, thus the set K given by (3.38) is also measurable. Therefore, $K\in\mathcal{D}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ. This completes the proof. □

We next establish some uniform estimates on the tails of solutions for large space and time variables, which will play an important role in proving the asymptotic compactness of solutions.

Lemma 3.4. *Suppose that* (3.1)–(3.6) *and* (3.36) *are satisfied. Then for every* $\tau\in\mathbb{R}$, $\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\}\in D$ *and for any* $\epsilon>0$, *there exist* $T=T(\tau,\omega,D,\epsilon)\geq 1$ *and* $N=N(\tau,\omega,\epsilon)>0$ *such that for all* $t\geq T$ *and* $\sigma\in[\tau-1,\tau]$,

$$\int_{\mathbb{R}^N\setminus\mathcal{O}_N}|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2dx\leq\epsilon, \tag{3.51}$$

where $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ *and* $\mathcal{O}_N=B_1(0,N^{\epsilon_1})\times\ldots\times B_N(0,N^{\epsilon_N})$.

Proof. We first consider N functions $\varphi_{1,R},\varphi_{2,R},\ldots,\varphi_{N,R}$ such that

$$\varphi_{1,R}=\varphi_1\left(\frac{|x_1|^{\epsilon_1}}{R^{\epsilon_1}}\right),\ldots,\varphi_{N,R}=\varphi_N\left(\frac{|x_N|^{\epsilon_N}}{R^{\epsilon_N}}\right),$$

where $x=(x_1,\ldots,x_N)\in\mathbb{R}^N$, with

$$0\leq\varphi_i\leq1,\qquad \varphi_i=\begin{cases}1 & \text{in } [0,\frac{1}{2}],\\ 0 & \text{in } [1,+\infty],\end{cases}\qquad i=1,\ldots,N,$$

and satisfy

$$\left|\frac{\partial\varphi_{1,R}}{\partial x_1}\right|\leq\frac{c}{R^{\epsilon_1}},\ldots,\left|\frac{\partial\varphi_{N,R}}{\partial x_N}\right|\leq\frac{c}{R^{\epsilon_N}}, \tag{3.52}$$

for some constant $c>0$. Denoting by $\varphi_R=\varphi_{1,R}\times\varphi_{2,R}\times\cdots\times\varphi_{N,R}$ and taking the inner product of (3.7) with $\varphi_R u$ in $L^2(\mathbb{R}^N)$, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}\varphi_R|u|dx-2\int_{\mathbb{R}^N}\varphi_R\Delta_\lambda u\,udx\\ &=-2\gamma\int_{\mathbb{R}^N}\varphi_R|u|^2dx+2\int_{\mathbb{R}^N}\varphi_R f(t,x,u)udx\\ &\quad+2\int_{\mathbb{R}^N}g(t,x)udx+2\mathcal{W}_\rho(\theta_t\omega)\int_{\mathbb{R}^N}\varphi_R h(t,x,u)udx.\end{aligned}$$

We first observe that

$$-\int_{\mathbb{R}^N}\varphi_R\Delta_\lambda uudx=\int_{\mathbb{R}^N}\varphi_R\|\nabla_\lambda u\|^2dx+\int_{\mathcal{O}_{2R}\setminus\mathcal{O}_R}\nabla_\lambda(\varphi_R u)\nabla_\lambda udx,$$

where $\mathcal{O}_R=B_1(0,R^{\epsilon_1})\times\ldots\times B_N(0,R^{\epsilon_N})$.

Since $\nabla_\lambda\varphi_R=(\lambda_1(x),\partial_{x_1}\varphi_R,\ldots,\lambda_N(x),\partial_{x_N}\varphi_R)$, hence on $\mathcal{O}_{2R}\setminus\mathcal{O}_R$, by (3.52) we have

$$\begin{aligned}|\nabla_\lambda\varphi_R|&\leq|\lambda_1(x)||\partial_{x_1}\varphi_R|+\cdots+|\lambda_N(x)||\partial_{x_N}\varphi_R|\\ &\leq 2R^{\epsilon_1-1}|\varphi_{1,R}'\varphi_{2,R}\cdots\varphi_{N,R}|+\cdots+2R^{\epsilon_N-1}|\varphi_{1,R}\cdots\varphi_{N-1,R}\varphi_{N,R}'|\\ &\leq R^{\epsilon_1-1}\cdot\frac{c}{R^{\epsilon_1}}+\cdots+R^{\epsilon_N-1}\cdot\frac{c}{R^{\epsilon_N}}\\ &=\frac{c}{R},\end{aligned}$$

where we used the fact that $|\lambda_i(x)| \le cR^{\epsilon_i - 1}$, $\forall x \in \mathcal{O}_R, i = 1, \dots, N$, (see, e.g. [2, 36, 37, 38, 39]). Thus, we have

$$\begin{aligned} -\int_{\mathbb{R}^N} \varphi_R \Delta_\lambda u u dx &\le \int_{\mathbb{R}^N} \varphi_R \|\nabla_\lambda u\|^2 dx + \frac{c}{R}\int_{\mathcal{O}_{2R}\setminus\mathcal{O}_R} |u\nabla_\lambda u| dx \\ &\le \int_{\mathbb{R}^N} \varphi_R \|\nabla_\lambda u\|^2 dx + \frac{c}{R}(\|\nabla_\lambda u\|^2 + \|u\|^2). \end{aligned} \tag{3.53}$$

By (3.1), we obtain

$$\int_{\mathbb{R}^N} \varphi_R f(t,x,u)u\, dx \le -c_1 \int_{\mathbb{R}^N} \varphi_R(x)|u|^p dx + \int_{\mathbb{R}^N} \varphi_R(x)|f_1(t,x)| dx. \tag{3.54}$$

And by Young's inequality, we find

$$\int_{\mathbb{R}^N} \varphi_R(x) g(t,x) u dx \le \frac{\gamma}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{2\gamma}\int_{\mathbb{R}^N} \varphi_R(x)|g(t,x)|^2 dx. \tag{3.55}$$

Using (3.5), we also have

$$\begin{aligned} &\mathcal{W}_\rho(\theta_t\omega)\int_{\mathbb{R}^N} \varphi_R(x) h(t,x,u) u dx \\ &\le |\mathcal{W}_\delta(\theta_t\omega)| \int_{\mathbb{R}^N} \varphi_R(x)(h_1(t,x)|u|^q + h_2(t,x)|u|) dx \\ &\le \frac{c_1}{2}\int_{\mathbb{R}^N} \varphi_R(x)|u|^p dx + c|\mathcal{W}_\delta(\theta_t\omega)|^{\frac{p}{p-q}} \int_{\mathbb{R}^N} \varphi_R(x)|h_1(t,x)|^{\frac{p}{p-q}} dx \\ &\quad + c|\mathcal{W}_\delta(\theta_t\omega)|^{p_1}\int_{\mathbb{R}^N} \varphi_R(x)|h_2(t,x)|^{p_1} dx. \end{aligned} \tag{3.56}$$

Thus, from (3.53)–(3.56), there exists $N_1 = N_1(\epsilon) > 0$ such that for all $R \ge N_1$,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^N} \varphi_R |u|^2 dx + c_1 \int_{\mathbb{R}^N} |u|^p dx + \gamma \int_{\mathbb{R}^N} \varphi_R |u|^2 dx \\ &\le \epsilon\|u\|_{H^1_\lambda} + c\int_{\mathbb{R}^N\setminus\mathcal{O}_R} (|g(t,x)|^2 + |f_1(t,x)|) dx \\ &\quad + c|\mathcal{W}_\delta(\theta_t\omega)|^{\frac{p}{p-q}} dx \int_{\mathbb{R}^N} \varphi_R |h_1(t,x)|^{\frac{p}{p-q}} dx \\ &\quad + c|\mathcal{W}_\delta(\theta_t\omega)|^{p_1} \int_{\mathbb{R}^N} \varphi_R |h_2(t,x)|^{p_1} dx. \end{aligned} \tag{3.57}$$

Now, given $t \ge 1, \tau \in \mathbb{R}$ and $\omega \in \Omega$, multiplying (3.57) by $e^{\gamma t}$ and then integrating over $(\tau - t, \sigma)$, where $\sigma \in [\tau - 1, \tau]$ we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \varphi_R(x)|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \\ &\le e^{\gamma(\tau - t - \sigma)}\|u_{\tau-t}\|^2 + \epsilon\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{H^1_\lambda(\mathbb{R}^N)} ds \\ &\quad + \int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)} \int_{\overline{\Omega}_k^c} (|g(s+\tau,x)|^2 + f_1(s+\tau,x)) dx ds \\ &\quad + c\int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)} \Bigg(|\mathcal{W}_\rho(\theta_s\omega)|^{\frac{p}{p-q}} \int_{\overline{\Omega}_k^c} |h_1(s,x)|^{\frac{p}{p-q}} dx \\ &\qquad\qquad + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1} \int_{\overline{\mathcal{O}}_k^c} |h_2(s,x)|^{p_1} dx \Bigg) ds. \end{aligned} \tag{3.58}$$

By the fact that $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and D is tempered, we obtain

$$\limsup_{t\to+\infty} e^{\gamma(\tau-t-\sigma)}\|u_{\tau-t}\|^2 \le e^{\gamma}\limsup_{t\to+\infty} e^{-\gamma t}\|D(\tau - t, \theta_{-t}\omega)\|^2 = 0,$$

which means that we can choose $T_1 = T_1(\tau, \omega, D, \epsilon) \geq 1$ such that for all $t \geq T_1$,

$$e^{\gamma(\tau-t-\sigma)}\|u_{\tau-t}\|^2 \leq \epsilon. \tag{3.59}$$

By (3.36), there exists a constant $N_2 = N_2(\tau, \gamma, \epsilon) \geq N_1$, such that for all $k \geq N_2$,

$$\begin{aligned}
&\int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)} \int_{\overline{\Omega}_k^c} (|g(s+\tau,x)|^2 + f_1(s+\tau,x))dxds \\
&\leq e^{\gamma} \int_{-\infty}^{0} e^{\gamma s} \int_{\overline{\Omega}_k^c} (|g(s+\tau,x)|^2 + f_1(s+\tau,x))dxds \leq \epsilon.
\end{aligned} \tag{3.60}$$

By (2.5), (2.6) and (2.7), we find

$$\begin{aligned}
&\int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)} \bigg(|\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} \int_{\mathbb{R}^N} |h_1(s,x)|^{\frac{p}{p-q}} dx \\
&\qquad\qquad + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1} \int_{\mathbb{R}^N} |h_2(s,x)|^{p_1} dx \bigg) ds \\
&\leq c \int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)} \left(|\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1} \right) ds < +\infty,
\end{aligned}$$

which implies that there is a $N_3(\tau, \gamma, \omega, \epsilon) \geq N_2$ such that for all $k \geq N_3$,

$$\begin{aligned}
&\int_{-\infty}^{\sigma-\tau} e^{\gamma(s+\tau-\sigma)} \bigg(|\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} \int_{\mathbb{R}^N} |h_1(s,x)|^{\frac{p}{p-q}} dx \\
&\qquad\qquad + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1} \int_{\mathbb{R}^N} |h_2(s,x)|^{p_1} dx \bigg) ds \leq \epsilon.
\end{aligned} \tag{3.61}$$

Combining (3.58)–(3.61) and Lemma 3.3, we have that for all $\sigma \in [\tau-1, \tau], t \geq T_1(\tau, \omega, D, \epsilon)$ and $k \geq N_3$,

$$\begin{aligned}
&\int_{\overline{\mathcal{O}}_{\sqrt{2}k}^c} |u(\sigma, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \\
&\leq \int_{\mathbb{R}^N} \varphi_R(x)|u(\sigma, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \\
&\leq c\epsilon \bigg(1 + \int_{-\infty}^{0} e^{\gamma s}(\|g(s+\tau,x)\|^2 + \|f_1(s+\tau,x)\|_{L^1} \\
&\qquad\qquad + |\mathcal{W}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{W}_\delta(\theta_s\omega)|^{p_1})ds \bigg),
\end{aligned}$$

which concludes the proof. □

Lemma 3.5. *Suppose that* (3.1)–(3.6) *and* (3.36) *are satisfied. Then the continuous cocycle* Φ *of problem* (3.7) *is* $\mathcal{D}$*-pullback asymptotically compact in* $L^2(\mathbb{R}^N)$.

Proof. By Lemma 3.3, there exist $T = T(\tau, \omega, D) > 0$ and $c = c(\tau, \omega) > 0$ such that for all $t \geq T$ and $u_0 \in D(\tau - t, \theta_{-t}\omega)$,

$$\|u(\tau-1, \tau-t, \theta_{-\tau}\omega, u_0)\| \leq c(\tau, \omega). \tag{3.62}$$

Since $t_n \to +\infty$ and $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, from (3.62) there exists a $N_1 = N_1(\tau, \omega, D) > 0$ such that

$$\|u(\tau-1, \tau-t_n, \theta_{-\tau}\omega, u_{0,n})\| \leq c(\tau, \omega).$$

hence, the sequence

$$\{u(\tau-1, \tau-t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty} \quad \text{is bounded in } L^2(\mathbb{R}^N).$$

Thus, by Lemma 3.2, there exist $s \in (\tau-1,\tau), u_0 \in L^2(\mathbb{R}^N)$ and a subsequence (we do not relabel) such that for every $k \in \mathbb{N}$,

$$u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}) = u(s,\tau-1,\theta_{-\tau}\omega,u(\tau-1,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})) \to u_0 \tag{3.63}$$

in $L^2(\mathcal{O}_k)$ as $n \to \infty$. Moreover, since $u_0 \in L^2(\mathbb{R}^N)$, we have for any $\epsilon > 0$, there exists $k_1 = k_1(\epsilon) > 0$ such that for all $k \geq k_1$,

$$\int_{\mathbb{R}^N\setminus\mathcal{O}_k} |u_0|^2 dx \leq \epsilon. \tag{3.64}$$

On the other hand, by Lemma 3.4, there exist $N_1 = N_2(\tau,\omega,D,\epsilon) \geq 1$ and $k_2 = k_2(\tau,\omega,\epsilon) \geq k_1$ such that for all $n \geq N_2$ and $k \geq k_2$,

$$\int_{\mathbb{R}^N\setminus\mathcal{O}_k} |u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})|^2 dx \leq \epsilon. \tag{3.65}$$

From (3.63), there exists $N_3 = N_3(\tau,\omega,D,\epsilon) \geq N_2$ such that for all $n \geq N_3$,

$$\int_{\mathcal{O}_{k_2}} |u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}) - u_0|^2 dx \leq \epsilon. \tag{3.66}$$

By (3.34), we have

$$\begin{aligned}
&\|u(\tau,s,\theta_{-\tau}\omega,u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})) - u(\tau,s,\theta_{-\tau}\omega,u_0)\|^2 \\
&\leq c\int_{\mathbb{R}^N\setminus\mathcal{O}_{k_2}} |u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}) - u_0|^2 dx \\
&\quad + c\int_{\mathcal{O}_{k_2}} |u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}) - u_0|^2 dx \\
&\leq c\int_{\mathbb{R}^N\setminus\mathcal{O}_{k_2}} \left(|u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})|^2 + |u_0|^2\right) dx \\
&\quad + c\int_{\mathcal{O}_{k_2}} |u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n}) - u_0|^2 dx,
\end{aligned}$$

which together with (3.64)–(3.66) implies that

$$\|u(\tau,s,\theta_{-\tau}\omega,u(s,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})) - u(\tau,s,\theta_{-\tau}\omega,u_0)\|^2 \to 0 \quad \text{in } L^2(\mathbb{R}^N)$$

as $n \to \infty$. Moreover, since $\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,u_{0,n}) = u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{0,n})$, thus we obtain the conclusion as stated in the lemma. □

We now give the main result of this section, namely, the existence of $\mathcal{D}$-pullback attractors for Φ with general noise.

Theorem 3.1. *Suppose that* (3.1)–(3.6) *and* (3.36) *hold. Then the process* Φ *has a unique* $\mathcal{D}-$*pullback random attractor* $\mathcal{A}$, *which is given by for each* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\begin{aligned}
\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega) &= \bigcup_{B\in\mathcal{D}} \Omega(B,\tau,\omega) \\
&= \{\Gamma(0,\tau,\omega) : \Gamma \textit{ is a } \mathcal{D} - \textit{complete orbit of } \Phi\},
\end{aligned}$$

where K *is given by* (3.38). *In addition, if* f, g, h *and* f_1 *are* T*-periodic functions with respect to* t, *then the* $\mathcal{D}$*-pullback random attractor* $\mathcal{A}(\tau,\omega)$ *is also* T*-periodic.*

Proof. We first see that from Lemma 3.3 that, the cocycle Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K as in (3.38), and by Lemma 3.5 we obtain Φ is asymptotically compact in $L^2(\mathbb{R}^N)$. Therefore, all conditions of Proposition 2.1 are satisfied, thus we obtain the existence and uniqueness of $\mathcal{D}$-pullback random attractor $\mathcal{A}(\tau,\omega)$ for Φ. Moreover, if the functions f, h, g and f_1 are T-periodic with respect to t, then Φ is also T-periodic, i.e.,

$$\Phi(t,\tau+T,\omega,\cdot)=\Phi(t,\tau,\omega,\cdot) \quad t\in\mathbb{R}^+,\tau\in\mathbb{R},\omega\in\Omega.$$

Thus, by Lemma 3.3 we have

$$K(\tau+T,\omega)=K(\tau,\omega), \quad \forall\tau\in\mathbb{R},\omega\in\Omega,$$

this shows that the T-periodicity of $\mathcal{A}$ follows from the periodicity of Φ and K by Proposition 2.1. □

4. Random attractors: multiplicative noise

In this section, we consider a class of stochastic degenerate parabolic equations involving Δ_λ-Laplace operator driven by multiplicative noise. More precisely, we consider problem (1.1) in the case of multiplicative noise,

$$\frac{\partial u}{\partial t}-\Delta_\lambda u+\gamma u-f(t,x,u)=g(t,x)+u\circ\frac{dW(t)}{dt}, \quad t>\tau, x\in\mathbb{R}^N, \tag{4.1}$$

with the initial condition

$$u(\tau,x)=u_\tau(x). \tag{4.2}$$

To show the existence of random attractors of problem (4.1)–(4.2), we will use the change of variables

$$v(t,\tau,\omega)=e^{-\omega(t)}u(t,\tau,\omega)$$

to convert (4.1) into a pathwise deterministic equation

$$\frac{dv}{dt}-\Delta_\lambda v+\gamma v-e^{-\omega(t)}f(t,x,e^{-\omega(t)}v)+e^{-\omega(t)}g(t,x), \quad t>\tau, \tag{4.3}$$

with the initial condition

$$v(\tau,x)=v_\tau(x), \tag{4.4}$$

where $v_\tau(x)=e^{-\omega(\tau)}u_\tau(x)$. Then, given $\omega\in\Omega,\tau\in\mathbb{R}$ and $v_\tau\in L^2(\mathbb{R}^N)$, system (4.3)–(4.4) is a deterministic system. Thus, by a standard Faedo-Galerkin approximation technique (see e.g., [43]), one can show that for a.e. $\omega\in\Omega$, there exists unique solution $v(\cdot,\tau,\omega,v_\tau)\in C([\tau,\infty);L^2(\mathbb{R}^N))\cap L^2_{\text{loc}}(\tau,\infty;H^1_\lambda(\mathbb{R}^N))$ if f satisfies (4.3) and (4.4). Moreover, $v(\cdot,\tau,\omega,v_\tau)$ is continuous in v_τ with respect to the norm of $L^2(\mathbb{R}^N)$ and is $(\mathcal{F},\mathcal{B}(L^2(\mathbb{R}^N)))$-measurable in $\omega\in\Omega$. Thus, we can define a continuous cocycle $\Phi_0:\mathbb{R}^+\times\mathbb{R}\times\Omega\times L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$ over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ by

$$\Phi_0(t,\tau,\omega,u_\tau)=u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau)=e^{\omega(t)-\omega(-\tau)}v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau), \tag{4.5}$$

where $t\in\mathbb{R}^+,\tau\in\mathbb{R}$ and $\omega\in\Omega$.

4.1. Existence of pullback random attractors: multiplicative noise.

We first show that system (4.1)–(4.2) has a $\mathcal{D}$-pullback random attractor in $L^2(\mathbb{R}^N)$.

Lemma 4.1. *Assume that* (3.1)*–*(3.3) *and* (3.36) *hold. Then for every* $\sigma,\tau\in\mathbb{R},\omega\in\Omega$ *and* $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T=T(\sigma,\tau,\omega,D)>0$ *such that for all* $t\geq T$,

$$\begin{aligned}&e^{2\omega(\sigma-\tau)}\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\&\quad+\int_{\tau-t}^{\sigma}e^{\frac{3}{2}\gamma(s-\sigma)-2\omega(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^1_\lambda}ds\\&\leq M\int_{-\infty}^{0}e^{\frac{3}{2}\gamma s-2\omega(s+\sigma-\tau)}(\|g(s+\sigma)\|^2+\|f_1(s+\sigma)\|_{L^1})ds,\end{aligned}$$

where $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ *and* M *is a positive constant independent of* σ,τ,ω *and* D.

Proof. Multiplying v on both sides of (4.3),

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\|\nabla_\lambda v\|^2+\gamma\|v\|^2=e^{-\omega(t)}\int_{\mathbb{R}^N}f(t,x,u)vdx+e^{-\omega(t)}\int_{\mathbb{R}^N}g(t,x)vdx. \tag{4.6}$$

Next we give some estimates on the above equation, by (3.1) we have

$$e^{-\omega(t)}\int_{\mathbb{R}^N}f(t,x,u)vdx\le -c_1e^{-2\omega(t)}\|u\|^p_{L^p}+e^{-2\omega(t)}\|f_1(t)\|_{L^1}, \tag{4.7}$$

and by Young's inequality

$$e^{-\omega(t)}\int_{\mathbb{R}^N}g(t,x)vdx\le \frac{\gamma}{8}\|v\|^2+\frac{2}{\gamma}e^{-\omega(t)}\|g(t)\|^2. \tag{4.8}$$

Thus, we could deduce from (4.6)–(4.8) that

$$\begin{aligned}&\frac{d}{dt}\|v\|^2+2\|\nabla_\lambda v\|^2+\frac{\gamma}{4}\|v\|^2+2c_1e^{(p-2)\omega(t)}\|v\|^p_{L^p}\\&\le -\frac{3}{2}\gamma\|v\|^2+\frac{4}{\gamma}e^{-2\omega(t)}\|g(t)\|^2+2e^{-2\omega(t)}\|f_1(t)\|_{L^1}.\end{aligned} \tag{4.9}$$

Multiplying (4.9) by $e^{\frac{3}{2}\gamma t}$ and then integrating over $(-\tau-t,\sigma)$ with $\sigma\ge\tau-t$, we have for every $\omega\in\Omega$,

$$\begin{aligned}&\|v(\sigma,\tau-t,\omega,v_{\tau-t})\|^2+2\int_{\tau-t}^{\sigma}e^{\gamma(s-\sigma)}\|\nabla_\lambda v(s,\tau-t,\omega,v_{\tau-t})\|^2ds\\&+\frac{\gamma}{4}\int_{\tau-t}^{\sigma}e^{\gamma(s-\sigma)}\|v(s,\tau-t,\omega,v_{\tau-t})\|^2ds\\&\le e^{\frac{3}{2}\gamma(\tau-t-\sigma)}\|v_{\tau-t}\|^2+\int_{\tau-t}^{\sigma}e^{\frac{3}{2}(s-\sigma)-2\omega(s)}\left(\frac{4}{\gamma}\|g(s)\|^2+2\|f_1(s)\|_{L^1}\right)ds.\end{aligned} \tag{4.10}$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.10), by the fact that for any $s\ge\tau-t$,

$$u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})=e^{\omega(s-\tau)-\omega(-\tau)}v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}), \tag{4.11}$$

thus we have

$$\begin{aligned}&e^{2\omega(\sigma-\tau)}\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\&+2\int_{\tau-t}^{\sigma}e^{\frac{3}{2}\gamma(s-\sigma)-2\omega(s-\tau)}\|\nabla_\lambda u(s,\tau,\theta_{-\tau}\omega,u_{\tau-t})\|^2ds\\&+\frac{\gamma}{4}\int_{\tau-t}^{\sigma}e^{\frac{3}{2}\gamma(s-\sigma)-2\omega(s-\tau)}\|u(s,\tau,\theta_{-\tau}\omega,u_{\tau-t})\|^2ds\\&\le e^{\frac{3}{2}\gamma(\tau-t\sigma)-2\omega(-t)}\|u_{\tau-t}\|^2+\int_{\tau-t}^{\sigma}e^{\frac{3}{2}\gamma(s-\sigma)-2\omega(s-\tau)}\left(\frac{4}{\gamma}\|g(s)\|^2+2\|f_1(s)\|_{L^1}\right)ds\\&\le e^{\frac{3}{2}\gamma(\tau-t\sigma)-2\omega(-t)}\|u_{\tau-t}\|^2\\&+\int_{-\infty}^{0}e^{\frac{3}{2}\gamma s-2\omega(s+\sigma-\tau)}\left(\frac{4}{\gamma}\|g(s+\sigma)\|^2+2\|f_1(s+\sigma)\|_{L^1}\right)ds.\end{aligned} \tag{4.12}$$

By (2.5) and (3.36), we obtain

$$\int_{-\infty}^{0}e^{\frac{3}{2}\gamma s-2\omega(s+\sigma-\tau)}\left(\frac{4}{\gamma}\|g(s+\sigma)\|^2+2\|f_1(s+\sigma)\|_{L^1}\right)ds<+\infty. \tag{4.13}$$

Since $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ and D is tempered, we obtain

$$\begin{aligned}&\limsup_{t\to\infty}e^{\frac{3}{2}\gamma(\tau-t-\sigma)-2\omega(-t)}\|u_{\tau-t}\|^2\\&\le\limsup_{t\to\infty}e^{\frac{3}{2}\gamma(\tau-t-\sigma)-2\omega(-t)}\|D(\tau-t),\theta_{-t}\omega\|^2=0,\end{aligned}$$

from this, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\begin{aligned} & e^{\frac{3}{2}\gamma(\tau-t-\sigma)-2\omega(-\tau)}\|u_{\tau-t}\|^2 \\ & \leq \int_{-\infty}^{0} e^{\frac{3}{2}\gamma s-2\omega(s+\sigma-\tau)}\left(\frac{4}{\gamma}\|g(s+\sigma)\|^2+2\|f_1(s+\sigma)\|_{L^1}\right)ds<+\infty, \end{aligned}$$

which together with (4.12) and (4.13) gives the desired estimates. ☐

Corollary 4.1. *For every $\tau \in \mathbb{R}, \omega \in \Omega, T > 0$ there exists $c = c(\tau, \omega, T) > 0$ such that for all $t \in [\tau, \tau + T]$, the solution v satisfies*

$$\|v(t,\tau,\omega,v_\tau)\|^2+\int_\tau^t \|v(s,\tau,\omega,v_\tau)\|_{L^p}^p ds \leq c\|v_\tau\|^2+\int_\tau^t (\|g(s)\|^2+\|f_1(s)\|_{L^1})ds.$$

As in Lemma 3.2, we have the asymptotic compactness of solutions of (4.1) on bounded domains.

Lemma 4.2. *Assume that (3.1)–(3.6) are satisfied and $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in $L^2(\mathbb{R}^N)$. Then for every $\tau \in \mathbb{R}, t > \tau$ and $\omega \in \Omega$, there exist $u_0 \in L^2(\tau, t; L^2(\mathbb{R}^N))$ and a subsequence $\{u(\cdot, \tau, \omega, u_{n_\ell})\}_{\ell=1}^{\infty}$ of $\{u(\cdot, \tau, \omega, u_n)\}_{n=1}^{\infty}$ such that $u(s, \tau, \omega, u_{n_\ell}) \to u_0(s)$ in $L^2(\mathcal{O}_k)$ as $\ell \to \infty$ for every $k \in \mathbb{N}$ and for almost all $s \in (\tau, t)$.*

We next establish some uniform estimates on the tails of solutions.

Lemma 4.3. *Suppose that (3.1)–(3.3) and (3.36) are satisfied. Then, for every $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for any $\epsilon > 0$, there exists $T = T(\tau, \omega, D, \epsilon) \geq 1$ and $N = N(\tau, \omega, \epsilon) > 0$ such that for all $t \geq T$ and $\sigma \in [\tau - 1, \tau]$, the solutions u of system (4.1) satisfies*

$$\int_{\mathcal{O}_N^c} |u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \leq \epsilon,$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Let k be a fixed positive integer which will be specified later and take $\varphi_k(x)$ as in Lemma 3.4. Multiplying (4.3) by $\varphi_k(x)v$ and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\varphi_k(x)|v|^2dx = & \int_{\mathbb{R}^N}\varphi_k(x)(\Delta_\lambda v)vdx-\gamma\int_{\mathbb{R}^N}\varphi_k(x)|v|^2dx \\ & + e^{-\omega(t)}\int_{\mathbb{R}^N}\varphi_k(x)f(t,x,e^{\omega(t)}v)vdx \\ & + e^{-\omega(t)}\int_{\mathbb{R}^N}\varphi_k(x)g(t,x)vdx. \end{aligned} \tag{4.14}$$

Similarly to (3.53), we obtain

$$\int_{\mathbb{R}^N}\varphi_k(x)(\Delta_\lambda v)vdx \leq -\int_{\mathbb{R}^N}\varphi_k(x)\|\nabla_\lambda v\|^2dx+\frac{c}{k}(\|\nabla_\lambda v\|^2+\|v\|^2). \tag{4.15}$$

From (3.1), we also have that

$$\begin{aligned} e^{-\omega(t)}\int_{\mathbb{R}^N}\varphi_k(x)f(t,x,e^{\omega(t)}v)vdx \leq & -c_1e^{(p-2)\omega(t)}\int_{\mathbb{R}^N}\varphi_k(x)|v|^pdx \\ & + e^{-2\omega(t)}\int_{\mathbb{R}^N}\varphi_k(x)|f_1(t,x)|dx. \end{aligned} \tag{4.16}$$

By Young's inequality, one has

$$e^{-\omega(t)}\int_{\mathbb{R}^N}\varphi_k(x)g(t,x)vdx \leq \frac{\gamma}{4}\int_{\mathbb{R}^N}\varphi_k(x)|v|^2dx+\frac{1}{\gamma}e^{-2\omega(t)}\int_{\mathbb{R}^N}\varphi_k(x)|g(t,x)|^2dx. \tag{4.17}$$

Hence, from (4.14)–(4.17), we infer that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}\varphi_k(x)|v|^2dx+\frac{3}{2}\gamma\int_{\mathbb{R}^N}\varphi_k(x)|v|^2dx\\&\le\frac{c}{k}\|v\|^2_{H^1_\lambda}+ce^{-2\omega(t)}\int_{\mathbb{R}^N}\varphi_k(x)(|g|^2+|f_1|)dx.\end{aligned}\tag{4.18}$$

Therefore, by (4.18), there is a $N_1=N_1(\epsilon)>0$ such that for all $k\ge N_1$,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}\varphi_k(x)|v|^2dx+\frac{3}{2}\gamma\int_{\mathbb{R}^N}\varphi_k(x)|v|^2dx\\&\le\epsilon\|v\|^2_{H^1_\lambda}+ce^{-2\omega(t)}\int_{\mathcal{O}_k^c}(|g|^2+|f_1|)dx.\end{aligned}\tag{4.19}$$

Given $t\ge1,\tau\in\mathbb{R}$ and $\omega\in\Omega$, multiplying (4.19) by $e^{\frac{3}{2}t}$ and integrating over $(\tau-t,\sigma)$ with $\sigma\in[\tau-1,\tau]$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\varphi_k(x)|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2dt\\&\le e^{\frac{3}{2}\gamma(\tau-t-\sigma)}\|v_{\tau-t}\|^2+\epsilon\int_{\tau-t}^{\sigma}e^{\frac{3}{2}\gamma(s-\sigma)}\|v(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^1_\lambda}ds\\&\quad+ce^{2\omega(-\tau)}\int_{-\infty}^{\sigma-\tau}e^{\frac{3}{2}\gamma(s+\tau-\sigma)-2\omega(s)}\int_{\mathcal{O}_k^c}(|g(s+\tau,x)|^2+|f_1(s+\tau,x)|)dxds.\end{aligned}\tag{4.20}$$

Combining (4.11) and (4.20), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\varphi_k(x)|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2dx\\&\le e^{2\omega(\sigma-\tau)}e^{\frac{3}{2}\gamma(\tau-t-\sigma)-2\omega(-t)}\|u_{\tau-t}\|^2\\&\quad+\epsilon e^{2\omega(\sigma-\tau)}\int_{\tau-t}^{\sigma}e^{\frac{3}{2}\gamma(s-\sigma)-2\omega(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^1_\lambda}ds\\&\quad+ce^{2\omega(\sigma-\tau)}\int_{-\infty}^{\sigma-\tau}e^{\frac{3}{2}\gamma(s+\tau-\sigma)-2\omega(s)}\int_{\mathbb{R}^N\setminus\mathcal{O}_k}(|g(s+\tau,x)|^2+|f_1(s+\tau,x)|)dxds.\end{aligned}\tag{4.21}$$

Note that $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ and D is tempered, thus we have

$$\begin{aligned}&\limsup_{t\to\infty}e^{\frac{3}{2}\gamma(\tau-t-\sigma)-2\omega(-t)}\|u_{\tau-t}\|^2\\&\le e^{\frac{3}{2}\gamma}\limsup_{t\to\infty}e^{-\frac{3}{2}\gamma t-2\omega(-t)}\|D(\tau-t,\theta_{-t}\omega)\|^2=0,\end{aligned}$$

which implies that there exists $T_1=T_1(\tau,\omega,D,\epsilon)\ge1$ such that for all $t\ge T_1$,

$$e^{2\omega(\sigma-\tau)}e^{\frac{3}{2}\gamma(\tau-t-\sigma)-2\omega(-t)}\|u_{\tau-t}\|^2\le\epsilon.\tag{4.22}$$

On the other hand, by (4.13) and Lemma 4.1, there is a $T_2=T_2(\tau,\omega,D)\ge T_1$ such that for all $t\ge T_2$,

$$\begin{aligned}&\int_{\tau-t}^{\sigma}e^{\frac{3}{2}\gamma(s-\sigma)-2\omega(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^1_\lambda}ds\\&\le e^{\frac{3}{2}\gamma}\int_{\tau-t}^{\tau}e^{\frac{3}{2}\gamma(s-\tau)-2\omega(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^1_\lambda}ds\\&\le c\int_{-\infty}^{0}e^{\frac{3}{2}\gamma s-2\omega(s)}(\|g(s+\tau)\|^2+\|f_1(s+\tau)\|^2_{L^1})ds<+\infty.\end{aligned}\tag{4.23}$$

Using (3.36), there exists a $N_2=N_2(\tau,\epsilon)\ge N_1$ such that for all $k\ge N_2$,

$$\int_{-\infty}^{\sigma-\tau}e^{\frac{3}{2}\gamma(s+\tau-\sigma)-2\omega(s)}\int_{\mathbb{R}^N\setminus\mathcal{O}_k}(|g(s+\tau,x)|^2+|f_1(s+\tau,x)|)dxds$$

$$\leq e^{\frac{3}{2}\gamma}\int_{-\infty}^{0} e^{\frac{3}{2}\gamma s-2\omega(s)}\int_{\mathbb{R}^N\setminus\mathcal{O}_k}(|g(s+\tau,x)|^2+|f_1(s+\tau,x)|)dxds \leq \epsilon. \tag{4.24}$$

Summing up from (4.21) to (4.24), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\mathcal{O}_{\sqrt{2}k}} |u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2dx\\
&\leq \int_{\mathbb{R}^N}\varphi_k(x)|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2dx\\
&\leq c\epsilon + c\epsilon\int_{-\infty}^{0} e^{\frac{3}{2}\gamma s-2\omega(s)}(\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1})ds,
\end{aligned}$$

this gives us the conclusion of the lemma. □

We now show the existence of $\mathcal{D}$-pullback attractors for Φ_0.

Theorem 4.2 (**Existence of random attractors with muliplicative noise).** *Let the assumptions* (3.1)–(3.3) *and* (3.36)–(3.37) *hold. Then, the continuous cocycle* Φ_0 *of equation* (4.1) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}_0=\{\mathcal{A}_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ *in* $L^2(\mathbb{R}^N)$*. In addition, if* f,g *and* f_1 *are periodic functions with respect to* t, *then the* $\mathcal{D}$*-pullback attractor* $\mathcal{A}_0$ *is also* T*-periodic.*

Proof. By Lemma 3.3, we see that for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, there exists an absorbing set of Φ_0 given by

$$K_0(\tau,\omega)=\{u\in L^2(\mathbb{R}^N):\|u\|^2\leq R_0(\tau,\omega)\} \tag{4.25}$$

where

$$R_0(\tau,\omega)=M\int_{-\infty}^{0} e^{\frac{3}{2}\gamma s-2\omega(s)}(\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1})ds. \tag{4.26}$$

Indeed, by Lemma 4.1, there exists $T=T(\tau,\omega,D)>0$ such that for all $t\geq T$

$$\Phi_0(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))\subset K_0(\tau,\omega).$$

Similar to previous section, by (2.5) and (3.37) we can show that K_0 in 4.25 is tempered. Therefore, $K_0\in\mathcal{D}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ_0. By Lemmas 4.2 and 4.3, using similar arguments as in Lemma 3.5, one can obtain the $\mathcal{D}$-pullback asymptotically compact for Φ_0 in $L^2(\mathbb{R}^N)$. Thus, by Proposition 2.1 we obtain the desired result. □

4.2. **Convergence of the approximation solutions in the case of multiplicative noise.** We consider the approximation equation of (4.3) by

$$\frac{d}{dt}u_\delta=\Delta_\lambda u_\delta-\gamma u_\delta+f(t,x,u_\delta)+g(t,x)+u_\delta\mathcal{W}_\delta(\theta_t\omega),\quad t>\tau, \tag{4.27}$$

subject to the initial condition

$$u_\delta(\tau,x)=u_{\delta,\tau}(x),\quad x\in\mathbb{R}^N, \tag{4.28}$$

where $\mathcal{W}_\delta$ is the Wong-Zakai approximation (see Subsection 2.3).

We first observe that, for every $\delta\neq 0$, problem (4.27)–(4.28) is pathwise deterministic equation, and as previous section we can see that problem (4.27)–(4.28) defines a continuous cocycle Φ_δ in $L^2(\mathbb{R}^N)$ which has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_\delta$.

Let

$$v_\delta(t,\tau,\omega,v_{\delta,\tau})=e^{-\int_0^t\mathcal{W}_\delta(\theta_r\omega)dr}u_\delta(t,\tau,\omega,u_{\delta,\tau}), \tag{4.29}$$

where $v_{\delta,\tau}=e^{-\int_0^t\mathcal{W}_\delta(\theta_r\omega)dr}u_{\delta,\tau}$.

Thus, we obtain that

$$\frac{dv_\delta}{dt}-\Delta_\lambda v_\delta+\gamma v_\delta=e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}f(t,x,u_\delta)+e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}g(t,x), \quad t>\tau, \tag{4.30}$$

subject to the initial condition

$$v_\delta(\tau,x)=v_{\delta,\tau}(x), \quad x\in\mathbb{R}^N. \tag{4.31}$$

We first show the uniform estimates for the solutions of (4.30).

Lemma 4.4. *Assume that* (3.1)–(3.3) *and* (3.36) *are satisfied. Then for every* $\tau\in\mathbb{R},\omega\in\Omega$ *and* $T>0$, *there exist* $\delta_0=\delta_0(\tau,\omega,T)>0$ *and* $c=c(\tau,\omega,T)>0$ *such that for all* $0<|\delta|<\delta_0$ *and* $t\in[\tau,\tau+T]$, *the solution* v_δ *of equation* (4.30) *satisfies*

$$\begin{aligned}\|v_\delta(t,\tau,\omega,v_{\delta,\tau})\|^2&+\int_\tau^t\|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^2_{H^1_\lambda}ds+\int_\tau^t\|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^p_{L^p}ds\\&\le c\|v_{\delta,\tau}\|^2+c\int_\tau^t(\|g(s)\|^2+\|f_1(s)\|_{L^1})ds.\end{aligned}$$

Proof. For every $\omega\in\Omega$, multiplying (4.30) by v_δ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_\delta\|^2+\|\nabla_\lambda v_\delta\|^2+\gamma\|v_\delta\|^2&=e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\int_{\mathbb{R}^N}f(t,x,u_\delta)v_\delta dx\\&\quad+e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\int_{\mathbb{R}^N}g(t,x)v_\delta dx.\end{aligned}\tag{4.32}$$

Using (3.1) we have that

$$\begin{aligned}&e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\int_{\mathbb{R}^N}f(t,x,u_\delta)v_\delta dx\\&=e^{-2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\int_{\mathbb{R}^N}f(t,x,u_\delta)u_\delta dx\\&\le -c_1e^{(p-2)\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|v_\delta\|^p_{L^p}+e^{-2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|f_1(t)\|_{L^1}.\end{aligned}\tag{4.33}$$

And by Young's inequality we have

$$e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\int_{\mathbb{R}^N}g(t,x)v_\delta dx\le\frac{\gamma}{8}\|v_\delta\|^2+\frac{2}{\gamma}e^{-2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|g(t)\|^2. \tag{4.34}$$

Hence, from (4.32)–(4.34), we obtain

$$\begin{aligned}&\frac{d}{dt}\|v_\delta\|^2+\frac{\gamma}{4}\|v_\delta\|^2+2\|v_\delta\|^2\\&\le-\frac{3}{2}\gamma\|v_\delta\|^2-2c_1e^{(p-2)\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|v_\delta\|^p_{L^p}\\&\quad+\frac{\gamma}{4}e^{-2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|g(t)\|^2+2e^{-2\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}\|f_1(t)\|_{L^1}.\end{aligned}\tag{4.35}$$

Next, for every $\omega\in\Omega$, multiplying the above inequality by $e^{\frac{3}{2}\gamma t}$, then integrating over (τ,t) with $t\ge\tau$,

$$\begin{aligned}&\|v_\delta(t,\tau,\omega,v_{\delta,\tau})\|^2+\frac{\gamma}{4}\int_\tau^t e^{\frac{3}{2}\gamma(s-t)}\|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^2ds\\&+2\int_\tau^t e^{\frac{3}{2}\gamma(s-t)}\|\nabla_\lambda v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^2ds\\&+2c_1\int_\tau^t e^{\frac{3}{2}\gamma(s-t)+(p-2)\int_0^s \mathcal{W}_\delta(\theta_r\omega)dr}\|v_\delta(s,\tau,\omega,v_{\delta,\tau})\|^p_{L^p}ds\\&\le e^{-\frac{3}{2}(t-\tau)}\|v_{\delta,\tau}\|^2\end{aligned}$$

$$+ 2\int_\tau^t e^{\frac{3}{2}\gamma(s-t)+(p-2)\int_0^s \mathcal{W}_\delta(\theta_r\omega)dr}\left(\frac{2}{\gamma}\|g(s)\|^2+\|f_1(s)\|_{L^1}\right)ds, \tag{4.36}$$

which together with (2.10) and (4.36) gives us the desired estimates. □

Lemma 4.5. *For every $\delta \neq 0, \tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\delta) > 0$ such that for all $t \geq T$, the solutions u_δ of* (4.27) *satisfies*

$$\begin{aligned}
&\|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2\\
&\quad+\frac{\gamma}{4}\int_{-t}^0 e^{\frac{3}{2}\gamma s+2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|u_\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2ds\\
&\quad+2\int_{-t}^0 e^{\frac{3}{2}\gamma s+2\int_t^0 \mathcal{W}_\delta(\theta_r\omega)dr}\|\nabla_\lambda u_\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2ds\\
&\leq 4\int_{-\infty}^0 e^{\frac{3}{2}\gamma s+2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\left(\frac{2}{\gamma}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds,
\end{aligned}$$

where $u_{\delta,\tau-t} \in D(\tau-t,\theta_{-t}\omega)$.

Proof. Using (4.29) and (4.35) we obtain that

$$\begin{aligned}
&\|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2\\
&+\frac{\gamma}{4}\int_{\tau-t}^\tau e^{\frac{3}{2}\gamma(s-\tau)-2\int_s^\tau \mathcal{W}_\delta(\theta_{r-\tau}\omega)dr}\|u_\delta(s,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2ds\\
&+2\int_{\tau-t}^\tau e^{\frac{3}{2}\gamma(s-\tau)-2\int_s^\tau \mathcal{W}_\delta(\theta_{r-\tau}\omega)dr}\|\nabla_\lambda u_\delta(s,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2ds\\
&\leq e^{-\frac{3}{2}\gamma t+2\int_{\tau-t}^\tau \mathcal{W}_\delta(\theta_{r-\tau}\omega)dr}\|u_{\delta,\tau-t}\|^2\\
&+2\int_{\tau-t}^\tau e^{\frac{3}{2}\gamma(s-\tau)+2\int_s^\tau \mathcal{W}_\delta(\theta_{r-\tau}\omega)dr}\left(\frac{2}{\gamma}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds,
\end{aligned} \tag{4.37}$$

which implies the estimates as desired. □

As an immediately consequence of Lemma 4.5, we obtain the existence of a tempered pullback absorbing set for Φ_δ.

Lemma 4.6. *For every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the continuous cocycle Φ_δ has a closed measurable $\mathcal{D}$-pullback absorbing set $K_\delta = \{K_\delta(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ given by*

$$K_\delta(\tau,\omega) = \{u \in L^2(\mathbb{R}^N) : \|u\|^2 \leq R_\delta(\tau,\omega)\}, \tag{4.38}$$

where $R_\delta(\tau,\omega)$ is defined by

$$R_\delta(\tau,\omega) = 4\int_{-\infty}^0 e^{\frac{3}{2}\gamma s+2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\left(\frac{2}{\gamma}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds. \tag{4.39}$$

Moreover, we have for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\delta\to 0} R_\delta(\tau,\omega) = R_0(\tau,\omega), \tag{4.40}$$

where $R_0(\tau,\omega)$ is defined in (4.26).

Proof. For each $\delta \neq 0$, we known that K_δ given by (4.38) is a closed measurable random set in $L^2(\mathbb{R}^N)$. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$, by Lemma 4.5 there exists $T_0 = T_0(\tau,\omega,D,\delta) > 0$ such that for all $t \geq T_0$,

$$\Phi_\delta(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega)) \subseteq K_\delta(\tau,\omega).$$

This shows that K_δ pullback attracts all elements in $\mathcal{D}$. By (2.5) we can check that K_δ is tempered. In addition, by using the Lebesgue dominated convergence theorem, we obtain the convergence in (4.40) . □

We next derive uniform estimates on the tails of solutions with respect to δ.

Lemma 4.7. *Let* (3.1)–(3.3) *and* (3.36) *hold. Then for every* $\tau \in \mathbb{R}, \omega \in \Omega$ *and* $\epsilon > 0$, *there exist* $\delta_0 = \delta_0(\omega) > 0, T = T(\tau, \omega, \epsilon) > 0$ *and* $N = N(\tau, \omega, \epsilon) > 0$ *such that for all* $t \geq T$ *and* $0 < |\delta| < \delta_0$, *the solution* u_δ *of* (4.27) *satisfies*

$$\int_{\mathcal{O}_N^c} |u_\delta(\tau, \tau - t, \theta_{-\tau}\omega), u_{\delta,\tau-t}|^2 dx \leq \epsilon,$$

where $u_{\delta,\tau-t} \in K_\delta(\tau - t, \theta_{-t}\omega)$ *with* K_δ *given by* (4.38).

Proof. Let φ_k be the function defined as in Lemma 3.4. From (4.38) we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \varphi_k(x)|v_\delta|^2 dx - \int_{\mathbb{R}^N} \varphi_k(x)(\Delta_\lambda v_\delta)v_\delta dx + \gamma \int_{\mathbb{R}^N} \varphi_k(x)|v_\delta|^2 dx \\
&= e^{-\int^t \mathcal{W}_\delta(\theta_r\omega)dr} \int_{\mathbb{R}^N} \varphi_k(x) f(t, x, u_\delta)v_\delta dx + e^{-\int^t \mathcal{W}_\delta(\theta_r\omega)dr} \int_{\mathbb{R}^N} \varphi_k(x)g(t, x)v_\delta dx.
\end{aligned}$$

By the same steps used to derive (4.21), we also have that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \varphi_k(x)|u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})|^2 dx \\
&\leq e^{-\frac{3}{2}\gamma t + 2\int_{-t}^0 \mathcal{W}_\delta(\theta_r\omega)dr} \int_{\mathbb{R}^N} \varphi_k(x)|u_{\delta,\tau-t}|^2 dx \\
&+ \frac{2c_0}{k}\int_{-t}^0 e^{\frac{3}{2}\gamma s + 2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr} \|u_\delta(s + \tau, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|_{H_\lambda^1}^2 ds \\
&+ 2\int_{-\infty}^0 e^{\frac{3}{2}\gamma s + 2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr} \int_{\mathcal{O}_k^c} \left(\frac{1}{\gamma}|g(s + \tau)|^2 + |f_1(s + \tau)|\right) ds. \qquad (4.41)
\end{aligned}$$

By property of $\mathcal{W}_\delta$, we have

$$2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr = -2\int_s^{s+\delta} \frac{\omega(r)}{\delta}dr + 2\int_0^\delta \frac{\omega(r)}{\delta}dr. \qquad (4.42)$$

Using the continuity of ω, it follows that $\int_0^\delta \frac{w(r)}{\delta}dr \to 0$ as $\delta \to 0$, thus there exists $\delta_1 = \delta_1(\omega) > 0$ such that for all $0 < |\delta| < \delta_1$,

$$\left|2\int_0^\delta \frac{w(r)}{\delta}dr\right| \leq 1. \qquad (4.43)$$

Moreover, by the mean value theorem, there is $r_1 \in (s, s+\delta)$ such that $\int_s^{s+\delta} \frac{\omega(s)}{s}ds = 2\omega(r_1)$, and since $\frac{\omega(s)}{s} \to 0$ as $|s| \to \infty$, there exists $T_1 = T_1(\omega) < 0$ such that for all $s \leq T_1$ and $|\delta| \leq 1$, we have

$$2\left|\int_0^\delta \frac{\omega(r)}{\delta}dr\right| \leq \frac{\gamma}{8} - \frac{\gamma}{8}s. \qquad (4.44)$$

Thus, from (4.42)–(4.44), for $\delta_2 = \min\{\delta_1, 1\}$ we have for all $0 < |\delta| < \delta_2$ and $s \leq T_1$,

$$2\left|\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr\right| \leq \frac{\gamma}{8} - \frac{\gamma}{8}s + 1. \qquad (4.45)$$

On the other hand, using a similar argument as in (2.10), we also obtain there exist $\delta_0 = \delta_0(\omega) \in (0, \delta_2)$ and $c_1(\omega) > 0$ such that $\forall 0 < |\delta| < \delta_0$ and $T_1 \leq s \leq 0$,

$$2\left|\int_s^0 \mathcal{W}_\delta(\theta_s\omega)ds\right| \leq c_1(\omega),$$

which together with (4.45) shows that

$$2\left|\int_s^0 \mathcal{W}_\delta(\theta_s\omega)ds\right| \leq \frac{\gamma}{8} - \frac{\gamma}{8}s + c_2(\omega), \qquad \forall 0 < |\delta| < \delta_0 \text{ and } s \leq 0, \qquad (4.46)$$

where $c_2(\omega) = c_1(\omega) + 1$. Thus, by (4.39) and since $u_{\delta,\tau-1} \in K_\delta(\tau - t, \theta_{-t}\omega)$, we obtain

$$\begin{aligned}
&e^{-\frac{3}{2}\gamma t+2\int_{-t}^{0}\mathcal{W}_\delta(\theta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\\
&\le e^{-\frac{3}{2}\gamma t+2\int_{-t}^{0}\mathcal{W}_\delta(\theta_r\omega)dr}\|K_\delta(\tau-t,\theta_{-t}\omega)\|^2\\
&\le 4e^{-\frac{3}{2}\gamma t+2\int_{-t}^{0}\mathcal{W}_\delta(\theta_r\omega)dr}\int_{-\infty}^{0} e^{\frac{3}{2}+2\int_s^0\mathcal{W}_\delta(\theta_{r-t}\omega)dr}\\
&\qquad\qquad\times\left(\frac{2}{\gamma}\|g(s+\tau-t)\|^2+\|f_1(s+\tau-t)\|_{L^1}\right)ds\\
&\le 4e^{-\frac{3}{2}\gamma t+2\int_{-t}^{0}\mathcal{W}_\delta(\theta_r\omega)dr}\int_{-\infty}^{-t} e^{\frac{3}{2}\gamma(s+t)+2\int_s^{-t}\mathcal{W}_\delta(\theta_r\omega)dr}\\
&\qquad\qquad\times\left(\frac{2}{\gamma}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds. \qquad (4.47)
\end{aligned}$$

Hence, by (4.46) we deduce that for all $0 < |\delta| < \delta_0, s \le 0$ and $t \ge 0$,

$$\begin{aligned}
2\left|\int_s^{-t}\mathcal{W}_\delta(\theta_r\omega)dr\right| &\le 2\left|\int_s^{0}\mathcal{W}_\delta(\theta_r\omega)dr\right| + 2\left|\int_{-t}^{0}\mathcal{W}_\delta(\theta_r\omega)dr\right|\\
&\le \frac{\gamma}{4} + 2c_2 + \frac{\gamma}{8}t - \frac{\gamma}{8}s,
\end{aligned}$$

from this and by (3.36) and (4.47), for all $0 < |\delta| < \delta_0$, we have

$$\begin{aligned}
&e^{-\frac{3}{2}\gamma t+2\int_{-t}^{0}\mathcal{W}_\delta(\theta_r\omega)dr}\|u_{\delta,\tau-t}\|^2\\
&\le 4e^{\frac{3}{8}\gamma+3c_2}e^{-\frac{11}{8}\gamma t}\int_{-\infty}^{-t} e^{\frac{3}{2}\gamma(s+t)+\frac{1}{8}\gamma t-\frac{1}{8}\gamma s}\left(\frac{\gamma}{2}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds\\
&\le 4e^{\frac{3}{8}\gamma+3c_2}e^{-\frac{11}{8}\gamma t}\int_{-\infty}^{-t} e^{\frac{9}{8}\gamma(s+t)+\frac{1}{8}\gamma t-\frac{1}{8}\gamma s}\left(\frac{\gamma}{2}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds\\
&\le 4e^{\frac{3}{8}\gamma+3c_2}e^{-\frac{1}{8}\gamma t}\int_{-\infty}^{-t} e^{\gamma s}\left(\frac{\gamma}{2}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds \qquad (4.48)\\
&\to 0 \text{ as } t\to\infty.
\end{aligned}$$

Thus, a positive $T_2 = T_2(\tau, \omega, \epsilon) > 0$ exists such that for all $t \ge T_2$ and $0 < |\delta| < \delta_0$,

$$e^{-\frac{3}{2}\gamma t+2\int_{-t}^{0}\mathcal{W}_\delta(\theta_r\omega)dr}\|u_{\delta,\tau-t}\|^2 \le \frac{\epsilon}{3}. \qquad (4.49)$$

Combining (4.37) and (4.46), there exists $T_3 = T_3(\tau, \omega) > 0$ such that for all $t \ge T_3$ and $0 < |\delta| < \delta_0$, we have

$$\begin{aligned}
&\int_{-t}^{0} e^{\frac{3}{2}\gamma s+2\int_s^0\mathcal{W}_\delta(\theta_r\omega)dr}\|u_\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2_{H^1_\lambda}ds\\
&\le 1+2\int_{-\infty}^{s} e^{\frac{3}{2}\gamma s+2\int_s^0\mathcal{W}_\delta(\theta_r\omega)dr}\left(\frac{2}{\gamma}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds\\
&\le 1+2e^{\frac{\gamma}{8}+c_2}\int_{-\infty}^{0} e^{\gamma s}\left(\frac{2}{\gamma}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}\right)ds,
\end{aligned}$$

and from this, there exists $N_1 = N_1(\tau, \omega, \epsilon) > 0$ such that for all $k \ge N_1$, $t \ge T_3$ and $0 < |\delta| < \delta_0$, we have

$$\frac{2c_0}{k}\int_{-t}^{0} e^{\frac{3}{2}\gamma s+2\int_s^0\mathcal{W}_\delta(\theta_r\omega)dr}\|u_\delta(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2_{H^1_\lambda}ds \le \frac{\epsilon}{3}. \qquad (4.50)$$

Moreover, by (4.46) we have for all $0<|\delta|<\delta_0$,

$$\begin{aligned}&2\int_{-\infty}^{0} e^{\frac{3}{2}\gamma s+2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\int_{\mathcal{O}_k^c}\left(\frac{1}{\gamma}|g(s+\tau)|^2+|f_1(s+\tau)|\right)dxds\\ &\le 2e^{\frac{\gamma}{8}+c_2(\omega)}\int_{-\infty}^{0} e^{\gamma s}\int_{\mathcal{O}_k^c}\left(\frac{1}{\gamma}|g(s+\tau)|^2+|f_1(s+\tau)|\right)dxds,\end{aligned}$$

which implies that there exists $N_2=N_2(\tau,\omega,\epsilon)>0$ such that for all $k\ge N_2$ and $0<|\delta|<\delta_0$,

$$2\int_{-\infty}^{0} e^{\frac{3}{2}\gamma s+2\int_s^0 \mathcal{W}_\delta(\theta_r\omega)dr}\int_{\mathcal{O}_k^c}\left(\frac{1}{\gamma}|g(s+\tau)|^2+|f_1(s+\tau)|\right)dxds\le\frac{\epsilon}{3}. \tag{4.51}$$

Let $N=\max\{N_1,N_2\}$ and $T=\max\{T_1,T_2\}$, then from (4.41)–(4.51), we obtain for all $t\ge T, k\ge N$ and $0<|\delta|<\delta_0$,

$$\int_{\mathcal{O}^c_{\sqrt{2}k}}|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})|^2dx\le\int_{\mathbb{R}^N}\varphi_k(x)|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})|^2dx\le\epsilon.$$

The proof is completed. □

We next show the uniformly compactness of the attractor $\mathcal{A}_\delta$ for Φ_δ.

Lemma 4.8. *Let* (3.1)–(3.3), (3.36) *and* (3.37) *hold. Then for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, the sequence $\{u_n\}$ is precompact in $L^2(\mathbb{R}^n)$ whenever $\delta_n\to 0$ and $u_n\in\mathcal{A}_{\delta_n}(\tau,\omega)$.*

Proof. For $\epsilon>0$ is given and $\delta_0=\delta_0(\omega)$ as in Lemma 4.7, we will prove that $\bigcup_{0<|\delta|<\delta_0}\mathcal{A}_\delta(\tau,\omega)$ has a finite covering of balls of radius less than ϵ. Indeed, it follows from (4.39) and (4.46) that for all $0<|\delta|<\delta_0$,

$$R_\delta(\tau,\omega)\le 4e^{\frac{1}{8}\gamma+c_2}\int_{-\infty}^{0}e^{\gamma s}\left(\frac{2}{\gamma}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}ds\right)ds. \tag{4.52}$$

Set

$$B(\tau,\omega)=\{u\in L^2(\mathbb{R}^n):\|u\|^2\le R(\tau,\omega)\}, \tag{4.53}$$

with

$$R(\tau,\omega)=4e^{\frac{1}{8}\gamma+c_2}\int_{-\infty}^{0}e^{\gamma s}\left(\frac{2}{\gamma}\|g(s+\tau)\|^2+\|f_1(s+\tau)\|_{L^1}ds\right)ds. \tag{4.54}$$

Hence, from (4.52)–(4.54) we obtain that

$$K_\delta(\tau,\omega)\subseteq B(\tau,\omega)\qquad\forall 0<|\delta|<\delta_0,\tau\in\mathbb{R},\omega\in\Omega.$$

Thus, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, we have

$$\bigcup_{0<|\delta|<\delta_0}\mathcal{A}_\delta(\tau,\omega)\subseteq\bigcup_{0<|\delta|<\delta_0}K_\delta(\tau,\omega)\subseteq B(\tau,\omega).$$

Moreover, by Lemma 4.7, there exist $T=T(\tau,\omega,\epsilon)>0$ and $N=N(\tau,\omega,\epsilon)>0$ such that $\forall t\ge T$ and $0<|\delta|<\delta_0$,

$$\int_{\mathcal{O}_N^c}|u_\delta(\tau,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})|^2dx\le\frac{\epsilon}{2}, \tag{4.55}$$

whenever $u_{\delta,\tau-t}\in K_\delta(\tau-t,\theta_{-t}\omega)$. Hence, by the invariance of $\mathcal{A}_\delta$ we infer from (4.55) that

$$\int_{\mathcal{O}_N^c}|u|^2dx\le\frac{\epsilon}{2}\qquad\forall u\in\bigcup_{0<|\delta|<\delta_0}\mathcal{A}_\delta(\tau,\omega). \tag{4.56}$$

Moreover, on each bounded domain $\mathcal{O}_N \subset \mathbb{R}^N$, we can repeat exactly the same techniques as in [40] to show that the sequence $\{u_n\}_{n=1}^{\infty}$ is precompact in $L^2(\mathcal{O}_N)$. This together with (4.56) yields the conclusion as desired. ☐

We next show the convergence of solutions of (4.27) as $\delta \to 0$.

Lemma 4.9. *Suppose that* (3.1)–(3.4) *are satisfied. Let* u *and* u_δ *are the solutions of* (4.1) *and* (4.27), *respectively. Then, for every* $\tau \in \mathbb{R}, \omega \in \Omega, T > 0$ *and* $\epsilon \in (0,1)$, *there exists* $\delta_0 = \delta_0(\tau, \omega, T, \epsilon) > 0$ *and* $c = c(\tau, \omega, T) > 0$ *such that for all* $0 < |\delta| < \delta_0$ *and* $t \in [\tau, \tau + T]$,

$$\begin{aligned} &\|u_\delta(t,\tau,\omega,u_{\delta,\tau}) - u(t,\tau,\omega,u_\tau)\|^2 \le c\|u_{\delta,\tau} - u_\tau\|^2 \\ &+ c\epsilon \left(1 + \|u_{\delta,\tau}\|^2 + \|u_\tau\|^2 + \int_\tau^t \left(\|f_2(s)\|_{L^{p_1}}^{p_1} + \|g(s)\|^2 + \|f_1(s)\|_{L^1}\right) ds\right). \quad (4.57) \end{aligned}$$

Proof. Using Corollary 4.1 and Lemma 4.4, we first observe that for $\bar{v} = v_\delta - v$, by (4.3) and (4.27) one has

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\bar{v}\|^2 + \|\bar{v}\|_{H_\lambda^1}^2 + \gamma\|\bar{v}\|^2 = \int_{\mathbb{R}^N} \left(e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{-\omega(t)}\right) g(t)\bar{v}dx \\ &+ \int_{\mathbb{R}^N} \left(e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v_\delta) - e^{-\omega(t)} f(t,x,e^{\omega(t)}v)\right)\bar{v}dx. \quad (4.58) \end{aligned}$$

By (3.1)–(3.3) and (3.4), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v_\delta) - e^{-\omega(t)} f(t,x,e^{\omega(t)}v)\right)\bar{v}dx \\ &= \int_{\mathbb{R}^N} e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} \left(f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v_\delta) - f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v)\right)\bar{v}dx \\ &+ \int_{\mathbb{R}^N} \left(e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v_\delta) - e^{-\omega(t)}\right) f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v)\bar{v}dx \\ &+ \int_{\mathbb{R}^N} e^{-\omega(t)} \left(f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v) - e^{-\omega(t)} f(t,x,e^{\omega(t)}v\right)\bar{v}dx \\ &= \int_{\mathbb{R}^N} \bar{v}^2 \frac{\partial f}{\partial s}(t,x,s)dx \\ &+ \int_{\mathbb{R}^N} \left(e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v_\delta) - e^{-\omega(t)}\right) f(t,x,e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v)\bar{v}dx \\ &+ \int_{\mathbb{R}^N} \left(e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr - \omega(t)}\right)\bar{v}v\frac{\partial f}{\partial s}(t,x,s)dx \\ &\le \|f_3(t)\|_{L^\infty}\|\bar{v}\|_{H_\lambda^1}^2 + \left|e^{-\int_0^t W_\delta(\theta_r\omega)dr} - e^{-\omega(t)}\right| \\ &\times \int_{\mathbb{R}^N} \left(c_1 e^{(p-1)\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr}|v|^{p-1}|\bar{v}| + f_2(t,x)|\bar{v}|\right)dx \\ &+ |e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr - \omega(t)} - 1|\|f_4(t)\|_{L^\infty} \\ &\times \int_{\mathbb{R}^N} \left(\left(e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} + e^{\omega(t)}\right)^{p-2}|v|^{p-2}|\bar{v}| + |v|\|\bar{v}\|\right)dx. \quad (4.59) \end{aligned}$$

Thus, from (3.7)–(3.1) and by Lemma 4.1, for every $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon,\tau,\omega,T) > 0$ such that for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$, we have

$$\left|e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{-\omega(t)}\right| < \epsilon \quad \text{and} \quad \left|e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr - \omega(t)}\right| < \epsilon. \quad (4.60)$$

Hence, from (4.59) and (4.60) there exists $c_1 = c_1(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,

$$\int_{\mathbb{R}^N} \left(e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} f(t, x, e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v_\delta) - e^{-\omega(t)} f(t, x, e^{\omega(t)} v) \right) \bar{v} dx$$
$$\le c_1 \|\bar{v}\|^2 + c_1\epsilon + c_1\epsilon \int_{\mathbb{R}^N} (|v|^p + |v_\delta|^p + |f_2(t,x)|^{p_1}) dx. \tag{4.61}$$

Moreover, from (4.60) for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$, we have

$$\left| \left(e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{-\omega(t)}\right)(g(t), \bar{v}) \right| < \frac{1}{2}\epsilon\|\bar{v}\|^2 + \frac{1}{2}\epsilon\|g(t)\|^2. \tag{4.62}$$

Combining (4.58), (4.61) and (4.62), for all $\epsilon \in (0,1)$, $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,

$$\frac{d}{dt}\|\bar{v}\|^2 \le c_2\|\bar{v}\|^2 + c_2\epsilon\big(1 + \|v\|^p_{L^p} + \|v_\delta\|^p_{L^p} + \|g(t)\|^2 + \|f_2(t)\|^{p_1}_{L^{p_1}}\big).$$

Applying Gronwall's inequality, we find that for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,

$$\|\bar{v}\|^2 \le e^{c_2(t-\tau)}\|\bar{v}\|^2$$
$$+ c_2\epsilon e^{c_2(t-\tau)} \int_\tau^t \big(1 + \|v_\delta(s, \tau, \omega, v_{\delta,\tau})\|^p_{L^p} + \|v\|^p_{L^p} + \|g(s)\|^2 + \|f_2(s)\|^{p_1}_{L^{p_1}}\big) ds.$$

By Corollary 4.1 and Lemma 4.4, we have that there exist $\delta_2 \in (0, \delta_1)$ and $c_3 = c_3(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_2$ and $t \in [\tau, \tau + T]$,

$$\|v_\delta(t, \tau, \omega, v_{\delta,\tau}) - v(t, \tau, \omega, v_\tau)\|^2 \le e^{c_2(t-\tau)}\|v_{\delta,\tau} - v_\tau\|^2$$
$$+ c_3\epsilon e^{c_2(t-\tau)} \left(1 + \|v_\tau\|^2 + \|v_{\delta,\tau}\|^2 + \int_\tau^t \big(\|f_1(s)\|_{L^1} + \|g(s)\|^2 + \|f_2(s)\|^{p_1}_{L^{p_1}}\big) ds \right). \tag{4.63}$$

Since

$$\begin{aligned} u_\delta(t, \tau, \omega, u_{\delta,\tau}) - u(t, \tau, \omega, u_\tau) &= e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} v_\delta(t, \tau, \omega, v_{\delta,\tau}) - e^{\omega(t)} v(t, \tau, \omega, v_\tau) \\ &= e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} \big(v_\delta(t, \tau, \omega, v_{\delta,\tau}) - v(t, \tau, \omega, v_\tau)\big) \\ &\quad - \left(e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} - e^{\omega(t)}\right) v(t, \tau, \omega, v_\tau) \end{aligned} \tag{4.64}$$

where $v_{\delta,\tau} = e^{-\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr} u_{\delta,\tau}$ and $v_\tau = e^{-\omega(t)} u_\tau$. Therefore, from (2.9)–(2.10) and (4.64) there exist $\delta_3 \in (0, \delta_2)$ and $c_4 = c_4(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_3$ and $t \in [\tau, \tau + T]$,

$$\|u_\delta(t, \tau, \omega, u_{\delta,\tau}) - u(t, \tau, \omega, u_\tau)\| \le c_4\|v_\delta(t, \tau, \omega, v_{\delta,\tau}) - v(t, \tau, \omega, v_\tau)\|$$
$$+ c_4\left|e^{\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr - \omega(t)} - 1\right| \|v(t, \tau, \omega, v_\tau)\|.$$

Hence, by Corollary 4.1, (4.60) and (4.63), we obtain (4.57) as desired. □

From (4.57), we immediately have the following convergence of solutions of (4.30) as $\delta \to 0$.

Corollary 4.3. *Let (3.1)–(3.4) hold and $\delta_n \to 0$. Let u_{δ_n} and u be the solutions of (4.30) and (4.1) with initial data $u_{\delta_n,\tau}$ and u_τ, respectively. If $u_{\delta_n,\tau} \to u_\tau$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$, then for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $t > \tau$,*

$$u_{\delta_n}(t, \tau, \omega, u_{\delta_n,\tau}) \to u(t, \tau, \omega, u_\tau) \quad \text{in } L^2(\mathbb{R}^N),$$

as $n \to \infty$.

We are now ready prove convergence of random attractors in the multiplicative noise case.

Theorem 4.4. *Suppose that* (3.1)–(3.4) *and* (3.36)–(3.37) *are satisfied. Then, for every fixed* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{\delta \to 0} \operatorname{dist}_{L^2(\mathbb{R}^N)}(\mathcal{A}_\delta(\tau,\omega), \mathcal{A}_0(\tau,\omega)) = 0, \tag{4.65}$$

where dist *denotes the Hausdorff semi-distance.*

Proof. Let $\delta_n \to 0$ and $u_{n,\tau} \to u_\tau$ in $L^2(\mathbb{R}^N)$. Then, by Corollary 4.3, for all $\tau \in \mathbb{R}, t \geq 0$ and $\omega \in \Omega$, we have

$$\Phi_{\delta_n}(t,\tau,\omega,u_{\delta_n,\tau}) \to \Phi_0(t,\tau,\omega,u_\tau) \quad \text{in } L^2(\mathbb{R}^N). \tag{4.66}$$

Thus, by (4.40) and (4.52) one has for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\delta \to 0} \|K_\delta(\tau,\omega)\| \leq R_0(\tau,\omega),$$

which together with (4.66) and Lemma 4.8 verifies all the conditions in Proposition 2.2. Therefore, we obtain (4.65) as claimed. ☐

5. Random attractors: additive noise

In analogy with Section 4, in this section we investigate the stochastic equation driven by additive white noise together with its approximation

$$\frac{\partial u}{\partial t} = \Delta_\lambda u - \gamma u + f(t,x,u) + g(t,x) + h(x)\frac{dW(t)}{dt}, \quad t > \tau, \tag{5.1}$$

with the initial condition (4.2), and the approximation equation

$$\frac{\partial u}{\partial t} = \Delta_\lambda u - \gamma u + f(t,x,u) + g(t,x) + h(x)\mathcal{W}_\delta(\theta_t\omega), \quad t > \tau, \tag{5.2}$$

with the initial condition (4.27), where $h \in W_\lambda^{2,2}(\mathbb{R}^N) \cap W_\lambda^{2,p}(\mathbb{R}^N)$.

As previous sections, we use the change of variables

$$v(t,\tau,\omega) = u(t,\tau,\omega) - h(x)\omega(t).$$

Then equation (5.1) becomes the following pathwise deterministic equation,

$$\frac{\partial v}{\partial t} - \Delta_\lambda v + \gamma v = f(t,x,v + h(x)\omega(t)) + g(t,x) - \gamma h(x)\omega(t) + \omega(t)\Delta_\lambda h(x), \ t > \tau, \tag{5.3}$$

supplement the initial condition

$$v(\tau,x) = v_\tau(x), \quad x \in \mathbb{R}^N, \tag{5.4}$$

where $v_\tau = u_\tau - h(x)\omega(\tau)$. Given $\omega \in \Omega, \tau \in \mathbb{R}$ and $v_\tau \in L^2(\mathbb{R}^N)$, we can argument as in [43] to show system (5.3)–(5.4) has a unique solution $v(\cdot,\tau,\omega,v_\tau) \in C([\tau,\infty); L^2(\mathbb{R}^N))$ under assumptions (3.1)–(3.3). In addition, $v(\cdot,\tau,\omega,v_\tau)$ is continuous in v_τ with respect to the norm of $L^2(\mathbb{R}^N)$ and is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^N)))$-measurable in $\omega \in \Omega$. This allows us define a continuous cocycle $\tilde{\Phi}_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ for system (5.1)–(5.2) by

$$\begin{aligned}\tilde{\Phi}_0(t,\tau,\omega,u_\tau) &= u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) \\ &= v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau) + h(x)(\omega(t) - \omega(-\tau)),\end{aligned} \tag{5.5}$$

where $v_\tau = u_\tau + h(x)\omega(-\tau)$.

To obtain the asymptotic behavior of solutions in the case of additive noise, we assume the following conditions for the function f_2 in (3.2):

$$\int_{-\infty}^{\tau} e^{\gamma s}\|f_2(s)\|_{L^{p_1}}^{p_1} ds < +\infty \quad \text{for every } \tau \in \mathbb{R}, \tag{5.6}$$

and for any $c > 0$

$$\lim_{t\to-\infty} e^{ct} \int_{-\infty}^{\tau} e^{\gamma s} \|f_2(s+t)\|^{p_1}_{L^{p_1}} ds = 0, \tag{5.7}$$

where $\gamma > 0$.

We first show the existence of a closed measurable $\mathcal{D}$-pullback absorbing set for the cocycle $\tilde{\Phi}_0$ in (5.5).

Lemma 5.1. *Suppose that* (3.1)–(3.3), (3.36)–(3.37) *and* (5.6)–(5.7) *are satisfied. Then the continuous cocycle* Ψ_0 *has a closed measurable* $\mathcal{D}$*-pullback absorbing set*

$$\tilde{B}_0 = \{\tilde{B}_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$$

which is given by for every $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$

$$\tilde{B}_0(\tau,\omega) = \{u \in L^2(\mathbb{R}^N) : \|u\|^2_{L^2} \leq \tilde{M}_0(\tau,\omega)\}, \tag{5.8}$$

where $M_0(\tau,\omega)$ *is given by*

$$\begin{aligned}\tilde{M}_0(\tau,\omega) = c + c\int_{-\infty}^{0} e^{\gamma s}(\|g(s+\tau)\|^2_{L^2} + \|f_1(s+\tau)\|_{L^1} + \|f_2(s+\tau)\|^{p_1}_{L^{p_1}})ds \\ + c\int_{-\infty}^{0} e^{\gamma s}|\omega(s) - \omega(-\tau)|^p ds + c|\omega(-\tau)|^2,\end{aligned} \tag{5.9}$$

with c *is a positive number independent of* τ, ω *and* $\mathcal{D}$.

Proof. Multiplying (5.3) by v, then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\nabla_\lambda v\|^2 + \gamma\|v\|^2 \\ &= (f(t,x,v+h(x)\omega(t)),v) + (g(t,x),v) - \gamma\omega(t)(h(x),v) + (\Delta_\lambda h, v)\omega(t).\end{aligned} \tag{5.10}$$

From (3.1) and (3.2), there exists $c_1 > 0$ such that

$$\begin{aligned}&(f(v+h\omega(t),t),v) = (f(t,x,v+h(x)\omega(t)), v+h\omega) - \omega(f(t,x,v+h\omega),h) \\ &\qquad \leq -\frac{1}{2}c_1\|v+h\omega\|^p_{L^p} + c_1|\omega(t)|^p\|h\|^p_{L^p} + \|f_1(t)\|_{L^1} + \|f_2(t)\|^{p_1}_{L^{p_1}}.\end{aligned}$$

And by Young's inequality, we have

$$\begin{aligned}&|(g(t,x),v)| \leq \frac{\gamma}{4}\|v\|^2 + \frac{2}{\gamma}\|g(t)\|^2, \\ &\gamma|\omega(t)(h(x),v)| \leq 2\gamma|\omega(t)|^2\|h\|^2 + \frac{1}{2}\|v\|^2, \\ &|(\Delta_\lambda h, v)\omega(t)| \leq \frac{1}{2}\|\nabla_\lambda v\|^2 + \frac{1}{2}|\omega(t)|^2\|h\|^2_{H^1_\lambda}.\end{aligned}$$

Thus, by above estimates we obtain from (5.10) that

$$\begin{aligned}&\frac{d}{dt}\|v\|^2 + \|\nabla_\lambda v\|^2 + \frac{\gamma}{2}\|v\|^2 + c_1\|u\|^p_{L^p} \\ &\leq -\gamma\|v\|^2 + \frac{4}{\gamma}\|g(t)\|^2 + 2\|f_1(t)\|_{L^1} + 2\|f_2(t)\|^{p_1}_{L^{p_1}} + c_2(1+|\omega(t)|^p),\end{aligned} \tag{5.11}$$

where c_2 is a positive constant.

Multiplying (5.11) by $e^{\gamma t}$ and replacing ω by $\theta_{-\tau}\omega$ and then integrating over $(\tau - t, \sigma)$ with $\sigma \geq \tau - t$, we obtain

$$\begin{aligned}&\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \frac{\gamma}{2}\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds \\ &+ \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|\nabla_\lambda v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds\end{aligned}$$

$$
\begin{aligned}
&+ c_1 \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)} \|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_{L^p}^p ds \\
&\le e^{\gamma(\tau-t-\sigma)}\|v_{\tau-t}\|^2 + \frac{c_2}{\gamma} + c_2\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}|\omega(s-\tau)-\omega(-\tau)|^p ds \\
&+ 2 + \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\bigg(\frac{2}{\gamma}\|g(s)\|^2 + \|f_1\|_{L^1} + \|f_2(s)\|_{L^{p_1}}^{p_1}\bigg) ds. \qquad (5.12)
\end{aligned}
$$

Since $s \ge \tau - t$, then we have

$$
u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}) = v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}) + h(\omega(s-\tau)-\omega(-\tau)),
$$

which together with (5.12) implies

$$
\begin{aligned}
&\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 + \frac{\gamma}{2}\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 ds \\
&+ \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|\nabla_\lambda u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 ds \\
&+ c_1\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_{L^p}^p ds \\
&\le 2\|v(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \gamma\int_{\tau-t}^{\sigma}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds \\
&+ 2\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|\nabla_\lambda v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 ds \\
&+ c_1\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_{L^p}^p ds \\
&+ 2\|h\|^2|\omega(\sigma-\tau)-\omega(-\tau)|^2 + \gamma\|h\|^2\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}|\omega(s-\tau)-\omega(-\tau)|^2 ds \\
&+ 2\|h\|_{H^1_\lambda}^2\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}|\omega(s-\tau)-\omega(-\tau)|^2 ds \\
&\le 4e^{\gamma(\tau-t-\sigma)}(\|u_{\tau-t}\|^2 + \|h\|^2|\omega(-t)-\omega(-\tau)|^2) + c_3 \\
&+ c_3\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}|\omega(s-\tau)-\omega(-\tau)|^p ds \\
&+ 4\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\bigg(\|g(s)\|^2 + \|f_1(s)\|_{L^1} + \|f_2(s)\|_{L^{p_1}}^{p_1}\bigg) ds \\
&+ 2\|h\|^2|\omega(\sigma-\tau)-\omega(-\tau)|^2. \qquad (5.13)
\end{aligned}
$$

Thus, from (5.13) there exists $T_1 = T_1(\tau,\omega,D) > 0$ such that for all $t \ge T_1$,

$$
\begin{aligned}
&\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 + \frac{\gamma}{2}\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 ds \\
&+ \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|\nabla_\lambda u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 ds \\
&+ c_1\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_{L^p}^p ds \\
&\le c + c\int_{-\infty}^{0} e^{\gamma s}|\omega(s+\sigma-\tau)-\omega(-\tau)|^p ds \\
&+ c\int_{-\infty}^{0} e^{\gamma s}(\|g(s+\sigma)\|^2 + \|f_1(s+\sigma)\|_{L^1} + \|p_2(s+\sigma)\|_{L^{p_1}}^{p_1}) ds \\
&+ c|\omega(\sigma-\tau)-\omega(-\tau)|^2, \qquad (5.14)
\end{aligned}
$$

where $c > 0$ is a constant independent of τ, ω and D. Thus, from (5.14) we infer that

$$u(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq \tilde{B}_0(\tau, \omega), \tag{5.15}$$

where $\tilde{B}_0(\tau, \omega)$ is given by (5.8). By using (2.5), (3.37) and (5.7), we can check that $\tilde{B}_0$ is tempered in $L^2(\mathbb{R}^N)$, which along with (5.15) completes the proof. □

We now prove the compactness of the cocycle $\tilde{\Phi}_0$ by using the method estimate the tails of solutions.

Lemma 5.2. *Suppose that* (3.1)–(3.3), (3.36)–(3.37) *and* (5.6)–(5.7) *are satisfied. Then for every* $\tau \in \mathbb{R}, \omega \in \Omega, D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *and for any* $\epsilon > 0$, *there exist* $T = T(\tau, \omega, D, \epsilon) > 0$ *and* $N = N(\tau, \omega, \epsilon) > 0$ *such that for all* $t \geq T$ *and* $\sigma \in [\tau - 1, \tau]$, *the solution* u *of* (5.1) *satisfies*

$$\int_{\mathbb{R}^N \setminus \mathcal{O}_N} |u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau - t})|^2 dx \leq \epsilon,$$

where $u_{\tau - t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Taking the inner product of (5.3) with $\varphi_R(x)v$ in $L^2(\mathbb{R}^N)$, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \varphi_R(x)|v|^2 dx - \int_{\mathbb{R}^N} \varphi_R(x) v \Delta_\lambda v dx + \gamma \int_{\mathbb{R}^N} \varphi_R(x)|v|^2 dx \\ &= \int_{\mathbb{R}^N} \varphi_R(x) f(t, x, v + h\omega) v dx + \int_{\mathbb{R}^N} \varphi_R(x) g(t) v dx \\ &\quad - \gamma\omega(t) \int_{\mathbb{R}^N} \varphi_R(x) h v dx + 2\omega(t) \int_{\mathbb{R}^N} \varphi_R(x) \Delta_\lambda h v dx. \end{aligned} \tag{5.16}$$

By (3.1) and (3.2), we have

$$\begin{aligned} &2\int_{\mathbb{R}^N} \varphi_R(x) f(t, x, v + h\omega) v dx \\ &= 2\int_{\mathbb{R}^N} \varphi_R(x) f(t, x, v + h\omega)(v + \omega h) dx - 2\omega(t) \int_{\mathbb{R}^N} \varphi_R(x) f(t, x, v + h\omega) h dx \\ &\leq -2c_1 \int_{\mathbb{R}^N} \varphi_R(x)|v + h\omega|^p dx + 2\int_{\mathbb{R}^N} \varphi_R(x)|f_1(t, x)| dx \\ &\quad + 2|\omega(t)| \int_{\mathbb{R}^N} \varphi_R(x)|h|(c_2|v + h\omega|^{p-1} + f_2(t)) dx \\ &\leq -c_1 \int_{\mathbb{R}^N} \varphi_R(x)|v + h\omega|^p dx + 2\int_{\mathbb{R}^N} \varphi_R(x)|f_1(t, x)| dx \\ &\quad + 2|\omega(t)|^p \int_{\mathbb{R}^N} \varphi_R(x)|h|^p dx + c_1 \int_{\mathbb{R}^N} \varphi_R(x)|f_2(t, x)|^{p_1} dx, \end{aligned} \tag{5.17}$$

where c_1 is a positive constant. The last three terms in (5.16) are bounded by

$$\frac{\gamma}{2}\int_{\mathbb{R}^N} \varphi_R(x)|v|^2 dx + c_2|\omega(t)|^2 \int_{\mathbb{R}^N} \varphi_R(x)(|h|^2 + |\Delta_\lambda h|^2) dx + c_2 \int_{\mathbb{R}^N} \varphi_R(x)|g(t)|^2 dx. \tag{5.18}$$

Then, from (5.16)–(5.18) we deduce that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^N} \varphi_R(x)|v|^2 dx + \frac{3}{2}\gamma \int_{\mathbb{R}^N} \varphi_R(x)|v|^2 dx \\ &\leq \frac{c}{R}\|v\|^2_{H^1_\lambda} + c|\omega(t)|^p \int_{\mathbb{R}^N \setminus \mathcal{O}_R} |h|^p dx + c|\omega(t)|^2 \int_{\mathbb{R}^N \setminus \mathcal{O}_R} (|h|^2 + |\Delta_\lambda h|^2) dx \\ &\quad + c\int_{\mathbb{R}^N \setminus \mathcal{O}_R} (|g(t)|^2 + |f_1(t)| + |f_2(t)|^{p_1}) dx. \end{aligned} \tag{5.19}$$

Thus, by (5.14), (5.19) and repeating the arguments of Lemma 4.3 we obtain (5.9) as claimed. ☐

The existence of $\mathcal{D}$-pullback random attractors for Φ_0 is given in the following theorem.

Theorem 5.1 (**Existence of random attractors with additive noise).** *Assume that* (3.1)–(3.3), (3.36)–(3.37) *and* (5.6)–(5.7) *are satisfied. Then, the continuous cocycle* $\tilde{\Phi}_0$ *has a unique* $\mathcal{D}$*-pullback random attractor* $\tilde{\mathcal{A}}_0 = \tilde{\mathcal{A}}_0(\tau, \omega) := \{\tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *in* $L^2(\mathbb{R}^N)$. *In addition, if* f, g, f_1 *and* f_2 *are* T*-periodic functions in* t, *then the* $\mathcal{D}$*-pullback random attractor* $\tilde{\mathcal{A}}_0(\tau, \omega)$ *is also* T*-periodic.*

Proof. By Lemmas 5.1-5.2 and repeating the arguments as in Section 4, we also obtain the existence of a $\mathcal{D}$-pullback random attractor $\tilde{\mathcal{A}}_0$ for $\tilde{\Phi}_0$ as claimed. ☐

We next consider the approximation equation (5.2). We observe that, for every $\delta \neq 0$, equation (5.2) defines a continuous cocycle $\tilde{\Phi}_\delta$ in $L^2(\mathbb{R}^N)$ which possesses a unique $\mathcal{D}$-pullback random attractor $\tilde{\mathcal{A}}_\delta(\tau, \omega)$. We now show the convergence of $\tilde{\mathcal{A}}_\delta \to \tilde{\mathcal{A}}_0$ as $\delta \to 0$. To do this, as in previous section, we denote by

$$v_\delta(t, \tau, \omega, v_{\delta,\tau}) = u_\delta(t, \tau, \omega, u_{\delta,\tau}) - h(x) \int_0^t \mathcal{W}_\delta(\theta_r \omega) dr, \tag{5.20}$$

where $v_{\delta,\tau} = u_{\delta,\tau} - h(x) \int_0^\tau \mathcal{W}_\delta(\theta_r \omega) dr$.

From this and by (5.2), we obtain

$$\begin{aligned} \frac{dv_\delta}{dt} - \Delta_\lambda v_\delta + \gamma v_\delta &= f(x, t, v_\delta + h(x) \int_0^t \mathcal{W}_\delta(\theta_r \omega) dr) + g(t, x) \\ &\quad + \Delta_\lambda h \int_0^t \mathcal{W}_\delta(\theta_r \omega) dr - \gamma h(x) \int_0^t \mathcal{W}_\delta(\theta_r \omega) dr, \end{aligned} \tag{5.21}$$

with the initial condition

$$v_\delta(\tau, x) = v_{\delta,\tau}(x), \qquad x \in \mathbb{R}^N. \tag{5.22}$$

Lemma 5.3. *Let* (3.1)–(3.3) *and* (3.36)–(5.7) *hold. Then, the continuous cocycle* $\tilde{\Phi}_\delta$ *associated with system* (5.21)–(5.22) *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* $\tilde{B}_\delta = \{\tilde{B}_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *where*

$$\tilde{B}_\delta(\tau, \omega) = \{u \in L^2(\mathbb{R}^N) : \|u\|^2 \leq M_\delta(\tau, \omega)\}, \tag{5.23}$$

with

$$\begin{aligned} \tilde{M}_\delta(\tau, \omega) &= c \int_{-\infty}^0 e^{\gamma s} \big(\|g(s+\tau)\|^2 + \|f_1(s+\tau)\|_{L^1} + \|f_2(s+\tau)\|_{L^{p_1}}^{p_1}\big) ds \\ &\quad + c \int_{-\infty}^0 e^{\gamma s} \left| \int_{-\tau}^s \mathcal{W}_\delta(\theta_r \omega) dr \right|^p ds + c \left| \int_{-\tau}^s \mathcal{W}_\delta(\theta_r \omega) dr \right|^2 ds + c, \end{aligned} \tag{5.24}$$

here c *is a positive constant independent of* τ, ω *and* δ. *Moreover, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *we have*

$$\lim_{\delta \to 0} \tilde{M}_\delta(\tau, \omega) = \tilde{M}_0(\tau, \omega), \tag{5.25}$$

where $\tilde{M}_0(\tau, \omega)$ *is given by* (5.9).

Proof. From (5.21), it follows that for every $\omega \in \Omega$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_\delta\|^2 + \|\nabla_\lambda v_\delta\|^2 + \gamma \|v_\delta\|^2 &= \int_{\mathbb{R}^N} f(t, x, v_\delta + h(x) \int_0^t \mathcal{W}_\delta(\theta_r \omega) dr) v_\delta dx \\ + (g(t), v_\delta) - \int_0^t \mathcal{W}_\delta(\theta_r \omega) dr (\nabla_\lambda h, \nabla_\lambda v_\delta) &+ \gamma \int_0^t \mathcal{W}_\delta(\theta_r \omega) dr (h, v_\delta). \end{aligned} \tag{5.26}$$

Using (3.1), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N} f(t,x,v_\delta + h(x)\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr)v_\delta dx\\ &\le -\frac{c_1}{2}\|u_\delta\|^p_{L^p} + \|f_1(t)\|_{L^1} + \|f_2(t)\|^{p_1}_{L^{p_1}} + c_1\left|\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr\right|^p, \end{aligned}\tag{5.27}$$

and by Young's inequality, we obtain

$$(g(t),v_\delta) \le \frac{\gamma}{8}\|v_\delta\|^2 + \frac{2}{\gamma}\|g(t)\|^2. \tag{5.28}$$

For the last two terms in (5.26), we have

$$\begin{aligned}&\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr(\nabla_\lambda h, \nabla_\lambda v_\delta) + \gamma\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr(h,v_\delta)\\ &\le \frac{\gamma}{8}\|v_\delta\|^2 + \frac{1}{2}\|\nabla_\lambda v_\delta\|^2 + c_2\left(\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr\right)^2. \end{aligned}\tag{5.29}$$

Thus, we obtain from (5.26)–(5.29) that

$$\begin{aligned}&\frac{d}{dt}\|v_\delta\|^2 + \frac{3}{2}\gamma\|v_\delta\|^2 + \|\nabla_\lambda v_\delta\|^2 + c_1\|u_\delta\|^p_{L^p}\\ &\le \frac{4}{\gamma}\|g(t)\|^2 + 2\|f_1(t)\|^{p_1}_{L^{p_1}} + c_3\left|\int_0^t \mathcal{W}_\delta(\theta_r\omega)dr\right|^p + c_3.\end{aligned}$$

Multiplying the above inequality by $e^{\gamma t}$ and replacing ω by $\theta_{-\tau}\omega$, and integrating over $(\tau-t,\sigma)$ with $\sigma\ge\tau-t$, we have

$$\begin{aligned}&\|v_\delta(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2 + \frac{\gamma}{2}\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|v_\delta(s,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2 ds\\ &+\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|\nabla_\lambda v_\delta(s,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2 ds\\ &+c_1\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|u_\delta(s,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^p_{L^p} ds\\ &\le e^{\gamma(\tau-t-\sigma)}\|v_{\delta,\tau-t}\|^2 + \frac{c_3}{\gamma} + c_3\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\left|\int_{-\tau}^{s-\tau}\mathcal{W}_\delta(\theta_r\omega)dr\right|^p ds\\ &+2\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\left(\frac{2}{\gamma}\|g(s)\|^2 + \|f_1(s)\|_{L^1} + \|f_2(s)\|^{p_1}_{L^{p_1}}\right)ds. \end{aligned}\tag{5.30}$$

On the other hand, by (5.20) and (5.30) we have

$$\begin{aligned}&\|u_\delta(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2 + \frac{\gamma}{2}\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|u_\delta(s,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2 ds\\ &+\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|\nabla_\lambda u_\delta(s,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^2 ds\\ &+c_1\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|u_\delta(s,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^p_{L^p} ds\\ &\le 2\|v_\delta(\sigma,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2 + \gamma\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|v_\delta(s,\tau-t,\theta_{-\tau}\omega,v_{\delta,\tau-t})\|^2 ds\\ &+2\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|\nabla_\lambda v_\delta(s,\tau-t,\theta_{-\tau},v_{\delta,\tau-t})\|^2 ds\\ &+c_1\int_{\tau-t}^\sigma e^{\gamma(s-\sigma)}\|u_\delta(s,\tau-t,\theta_{-\tau}\omega,u_{\delta,\tau-t})\|^p_{L^p} ds\end{aligned}$$

$$
\begin{aligned}
&+ 2\|h\|^2\left|\int_{-\tau}^{\sigma-\tau} \mathcal{W}_\delta(\theta_r\omega)dr\right|^2 + \gamma\|h\|^2\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\left|\mathcal{W}_\delta(\theta_r\omega)dr\right|^2 ds \\
&+ 2\|\nabla_\lambda h\|^2\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\left|\int_{-\tau}^{s-\tau} \mathcal{W}_\delta(\theta_r\omega)dr\right|^2 ds \\
\le\ & 4e^{\gamma(\tau-t-\sigma)}\left(\|u_{\delta,\tau-t}\|^2 + \|h\|^2\left|\int_{-\tau}^{-t} \mathcal{W}_\delta(\theta_r\omega)dr\right|^2\right) \\
&+ c_4 + c_4\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\left|\int_{-\tau}^{s-\tau} \mathcal{W}_\delta(\theta_r\omega)dr\right|^p ds \\
&+ 4\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\left(\frac{2}{\gamma}\|g(s)\|^2 + \|f_1(s)\|_{L^1} + \|f_2(s)\|_{L^{p_1}}^{p_1}\right) ds \\
&+ c_4\left|\int_{-\tau}^{\sigma-\tau} \mathcal{W}_\delta(\theta_r\omega)dr\right|^2,
\end{aligned}
$$

where c_4 is a positive constant independent of τ, ω and D. Hence, we obtain that for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $u_{\delta,\tau-t} \in D(\tau - t, \theta_{-\tau}\omega) \in \mathcal{D}$, there exists $T_1 = T_1(\tau, \omega, D, \delta) > 0$ such that for all $t \ge T_1$,

$$
\begin{aligned}
&\|u_\delta(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|^2 + \frac{\gamma}{2}\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|^2 ds \\
&+ \int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|\nabla_\lambda u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|^2 ds \\
&+ c_1\int_{\tau-t}^{\sigma} e^{\gamma(s-\sigma)}\|u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|_{L^p}^p ds \\
\le\ & c + c\int_{-\infty}^{0} e^{\gamma s}\left|\int_{-\tau}^{s+\sigma-\tau} \mathcal{W}_\delta(\theta_r\omega)dr\right|^p ds + c\left|\int_{-\tau}^{\sigma-\tau} \mathcal{W}_\delta(\theta_r\omega)dr\right|^2 \\
&+ c\int_{-\infty}^{0} e^{\gamma s}\left(\|g(s+\sigma)\|^2 + \|f_1(s+\sigma)\|_{L^1} + \|f_2(s+\sigma)\|_{L^{p_1}}^{p_1}\right) ds,
\end{aligned}
$$

where c is a positive constant independent of τ, ω and D. Hence, we obtain that for all $t \ge T_1$,

$$
u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-\tau}\omega)) \subseteq \tilde{B}_\delta(\tau, \omega),
$$

where $\tilde{B}_\delta(\tau, \omega)$ is given by (5.23). Moreover, combining (2.5), (2.7), (3.37) and (5.7) we have that $\tilde{B}_\delta$ is tempered. It remains to prove (5.25), but this is similar to (4.40), so we omit it here. The proof is completed. □

The following lemma shows the convergence of solutions of (5.2) to solutions of (5.1) as $\delta \to 0$.

Lemma 5.4. *Let* (3.1)–(3.3) *hold, and u, u_{δ_n} be the solutions of* (5.1) *and* (5.2) *with initial data u_τ and $u_{\delta_n,\tau}$, respectively. If $u_{\delta_n,\tau} \to u_\tau$ in $L^2(\mathbb{R}^N)$ whenever $\delta_n \to 0$ as $n \to \infty$, then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $t \in [\tau, \tau + T]$, we have*

$$
u_{\delta_n}(t, \tau, \omega, u_{\delta_n,\tau}) \to u(t, \tau, \omega, u_\tau) \quad \text{in } L^2(\mathbb{R}^N) \qquad \text{as } n \to \infty.
$$

Proof. The proof can be completed by using the same arguments as in Lemma 4.9, therefore we omit it here. □

We now establish the results on upper semicontinuity of random $\tilde{\mathcal{A}}_\delta$ for system (5.1).

Lemma 5.5. *Let* (3.1)–(3.3), (3.36)–(3.37) *and* (5.6)–(5.7) *hold. Then, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence $\{u - n\}_{n=1}^\infty$ is precompact in $L^2(\mathbb{R}^N)$ whenever $\delta_n \to 0$ $u_n \in \tilde{\mathcal{A}}_{\delta_n}(\tau, \omega)$.*

Proof. By the same calculations as in Lemma 4.8 together with the uniform smallness of the tails of solutions as in Lemma 4.8, we also obtain the conclusion of the lemma. □

Finally, we present the main result of this section, namely, the upper semicontinuity of random pullback attractors for system (5.1).

Theorem 5.2. *Let* (3.1)–(3.3)*,* (3.36)–(3.37) *and* (5.6)–(5.7) *hold. Then, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*,*

$$\lim_{\delta \to 0} \operatorname{dist}_{L^2(\mathbb{R}^N)}(\tilde{\mathcal{A}}_\delta(\tau,\omega), \tilde{\mathcal{A}}_0(\tau,\omega)) = 0.$$

Proof. To this end, we can repeat the same arguments as in Theorem 4.2, using Lemmas 5.3–5.5, and hence we obtain the result as desired. □

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