CONTINUOUS FIELD OF ORLICZ SPACE ON LOCALLY COMPACT GROUPOIDS AND RELATED RESULTS

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Abstract. Let G be a locally compact second countable groupoid with a fixed Haar system $\lambda = \{\lambda^u\}_{u\in G^0}$ and (Φ, Ψ) be a complementary pair of N-functions satisfying the Δ_2-condition. In this paper, we introduce the continuous field of Orlicz space (L_0^Φ, Δ_1) over the unit space G^0 and provide a sufficient condition for the space of continuous sections vanishing at infinity, denoted E_0^Φ, to form a Banach algebra under a suitable convolution. Additionally, we establish the condition for a closed $C_b(G^0)$-submodule I of E_0^Φ to be a left ideal. Furthermore, we present a groupoid analogue of the characterization of the space of convolutors of Morse-Transue space for locally compact groups.

1. Introduction

Harmonic analysis of groupoids has been extensively studied over the past two to three decades, with many classical results from group theory generalized to the groupoid setting. Some of the key contributions to the field include the construction of groupoid C^*-algebras [1] and Renault's Disintegration Theorem [2]. In 1997, Renault [3] introduced and investigated the Fourier algebra on measured groupoids, and later, Paterson [4] extended this study to locally compact groupoids. Additionally, further results related to continuous representations of locally compact groupoids and the Fourier transform of compact groupoids can be found in [5, 6, 7].

The algebraic structure of the Orlicz space on locally compact groups is discussed in [8], and specifically for locally compact abelian groups in [9]. Additionally, a sufficient condition for the weighted Orlicz space on locally compact groups to form a Banach algebra is provided in [10]. In 2019, Kumar et al. [11] generalized these results to hypergroup K, presenting a sufficient condition for the Orlicz space to form a Banach algebra. This paper also proves the existence of an approximate identity and provides a necessary and sufficient condition for a closed subspace to be a left ideal of the Banach algebra $L^\Phi(K)$.

It is well known that the group von Neumann algebra $VN(G)$ can be identified with the dual of the Fourier algebra $A(G)$ for any locally compact group G. In 2004, Paterson [4] generalized this result to the groupoid setting. Herz introduced

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and studied the p-analogue of the Fourier algebra, denoted by $A_p(G)$, for any locally compact group G. He proved that the dual of $A_p(G)$ can be identified with a subspace of the convolutors in $L^p(G)$, denoted as $PM_p(G)$. Furthermore, in 2013, Aghababa [12] introduced and studied the convolutor space $C_{V_\Phi}(G)$ of the Morse-Transue space $M^\Phi(G)$ and showed that $C_{V_\Phi}(G)$ can be identified with the dual of a space denoted by $\check{A}_\Phi(G)$. This was inspired from the work of M. Cowling [13] in $L_p(G)$ spaces, where $1 < p < \infty$. In 2019, Kumar et al. [11] generalized this result to hypergroups.

Motivated by these works, we introduce a continuous field of Orlicz spaces over the unit space G^0 of a locally compact groupoid G and investigate the groupoid analogue of various properties of Orlicz spaces on locally compact groups and hypergroups. In this paper, we focus on the continuous sections vanishing at infinity of a field of Orlicz spaces over the unit space of groupoid and explore some of their properties.

The paper is organized as follows: In Section 2, we begin with some basic definitions and results related to locally compact groupoids and Orlicz spaces. In Section 3, we introduce a continuous field of Orlicz spaces over the unit space of the groupoid, denoted by G^0. Using the properties of the Haar system, we prove that the family $(L^\Phi(G^u), \lambda^u)_{u \in G^0}$ forms a continuous field of Banach spaces (L_0^Φ, Δ_1) or a continuous Banach bundle (L_0^Φ, p, G^0). We further shows that the continuous field of Orlicz spaces $\left(L_0^\Phi, \Delta_1\right)$ and $\left(L^\Phi, \Delta_1'\right)$ with respect to both Gauge and Orlicz norm respectively are isomorphic.

Next, we define the concept of a strongly continuous representation of the groupoid G over a continuous field of Banach spaces and show that the left-regular representation of G over (L_0^Φ, Δ_1) is a strongly continuous representation. Additionally, we discuss the algebraic structure of the space of continuous sections vanishing at infinity, E_0^Φ, and provide a sufficient condition for E_0^Φ to be a Banach algebra. Furthermore, we establish the existence of approximate identity and show that the closed $C_b\left(G^0\right)$-submodule I of the Banach algebra E_0^Φ is a left ideal if and only if the subbundle $(I_0^\Phi, p_{I_0^\Phi}, G^0)$ is invariant under left-regular representation of G.

As previously noted, Aghababa [12] introduced and studied the convolutor space $C_{V_\Phi}(G)$ of the Morse-Transue space $M^\Phi(G)$ and showed that $C_{V_\Phi}(G)$ can be identified with the dual of a space denoted $\check{A}_\Phi(G)$. In Section 4, we extend this result to the groupoid context. Unlike the case of groups, for a locally compact groupoid G, we prove that the convolutor space $C_{V_\Phi}(G)$ is a closed subspace of the space of all bounded right-module maps denoted $B_D^\Phi\left(\check{A}_\Phi(G), C_0(G^0)\right)$. More specifically, we show that the closed subspace $\check{A}_\Phi(G)'$ of the space of all bounded right-module maps from the $C_0(G^0)$-module $\check{A}_\Phi(G)$ to $C_0(G^0)$ can be identified with the space of convolutors $C_{V_\Phi}(G)$.

2. Preliminaries

Let us begin this section by recalling the notion of Groupoids.

A groupoid is a set G endowed with a product map $G^2 \to G : (x, y) \to xy$, where G^2 is a subset of $G \times G$ called the set of composable pairs, and an inverse map $G \to G : x \to x^{-1}$ such that the following relations are satisfied:

(i) $(x^{-1})^{-1} = x$,
(ii) $(x, y), (y, z) \in G^2$ implies $(xy, z), (x, yz) \in G^2$ and $(xy)z = x(yz)$,
(iii) $(x^{-1}, x) \in G^2$ and if $(x, y) \in G^2$, then $x^{-1}(xy) = y$,
(iv) $(x, x^{-1}) \in G^2$ and if $(z, x) \in G^2$, then $(zx)x^{-1} = z$.

If $x \in G,\ d, r : G \to G$, defined as $d(x) = x^{-1}x$ and $r(x) = xx^{-1}$ are its domain and range maps respectively. The set G^0, the image of range and domain maps is called the unit space of G. Its elements are units because $xd(x) = x = r(x)x$. For $u \in G^0, G^u = r^{-1}(u)$ and $G_u = d^{-1}(u)$. It is also known that $(x, y) \in G^2$ if and only if $d(x) = r(y)$. Observe that $G = \sqcup_{u \in G^0} G^u$. For $A, B \subset G$,

$$AB = \Big\{ab \in G : a \in A, b \in B, d(a) = r(b)\Big\},\ A^{-1} = \Big\{a^{-1} \in G : a \in A\Big\}.$$

Example 1. *(i) Equivalence Relations: Let R be an equivalence relation on a set X. Let $R^2 = \{\big((x, y), (y, z)\big) : (x, y), (y, z) \in R\}$. Define a partial product and inversion on R: $(x, y)(y, z) = (x, z)$, $(x, y)^{-1} = (y, x)$. Then R becomes a groupoid, with unit space $R^0 = \{(x, x) : x \in X\}$.*

(ii) Action Groupoid: Let X be a set and let Γ be a group acting on X on the left. Define $G = \Gamma \times X$ and $G^2 = \{((g, y), (h, x)) : y = h \cdot x,\ x, y \in X,\ g, h \in \Gamma\}$ where $h \cdot x$ denote the action of Γ. Then G becomes a groupoid under the given product and inversion: $(g, h \cdot x)(h, x) = (gh, x)$, $(g, x)^{-1} = (g^{-1}, g \cdot x)$. The unit space $G^0 = \{(e_\Gamma, x) : x \in X\}$, where e_Γ is the identity element of Γ.

(iii) Group bundles:The disjoint union of groups, known as a group bundle, forms a groupoid. For a collection $G_{i i \in I}$ of groups, define $G := \sqcup_{i \in I}(\{i\} \times G_i)$. The composable sets G^2 consists of those elements having same indices in their first coordinates. The product and the inverse are defined as $(i, x)(i, y) = (i, xy)$ and $(i, x)^{-1} = (i, x^{-1})$, respectively, where xy and x^{-1} are the product and inverse in the group G_i. The unit space $G^0 = \{(i, e_{G_i}) : i \in I\}$, where e_{G_i} is the identity of G_i.

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure such that:

(i) $x \to x^{-1} : G \to G$ is continuous,
(ii) $(x, y) \to xy : G^2 \to G$ is continuous where G^2 has the induced topology from $G \times G$.

One observation is that the range and domain maps are continuous in a topological groupoid. Here we consider topological groupoids whose topology is Hausdorff, locally compact and we call such a groupoid G a locally compact groupoid. The unit space G^0 is a locally compact Hausdorff space under the subspace topology.

Like the notion of haar measure in locally compact groups, there is the notion of haar system on locally compact groupoids. The following is the definition of a haar system on locally compact groupoids.

Let G be a locally compact groupoid. A left Haar system for G consists of a family $\{\lambda^u : u \in G^0\}$ of positive radon measures on G such that,

(i) the support of the measure λ^u is G^u,
(ii) for any $f \in C_c(G), u \to \lambda(f)(u) := \int f d\lambda^u$ is continuous and
(iii) for any $x \in G$ and $f \in C_c(G)$,

$$\int_{G^{d(x)}} f(xy) d\lambda^{d(x)}(y) = \int_{G^{r(x)}} f(y) d\lambda^{r(x)}(y).$$

According to [1, Proposition 2.4], if G is a locally compact groupoid with a left haar system, then the range map r is an open map. The same is the case for domain map d. This paper assumes the locally compact groupoid to be second countable. For more details on groupoid, refer to [1, 14].

Let $\{E(u)\}_{u\in G^0}$ be a family of Banach spaces. Every element $\prod_{u\in G^0} E(u)$, i.e. every function ξ defined on G^0 such that $\xi(u) \in E(u)$ for each $u \in G^0$, is called a vector field [15, Section 10.1.1].The vector fields are also called sections or cross-sections in various literature.

A continuous field of Banach spaces over G^0 is a family $\{E(u)\}_{u\in G^0}$ of Banach spaces, with a set $\Gamma \subset \prod_{u\in G^0} E(u)$ of vector fields such that:

(i) Γ is a complex linear subspace of $\prod_{u\in G^0} E(u)$,
(ii) For every $u \in G^0$, the set $\xi(u)$ for $\xi \in \Gamma$ is dense in $E(u)$,
(iii) For every $\xi \in \Gamma$, the function $u \to \|\xi(u)\|$ is continuous,
(iv) Let $\xi \in \prod_{u\in G^0} E(u)$ be a vector field; if for every $u \in G^0$ and every $\epsilon > 0$, there exists an $\xi' \in \Gamma$ such that $\|\xi(s) - \xi'(s)\| < \epsilon$ on a neighbourhood of u, then $\xi \in \Gamma$.

For more details, one can refer to [7, Chapter 3] and [15, Section 10.1]. For a continuous field of Banach spaces, we can define a topology on $\mathcal{E} = \sqcup_{u\in G^0} E(u)$, generated by the sets of the form

$$U(V, \xi, \epsilon) = \{h \in \mathcal{E} : \|h - \xi(p(h))\| < \epsilon, \xi \in \Gamma, p(h) \in V\},$$

where V is an open set in G^0, $\epsilon > 0$, and $p : \mathcal{E} \to G^0$ is the projection of the total space $\mathcal{E}$ to base space G^0 such that fiber $p^{-1}(u) = E(u), u \in G^0$. This map is a surjective continuous open map under the above topology. We denote such a continuous field of Banach spaces as $(\mathcal{E}, \Gamma)$. Under the above topology, it is also referred to as a continuous Banach bundle $(\mathcal{E}, p, G^0)$ whose continuous sections are elements of Γ[16, Definition 13.4, Theorem 13.18]. Here, $\mathcal{E}$ is called bundle space.

For $\xi \in \Gamma$, we denote $\xi(u)$ as ξ^u also, in the rest of the paper. Here, we call the sections or vector fields that are bounded but need not be continuous as bounded sections.

A convex function $\Phi : \mathbb{R} \to [0, \infty]$ which satisfy the conditions: $\Phi(-x) = \Phi(x), \Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = +\infty$ is called a *Young function.* For any *Young function* Φ, one can associate another convex function $\Psi : \mathbb{R} \to [0, \infty]$ having similar properties, called *complementary function* to Φ, defined by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\},\ y \in \mathbb{R}.$$

The pair (Φ, Ψ) is called a *complementary pair* of Young functions.

A *(nice) Young function* Φ, termed as N-function, is a continuous Young function such that $\Phi(x) = 0$ if and only if $x = 0$ and satisfies the following conditions:

$$\lim_{x\to 0} \frac{\Phi(x)}{x} = 0 \text{ and } \lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty.$$

We say that a Young function $\Phi : \mathbb{R} \to \mathbb{R}^+$ is Δ_2*-regular* if there exist $k > 0$ and $x_0 > 0$ such that

$$\Phi(2x) \leq k\Phi(x) \text{ for all } x \geq x_0, \text{ when } \sup_{u\in G^0} \lambda^u(G) < \infty,$$

and

$$\Phi(2x) \leq k\Phi(x) \text{ for all } x \geq 0, \text{ when } \sup_{u\in G^0} \lambda^u(G) = \infty.$$

We write $\Phi \in \Delta_2$ if Φ is Δ_2-regular. This definition is similar to the one defined in the case of locally compact groups [17, Definition 1 (Pg. 22) and Remark (Pg. 46)].

For $u \in G^0$, the *Orlicz space*, $L^\Phi(G^u)$, is defined as,

$$L^\Phi(G^u) = \{f : G^u \to \mathbb{C} : f \text{ is measurable and } \int_{G^u} \Phi(\alpha|f|)d\lambda^u < \infty, \text{ for some } \alpha > 0\}.$$

The set $L^\Phi(G^u)$ is a Banach space with respect to two norms:
Gauge Norm:$\|f\|_\Phi^0 = \inf\{k > 0 : \int_{G^u} \Phi\Big(\frac{|f|}{k}\Big)d\lambda^u \leq 1\}$.
Orlicz Norm:$\|f\|_\Phi = \sup\{\int_{G^u} |fg|d\lambda^u : \int_{G^u} \Psi(|g|)d\lambda^u \leq 1\}$ where Ψ denotes the complementary function to Young function Φ.

It is known that the two norms are equivalent. Infact, $\|\cdot\|_\Phi^0 \leq \|\cdot\|_\Phi \leq 2\|\cdot\|_\Phi^0$ and $\|f\|_\Phi^0 \leq 1$ if and only if $\int_{G^u} \Phi(|f|)d\lambda^u \leq 1$. Suppose f is a real-valued measurable function on G^u, and $\int_{G^u} f d\nu^u$ and $\int_{G^u} \Phi(f)d\nu^u$ exist, where ν^u is a probability measure on G^u, then the following Jensen's inequality[17, Chapter 3, Proposition 5] holds:

$$\Phi\Big(\int_{G^u} f d\nu^u\Big) \leq \int_{G^u} \Phi(f)d\nu^u.$$

The set of all complex measurable function f such that:

$$\int_{G^u} \Phi(a|f|)d\lambda^u < \infty, \text{ for all } a > 0.$$

is denoted as $M^\Phi(G^u)$. This space is called *Morse-Transue space* and is a closed subspace of $L^\Phi(G^u)$.The space of continuous functions on G^u with compact support, denoted $C_c(G^u)$, is dense in $M^\Phi(G^u)$. If $\Phi \in \Delta_2$, then $L^\Phi(G^u) = M^\Phi(G^u)$. Suppose (Φ, Ψ) are complimentary pair of N-functions and $(G^u, \mathfrak{B}^u, \lambda^u)$ is a σ-finite measure space, where $\mathfrak{B}^u$ denotes the σ-algebra of subsets of G^u on which λ^u is defined, then by [17, Chapter 4, Theorem 7], $M^\Phi(G^u)^* = L^\Psi(G^u)$ and $M^\Psi(G^u)^* = L^\Phi(G^u)$. For $g \in L^\Phi(G^u)$,

$$\|g\|_\Phi = \sup\left\{\left|\int_{G^u} gf d\lambda^u\right| : \|f\|_\Psi^0 \leq 1, f \in M^\Psi(G^u)\right\},$$
$$\|g\|_\Phi^0 = \sup\left\{\left|\int_{G^u} gf d\lambda^u\right| : \|f\|_\Psi \leq 1, f \in M^\Psi(G^u)\right\}.$$

If $f \in L^\Phi(G^u), g \in L^\Psi(G^u)$, then $fg \in L^1(G^u)$ and the following Holder's inequality holds:

$$\int_{G^u} |f(t)g(t)|d\lambda^u(t) \leq \|f\|_\Phi^0 \|g\|_\Psi.$$

For more details, refer[17]. Throughout this paper, $C_c(G), C_b(G)$ and $C_0(G)$ denote the spaces of continuous functions on G with compact support, bounded continuous functions on G and continuous functions on G that vanish at infinity, respectively.

3. Continuous Field of Orlicz Space

Let $\{L^\Phi(G^u)\}_{u\in G^0}$ be a family of Orlicz spaces over G^0 where (Φ, Ψ) is a complementary pair of N-functions with $\Phi \in \Delta_2$. We will prove that the above family forms a continuous field of Orlicz space over G^0.

Theorem 3.1. *For $f \in C_c(G)$, the function $g : G^0 \to \mathbb{R}$ defined as $g(u) = \|f^u\|_\Phi^0$, where $f^u = f_{|G^u}$, is continuous over G^0 with compact support.*

Proof. Let $f \in C_c(G)$. The function $\Phi(|f|)$ is continuous with compact support. Hence the map $\lambda(\Phi(f))(u) = \int_{G^u} \Phi(|f|)d\lambda^u$ is continuous over G^0. Let $\epsilon > 0$ and $u_0 \in G^0$. Assume $\|f^{u_0}\|_\Phi^0 \neq 0$ and we can see that $\int_{G^{u_0}} \Phi(\frac{|f^{u_0}|}{\|f^{u_0}\|_\Phi^0})d\lambda^{u_0} = 1$.
If $a, b \in I = (\|f^{u_0}\|_\Phi^0 - \epsilon, \|f^{u_0}\|_\Phi^0 + \epsilon)$, such that $0 < a < \|f^{u_0}\|_\Phi^0 < b$, then

$$\lambda\Big(\Phi\Big(\frac{|f|}{a}\Big)\Big)(u_0) > 1 \text{ and } \lambda\Big(\Phi\Big(\frac{|f|}{b}\Big)\Big)(u_0) < 1.$$

By continuity of both these functions, there exist open neighbourhood U of u_0 such that $\lambda\Big(\Phi\Big(\frac{|f|}{a}\Big)\Big)(u) > 1$ and $\lambda\Big(\Phi\Big(\frac{|f|}{b}\Big)\Big)(u) < 1$ for all $u \in U$. Thus,

$$a \leq \|f^u\|_\Phi^0 \leq b \text{ for all } u \in U, \text{ i.e. } \|f^u\|_\Phi^0 \in I, \text{ for all } u \in U.$$

Assume $\|f^{u_0}\|_\Phi^0 = 0$, then $f^{u_0} = 0$ and note that $\lambda\Big(\Phi\Big(\frac{|f|}{a}\Big)\Big)(u_0) = 0$, for all $a > 0$. So for $\epsilon > 0$, by continuity of $\lambda\Big(\Phi\Big(\frac{2|f|}{\epsilon}\Big)\Big)$, there exist open neighbourhood U of u_0 such that $\lambda\Big(\Phi\Big(\frac{2|f|}{\epsilon}\Big)\Big)(u) < 1$ which implies $\|f^u\|_\Phi^0 < \epsilon$, for all $u \in U$. Hence proved. $\square$

Let $L_0^\Phi = \sqcup_{u\in G^0} L^\Phi(G^u)$. Any $f \in C_c(G)$ can be identified as section $t_f : G^0 \to L_0^\Phi$ where $t_f(u) = f_{|G^u}$. Also for $g \in C_c(G^u)$, by partition of unity argument, we can find g' in $C_c(G)$, such that $g'_{|G^u} = g$. These sections satisfy axioms (i), (ii) and (iii) of [15, Definition 10.1.2] and by [15, Proposition 10.2.3], $\big(\{L^\Phi(G^u)\}_{u\in G^0}, \Delta_1\big)$ becomes a continuous field of Orlicz space with continuous sections being sections locally close to $C_c(G)$, denoted by Δ_1. Thus we can define a topology on the total space $L_0^\Phi = \sqcup_{u\in G^0} L^\Phi(G^u)$ generated by the following subsets:

$$U(V, \eta, \epsilon) = \{h \in L_0^\Phi : \|h - \eta(p(h))\|_\Phi^0 < \epsilon, p(h) \in V\},$$

where $\eta \in \Delta_1$ and $p : L_0^\Phi \to G^0$, is the projection of the total space on the base G^0. We can also denote this continuous field of Orlicz space as (L_0^Φ, Δ_1). Denote the set of bounded sections of L_0^Φ as $I_0^\Phi(G, \lambda)$ and continuous sections vanishing at infinity as E_0^Φ. Both are Banach spaces under the following norm:

$$\|\xi\|_\Phi^0 = \sup_{u\in G^0} \|\xi^u\|_\Phi^0, \quad \xi^u \in L^\Phi(G^u).$$

Now we have the following corollary. As the corollary follows similar lines to Theorem 3.1, we skip it.

Corollary 3.2. *The family $\{M^\Psi(G^u)\}_{u\in G^0}$ forms a continuous field of Morse-Transue space (M_0^Ψ, Δ) with Δ being the continuous sections, under $\|\cdot\|_\Psi^0$-norm, locally close to $C_c(G)$ and M_0^Ψ being the total space. E_0^Ψ forms the continuous section vanishing at infinity.*

Note that E_0^Φ and E_0^Ψ are $C_b(G^0)$ and $C_0(G^0)$-module: for $\xi \in E_0^\Phi, b \in C_0(G^0)$ or $C_b(G^0)$ we set $(\xi b)(u) = b(u)\xi^u$.

The following proposition is the Orlicz space version of [4, Proposition 7] and the proof is in similar lines.

Proposition 3.3. *$C_c(G)$ is dense in E_0^Φ and E_0^Ψ.*

Now, we prove an important lemma which will be used for various results.

Lemma 3.4. *Given $g \in L_0^\Phi(G^u)$ for any $u \in G^0$ with $\|g\|_\Phi^0 \le k$, there exist $\eta \in E_0^\Phi$ such that $\eta^u = g$ and $\|\eta\|_\Phi^0 \le k$.*

Proof. Assume $r = \|g\|_\Phi^0 \neq 0$. If $r = 0$, then the zero section is the required one. There exist a sequence $\{f_n\} \in C_c(G^u)$, tending to g, such that $\|f_n\|_\Phi^0 \le k, \forall n \in \mathbb{N}$.

By passing to a subsequence if necessary we may suppose that $\|f_{n+1}-f_n\|_\Phi^0 < \frac{k}{2^n}$. Then there exist $\{g_n\} \in C_c(G)$ such that $g_0^u = f_1, g_n^u = f_{n+1} - f_n$, and $\|g_n\|_\Phi^0 < k2^{-n}, n \geq 1$. The section $\eta_1 = \sum_{n=0}^{\infty} g_n$ is in E_0^Φ and $\eta_1^u = \lim f_n = g$. By continuity of η_1, there exist an open neighbourhood U around u such that $\|\eta_1^v\|_\Phi^0 > 0, v \in U$. Take $b \in C_c(G^0)$ such that $b(u) = 1, 0 \leq b \leq 1$ and support of b is in U. Then $\eta = r\eta_1 \frac{b}{\|\eta_1\|_\Phi^0}$ is the required section. Here $\eta^v = r\eta_1^v \frac{b(v)}{\|\eta_1^v\|_\Phi^0}, v \in G^0$. □

Theorem 3.5. *Let (Φ, Ψ) be a complementary pair of N-function and $\Phi \in \Delta_2$. For $f \in C_c(G)$, the function $g : G^0 \to \mathbb{R}$ defined as $g(u) = \|f^u\|_\Phi$ is a continuous function on G^0 with compact support.*

Proof. Let $u_0 \in G^0, \epsilon > 0$. Since $L^\Phi(G^{u_0}) = M^\Psi(G^{u_0})^*$ and using [17, Chapter 3, Theorem 13], we can say that,

$$\begin{aligned}\|f^{u_0}\|_\Phi &= \sup\left\{\int_{G^{u_0}} |fg| d\lambda^{u_0} : \|g^{u_0}\|_\Psi^0 \leq 1, g \in M^\Psi(G^{u_0})\right\} \\ &= \inf\left\{\frac{1}{k}\left(1 + \int_{G^{u_0}} \Phi(k|f|) d\lambda^{u_0}\right) : k > 0\right\}.\end{aligned}$$

By density of $C_c(G^{u_0})$, there exist a $g' \in C_c(G^{u_0})$, such that $\|f^{u_0}\|_\Phi - \epsilon < \int |fg'|\, d\lambda^{u_0}$. Hence, using partition of unity, we can find $g \in C_c(G)$ such that $g_{|_{G^{u_0}}} = g'$ and $\|g\|_\Psi^0 \leq 1$. By continuity of $\int |fg| d\lambda^u$, we can find a neighbourhood U of u_0 such that $\|f^{u_0}\|_\Phi - \epsilon < \int |fg| d\lambda^u$ for all $u \in U$ which implies $\|f^{u_0}\|_\Phi - \epsilon < \|f^u\|_\Phi, \forall u \in U$.

By definition, $\exists\ k > 0$ such that $\frac{1}{k}\left(1 + \int_{G^{u_0}} \Phi(k|f|) d\lambda^{u_0}\right) < \|f^{u_0}\|_\Phi + \epsilon$. So, by continuity, there exist an open neighbourhood V of u_0, such that

$$\frac{1}{k}\left(1 + \int_{G^u} \Phi(k|f|) d\lambda^u\right) < \|f^{u_0}\|_\Phi + \epsilon, \forall u \in V.$$

Thus, $\forall u \in V \cap U$,

$$\|f^u\|_\Phi \in I = (\|f^{u_0}\|_\Phi - \epsilon, \|f^{u_0}\|_\Phi + \epsilon).$$

Hence proved. □

Here also we see $C_c(G)$ as sections satisfy axioms (i), (ii) and (iii) of [15, Definition 10.1.2], under $\|\cdot\|_\Phi$-norm and by [15, Proposition 10.2.3], $\left(\{L^\Phi(G^u)\}_{u \in G^0}, \Delta_1'\right)$ becomes a continuous field of Orlicz space with continuous sections being sections locally close to $C_c(G)$, denoted by Δ_1'. The continuous section vanishing at infinity is denoted by E^Φ. It is a $C_b(G^0)$ and $C_0(G^0)$-module.

We can define a topology on the total space $L^\Phi = \sqcup_{u \in G^0} L^\Phi(G^u)$ generated by the following subsets:

$$U(V, \eta, \epsilon) = \left\{h \in L^\Phi : \|h - \eta(p(h))\|_\Phi < \epsilon, p(h) \in V\right\},$$

where $\eta \in \Delta_1'$ and $p : L^\Phi \to G^0$, is the projection of the total space on the base G^0. We denote this continuous field of Orlicz space as $\left(L^\Phi, \Delta_1'\right)$ also.

With the same idea as above, we can prove the following corollary.

Corollary 3.6. *The family* $\left\{M^\Psi\left(G^u\right)\right\}_{u\in G^0}$ *becomes a continuous field of Morse-Transue space* $\left(M^\Psi, \Delta'\right)$ *with* Δ' *being the continuous sections, under* $\|\cdot\|_\Psi$*-norm, locally close to* $C_c(G)$ *and* M^Ψ *being the total space.* E^Ψ *forms the continuous section vanishing at infinity. It is a* $C_b\left(G^0\right)$ *and* $C_0\left(G^0\right)$*-module.*

Using [18, Lemma 2.3] , the continuous field of Orlicz spaces $\left(L_0^\Phi, \Delta_1\right)$ and $\left(L^\Phi, \Delta_1'\right)$ are isomorphic. Here we can take the identity map as the morphism between two continuous fields of Banach spaces. Similarly, $\left(M^\Psi, \Delta'\right)$ and $\left(M_0^\Psi, \Delta\right)$ are isomorphic. Note that Lemma 3.4 can also be proved in the context of continuous field of Banach spaces $\left(L^\Phi, \Delta_1'\right)$,$\left(M^\Psi, \Delta'\right)$ and $\left(M_0^\Psi, \Delta\right)$. We can easily see that $C_c(G)$ is dense in E^Φ and E^Ψ.

Definition 3.7. *A strongly continuous representation of groupoids on a continuous field of Banach space* $\left(B_\pi = \{B_u\}_{u\in G^0}, \Gamma\right)$ *over* G^0 *is a triple* $\left(B_\pi, \Gamma, \pi\right)$ *such that*

(i) $\pi(x) \in \mathcal{B}\left(B_{d(x)}, B_{r(x)}\right)$ *is a invertible isometry, for each* $x \in G$*,*

(ii) $\pi(u)$ *is the identity map on* B_u *for all* $u \in G^0$*,*

(iii) $\pi(x)\pi(y) = \pi(xy)$ *for all* $(x, y) \in G^2$*,*

(iv) $\pi(x)^{-1} = \pi\left(x^{-1}\right)$ *for all* $x \in G$*,*

(v) $x \to \pi(x)\eta^{d(x)}$ *is continuous for every* $\eta \in \Gamma$*.*

Theorem 3.8. *The left-regular representation* $\left(L_0^\Phi, \Delta_1, L\right)$ *is a strongly continuous representation.*

Proof. Let $x \in G, r(x) = u, d(x) = v$ and $f \in L^\Phi\left(G^v\right)$. Then, $\int \Phi(|L(x)f|)d\lambda^{r(x)} = \int L(x)\Phi(|f|)d\lambda^{r(x)} = \int \Phi(|f|)d\lambda^{d(x)}$. Thus it is an isometry. The conditions (i), (ii), (iii) and (iv) of Definition 3.7 can be easily verified.

We need to prove that $x \to L(x)\eta^{d(x)}$ is continuous for every $\eta \in \Delta_1$. Let $x_0 \in G$, then $L\left(x_0\right)\eta^{d(x_0)} \in L^\Phi\left(G^{r(x_0)}\right)$. For a given open set K containing $L\left(x_0\right)\eta^{d(x_0)}$ in L_0^Φ, by Lemma 3.4, we can find an open set of the form $U(V, \xi, \epsilon) \subset K$, where V is a neighbourhood of $r\left(x_0\right), \epsilon > 0$ and $\xi \in E_0^\Phi$ such that $L\left(x_0\right)\eta^{d(x_0)} = \xi^{r(x_0)}$. There exist $\eta_0, \xi_0 \in C_c(G), V_0 \subset V, r\left(x_0\right) \in V_0$ and an open set V_1 containing $d\left(x_0\right)$ such that:

$$\|\xi^w - \xi_0^w\|_\Phi^0 < \frac{\epsilon}{3}, \forall w \in V_0,\ \|\eta^u - \eta_0^u\|_\Phi^0 < \frac{\epsilon}{3}, \forall u \in V_1,\ L\left(x_0\right)\eta_0^{d(x_0)} = \xi_0^{r(x_0)}.$$

Define $f \in C_c\left(G^{(2)}\right)$ as $f\left(x', y'\right) = \Phi(\frac{3}{\epsilon}|\eta_0\left(x'^{-1}y'\right) - \xi_0(y')\,|)$. Using [18, Lemma 3.12], we can say that $F\left(x'\right) = \int_{G^{r(x')}} \Phi\left(\frac{3}{\epsilon}|\eta_0\left(x'^{-1}y'\right) - \xi_0\left(y'\right)|\right)d\lambda^{r(x')}\left(y'\right)$ is continuous on G. Note that $F\left(x_0\right) = 0$ and by continuity of F, there exist an open neighbourhood W of x_0 such that $F\left(x'\right)<1, \forall x' \in W$ which implies

$$\left\|L\left(x'\right) \eta_0^{d\left(x'\right)}-\xi_0^{r\left(x'\right)}\right\|_{\Phi}^0 \leq \frac{\epsilon}{3}$$

for every $x' \in W$. So for all $x \in W'=r^{-1}\left(V_0\right) \cap d^{-1}\left(V_1\right) \cap W$,

$$\begin{aligned}
\left\|L(x)\eta^{d(x)}-\xi^{r(x)}\right\|_{\Phi}^0 \leq & \left\|L(x)\eta^{d(x)}-L(x)\eta_0^{d(x)}\right\|_{\Phi}^0 \\
& +\left\|L(x)\eta_0^{d(x)}-\xi_0^{r(x)}\right\|_{\Phi}^0+\left\|\xi_0^{r(x)}-\xi^{r(x)}\right\|_{\Phi}^0 \\
& =\left\|\eta^{d(x)}-\eta_0^{d(x)}\right\|_{\Phi}^0+\left\|L(x)\eta_0^{d(x)}-\xi_0^{r(x)}\right\|_{\Phi}^0 \\
& +\left\|\xi_0^{r(x)}-\xi^{r(x)}\right\|_{\Phi}^0<\epsilon .
\end{aligned}$$

Thus we have showed that, $L(x)\eta^{d(x)} \in U(V, \xi, \epsilon), \forall x \in W'$. Hence we have proved that $x \rightarrow L(x)\eta^{d(x)}$ is continuous at x_0. □

Let $I(G, \lambda)$ denote the space of bounded sections of the Banach bundle $\{L^1(G)=L^1\left(G^u, \lambda^u\right)\}_{u \in G^0}$ and E^1 denote the continuous sections over G^0 vanishing at infinity. Both form Banach spaces under the following norm:

$$\|\eta\|_1=\sup_{u \in G^0}\left\|\eta^u\right\|_1, \ \eta^u \in L^1(G^u).$$

It can be easily seen that $(L^1(G), \Gamma')$ is a continuous field of Banach space where Γ' is the space of continuous sections locally close to $C_c(G)$ under $\|\cdot\|_1$-norm. Note that $(C_c(G), \|\cdot\|_1)$ is a normed algebra under the following convolution:

$$f * g(x)=\int_{G^{r(x)}} f(y) g\left(y^{-1} x\right) d\lambda^{r(x)}(y).$$

The density of $C_c(G)$ in E^1 by identifying $C_c(G)$ as sections, makes the space E^1 a Banach algebra.

Lemma 3.9. *Let G be a groupoid and (Φ, Ψ) be a complementary pair of N-functions with $\Phi \in \Delta_2$. Then $I_0^{\Phi}(G, \lambda) \subseteq I(G, \lambda)$, iff there exists $d>0$ such that $\|f\|_1 \leq d\|f\|_{\Phi}^0, \forall f \in I_0^{\Phi}(G, \lambda)$.*

The proof is similar to that of [11, lemma 3.3] using the Open Mapping Theorem. Similarly we can see that $E_0^{\Phi} \subseteq E^1$ iff there exists $d>0$ such that $\|f\|_1 \leq d\|f\|_{\Phi}^0, \forall f \in E_0^{\Phi}$.

Corollary 3.10. *Let (Φ, Ψ) be a complementary pair of N-functions with $\Phi \in \Delta_2$. If $I_0^{\Phi}(G, \lambda) \subseteq I(G, \lambda)$, then $E_0^{\Phi} \subseteq E^1$.*

Lemma 3.11. *Let (Φ, Ψ) be a complementary pair of N-functions with $\Phi \in \Delta_2$. If G is such that, $\sup_{u \in G^0} \lambda^u(G)<\infty$, then $I_0^{\Phi}(G, \lambda) \subseteq I(G, \lambda)$ and $E_0^{\Phi} \subseteq E^1$. In particular, the conclusion holds if G is a compact groupoid.*

Proof. Assume that $\sup_{u\in G^0} \lambda^u(G) < \infty$. Since Φ is convex, there exists $c, r_0 > 0$ such that $\Phi(r) \geq cr, \forall r \geq r_0$. $\xi \in I_0^\Phi(G,\lambda)$ implies $\sup_{u\in G^0} \int \Phi(a|\xi^u|)d\lambda^u < \infty$ for some $a > 0$.

For $u \in G^0$, let $N^u = \{s \in G^u : a|\xi^u(s)| \leq r_0\}$.

$$\begin{aligned}\int |\xi^u| d\lambda^u &= \frac{1}{a}\left(\int_{N^u} a|\xi^u|d\lambda^u + \int_{G/N^u} a|\xi^u|d\lambda^u\right)\\ &\leq \frac{1}{a}\left(r_0\lambda^u(G) + \int_{G/N^u} a|\xi^u|d\lambda^u\right)\\ &\leq \frac{1}{a}\sup_{u\in G^0}\left(r_0\lambda^u(G) + \frac{1}{c}\int \Phi(a|\xi^u|)d\lambda^u\right).\end{aligned}$$

Hence $\sup_{u\in G^0} \int |\xi^u| d\lambda^u < \infty$ implies $\xi \in I(G,\lambda)$. ☐

We will now show a sufficient condition for E_0^Φ to form a Banach algebra. It is the groupoid version of [11, Theorem 3.5]. A necessary and sufficient condition is proved in [9, Theorem 2], when G is a locally compact abelian group.

Theorem 3.12. *Let G be a locally compact groupoid and (Φ,Ψ) be a complementary pair of N-functions with $\Phi \in \Delta_2$. If $E_0^\Phi \subseteq E^1$, then the bilinear map $F : C_c(G)\times C_c(G) \to C_c(G), F(f,g) = f * g$ is a bounded map under $\|\cdot\|_\Phi^0$-norm and can be extended to $E_0^\Phi \times E_0^\Phi$. In particular, the convolution is commutative if G is a commutative group bundle.*

Proof. Since $E_0^\Phi \subset E^1$, by Lemma 3.9, there exists $d > 0$ such that $\|f\|_1 \leq d\|f\|_\Phi^0$, for all $f \in E_0^\Phi$. Let $f, g \in C_c(G)$ then $f * g(x) = \int f(y)g\left(y^{-1}x\right) d\lambda^{r(x)}(y) \in C_c(G)$.

Let $u \in G^0$. Using Fubini's theorem, Holder's Inequality, Lemma 3.4 in the context of M_0^Ψ and isometry of left translation $L(y) : L^\Phi(G^{d(y)}) \to L^\Phi(G^{r(y)})$, we get

$$\begin{aligned}\|(f*g)^u\|_\Phi &= \sup\left\{\int \left|f * g\right|\left|h\right| d\lambda^u : \|h\|_\Psi^0 \leq 1, h \in E_0^\Psi\right\}\\ &\leq \sup\left\{\iint \left|f(y)\right|\left|g\left(y^{-1}x\right)\right|\left|h(x)\right| d\lambda^{r(x)}(y)d\lambda^u(x) : \|h\|_\Psi^0 \leq 1\right\}\\ &= \sup\left\{\iint \left|f(y)\right|\left|g\left(y^{-1}x\right)\right|\left|h(x)\right| d\lambda^{r(y)}(x)d\lambda^u(y) : \|h\|_\Psi^0 \leq 1\right\}\\ &\leq 2\|f\|_1\|g\|_\Phi^0.\end{aligned}$$

Thus,

$$\|(f*g)\|_\Phi^0 = \sup_{u\in G^0} \|(f*g)^u\|_\Phi^0 \leq \sup_{u\in G^0} \|(f*g)^u\|_\Phi \leq 2\|f\|_1\|g\|_\Phi^0.$$

So, $\|F(f,g)\|_\Phi^0 \leq 2d\|f\|_\Phi^0\|g\|_\Phi^0$. Now by density of $C_c(G)$, we extend F to $E_0^\Phi \times E_0^\Phi$ and $\|F(\xi,\eta)\|_\Phi^0 \leq 2d\|\xi\|_\Phi^0\|\eta\|_\Phi^0$. Define an equivalent norm $\|\cdot\|'_\Phi = 2d\|\cdot\|_\Phi^0$, so that $\|F(f,g)\|'_\Phi \leq \|f\|'_\Phi\|g\|'_\Phi$.

If G is a commutative group bundle, $G^u = G_u = G(u)$ and $\lambda^u = \lambda_u$ for all $u \in G^0$. Hence for $f, g \in C_c(G)$,

$$\begin{aligned} f * g(x) &= \int_{G^{r(x)}} f(y) g\left(y^{-1}x\right) d\lambda^{r(x)}(y) \\ &= \int_{G_{r(x)}} f\left(y^{-1}\right) g(yx) d\lambda_{r(x)}(y) \\ &= \int_{G_{d(x)}} \check{f}\left(yx^{-1}\right) g(y) d\lambda_{d(x)}(y) \\ &= \int_{G_{d(x)}} f\left(xy^{-1}\right) g(y) d\lambda_{d(x)}(y) \\ &= \int_{G^{r(x)}} f\left(y^{-1}x\right) g(y) d\lambda^{r(x)}(y) = g * f(x). \end{aligned}$$

□

For $\xi, \eta \in E_0^\Phi$, we denote $F(\xi, \eta)$ as $\xi * \eta$. We showed the sufficient condition for E_0^Φ to become a Banach algebra, by taking the equivalent norm mentioned above. We denote it again as $\|\cdot\|_\Phi^0$ for convenience. Similarly, we can see that algebra structure exists for E^Φ also under the same sufficient condition.

Lemma 3.13. *Let G be a groupoid and (Φ, Ψ) be a complementary pair of N-function with $\Phi \in \Delta_2$. For $f \in C_c(G)$, the map $L_f : C_c(G) \to E_0^\Phi$, defined as $L_f(g) = f * g$, extends to a bounded linear map on E_0^Φ with $\|L_f\| \le \|f\|_1$. Moreover, the norm decreasing homomorphism $f \to L_f$ from $(C_c(G), \|\cdot\|_1)$ into $B(E_0^\Phi)$ can be extended to E^1.*

Proof. Let $f, g \in C_c(G)$ with $\|f\|_1 = 1$ and $\|g\|_\Phi^0 = 1$. We will show that $\int \Phi(|f * g|)\, d\lambda^u < 1, \forall u \in G^0$. Let $u \in G^0$ and $\|f^u\|_1 \neq 0$, using Jensen's inequality, Fubini's theorem and isometry of left translation,

$$\begin{aligned} \int \Phi(|f * g|)(x) d\lambda^u(x) &= \int \Phi\left(\left|\int f(y) g\left(y^{-1}x\right) d\lambda^{r(x)}(y)\right|\right) d\lambda^u(x) \\ &\le \int \Phi\left(\int |f(y)| \left|g\left(y^{-1}x\right)\right| d\lambda^u(y)\right) d\lambda^u(x) \\ &\le \int \Phi\left(\int \frac{|f(y)| \left|g\left(y^{-1}x\right)\right|}{\|f^u\|_1} d\lambda^u(y)\right) d\lambda^u(x) \\ &\le \iint \Phi\left(\left|g\left(y^{-1}x\right)\right|\right) d\nu^u(y) d\lambda^u(x) \\ &= \iint \Phi\left(\left|g\left(y^{-1}x\right)\right|\right) d\lambda^u(x) d\nu^u(y) \le 1, \end{aligned}$$

where $\nu^u(E) = \frac{1}{\|f^u\|_1}\int |f| d\lambda^u$, $E \subset G^u$. Thus L_f can be extended to E_0^Φ and $\|L_f\| \leq \|f\|_1$. Also by density of $C_c(G)$ in E^1 we can extend the norm decreasing homomorphism $f \to L_f$ and thus define L_ξ for every $\xi \in E^1$. □

The above lemma shows that E_0^Φ is an E^1-module and denote $L_\xi(g)$ as $\xi * g$ for $\xi \in E^1$ and $g \in E_0^\Phi$. This is analogous to [19, Corollary 3.4(i)].

In the following lemma, we show that the right convolution of $C_c(G)$ over E_0^Φ is a bounded operator.

Lemma 3.14. *Let G be a groupoid and (Φ, Ψ) be a complementary pair of N-functions both satisfying Δ_2-condition. For $F \in C_c(G)$, the map $R_F : C_c(G) \to C_c(G)$, defined as $R_F(g) = g * F$, extends to a bounded linear map on E_0^Φ with $\|R_F\| \leq 2K_F^2$, where $K_F = \max\{\sup_{u\in G^0} \|\Phi^{-1}\circ|F^u|\|_{\tilde{\Psi}}^0, \sup_{u\in G^0}\|\Psi^{-1}\circ|\check{F}^u|\|_{\tilde{\Psi}}^0\}$, for an N- function $\tilde{\Psi}$.*

Proof. Let $F, g, h \in C_c(G)$ and $u \in G^0$.

$$\left|\int g * F(x)h(x)d\lambda^u(x)\right| \leq \int\int |g(y)F(y^{-1}x)||h(x)|d\lambda^{r(x)}(y)d\lambda^u(x)$$

$$\leq \int\int \left[|g(y)|\Phi^{-1}(|F(y^{-1}x)|)\right]\left[|h(x)|\Psi^{-1}(|F(y^{-1}x)|)\right] d\lambda^u(y)d\lambda^u(x) \leq 2AB$$

where

$$A = \|k(x,y)\|_\Phi^0,\ k(x,y) = |g(y)|\Phi^{-1}(|F(y^{-1}x)|)$$
$$B = \|m(x,y)\|_\Psi^0,\ m(x,y) = |h(x)|\Psi^{-1}(|F(y^{-1}x)|),\ (x,y) \in G^u \times G^u.$$

According to [17, Proposition 2.13 and Proposition 2.4(ii)], there exist an N- function $\tilde{\Psi} = \max\{\Psi_1, \Psi_2\}$ where $\Psi_1(a) = \sup\{\frac{\Phi(ab)}{\Phi(b)} : b \geq 0\}$ and $\Psi_2(a) = \sup\{\frac{\Psi(ab)}{\Psi(b)} : b \geq 0\}$ $a, b \geq 0$, so that $\Phi(ba) \leq \Phi(b)\tilde{\Psi}(a)$ and $\Psi(ba) \leq \Psi(b)\tilde{\Psi}(a)$. Also note that the functions $\Phi^{-1}\circ|F|$ and $\Psi^{-1}\circ|\check{F}|$ are bounded and vanishes outside the supp(F) and supp$(\check{F})$ respectively. So,

$$\begin{aligned} &\int\int \Psi\left(\frac{|h(x)|\Psi^{-1}(|F(y^{-1}x)|)}{\|h\|_\Psi^0\|\Psi^{-1}\circ\check{F}\|_{\tilde{\Psi}}^0}\right)d\lambda^u(x)d\lambda^u(y) \\ \leq\ &\int\int \Psi\left(\frac{|h(x)|}{\|h\|_\Psi^0}\right)\tilde{\Psi}\left(\frac{\Psi^{-1}(|F(y^{-1}x)|)}{\|\Psi^{-1}\circ\check{F}\|_{\tilde{\Psi}}^0}\right)d\lambda^u(x)d\lambda^u(y) \\ =\ &\int \Psi\left(\frac{|h(x)|}{\|h\|_\Psi^0}\right)\int\tilde{\Psi}\left(\frac{\Psi^{-1}(|\check{F}(y)|)}{\|\Psi^{-1}\circ\check{F}\|_{\tilde{\Psi}}^0}\right)d\lambda^{d(x)}(y)d\lambda^u(x) \leq 1. \end{aligned}$$

Thus,

$$\|m(x,y)\|_\Psi^0 \leq \|h\|_\Psi^0\|\Psi^{-1}\circ|\check{F}|\|_{\tilde{\Psi}}^0.$$

Similarly,

$$\|k(x,y)\|_\Phi^0 \leq \|g\|_\Phi^0\|\Phi^{-1}\circ|F|\|_{\tilde{\Psi}}^0.$$

Hence, we can see that

$$\|g * F\|_\Phi^0 \leq 2\|g\|_\Phi^0 K_F^2.$$

□

Now, we will show the existence of the left approximate identity on E_0^Φ when it is a Banach algebra. The corresponding results are proved in [8, Proposition 2] for the group case and [11, Theorem 3.8] for the hypergroup case.

Theorem 3.15. *Let G be a groupoid, and let (Φ, Ψ) be a complementary pair of N-functions with $\Phi \in \Delta_2$, then there exist a sequence $\{e_n\}$ of continuous functions with compact support bounded in $\|\cdot\|_1$-norm such that $\|e_n * \xi - \xi\|_\Phi^0 \to 0$ for every $\xi \in E_0^\Phi$.*

Proof. Let $f \in C_c(G)$ with $\operatorname{supp}(f) = K$ and $\epsilon > 0$. By [4, Proposition 11], there exists a bounded approximate identity $\{e_n\} \geq 0$ for $C_c(G)$ under $\|\cdot\|_1$-norm with $\|e_n\|_1 \leq 2$, for all $n \in \mathbb{N}$. There exist a sequence $\{U_n\}$ of open neighbourhoods of G^0 in G such that each U_n is d-relatively compact, $U_n \subset U_1$ and is a fundamental sequence for G^0 in the sense that every neighbourhood V of G^0 in G contains U_n eventually. There is an increasing sequence $\{K_n\}$ of compact subsets of G^0 such that $\cup K_n = G^0$. Also $\{e_n\}$ is constructed such that $\operatorname{supp}(e_n) \subset U_n$ and $\int e_n d\lambda^u = 1$ for all $u \in K_n$.

Let $L = \overline{U_1 K}$, where $U_1 K = \left(U_1 \cap d^{-1}(r(K))\right) K$ is relatively compact such that $\operatorname{supp}(f)$ and $\operatorname{supp}(e_n * f)$ is contained in L. Since L is compact, $\sup_{u \in G^0} \lambda^u(L) < \infty$ and let that value be M. Choose $\delta > 0$ such that $\Phi(x) < \frac{1}{2M}$ for all x with $|x| < \delta$. This is possible by continuity of Φ. Note that $r(L) \subset K_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Also by continuity of f and product in G, there exist $n_1 \in \mathbb{N}, n_1 \geq n_0$ such that for $n \geq n_1$, $\frac{\left|f\left(y^{-1}x\right) - f(x)\right|}{\epsilon} < \delta$ for all $(x, y) \in (L \times U_n) \cap G^2$. Now choose e_n, where $n \geq n_1$. Let $u \in r(L)$,

$$\begin{aligned}
\int \Phi\left(\frac{|e_n * f - f|}{\epsilon}\right)(x) d\lambda^u(x) &= \int_L \Phi\left(\frac{1}{\epsilon}\left|\int e_n(y) f\left(y^{-1}x\right) d\lambda^{r(x)}(y) - f(x)\right|\right) d\lambda^u(x) \\
&\leq \int_L \Phi\left(\int e_n(y) \frac{\left|f\left(y^{-1}x\right) - f(x)\right|}{\epsilon} d\lambda^{r(x)}(y)\right) d\lambda^u(x) \\
&= \int_L \Phi\left(\int \frac{\left|f\left(y^{-1}x\right) - f(x)\right|}{\epsilon} d\nu_n^u(y)\right) d\lambda^u(x) \\
&\quad \text{where } \nu_n^u(E) = \int_E e_n(y) d\lambda^u(y),\ E \subset G^u \\
&\leq \int_L \int_{U_n} \Phi\left(\frac{\left|f\left(y^{-1}x\right) - f(x)\right|}{\epsilon}\right) d\nu_n^u(y) d\lambda^u(x) \\
&= \int_{U_n} \int_L \Phi\left(\frac{\left|f\left(y^{-1}x\right) - f(x)\right|}{\epsilon}\right) d\lambda^u(x) d\nu_n^u(y)
\end{aligned}$$

$$\leq \frac{1}{2M} M \int e_n(y) d\lambda^u \leq 1.$$

The above is true for all $u \in r(L)$ and all $n \geq n_1$. Also, for $u \notin r(L)$, $\int \Phi\left(\frac{|e_n * -f|}{\epsilon}\right) d\lambda^u = 0$, since $\operatorname{supp}(e_n * f - f) \subset L$.
So, $\|e_n * f - f\|_\Phi^0 \leq \epsilon$ for all $n \geq n_1$. The result follows by Lemma 3.13 and the density of $C_c(G)$. □

Using [20, Theorem 4.6], for a closed $C_b(G^0)$-submodule I of E_0^Φ, there exist a subbundle of (L_0^Φ, p, G^0), denoted by $(I_0^\Phi, p_{I_0^\Phi}, G^0)$, such that the set of all sections of $(I_0^\Phi, p_{I_0^\Phi}, G^0)$ vanishing at infinity on G^0 coincides with I. Here the fiber $I_0^\Phi(u) = \{\xi^u : \xi \in I\}$ is a closed subspace of $L^\Phi(G^u)$. We say that the subbundle $(I_0^\Phi, p_{I_0^\Phi}, G^0)$ is invariant under the groupoid representation π, if $\pi(x)(I_0^\Phi(d(x))) \subset I_0^\Phi(r(x))$ for all $x \in G$.

The next theorem provides the sufficient and necessary condition for a closed $C_b(G^0)$ submodule of E_0^Φ to become an ideal when E_0^Φ is a Banach algebra. This is the groupoid version of the result in [11, Theorem 3.9].

Theorem 3.16. *Let G be a groupoid, (Φ, Ψ) be a complementary pair of N-functions with $\Phi \in \Delta_2$ and E_0^Φ be an algebra. A closed $C_b(G^0)$-submodule I of E_0^Φ is a left ideal if and only if the subbundle $(I_0^\Phi, p_{I_0^\Phi}, G^0)$ is invariant under left-regular representation of G.*

Proof. Let $u \in G^0, f, g \in C_c(G)$. L_z denote the left translation from $L^\Phi(G^{d(z)})$ to $L^\Phi(G^{r(z)})$. For $x \in G^{r(z)}$,

$$\begin{aligned} L_z(f*g)^{d(z)}(x) = (f*g)^{d(z)}\left(z^{-1}x\right) &= \int_{G^{d(z)}} f(y) g\left(y^{-1}z^{-1}x\right) d\lambda^{d(z)}(y) \\ &= \int_{G^{d(z)}} f(y) \check{g}\left(x^{-1}zy\right) d\lambda^{d(z)}(y) \\ &= \int_{G^{r(z)}} L_z f(y) g\left(y^{-1}x\right) d\lambda^{r(z)}(y) \\ &= \left(L_z f * g\right)(x), \end{aligned}$$

where $(L_z f * g)$ denotes the function $(\tilde{f} * g)^{r(z)}$ for $\tilde{f} \in C_c(G)$ such that $\tilde{f}_{|_{G^{r(z)}}} = L_z f^{d(z)}$. So by density of $C_c(G)$, $L_z(f * \eta)^{d(z)} = (\tilde{f} * \eta)^{r(z)} \in L^\Phi(G^{r(z)})$ for $\eta \in E_0^\Phi, \tilde{f} \in C_c(G)$ and can be denoted as $(L_z f * \eta)$.

Let I be a closed left ideal of E_0^Φ. Suppose $\{e_n\}$ is the left approximate identity of E_0^Φ. Let $x \in G, d(x) = u, r(x) = v$ and $\eta^u \in I_0^\phi(u)$ for $\eta \in I$.

$$\begin{aligned} \|(L_x e_n * \eta - L_x \eta^u)\|_\Phi^0 &= \|L_x\left(e_n * \eta - \eta\right)^u\|_\Phi^0 \\ &= \|(e_n * \eta - \eta)^u\|_\Phi^0 \leq \|(e_n * \eta - \eta)\|_\Phi^0 \to 0. \end{aligned}$$

As mentioned before $(L_x e_n * \eta)$ denotes the function $(e_n' * \eta)^v$ for some $e_n' \in C_c(G)$ with ${e_n'}_{|_{G^v}} = L_x e_n^u$ and since I is a left ideal, $e_n' * \eta \in I$ for every $n \in \mathbb{N}$. Hence $(e_n' * \eta)^v \in I_0^\Phi(v)$. So $L_x \eta^u \in I_0^\Phi(v)$. Since x was arbitrary we can say that $(I_0^\Phi, p_{I_0^\Phi}, G^0)$ is invariant under left regular representation.

Let $(I_0^\Phi, p_{I_0^\Phi}, G^0)$ is invariant under left regular representation. If I is not an ideal, by density of $C_c(G)$ in E_0^Φ, there exist $f' \in C_c(G)$ and $g \in I$ such that $f' * g \notin I$. Hence by [20, Theorem 4.6], there exist $u_0 \in G^0$, such that $(f' * g)^{u_0} \notin I_0^\Phi(u_0)$. So there exist $h \in L^\Psi(G^{u_0})$ such that $\int_{G^{u_0}} (f' * g)\, h d\lambda^{u_0} \neq 0$ and $\int_{G^{u_0}} k h d\lambda^{u_0} = 0$, for all $k \in I_0^\Phi(u_0)$.
$f' * g_n \to f' * g$ in E_0^Φ as $g_n \to g, g_n \in C_c(G)$. So in particular, by Holder's inequality, $\int_{G^{u_0}} (f' * g_n)\, h d\lambda^{u_0} \to \int_{G^{u_0}} (f' * g)\, h d\lambda^{u_0}$.

$$\begin{aligned}\int_{G^{u_0}} (f' * g_n)\, h d\lambda^{u_0} &= \int_{G^{u_0}} \left(\int_{G^{u_0}} f'(y) g_n\left(y^{-1}x\right) d\lambda^{u_0}(y) \right) h(x) d\lambda^{u_0}(x) \\ &= \iint_{G^{u_0}} f'(y) g_n(y^{-1}x) h(x) d\lambda^{u_0}(y) d\lambda^{u_0}(x) \\ &= \iint_{G^{u_0}} g_n(y^{-1}x) h(x) f'(y) d\lambda^{u_0}(x) d\lambda^{u_0}(y).\end{aligned}$$

Note that since $L_y : L^\Phi(G^{d(y)}) \to L^\Phi(G^{u_0})$ is isometry, the above integral converges to $\iint_{G^{u_0}} L_y g(x) h(x) f'(y) d\lambda^{u_0}(x) d\lambda^{u_0}(y)$. Since $L_y g^{d(y)} \in I_0^\Phi(u_0)$, $\iint_{G^{u_0}} L_y g(x) h(x) f'(y) d\lambda^{u_0}(x) d\lambda^{u_0}(y) = 0$, for all $y \in G^{u_0}$.
So $\int_{G^{u_0}} (f' * g)\, h d\lambda^{u_0} = 0$, which is a contradiction. Hence proved. ☐

4. The space of Convolutors of E_0^Φ

In locally compact groups and hypergroups, the space of convolutors of Morse-Transue space $M^\Phi(G)$, is identified with the dual of a space, denoted by $\check{A}_\Phi(G)$. This well-known result can be found in [12] and [11]. In this section, we provide a groupoid version of this result. Here we assume (Φ, Ψ) as complementary pair of N-functions both satisfying Δ_2-condition.

A bounded linear operator T on E_0^Φ is called a convolutor if $T(f * g) = Tf * g$, for all $f, g \in C_c(G)$. This space is denoted as $C_{V_\Phi}(G)$. It can be easily verified that $C_{V_\Phi}(G)$ is a closed subspace of $B(E_0^\Phi)$.

Lemma 4.1. *Let G be a groupoid and (Φ, Ψ) be a complementary pair of N-functions. If $T \in C_{V_\Phi}(G)$, then there exist $\{e_n\} \in C_c(G)$ with $\|e_n\|_1 \leq 2$ such that if we set $T_n(f) = T(e_n * f), f \in E_0^\Phi$,*

(i) $\|T_n\| \leq 2\|T\|$,
(ii) $\lim_{n\to\infty} \|T_n f - Tf\|_\Phi^0 = 0$, *for* $f \in E_0^\Phi$.

Proof. We proved that there exist $\{e_n\} \in C_c(G)$ with $\|e_n\|_1 \leq 2$ and $\|e_n * f - f\|_\Phi^0 \to 0$ for all $f \in E_0^\Phi$. Take $f \in C_c(G)$,

$$\|T(e_n * f)\|_\Phi^0 \leq \|T\| \, \|e_n\|_1 \, \|f\|_\Phi^0 \leq 2\|T\|\|f\|_\Phi^0.$$

We can see that T_n is contained in $C_{V_\Phi}(G)$ and $\|T_n(f)\|_\Phi^0 \leq 2\|T\|\|f\|_\Phi^0$ for all $f \in E_0^\Phi$. Thus, $\|T_n\| \leq 2\|T\|$ for every $n \in \mathbb{N}$. For $f \in C_c(G)$,

$$\begin{aligned}\lim_{n\to\infty} \|T_n(f) - T(f)\|_\Phi^0 &= \lim_{n\to\infty} \|T(e_n * f - f)\|_\Phi^0 \\ &\leq \|T\| \lim_{n\to\infty} \|e_n * f - f\|_\Phi^0 = 0.\end{aligned}$$

□

Let $\mathcal{K}(G)$ be the collection of compact sets with nonempty interior intersecting G^0. For $P \in \mathcal{K}(G)$, define

$$E_0^\Phi(P) = \overline{\{f \in C_c(G) : \operatorname{supp}(f) \subset P\}}^{\|\cdot\|_\Phi^0}.$$

$$E_0^\Psi(P) = \overline{\{f \in C_c(G) : \operatorname{supp}(f) \subset P\}}^{\|\cdot\|_\Psi^0}.$$

By the density of $C_c(G)$ and Holder's inequality, we can define a function in $C_c(G)$ denoted as $\xi * \check{\eta}$, for $\xi \in E_0^\Psi(P)$ and $\eta \in E_0^\Phi(P)$.

$$\check{A}_{\Phi,P}(G) = \Big\{h \in C_c(G) : h = \sum_{n=1}^\infty g_n * \check{f}_n, f_n \in E_0^\Phi(P),\ g_n \in E_0^\Psi(P), \\ \sum_{n=1}^\infty \|f_n\|_\Phi^0 \, \|g_n\|_\Psi^0 < \infty\Big\}.$$

The norm is defined as follows,

$$\|h\|_{\check{A}_{\Phi,P}(G)} = \inf\left\{\sum_{n=1}^\infty \|f_n\|_\Phi^0 \, \|g_n\|_\Psi^0 : h = \sum_{n=1}^\infty g_n * \check{f}_n\right\}.$$

Set $\check{A}_\Phi(G) = \cup_{P\in\mathcal{K}(G)} \check{A}_{\Phi,P}(G)$. Then $\check{A}_\Phi(G)$ is a subspace of $C_c(G)$ and for $h \in \check{A}_\Phi(G)$,

$$\|h\|_{\check{A}_\Phi(G)} = \inf\Big\{\|h\|_{\check{A}_{\Phi,P}(G)} : h \in \check{A}_{\Phi,P}(G), P \in \mathcal{K}(G)\Big\}.$$

For $f, g \in C_c(G)$, since $|f * \check{g}(x)| \leq 2\|f\|_\Phi^0 \|g\|_\Psi^0$, for all $x \in G$, $\|h\|_\infty \leq 2\|h\|_{\check{A}_\Phi(G)}$. $\bar{A}_\Phi(G)$ denotes the norm completion of $\check{A}_\Phi(G)$. When G is a locally compact group, this space coincides with the norm completion of $\check{A}_\Phi(G)$, defined in [12, Section 4, Page 27].

Lemma 4.2. *$\check{A}_\Phi(G)$ is a left and right $C_0(G^0)$-module with following action:*

For $b \in C_0(G^0), h \in \check{A}_\Phi(G), (bh)(x) = b(d(x))h(x)$ and $(hb)(x) = h(x)b(r(x))$.

Proof. Let $f, g \in C_c(G)$, having support on $P \in \mathcal{K}(G)$, then $h = f * \check{g} \in \check{A}_\Phi(G)$.

$$\begin{aligned}(hb)(x) &= b(r(x))h(x) = b(r(x))f * \check{g}(x)\\ &= b(r(x)) \int f(y)\check{g}\left(y^{-1}x\right) d\lambda^{r(x)}(y)\\ &= \int f(y)\check{g}\left(y^{-1}x\right) b(r(y)) d\lambda^{r(x)}(y)\\ &= fb * \check{g}(x) \in \check{A}_\Phi(G).\end{aligned}$$

Also $(fb) \in E_0^\Psi(P)$. Hence $hb \in \check{A}_\Phi(G)$ for any $h \in \check{A}_\Phi(G)$. Similarly, we can see that $bh(x) = f * (\check{gb})(x)$. Here, $\|hb\|_{\check{A}_\Phi(G)} \le \|b\|_\infty \|h\|_{\check{A}_\Phi(G)}$ and $\|bh\|_{\check{A}_\Phi(G)} \le \|b\|_\infty \|h\|_{\check{A}_\Phi(G)}$. ☐

Let $B_D^\Phi\left(\check{A}_\Phi(G), C_0(G^0)\right)$ be the space of all bounded right-module linear maps from $\check{A}_\Phi(G)$ to $C_0(G^0)$. For $\alpha \in B_D^\Phi\left(\check{A}_\Phi(G), C_0(G^0)\right)$, $f \in C_c(G)$,
define $f\alpha : C_c(G) \to C_0(G^0)$ as, $f\alpha(g) = \alpha(g * \check{f})$. It is a linear map. Also,

$$|f\alpha(g)(u)| = |\alpha(g * \check{f})(u)| \le \|\alpha\| \|g * \check{f}\|_{\check{A}_\Phi(G)} \le \|\alpha\| \|g\|_\Psi^0 \|f\|_\Phi^0 \le \|\alpha\| \|g\|_\Psi \|f\|_\Phi^0.$$

Hence, $\|f\alpha(g)\|_\infty \le \|\alpha\| \|f\|_\Phi^0 \|g\|_\Psi$. We can extend $f\alpha$ to E^Ψ.

$f\alpha(gb) = \alpha(gb * \check{f}) = \alpha((g * \check{f})b) = \alpha((g * \check{f}))b$. Thus, $\alpha \to f\alpha$ is a bounded linear map from $B_D^\Phi\left(\check{A}_\Phi(G), C_0(G^0)\right)$ to $B_D^\Phi\left(E^\Psi, C_0(G^0)\right)$, where $B_D^\Phi\left(E^\Psi, C_0(G^0)\right)$ is the space of bounded linear right-module maps from E^Ψ to $C_0(G^0)$.
For $\xi \in E_0^\Phi$, $\eta \in E^\Psi$, define:

$$\langle \xi, \eta \rangle(u) = \int_{G^u} \xi^u \eta^u d\lambda^u.$$

Note that the function $\langle \xi, \eta \rangle$ is in $C_0\left(G^0\right)$ such that,

$$\|\langle \xi, \eta \rangle\|_\infty \le \|\xi\|_\Phi^0 \|\eta\|_\Psi \le 2\|\xi\|_\Phi^0 \|\eta\|_\Psi^0.$$

This is possible due to denseness of $C_c(G)$ and Holder's inequality.
Let,

$$R\left(E^\Psi, C_0(G^0)\right) = \left\{S \in B_D\left(E^\Psi, C_0(G^0)\right) : S(\xi)(u) = \langle g, \xi \rangle(u) \,\text{for some}\, g \in E_0^\Phi(G)\right\}.$$

Since $M^\Psi\left(G^u\right)^* = L^\Phi\left(G^u\right)$ for every $u \in G^0$ and using lemma 3.4 in the context of M^Ψ, if $S \in R\left(E^\Psi, C_0(G^0)\right)$, there exist a unique $g \in E_0^\Phi$ such that $S(\xi)(u) = \langle g, \xi \rangle(u)$. Also,

$$\begin{aligned}\|g^u\|_\Phi^0 &= \sup\left\{\left|\int g^u f d\lambda^u\right| : f \in M^\Psi\left(G^u\right), \|f\|_\Psi \le 1\right\},\\ &= \sup\left\{|S(\xi)(u)| : \xi \in E^\Psi, \|\xi\|_\Psi \le 1\right\}.\end{aligned}$$

So $\|g\|_\Phi^0 = \|S\|$. Hence we can see that $R\left(E^\Psi, C_0(G^0)\right)$ is closed in $B_D\left(E^\Psi, C_0(G^0)\right)$.

Let $\check{A}_\Phi(G)'$ be the set of $\alpha \in B_D^\Phi\left(\check{A}_\Phi(G), C_0(G^0)\right)$, such that $f\alpha \in R\left(E^\Psi, C_0(G^0)\right)$ for all $f \in C_c(G)$. It can be easily verified that $\check{A}_\Phi(G)'$ is a closed subspace of $B_D^\Phi\left(\check{A}_\Phi(G), C_0(G^0)\right)$.

The following is the main theorem of this section. For the proof we use some ideas from [4] and [12].

Theorem 4.3. *Let G be a groupoid, (Φ, Ψ) be a complementary pair of N-functions both satisfying Δ_2-condition. Then $\check{A}_\Phi(G)'$ can be identified with $C_{V_\Phi}(G)$.*

Proof. Let $T \in C_{V_\Phi}(G), h \in \check{A}_\Phi(G)$. Then for some $P \in \mathcal{K}(G)$,

$$h = \sum_{i=1}^{\infty} g_i * \check{f}_i,\ \ f_i \in E_0^\Phi(P), g_i \in E_0^\Psi(P).$$

Define $\phi_T : \check{A}_\Phi(G) \to C_0\left(G^0\right)$ by

$$\phi_T(h)(u) = \sum_{i=1}^{\infty} \left\langle T f_i, g_i\right\rangle(u).$$

It is clear that ϕ_T is linear. Further,

$$\|\phi_T(h)\|_\infty \le 2\sum_{i=1}^{\infty} \|Tf_i\|_\Phi^0 \|g_i\|_\Psi^0 \le 2\|T\| \sum_{i=1}^{\infty} \|f_i\|_\Phi^0 \|g_i\|_\Psi^0 < \infty.$$

We need to show that $\phi_T(h)$ is independent of the representation of h. For that, it is enough to show that, $\phi_T(h) = 0$, if $h = 0$. Suppose $h = \sum_{i=1}^\infty g_i * \check{f}_i = 0$. Also, $T_k(f) = T\left(e_k * f\right) = Te_k * f,\ f \in C_c(G)$. For all $u \in G^0$,

$$\begin{aligned}\left|\sum_{i=1}^{\infty} \left\langle T_k f_i, g_i\right\rangle(u)\right| &\le 2\sum_{i=1}^{\infty} \|T_k f_i\|_\Phi^0 \|g_i\|_\Psi^0 \le 2\left\|T_k\right\| \sum_{i=1}^{\infty} \|f_i\|_\Phi^0 \|g_i\|_\Psi^0 \\ &\le 4\|T\| \sum_{i=1}^{\infty} \|f_i\|_\Phi^0 \|g_i\|_\Psi^0 < \infty.\end{aligned}$$

So, using Lemma 4.1, the function $\sum_{i=1}^\infty \left\langle T_k f_i, g_i\right\rangle$ converges uniformly in k to $\sum_{i=1}^\infty \left\langle T f_i, g_i\right\rangle$ on $C_0\left(G^0\right)$.

Let $n_j \in \mathbb{N}$ such that $\sum_{n>n_j} \|f_n\|_\Phi^0 \|g_n\|_\Psi^0 < \frac{1}{2jM}$ where $M = \max\{4\|T\|, 2\}$. Since $\{f \in C_c(G) : \operatorname{supp}(f) \subset P\}$ is dense in $E_0^\Phi(P)$, there exist a family of sequences $\left\{(h_i^j)_{i\in\mathbb{N}} : h_i^j \in C_c(G), \operatorname{supp}(h_i^j) \subset P,\ j \in \mathbb{N}\right\}$ such that $h_i^j = 0\ \forall\ i > n_j$ and $\|f_i - h_i^j\|_\Phi^0 < \frac{1}{2jK_jM}$, for $1 \le i \le n_j$ where $K_j = \sum_{i=1}^{n_j} \|g_i\|_\Psi^0$. Hence,

$$\begin{aligned}\sum_{i=1}^{\infty} \left\langle T_k f_i, g_i\right\rangle(u) &= \lim_{j\to\infty} \sum_{i=1}^{n_j} \left\langle Te_k * h_i^j, g_i\right\rangle(u) \\ &= \lim_{j\to\infty} \sum_{i=1}^{n_j} \left\langle \chi_{P^*} Te_k, g_i * \check{h_i^j}\right\rangle(u)\end{aligned}$$

$$\begin{aligned}
&= \left\langle \chi_{P^*} Te_k, \lim_{j\to\infty} \sum_{i=1}^{n_j} g_i * \check{h_i^j} \right\rangle (u) \\
&= \left\langle \chi_{P^*} Te_k, \sum_{i=1}^{\infty} g_i * \check{f_i} \right\rangle (u)
\end{aligned}$$

Here $P^* = PP^{-1}$ is compact in G. Note that $\chi_{P^*} Te_k$ denotes the section whose value at u is the function:

$$\left(\chi_{P^*} Te_k\right)^u (t) = \chi_{P^* \cap G^u}(t) \left(Te_k\right)^u (t)$$

for $t \in G^u$. We can easily see that $\chi_{P^*} Te_k \in I(G, \lambda)$. So, $\sum_{n=1}^{\infty} \langle Tf_n, g_n \rangle (u) = 0$, for all $u \in G^0$. Hence $\phi_T(h)$ is well defined and $\|\phi_T(h)\|_\infty \le 2\|T\| \|h\|_{\check{A}_{\Phi(G)}}$. Further,

$$\begin{aligned}
\|T\| &= \sup \left\{ \|(Tf)^u\|_\Phi^0 : u \in G^0, f \in C_c(G), \|f\|_\Phi^0 \le 1 \right\}, \\
&= \sup \left\{ |\langle Tf, g \rangle (u)| : u \in G^0, g, f \in C_c(G), \|f\|_\Phi^0 \le 1, \|g\|_\Psi \le 1 \right\}, \\
&= \sup \left\{ \|\phi_T(h)\|_\infty : h = g * \check{f}, \|h\|_{\check{A}_{\Phi(G)}} \ \le 1 \right\} \le \|\phi_T\|.
\end{aligned}$$

So $\|T\| \le \|\phi_T\| \le 2\|T\|$. For $b \in C_0\left(G^0\right), h \in \check{A}_{\Phi(G)}$,

$$\phi_T(hb)(u) = \sum_{n=1}^{\infty} \langle Tf_n, g_n b \rangle (u) = \sum_{n=1}^{\infty} \langle Tf_n, g_n \rangle (u) b(u) = \left(\phi_T(h) b\right)(u).$$

Thus ϕ_T is a bounded linear right $C_0(G^0)$-module map. Now for $f \in C_c(G)$,

$$f\phi_T(g)(u) = \phi_T(g * \check{f})(u) = \langle Tf, g \rangle (u),$$

for all $g \in C_c(G)$. Hence $f\phi_T \in R\left(E^\Psi, C_0(G^0)\right)$ implies $\phi_T \in \check{A}_\Phi(G)'$.

It remains to show that $\phi : T \to \phi_T$ is surjective. Let $\alpha \in \check{A}_\Phi(G)'$, then for $f \in C_c(G)$, there exist $F_f \in E_0^\Phi$ such that,

$$(f\alpha)(g)(u) = \langle F_f, g \rangle (u), g \in E^\Psi.$$

Define $T(f) = F_f$, for all $f \in C_c(G)$. Note $\langle F_f, g \rangle (u) = \langle T(f), g \rangle (u) = \alpha(g * \check{f})(u)$, for all $g \in C_c(G)$. Further,

$$|\langle T(f), g \rangle (u)| = |\alpha(g * \check{f})(u)| \le \|\alpha\| \|f\|_\Phi^0 \|g\|_\Psi.$$

Hence $g \in C_c(G)$ implies $\|Tf\|_\Phi^0 \le \|\alpha\| \|f\|_\Phi^0$. So, T can be extended to E_0^Φ and $\|T\| \le \|\alpha\|$. Let $f_1, f_2, g \in C_c(G)$. Then,

$$\begin{aligned}
\langle T(f_1 * f_2), g \rangle &= \alpha(g * (f_1 \check{*} f_2)) \\
&= \alpha(g * (\check{f_2} * \check{f_1})) \\
&= \alpha\left(\left(g * \check{f_2}\right) * \check{f_1}\right) \\
&= \left\langle Tf_1, g * \check{f_2} \right\rangle = \langle Tf_1 * f_2, g \rangle.
\end{aligned}$$

Hence, $T \in C_{V_\Phi}(G)$. $\square$

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Data Availability

Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

COMPETING INTERESTS

The authors declare that they have no competing interests.

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