

GENERALISED WEIGHTED COMPOSITION OPERATORS ON WEIGHTED FOCK SPACES INDUCED BY A_∞ -TYPE WEIGHTS

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ABSTRACT. In this paper, we characterize the boundedness and compactness of generalised weighted composition operators on weighted Fock spaces $\mathcal{F}_{\alpha,\omega}^p$ with $\omega \in A_\infty^{restricted}$. We also study the essential norm of these operators on the weighted Fock spaces.

1. INTRODUCTION

Let $\mathcal{H}(\mathbb{C})$ be the space of entire functions on the complex plane \mathbb{C} . A function $\omega : \mathbb{C} \rightarrow [0, \infty)$ is a weight if ω belongs to $L_{loc}^1(\mathbb{C}, dA)$, where dA denotes the Lebesgue measure on \mathbb{C} . For $0 < p, \alpha < \infty$ and a weight ω , the weighted space $\mathcal{L}_{\alpha,\omega}^p$ consists of measurable functions f on \mathbb{C} satisfying

$$\|f\|_{\mathcal{L}_{\alpha,\omega}^p}^p = \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} \omega(z) dA(z) < \infty,$$

and the weighted Fock space $\mathcal{F}_{\alpha,\omega}^p = \mathcal{L}_{\alpha,\omega}^p \cap \mathcal{H}(\mathbb{C})$. Clearly, if $\omega \equiv 1$, then $\mathcal{F}_{\alpha,\omega}^p$ is the classical Fock spaces F_α^p (see, for instance, [16] and the references therein).

Throughout the paper, we use Q to denote a square in \mathbb{C} with sides parallel to the coordinate axes, and write $l(Q)$ for its side length. If $E \subset \mathbb{C}$ is measurable, we denote by $|E|$ its Lebesgue area measure and let $\omega(E) = \int_E \omega(z) dA(z)$. For $1 \leq p \leq \infty$, let p' denote the conjugate exponent of p , that is, $\frac{1}{p} + \frac{1}{p'} = 1$ with $1 \leq p \leq \infty$.

Let $1 < p < \infty$. We say a weight ω belongs to the class $A_p^{restricted}$ if $\omega(z) > 0$ a.e. on \mathbb{C} and, for some $r > 0$,

$$\mathcal{C}_{p,r}(\omega) := \sup_{Q, l(Q)=r} \left(\frac{1}{|Q|} \int_Q \omega dA \right) \left(\frac{1}{|Q|} \int_Q \omega^{-\frac{p'}{p}} dA \right)^{\frac{p}{p'}} < \infty.$$

The pioneering work about the weight class $A_p^{restricted}$ was done by Isralowitz [9]. In fact, it follows from [9, Theorem 3.1] that these classes of weights are independent of the choice of r , and $\omega \in A_p^{restricted}$ if and only if the orthogonal projection P_α is bounded on $\mathcal{L}_{\alpha,\omega}^p$. The boundedness of projections on L^p spaces is a classical topic, which has many applications on operator

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theory, such as [2, 12, 16]. After that, Cascante, Fàbrega and Peláez [3] investigated a new class of weight $A_1^{restricted}$. That is, for fixed $r > 0$, a weight $\omega \in A_1^{restricted}$ if $\omega(z) > 0$ a.e. on \mathbb{C} and

$$\mathcal{C}_{1,r}(\omega) := \sup_{Q, l(Q)=r} \frac{\omega(Q)}{|Q| \inf_{u \in Q} \omega(u)} < \infty.$$

Similarly to Muckenhoupt weights, we denote

$$A_\infty^{restricted} := \bigcup_{1 \leq p < \infty} A_p^{restricted}.$$

Furthermore, the authors in [3] characterize Littlewood-Paley formulas for weighted Fock spaces $\mathcal{F}_{\alpha,\omega}^p$ with $\omega \in A_\infty^{restricted}$. Using Littlewood-Paley formulas, the boundedness of the differentiation and integration operators are also obtained (see [3, Theorem 1.2]). We denote $D^n(f) = f^{(n)}$ with $n \in \mathbb{N}$ and $D^0(f) = f$. For a positive Borel measure μ on \mathbb{C} , we supplement the compactness of differentiation operator $D^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow L^q(\mu)$ with $0 < p, q < \infty$, see Theorems 10 and 11 below. For more interesting results on the weighted Fock spaces we refer to [4, 5, 15].

Let φ be an analytic self-map of \mathbb{C} , u an entire function on \mathbb{C} , and $n \in \mathbb{N} \cup \{0\}$. The generalized weighted composition operator is defined by

$$D_{\varphi,u}^n f = u \cdot f^{(n)} \circ \varphi.$$

Clearly, if $n = 0$ and $u(z) \equiv 1$, then the generalized weighted composition operator is the well-known composition operator $C_\varphi f = f \circ \varphi$. And if $n = 0$, then we get the weighted composition operator $uC_\varphi f = u \cdot f \circ \varphi$. These operators have been studied in various spaces (see, for example, [4, 6, 7, 10, 13] and references therein). In [17], Zhu characterized the boundedness and compactness of the operator $D_{\varphi,u}^n$ between different standard weighted Bergman spaces. Similar problems for the generalized weighted composition operator on Bergman spaces induced by doubling weights were considered in [11]. The bounded and compact properties of the generalized weighted composition operator on classical Fock spaces have been studied in [8].

In this article, we continue those lines of research for this class of operators. To state our main results, we need some more notations. Let μ be a positive Borel measure on \mathbb{C} and h a measurable function on \mathbb{C} . Let φ be an analytic self-map of \mathbb{C} , we define the measure related to h by

$$\varphi_*(h)(E) = \int_{\varphi^{-1}(E)} h d\mu$$

for every Borel subset E of \mathbb{C} . We denote by $D(z, r)$ the Euclidean disk of center z and radius $r > 0$. We write $\Re(z)$ for the real part of a complex number z , and $\omega_\gamma(z) := \frac{\omega(z)}{(1+|z|)^\gamma}$ with $\gamma \in \mathbb{R}$ (see (3) for more details).

For a positive Borel measure ν and $0 < q < p < \infty$, [11, Theorem 1.1] describes the bounded and compact operators $D_{\varphi,u}^n$ acting from weighted Bergman space A_ω^p to L_ν^q . As an application of Theorem 10, we deduce the

following consequence, which generalizes this result on weighted Fock spaces $\mathcal{F}_{\alpha,\omega}^p$ with $\omega \in A_\infty^{\text{restricted}}$.

Theorem 1. *Suppose that $0 < q < p < \infty$, $\alpha \in (0, \infty)$, $\omega \in A_\infty^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$ and ν is a positive Borel measure on \mathbb{C} . Let φ be an analytic self-map of \mathbb{C} and $u \in L_\nu^q$. Then the following conditions are equivalent:*

- (i) $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow L_\nu^q$ is bounded;
- (ii) $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow L_\nu^q$ is compact;
- (iii) The function

$$\frac{1}{\omega_{np}(D(u, 1))} \int_{D(u, 1)} e^{\frac{q\alpha|z|^2}{2}} d\varphi_*(|u|^q \nu)(z)$$

is in $L^{\frac{p}{p-q}}(\mathbb{C}, \omega_{np})$.

When $0 < p \leq q < \infty$, Liu studied the boundedness and compactness of the operator $D_{\varphi,u}^n$ acting between different weighted Bergman spaces A_ω^p (see [11, Theorem 1.3]). Recently, Chen characterized the boundedness and compactness of composition operator C_φ acting between different weighted Fock spaces $\mathcal{F}_{\alpha,\omega}^p$ with $\omega \in A_\infty^{\text{restricted}}$, see [4, Theorem 1.1]. We extend these results to our weighted Fock spaces as follows.

Theorem 2. *Suppose that $0 < p \leq q < \infty$, $\alpha, \beta > 0$, $\omega \in A_\infty^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$ and ν is a weight. Let φ be an analytic self-map of \mathbb{C} and $u \in \mathcal{F}_{\beta,\nu}^q$. Then:*

- (i) $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow \mathcal{F}_{\beta,\nu}^q$ is bounded if and only if

$$\sup_{a \in \mathbb{C}} \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha\Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) < \infty.$$

- (ii) $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow \mathcal{F}_{\beta,\nu}^q$ is compact if and only if

$$\lim_{|a| \rightarrow \infty} \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha\Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) = 0.$$

In addition to boundedness and compactness, we intend to consider the essential norm of the generalised weighted composition operator $D_{\varphi,u}^n$. For classical Fock spaces F_α^p , Tien and Khoi [13] provided lower and upper estimates for essential norm of weighted composition operator uC_φ . Later on, Hu, Li and Qu [8] extended these estimates for the operator $D_{\varphi,u}^n$ on the Fock spaces F_α^p . Inspired by these papers, we investigate lower and upper estimates for essential norm of the operator $D_{\varphi,u}^n$ on weighted Fock spaces $\mathcal{F}_{\alpha,\omega}^p$ via Littlewood-Paley formulas (see [3, Theorem 1.1]). Combining Theorem 16 with Corollary 17 below, we offer these estimates of $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow \mathcal{F}_{\alpha,\omega}^p$ whenever $\varphi(z) = az + b$ with $0 < |a| < 1$.

Theorem 3. *Suppose that $1 < p < \infty$, $\alpha > 0$, $\omega \in A_\infty^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$. Let $u \in \mathcal{F}_{\alpha,\omega}^p$ be a nonzero function and $\varphi(z) = az + b$ with $0 < |a| < 1$. If*

$D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow \mathcal{F}_{\alpha,\omega}^p$ is bounded, then

$$\limsup_{|z| \rightarrow \infty} m_{z,n}(u, \varphi) \lesssim \|D_{\varphi,u}^n\|_e \lesssim |a|^{-\frac{2}{p}} \limsup_{|z| \rightarrow \infty} \frac{m_{z,n}(u, \varphi)\omega(z)}{\omega(\varphi(z))}.$$

The rest of paper contains the proofs of the conclusions as above. In Section 2, we provide some auxiliary results. The proofs of Theorems 1 and 2 are given in Section 3. Finally, we establish lower and upper estimates for essential norm of the operator $D_{\varphi,u}^n$ on weighted Fock spaces $\mathcal{F}_{\alpha,\omega}^p$ in Section 4.

In the following, the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set of a question. Furthermore, we write $U(z) \asymp V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2. PRELIMINARIES

By [3, Remark 2.3], if $\omega \in A_{\infty}^{restricted}$, for any $t > 0$ and $N \in \mathbb{N}$ there is $C = C(\omega, N, p)$ such that

$$\omega(D(a, t)) \leq \omega(D(a, Nt)) \leq C\omega(D(a, t)), \quad a \in \mathbb{C},$$

and for any $z, w \in \mathbb{C}$ such that $|z - w| < L$ ($L > 0$),

$$(1) \quad \omega(D(z, r)) \asymp \omega(D(w, R)), \quad C^{-1} \leq r, R \leq C.$$

Combining these results with [9, Lemma 3.4], [3, Lemma B] and [4, Lemma 2.2], we have the following lemma. As mentioned earlier, the notation $Q_1(z)$ means the square of center $z \in \mathbb{C}$ with side length $l(Q) = 1$. Besides, since \mathbb{R}^2 can be canonically identified with \mathbb{C} , we will treat \mathbb{Z}^2 as a subset of \mathbb{C} .

Lemma 4. *Suppose that $\omega \in A_{\infty}^{restricted}$.*

(i) *There exists $C > 0$ such that for any $v, v' \in \mathbb{Z}^2$,*

$$\frac{\omega(Q_1(v))}{\omega(Q_1(v'))} \leq C^{|v-v'|}.$$

(ii) *For any $z, u \in \mathbb{C}$ such that $|z - u| < 1$,*

$$\omega(Q_1(z)) \asymp \omega(Q_1(u)) \asymp \omega(D(z, 1)) \asymp \omega(D(u, 1)).$$

Some preliminary results, which will be used to prove our main consequences in the later.

Lemma 5. ([3, Lemma 2.8]) *Assume $\omega \in A_{\infty}^{restricted}$. Then there exists r_0 such that for any $r \in (0, r_0)$ and for any $\beta > 0$,*

$$\int_{\mathbb{C}} e^{-\beta|z|^2} \omega(z) dA(z) \leq C(\omega, r, \beta) \omega(Q_r(0)) < \infty.$$

Lemma 6. ([3, Lemma 3.1]) *Let $0 < p, t < \infty$, $\alpha > 0$ and $\omega \in A_\infty^{\text{restricted}}$. Then, there exists a constant $C = C(\alpha, p, \omega, t)$ such that*

$$|f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} \leq \frac{C}{\omega(D(z, t))} \int_{D(z, t)} |f(u)|^p e^{-\frac{p\alpha}{2}|u|^2} \omega(u) dA(u),$$

for any $z \in \mathbb{C}$ and $f \in \mathcal{H}(\mathbb{C})$.

The following two lemmas generalize [3, Lemma 3.1 and Corollary 3.2] and [8, Lemmas 2.2 and 2.3] to the weighted Fock spaces $\mathcal{F}_{\alpha, \omega}^p$.

Lemma 7. *Let $0 < p, t < \infty$, $\alpha > 0$, $\omega \in A_\infty^{\text{restricted}}$ and $n \in \mathbb{N} \cup \{0\}$. Then there exists a constant $C = C(\alpha, p, \omega, t)$ such that*

$$|f^{(n)}(z)| \leq Cn! e^{\frac{\alpha}{2}|z|^2} \frac{(1 + |z|)^n}{\omega(D(z, t))^{\frac{1}{p}}} \|f\|_{\mathcal{F}_{\alpha, \omega}^p}$$

for any $z \in \mathbb{C}$ and $f \in \mathcal{H}(\mathbb{C})$.

Proof. Firstly, let n be a positive integer. For $|z| \leq 1$, Lemma 6 and Cauchy integral formula indicate

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{2\pi} \int_{|z-\zeta|=1} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| \leq n! \max_{|z-\zeta|=1} |f(\zeta)| \\ &\leq C_1 n! \|f\|_{\mathcal{F}_{\alpha, \omega}^p} \max_{|z-\zeta|=1} \frac{e^{\frac{\alpha}{2}|\zeta|^2}}{\omega(D(\zeta, t_1))^{\frac{1}{p}}} \leq C_1 C_2 n! e^{2\alpha} \|f\|_{\mathcal{F}_{\alpha, \omega}^p} \omega(D(z, t_2))^{-\frac{1}{p}}. \end{aligned}$$

For $|z| > 1$, we conclude from Cauchy integral formula again that

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{2\pi} \int_{|z-\zeta|=|z|^{-1}} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| \leq n! |z|^n \max_{|z-\zeta|=|z|^{-1}} |f(\zeta)| \\ &\leq C_1 n! |z|^n \|f\|_{\mathcal{F}_{\alpha, \omega}^p} \max_{|\zeta| \leq |z| + |z|^{-1}} \frac{e^{\frac{\alpha}{2}|\zeta|^2}}{\omega(D(\zeta, t_1))^{\frac{1}{p}}} \\ &\leq C_1 C_2 n! e^{2\alpha} (1 + |z|)^n \|f\|_{\mathcal{F}_{\alpha, \omega}^p} \frac{e^{\frac{\alpha}{2}|z|^2}}{\omega(D(z, t_2))^{\frac{1}{p}}}. \end{aligned}$$

The case $n = 0$ follows from [3, Corollary 3.2], which ends this proof. \square

Lemma 8. *Let $0 < p, t < \infty$, $\alpha > 0$, $\omega \in A_\infty^{\text{restricted}}$ and let n be a positive integer. Then there exists a constant $C = C(\alpha, p, \omega, t)$ such that*

$$\left| f^{(n)}(z) e^{-\frac{\alpha}{2}|z|^2} \right| \leq Cn! \frac{(1 + |z|)^n}{\omega(D(z, t))^{\frac{1}{p}}} \left(\int_{D(z, t)} |f(\xi) e^{-\frac{\alpha}{2}|\xi|^2}|^p \omega(\xi) dA(\xi) \right)^{\frac{1}{p}}$$

for any $z \in \mathbb{C}$ and $f \in \mathcal{H}(\mathbb{C})$.

Proof. Let first $|z| \leq 1$. By the joining of Lemma 6 and the proof of Lemma 7, we have

$$\begin{aligned} \left| f^{(n)}(z) e^{-\frac{\alpha}{2}|z|^2} \right| &\leq |f^{(n)}(z)| \leq n!(1+|z|)^n \max_{|z-\zeta|=1} |f(\zeta)| \\ &\leq C_1 n! (1+|z|)^n \max_{|\zeta| \leq 2} \left(\omega(D(\zeta, r_0))^{-1} \int_{D(\zeta, r_0)} |f(u) e^{-\frac{\alpha}{2}|u|^2}|^p \omega(u) dA(u) \right)^{\frac{1}{p}} \\ &\leq C_1 C_2 n! (1+|z|)^n \omega(D(z, 3))^{-\frac{1}{p}} \left(\int_{D(z, 3)} |f(u) e^{-\frac{\alpha}{2}|u|^2}|^p \omega(u) dA(u) \right)^{\frac{1}{p}}. \end{aligned}$$

If $|z| > 1$, then we obtain

$$\begin{aligned} \left| f^{(n)}(z) e^{-\frac{\alpha}{2}|z|^2} \right| &\leq n! |z|^n e^{-\frac{\alpha|z|^2}{2}} \max_{|z-\zeta| \leq |z|-1} |f(\zeta)| \\ &\leq C_1 n! |z|^n e^{-\frac{\alpha|z|^2}{2}} \max_{|z-\zeta| \leq |z|-1} \frac{e^{\frac{\alpha}{2}|\zeta|^2}}{\omega(D(\zeta, r_1))^{\frac{1}{p}}} \left(\int_{D(\zeta, r_1)} |f(u) e^{-\frac{\alpha}{2}|u|^2}|^p \omega(u) dA(u) \right)^{\frac{1}{p}} \\ &\lesssim n! \frac{(1+|z|)^n}{\omega(D(z, r_1))^{\frac{1}{p}}} \max_{|z-\zeta| \leq |z|-1} \left(\int_{D(\zeta, r_1)} |f(u) e^{-\frac{\alpha}{2}|u|^2}|^p \omega(u) dA(u) \right)^{\frac{1}{p}} \\ &\lesssim n! \frac{(1+|z|)^n}{\omega(D(z, r_2))^{\frac{1}{p}}} \left(\int_{D(z, r_1+1)} |f(u) e^{-\frac{\alpha}{2}|u|^2}|^p \omega(u) dA(u) \right)^{\frac{1}{p}}. \end{aligned}$$

This completes the proof. \square

The following lemma, see [1, Lemma 4.10] or [14, Lemma 3.7], plays a key role to study compactness of linear operators.

Lemma 9. *Let X and Y be two Banach (or quasi-Banach) spaces of entire functions on \mathbb{C} , and let $T : X \rightarrow Y$ be a linear operator. Suppose that the following conditions are satisfied:*

- (i) *The point evaluation functionals on Y are bounded.*
- (ii) *The closed unit ball of X is a compact subset of $\mathcal{H}(\mathbb{C})$, where $\mathcal{H}(\mathbb{C})$ is endowed with the topology of uniform convergence on compacta.*
- (iii) *$T : X \rightarrow Y$ is continuous, where both X and Y are endowed with the topology of uniform convergence on compacta.*

Then $T : X \rightarrow Y$ is a compact operator if and only if for any bounded sequence $\{f_j\}$ in X such that $f_j \rightarrow 0$ uniformly on compact subset of \mathbb{C} as $j \rightarrow \infty$, the sequence $\{Tf_j\}$ converges to zero in the norm of Y as $j \rightarrow \infty$.

3. PROOFS OF THEOREMS 1 AND 2

In this section, we will provide the boundedness and compactness of generalised weighted composition operator $D_{\varphi, u}^n$ between different spaces. At first, we consider the compactness of differentiation operator $D^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow L^q(\mu)$ with $0 < q < p < \infty$. This result is a supplement of [3, Theorem 1.2].

Theorem 10. Let $\alpha \in (0, \infty)$, $\omega \in A_\infty^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$ and let μ be a positive Borel measure on \mathbb{C} . For $0 < q < p < \infty$, the following conditions are equivalent:

- (i) $D^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow L^q(\mu)$ is bounded;
- (ii) $D^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow L^q(\mu)$ is compact;
- (iii) The function

$$\frac{1}{\omega_{np}(D(u, 1))} \int_{D(u, 1)} e^{\frac{q\alpha|z|^2}{2}} d\mu(z)$$

is in $L^{\frac{p}{p-q}}(\mathbb{C}, \omega_{np})$.

Proof. It is clear that (ii) \Rightarrow (i), and the proof of (i) \Rightarrow (iii) follows by [3, Theorem 1.2]. Next, we show (iii) implies (ii). It is enough to give that for any bounded sequences $\{f_k\} \in \mathcal{F}_{\alpha, \omega}^p$ which tends to zero uniformly on compact subsets of \mathbb{C} as $k \rightarrow \infty$, we deduce $\|D^n f_k\|_{L^q(\mu)}^q \rightarrow 0$ as $k \rightarrow \infty$. Set $\|f_k\|_{\mathcal{F}_{\alpha, \omega}^p} \leq M < \infty$. By Lemma 8 with $t = 1$, we have

$$\left| f^{(n)}(z) \right|^q \leq C(n!)^q \frac{(1 + |z|)^n e^{\frac{q\alpha}{2}|z|^2}}{\omega(D(z, 1))} \int_{D(z, 1)} |f(\xi) e^{-\frac{\alpha}{2}|\xi|^2}|^q \omega(\xi) dA(\xi).$$

Note that $1 + |\xi| \asymp 1 + |z|$ for $\xi \in D(z, 1)$, and then by Fubini's theorem and Lemma 4, we get

$$\|D^n f_k\|_{L^q(\mu)}^q \lesssim \int_{\mathbb{C}} |f_k(\xi) e^{-\frac{\alpha}{2}|\xi|^2}|^q \frac{(1 + |\xi|)^{nq}}{\omega(D(\xi, 1))} \int_{D(\xi, 1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z) \omega(\xi) dA(\xi).$$

For any $\epsilon > 0$. Duo to the hypothesis, there is an $R > 0$ such that

$$\int_{\mathbb{C} \setminus \overline{D(0, R)}} \left(\frac{(1 + |\xi|)^{nq}}{\omega(D(\xi, 1))} \int_{D(\xi, 1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z) \right)^{p/(p-q)} \omega(\xi) dA(\xi) \leq \epsilon^{p/(p-q)}.$$

Since $\{f_k\}$ uniformly converge on compact subsets of \mathbb{C} as $k \rightarrow \infty$, we may choose $k_0 \in \mathbb{N}$ such that $|f_k(\xi)|^q < \epsilon$ whenever $k \geq k_0$ and $\xi \in \overline{D(0, R)}$. Therefore,

$$\begin{aligned} \|D^n f_k\|_{L^q(d\mu)}^q &\lesssim \left(\int_{\overline{D(0, R)}} + \int_{\mathbb{C} \setminus \overline{D(0, R)}} \right) |f_k(\xi) e^{-\frac{\alpha}{2}|\xi|^2}|^q \\ &\quad \cdot \frac{(1 + |\xi|)^{nq}}{\omega(D(\xi, 1))} \int_{D(\xi, 1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z) \omega(\xi) dA(\xi) \\ &\lesssim \sup_{\xi \in \overline{D(0, R)}} \frac{(1 + |\xi|)^{nq}}{\omega(D(\xi, 1))} \int_{D(\xi, 1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z) \int_{\overline{D(0, R)}} |f_k(\xi)|^q \omega(\xi) dA(\xi) \\ &\quad + \left(\int_{\mathbb{C}} |f_k(\xi)|^p \omega(\xi) dA(\xi) \right)^{q/p} \\ &\quad \cdot \left(\int_{\mathbb{C} \setminus \overline{D(0, R)}} \left(\frac{(1 + |\xi|)^{nq}}{\omega(D(\xi, 1))} \int_{D(\xi, 1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z) \right)^{\frac{p}{p-q}} \omega(\xi) dA(\xi) \right)^{\frac{p-q}{p}} \\ &\lesssim \epsilon + \epsilon \|f_k\|_{\mathcal{F}_{\alpha, \omega}^p} \leq (1 + M)\epsilon. \end{aligned}$$

And hence, $D^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow L^q(\mu)$ is compact, which finishes the proof. \square

With the help of the above conclusion, we can now provide the proof of Theorem 1.

Proof of Theorem 1. Notice that the non-univalent change of variables formula indicates $\|D_{\varphi, u}^n f\|_{L^q_\nu} = \|f^{(n)}\|_{L^q_{\varphi_* (|u|^{q\nu})}}$. This together with Theorem 10 ends the proof. \square

The following theorem concerns the compactness of differentiation operator $D^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow L^q(\mu)$ with $0 < p \leq q < \infty$.

Theorem 11. *Let $\alpha \in (0, \infty)$, $\omega \in A_\infty^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$ and let μ be a positive Borel measure on \mathbb{C} . For $0 < p \leq q < \infty$, the following conditions are equivalent:*

- (i) $D^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow L^q(\mu)$ is compact;
- (ii)

$$\lim_{|a| \rightarrow \infty} \frac{1}{(\omega_{np}(D(a, 1)))^{\frac{q}{p}}} \int_{D(a, 1)} e^{\frac{q\alpha|z|^2}{2}} d\mu(z) = 0.$$

Proof. We begin with the case $n = 0$. Assume that $I_d : \mathcal{F}_{\alpha, \omega}^p \rightarrow L^q(\mu)$ is compact. Let

$$(2) \quad f_a(z) = \omega(D(a, 1))^{-1/p} e^{\alpha \bar{a}z - \frac{\alpha}{2}|a|^2}, \quad z \in \mathbb{C}.$$

It follows from [4, Lemma 2.3] that $\|f_a\|_{\mathcal{F}_{\alpha, \omega}^p} \asymp 1$ and f_a tends to zero uniformly on compact subsets of \mathbb{C} as $|a|$ goes to ∞ . Hence the closure of the set $\{f_a : a \in \mathbb{C}\}$ is compact in $L^q(\mu)$. For any $\epsilon > 0$, the open balls $B(f_a, \epsilon) = \{f \in L^q(\mu) : \|f_a - f\|_{L^q(\mu)} < \epsilon\}$ cover the closure of the set $\{f_a : a \in \mathbb{C}\}$. Then there is a finite subcover $\{B(f_{a_n}, \epsilon) : n = 1, \dots, N = N(\epsilon)\}$. For each $a \in \mathbb{C}$, let $j = j(a) \in \{1, \dots, N\}$ such that $f_a \in B(f_{a_j}, \epsilon)$. Therefore, for each $R > 0$,

$$\begin{aligned} & \int_{\mathbb{C} \setminus D(0, R)} |f_a(z)|^q d\mu(z) \\ & \lesssim \int_{\mathbb{C} \setminus D(0, R)} |f_a(z) - f_{a_j}(z)|^q d\mu(z) + \int_{\mathbb{C} \setminus D(0, R)} |f_{a_j}(z)|^q d\mu(z) \\ & \leq \|f_a - f_{a_j}\|_{L^q(\mu)}^q + \max_{n=1, \dots, N} \int_{\mathbb{C} \setminus D(0, R)} |f_{a_n}(z)|^q d\mu(z), \end{aligned}$$

and so

$$\lim_{R \rightarrow \infty} \int_{\mathbb{C} \setminus D(0, R)} |f_a(z)|^q d\mu(z) = 0$$

uniformly in a . Combining with the uniform convergence, we deduce

$$\begin{aligned} 0 &= \lim_{|a| \rightarrow \infty} \|f_a\|_{L^q(\mu)}^q \geq \lim_{|a| \rightarrow \infty} \int_{D(a,1)} |f_a(z)|^q d\mu(z) \\ &= \lim_{|a| \rightarrow \infty} \omega(D(a,1))^{-\frac{q}{p}} \int_{D(a,1)} e^{\frac{q\alpha}{2}|z|^2 - \frac{q\alpha}{2}|z-a|^2} d\mu(z) \\ &\gtrsim \lim_{|a| \rightarrow \infty} \omega(D(a,1))^{-\frac{q}{p}} \int_{D(a,1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z). \end{aligned}$$

Conversely, assume (ii) holds, let $\{f_k\}_k \in \mathcal{F}_{\alpha,\omega}^p$ be a sequence such that $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{F}_{\alpha,\omega}^p} = M < \infty$. Then we conclude from [3, Corollary 3.2] that $\{f_k\}_k$ is uniformly bounded on compact subsets of \mathbb{C} . With the help of Montel's theorem and Weierstrass' theorem, there is a subsequence $\{f_{k_j}\}_j$ converges uniformly on compact subsets of \mathbb{C} to a function f which is in $\mathcal{H}(\mathbb{C})$. And thus $f \in \mathcal{F}_{\alpha,\omega}^p$ by Fatou's lemma. Set $|a_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then the hypothesis gives that, for any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$\frac{1}{(\omega(D(a_n,1)))^{\frac{q}{p}}} \int_{D(a_n,1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z) < \epsilon.$$

This together with (1), Lemma 6 and Minkowski's inequality in continuous form (Fubini's theorem in the case $p = q$) shows

$$\begin{aligned} &\sum_{n=N}^{\infty} \int_{D(a_n,1)} |f(z) - f_{k_j}(z)|^q d\mu(z) \\ &\lesssim \sum_{n=N}^{\infty} \int_{D(a_n,1)} \frac{e^{\frac{q\alpha}{2}|z|^2}}{\omega(D(z,1))^{\frac{q}{p}}} \left(\int_{D(z,1)} |f(u) - f_{k_j}(u)|^p e^{-\frac{p\alpha}{2}|u|^2} \omega(u) dA(u) \right)^{\frac{q}{p}} d\mu(z) \\ &\leq \sum_{n=N}^{\infty} \left[\int_{\{u: D(z,1) \cap D(a_n,1) \neq \emptyset\}} |f(u) - f_{k_j}(u)|^p e^{-\frac{p\alpha}{2}|u|^2} \left(\frac{\int_{D(a_n,1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z)}{\omega(D(z,1))^{\frac{q}{p}}} \right)^{\frac{p}{q}} \omega(u) dA(u) \right]^{\frac{q}{p}} \\ &\lesssim \epsilon \sum_{n=N}^{\infty} \left(\int_{\{u: D(z,1) \cap D(a_n,1) \neq \emptyset\}} |f(u) - f_{k_j}(u)|^p e^{-\frac{p\alpha}{2}|u|^2} \omega(u) dA(u) \right)^{\frac{q}{p}} \\ &\lesssim \epsilon \|f - f_{k_j}\|_{\mathcal{F}_{\alpha,\omega}^p}^q \leq \epsilon M^q. \end{aligned}$$

Therefore,

$$\begin{aligned} &\limsup_{j \rightarrow \infty} \int_{\mathbb{C}} |f(z) - f_{k_j}(z)|^q d\mu(z) \\ &\leq \limsup_{j \rightarrow \infty} \left(\sum_{n=1}^{N-1} \int_{D(a_n,1)} |f(z) - f_{k_j}(z)|^q d\mu(z) + \sum_{n=N}^{\infty} \int_{D(a_n,1)} |f(z) - f_{k_j}(z)|^q d\mu(z) \right) \lesssim \epsilon, \end{aligned}$$

because of

$$\lim_{j \rightarrow \infty} \sum_{n=1}^{N-1} \int_{D(a_n,1)} |f(z) - f_{k_j}(z)|^q d\mu(z) = 0$$

by the uniform convergence in compact subsets. And hence, for any $\epsilon > 0$, we get

$$\limsup_{j \rightarrow \infty} \int_{\mathbb{C}} |f(z) - f_{k_j}(z)|^q d\mu(z) = 0,$$

which concludes $I_d : \mathcal{F}_{\alpha, \omega}^p \rightarrow L^q(\mu)$ is compact. Next, by [3, Lemma 2.1], we have

$$(3) \quad \omega \in A_{\infty}^{restricted} \Leftrightarrow \omega_{\gamma}(z) = \frac{\omega(z)}{(1 + |z|)^{\gamma}} \in A_{\infty}^{restricted}.$$

Now, combining Lemma 9, the proof for case $n \in \mathbb{N}$ is an analogue of above, we omit the details here, and the desired consequences are complete. \square

We use $\mu_{\varphi}^{\beta, \nu}(|u|^q)$ to denote the positive measure on \mathbb{C} defined by

$$\mu_{\varphi}^{\beta, \nu}(|u|^q)(E) = \int_{\varphi^{-1}(E)} |u(z)|^q e^{-\frac{q\beta}{2}|z|^2} \nu(z) dA(z)$$

for every Borel subset E of \mathbb{C} . Next, we study the boundedness and compactness of $D_{\varphi, u}^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow \mathcal{F}_{\beta, \nu}^q$ with $0 < p \leq q < \infty$. For this goal, we will divide Theorem 2 into two theorems as follows.

Theorem 12. *Suppose that $0 < p \leq q < \infty$, $\alpha, \beta > 0$, $\omega \in A_{\infty}^{restricted}$, $n \in \mathbb{N} \cup \{0\}$ and ν is a weight. Let φ be an analytic self-map of \mathbb{C} and $u \in \mathcal{F}_{\beta, \nu}^q$. Then $D_{\varphi, u}^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow \mathcal{F}_{\beta, \nu}^q$ is bounded if and only if*

$$(4) \quad \sup_{a \in \mathbb{C}} \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha \Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) < \infty.$$

Proof. Assume that $D_{\varphi, u}^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow \mathcal{F}_{\beta, \nu}^q$ is bounded. It follows that

$$\begin{aligned} \|D_{\varphi, u}^n f_a\|_{\mathcal{F}_{\beta, \nu}^q}^q &= (\alpha|a|)^{qn} \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega(D(a, 1))^{q/p}} e^{q\alpha \Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) \\ &\asymp \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha \Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) \leq C \|f_a\|_{\mathcal{F}_{\alpha, \omega}^p}^q < \infty, \end{aligned}$$

which concludes (4).

Conversely, suppose that (4) holds. Thus,

$$\begin{aligned} \infty &> \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha \Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) \\ &\geq \int_{\varphi^{-1}(D(a, 1))} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha \Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) \\ &= \int_{D(a, 1)} \omega_{np}(D(a, 1))^{-q/p} e^{\frac{q\alpha}{2}|z|^2 - \frac{q\alpha}{2}|z-a|^2} d\mu_{\varphi}^{\beta, \nu}(|u|^q)(z) \\ &\gtrsim \frac{1}{\omega_{np}(D(a, 1))^{q/p}} \int_{D(a, 1)} e^{\frac{q\alpha|z|^2}{2}} d\mu_{\varphi}^{\beta, \nu}(|u|^q)(z). \end{aligned}$$

Using Theorem [3, Theorem 1.2], we deduce $\|f^{(n)}\|_{L^q_{\mu_{\varphi}^{\beta,\nu}(|u|^q)}} \lesssim \|f\|_{\mathcal{F}_{\alpha,\omega}^p}$ with $f \in \mathcal{F}_{\alpha,\omega}^p$, and hence $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow \mathcal{F}_{\beta,\nu}^q$ is bounded since $\|D_{\varphi,u}^n f\|_{\mathcal{F}_{\beta,\nu}^q} = \|f^{(n)}\|_{L^q_{\mu_{\varphi}^{\beta,\nu}(|u|^q)}}$. This finishes the proof. \square

Theorem 13. *Suppose that $0 < p \leq q < \infty$, $\alpha, \beta > 0$, $\omega \in A_{\infty}^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$ and ν is a weight. Let φ be an analytic self-map of \mathbb{C} and $u \in \mathcal{F}_{\beta,\nu}^q$. Then $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow \mathcal{F}_{\beta,\nu}^q$ is compact if and only if*

$$(5) \quad \lim_{|a| \rightarrow \infty} \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha\Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) = 0.$$

Proof. Assume that $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow \mathcal{F}_{\beta,\nu}^q$ is compact. Set the function f_a as in (2). Then we obtain $\|f_a\|_{\mathcal{F}_{\alpha,\omega}^p} \asymp 1$ and f_a tends to zero uniformly on compact subsets of \mathbb{C} as $|a|$ goes to ∞ . And thus, $\|D_{\varphi,u}^n f_a\|_{\mathcal{F}_{\beta,\nu}^q} \rightarrow 0$ when $|a| \rightarrow \infty$. It follows that

$$\begin{aligned} & \lim_{|a| \rightarrow \infty} \|D_{\varphi,u}^n f_a\|_{\mathcal{F}_{\beta,\nu}^q}^q \\ &= \lim_{|a| \rightarrow \infty} (\alpha|a|)^{qn} \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega(D(a, 1))^{q/p}} e^{q\alpha\Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) \\ &\asymp \lim_{|a| \rightarrow \infty} \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha\Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) = 0, \end{aligned}$$

which indicates (5).

Conversely, suppose that (5) holds. By the joining of Theorem 11 and the proof of Theorem 12, it is enough to show

$$\lim_{|a| \rightarrow \infty} \frac{1}{\omega_{np}(D(a, 1))^{q/p}} \int_{D(a, 1)} e^{\frac{q\alpha|z|^2}{2}} d\mu_{\varphi}^{\beta,\nu}(|u|^q)(z) = 0.$$

Since

$$\begin{aligned} 0 &= \lim_{|a| \rightarrow \infty} \int_{\mathbb{C}} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha\Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) \\ &\geq \lim_{|a| \rightarrow \infty} \int_{\varphi^{-1}(D(a, 1))} \frac{|u(z)|^q}{\omega_{np}(D(a, 1))^{q/p}} e^{q\alpha\Re(\bar{a}\varphi(z)) - \frac{q\alpha}{2}|a|^2 - \frac{q\beta}{2}|z|^2} \nu(z) dA(z) \\ &= \lim_{|a| \rightarrow \infty} \int_{D(a, 1)} \omega_{np}(D(a, 1))^{-q/p} e^{\frac{q\alpha}{2}|z|^2 - \frac{q\alpha}{2}|z-a|^2} d\mu_{\varphi}^{\beta,\nu}(|u|^q)(z) \\ &\gtrsim \lim_{|a| \rightarrow \infty} \frac{1}{\omega_{np}(D(a, 1))^{q/p}} \int_{D(a, 1)} e^{\frac{q\alpha|z|^2}{2}} d\mu_{\varphi}^{\beta,\nu}(|u|^q)(z), \end{aligned}$$

we deduce that $D_{\varphi,u}^n : \mathcal{F}_{\alpha,\omega}^p \rightarrow \mathcal{F}_{\beta,\nu}^q$ is compact, and the proof is complete. \square

Suppose $0 < p, q < \infty$. We call μ a (p, q) -Carleson measure if the embedding operator $i : \mathcal{F}_{\alpha,\omega}^p \rightarrow L^q(\mu)$ is bounded, that is, there exists some

constant C such that for $f \in \mathcal{F}_{\alpha, \omega}^p$,

$$\left(\int_{\mathbb{C}} |f(z)|^q d\mu(z) \right)^{\frac{1}{q}} \leq C \|f\|_{\mathcal{F}_{\alpha, \omega}^p}.$$

For a positive Borel measure μ on \mathbb{C} and $0 < p, \alpha < \infty$, let $\tilde{\mu}_{\alpha, q}$ be the function

$$\tilde{\mu}_{\alpha, q}(z) = \int_{\mathbb{C}} e^{q\alpha \Re(\bar{z}u) - \frac{q\alpha}{2}|z|^2} d\mu(u), \quad z \in \mathbb{C}.$$

The following theorem establishes a new characterization about boundedness of the embedding operator with $0 < p \leq q < \infty$.

Theorem 14. *Let $\alpha \in (0, \infty)$, $\omega \in A_{\infty}^{\text{restricted}}$ and let μ be a positive Borel measure on \mathbb{C} . For $0 < p \leq q < \infty$, the following conditions are equivalent:*

- (i) μ is a (p, q) -Carleson measure;
- (ii) $\Psi_{\mu}(a) = \frac{\tilde{\mu}_{\alpha, q}(a)}{\omega(D(a, 1))^{\frac{q}{p}}} \in L^{\infty}$, $a \in \mathbb{C}$.

Proof. Let $0 < p \leq q < \infty$. Then

$$(6) \quad \int_{D(a, 1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z) \lesssim (\omega(D(a, 1)))^{\frac{q}{p}}, \quad a \in \mathbb{C},$$

if and only if μ is a (p, q) -Carleson measure by [3, Theorem 4.3].

For any $a \in \mathbb{C}$, it is easy to see that

$$\tilde{\mu}_{\alpha, q}(a) = \int_{\mathbb{C}} e^{\frac{q\alpha}{2}|z|^2 - \frac{q\alpha}{2}|a-z|^2} d\mu(z) \gtrsim \int_{D(a, 1)} e^{\frac{q\alpha}{2}|z|^2} d\mu(z).$$

This gives (6) if $\Psi_{\mu}(a) \in L^{\infty}$ for $a \in \mathbb{C}$, and thus μ is a (p, q) -Carleson measure.

Let $K_a(z) = e^{\alpha \bar{a}z}$. Assume that μ is a (p, q) -Carleson measure. By [3, Theorem 4.3] again, we have $\|K_a\|_{L^q(\mu)} \lesssim \|K_a\|_{\mathcal{F}_{\alpha, \omega}^p}$. For any $a \in \mathbb{C}$, it follows from [3, Proposition 4.1] that

$$e^{-\frac{q\alpha}{2}|a|^2} \int_{\mathbb{C}} e^{q\alpha \Re(\bar{a}z)} d\mu(z) = \int_{\mathbb{C}} e^{\frac{q\alpha}{2}|z|^2 - \frac{q\alpha}{2}|z-a|^2} d\mu(z) \lesssim (\omega(D(a, 1)))^{\frac{q}{p}},$$

which indicates $\Psi_{\mu}(a) \in L^{\infty}$ with $a \in \mathbb{C}$. This finishes the proof. \square

4. PROOF OF THEOREM 3

Following the ideas in [8, 13] with some modifications, we will offer the proof of Theorem 3 in this section.

Recall first that the essential norms for bounded linear operators. Let X and Y be Banach spaces, and let $\mathcal{K}(X, Y)$ be the set of all compact operators from X into Y . The essential norm of a bounded linear operator $L : X \rightarrow Y$ is denoted by $\|L\|_e$, where

$$\|L\|_e = \inf\{\|L - T\| : T \in \mathcal{K}(X, Y)\}.$$

It is clear that L is compact if and only if $\|L\|_e = 0$. The following auxiliary conclusion will be used later.

Lemma 15. ([10, Lemma 2.1]) *Let f and φ be two entire functions on \mathbb{C} such that $f \neq 0$. Suppose that there is a positive constant C such that*

$$|f(z)|^2 e^{|\varphi(z)|^2 - |z|^2} \leq C$$

for all $z \in \mathbb{C}$. Then $\varphi(z) = \varphi(0) + \lambda z$ for some $|\lambda| \leq 1$. If $|\lambda| = 1$, then $f(z) = f(0)e^{-\bar{\beta}z}$, where $\beta = \bar{\lambda}\varphi(0)$. Furthermore, if

$$\lim_{|z| \rightarrow \infty} |f(z)|^2 e^{|\varphi(z)|^2 - |z|^2} = 0,$$

then $\varphi(z) = \lambda z + b$ with $|\lambda| < 1$.

Before embarking on the proof of Theorem 3, we deduce some necessary conditions as follows.

Theorem 16. *Suppose that $0 < p, q < \infty$, $\alpha > 0$, $\omega \in A_\infty^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$, and φ is an analytic self-map of \mathbb{C} . Let u be a nonzero entire function on \mathbb{C} . If $D_{\varphi, u}^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow \mathcal{F}_{\alpha, \omega}^q$ is bounded, then $u \in \mathcal{F}_{\alpha, \omega}^q$. In this case, $\varphi(z) = az + b$ with $|a| < 1$ and*

$$|u(z)| |\varphi(z)|^n e^{\frac{3\alpha|\varphi(z)|^2}{4} - \frac{3\alpha|z|^2}{4}} \lesssim \|D_{\varphi, u}^n f \varphi(z)\|_{\mathcal{F}_{\alpha, \omega}^q} \leq \|D_{\varphi, u}^n\|$$

for any $z \in \mathbb{C}$.

Proof. If $f(z) = z^n$, then we have $u \in \mathcal{F}_{\alpha, \omega}^q$ by the boundedness of $D_{\varphi, u}^n$. Setting f_a as in (2). For any $a, z \in \mathbb{C}$, we conclude from Lemma 4 that there are constants $C_1, C_2 > 0$ such that

$$(7) \quad \omega(D(z, 1)) \geq C_1^{-|z|} \omega(D(0, 1)) \quad \text{and} \quad \omega(D(a, 1)) \leq C_2^{|a|} \omega(D(0, 1)).$$

This together with Lemma 7 yields

$$\begin{aligned} \|D_{\varphi, u}^n f_a\|_{\mathcal{F}_{\alpha, \omega}^q} &\gtrsim |D_{\varphi, u}^n f_a(z)| \omega(D(z, 1))^{\frac{1}{q}} e^{-\frac{\alpha|z|^2}{2}} \\ &= |u(z)| \frac{\omega(D(z, 1))^{\frac{1}{q}}}{\omega(D(a, 1))^{\frac{1}{p}}} (\alpha|a|)^n e^{\alpha\Re(\bar{a}\varphi(z)) - \frac{\alpha|a|^2}{2} - \frac{\alpha|z|^2}{2}} \\ &\geq C_1^{-\frac{|z|}{q}} C_2^{-\frac{|a|}{p}} |u(z)| \omega(D(0, 1))^{\frac{1}{q} - \frac{1}{p}} (\alpha|a|)^n e^{\alpha\Re(\bar{a}\varphi(z)) - \frac{\alpha|a|^2}{2} - \frac{\alpha|z|^2}{2}}. \end{aligned}$$

Setting $a = \varphi(z)$, we deduce

$$C_1^{-\frac{|z|}{q}} C_2^{-\frac{|\varphi(z)|}{p}} |u(z)| |\varphi(z)|^n e^{\frac{\alpha|\varphi(z)|^2}{2} - \frac{\alpha|z|^2}{2}} \lesssim \|D_{\varphi, u}^n f \varphi(z)\|_{\mathcal{F}_{\alpha, \omega}^q} \leq \|D_{\varphi, u}^n\|.$$

Note that

$$C_1^{-\frac{|z|}{q}} C_2^{-\frac{|\varphi(z)|}{p}} e^{\frac{\alpha|\varphi(z)|^2}{2} - \frac{\alpha|z|^2}{2}} \geq C_1^{-\frac{|z|}{q}} e^{-\frac{\alpha|z|^2}{4}} C_2^{-\frac{|\varphi(z)|}{p}} e^{-\frac{\alpha|\varphi(z)|^2}{4}} e^{\frac{3\alpha|\varphi(z)|^2}{4} - \frac{3\alpha|z|^2}{4}},$$

it follows that

$$|u(z)| |\varphi(z)|^n e^{\frac{3\alpha|\varphi(z)|^2}{4} - \frac{3\alpha|z|^2}{4}} \lesssim \|D_{\varphi, u}^n f \varphi(z)\|_{\mathcal{F}_{\alpha, \omega}^q} \leq \|D_{\varphi, u}^n\|.$$

Hence, by Lemma 15 we get $\varphi(z) = az + b$ with $|a| \leq 1$. However, if $|a| = 1$, then Liouville's theorem and Lemma 15 indicate $u(z)\varphi(z)^n \equiv 0$, and hence $u(z) = 0$ since $\varphi(z) \neq 0$. The proof of this theorem is complete. \square

Corollary 17. *Suppose that $0 < p, q < \infty$, $\alpha > 0$, $\omega \in A_\infty^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$, and u is a nonzero analytic function in $\mathcal{F}_{\alpha, \omega}^q$. If $\varphi(z) = b$, then $D_{\varphi, u}^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow \mathcal{F}_{\alpha, \omega}^q$ is compact, and there are constants $C_1, C_2 > 0$ such that*

$$C_1^{-\frac{|b|}{p}} |b|^n e^{\frac{\alpha|b|^2}{2}} \|u\|_{\mathcal{F}_{\alpha, \omega}^q} \lesssim \|D_{\varphi, u}^n\| \lesssim C_2^{\frac{|b|}{p}} (1 + |b|)^n e^{\frac{\alpha|b|^2}{2}} \|u\|_{\mathcal{F}_{\alpha, \omega}^q}.$$

Proof. For $g \in \mathcal{F}_{\alpha, \omega}^p$, it follows from Lemma 7 and (7) that

$$\|D_{\varphi, u}^n g\|_{\mathcal{F}_{\alpha, \omega}^q} \lesssim C_2^{\frac{|b|}{p}} (1 + |b|)^n e^{\frac{\alpha|b|^2}{2}} \|g\|_{\mathcal{F}_{\alpha, \omega}^p} \|u\|_{\mathcal{F}_{\alpha, \omega}^q}.$$

Letting f_b as in (2). Using (7) again, we obtain

$$\|D_{\varphi, u}^n f_b\|_{\mathcal{F}_{\alpha, \omega}^q} \gtrsim C_1^{-\frac{|b|}{p}} |b|^n e^{\frac{\alpha|b|^2}{2}} \|u\|_{\mathcal{F}_{\alpha, \omega}^q}.$$

Furthermore, $D_{\varphi, u}^n$ is a finite rank operator with rank 1, and so it is compact. This finishes the proof. \square

For entire functions u and φ on \mathbb{C} , we denote

$$m_{z, n}(u, \varphi) = |u(z)| |\varphi(z)|^n e^{\frac{|\varphi(z)|^2 - |z|^2}{2}}, \quad z \in \mathbb{C}.$$

In view of Theorem 16 and Corollary 17, we study the essential norm of $D_{\varphi, u}^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow \mathcal{F}_{\alpha, \omega}^q$ when $\varphi(z) = az + b$ with $0 < |a| < 1$.

Theorem 18. *Suppose that $1 < p \leq q < \infty$, $\alpha > 0$, $\omega \in A_\infty^{\text{restricted}}$, $n \in \mathbb{N} \cup \{0\}$. Let $u \in \mathcal{F}_{\alpha, \omega}^q$ be a nonzero function and $\varphi(z) = az + b$ with $0 < |a| < 1$. If $D_{\varphi, u}^n : \mathcal{F}_{\alpha, \omega}^p \rightarrow \mathcal{F}_{\alpha, \omega}^q$ is bounded, then*

$$\limsup_{|z| \rightarrow \infty} m_{z, n}(u, \varphi) \lesssim \|D_{\varphi, u}^n\|_e.$$

Proof. It follows from Theorem 16 that $\limsup_{|z| \rightarrow \infty} m_{z, n}(u, \varphi) < \infty$, and there is a constant $C_0 > 0$ so that $m_{z, n}(u, \varphi) \leq C_0 \|D_{\varphi, u}^n f_{\varphi(z)}\|_{\mathcal{F}_{\alpha, \omega}^q}$. Assume that

$$C_0 \|D_{\varphi, u}^n\|_e < \limsup_{|z| \rightarrow \infty} m_{z, n}(u, \varphi).$$

Then there exist positive constants C_1, C_2 and a compact operators $T : \mathcal{F}_{\alpha, \omega}^p \rightarrow \mathcal{F}_{\alpha, \omega}^q$ such that

$$C_0 \|D_{\varphi, u}^n - T\| < C_1 < C_2 < \limsup_{|z| \rightarrow \infty} m_{z, n}(u, \varphi).$$

In fact, there is a sequence $\{z_j\}$, satisfying $|z_j| \rightarrow \infty$ whenever $j \rightarrow \infty$, such that

$$(8) \quad \limsup_{j \rightarrow \infty} m_{z_j, n}(u, \varphi) = \limsup_{|z| \rightarrow \infty} m_{z, n}(u, \varphi) > C_2.$$

It follows that

$$(9) \quad \begin{aligned} \|D_{\varphi, u}^n - T\| &\geq \|D_{\varphi, u}^n f_{\varphi(z_j)} - T f_{\varphi(z_j)}\|_{\mathcal{F}_{\alpha, \omega}^q} \geq \|D_{\varphi, u}^n f_{\varphi(z_j)}\|_{\mathcal{F}_{\alpha, \omega}^q} - \|T f_{\varphi(z_j)}\|_{\mathcal{F}_{\alpha, \omega}^q} \\ &\geq C_0^{-1} m_{z_j, n}(u, \varphi) - \|T f_{\varphi(z_j)}\|_{\mathcal{F}_{\alpha, \omega}^q}. \end{aligned}$$

Notice that $|\varphi(z_j)| \rightarrow \infty$ as $j \rightarrow \infty$. Then by the similar reason as in [13, Lemmas 2.3 and 2.4], we deduce that $\|Tf_{\varphi(z_j)}\|_{\mathcal{F}_{\alpha,\omega}^q} \rightarrow 0$ when $j \rightarrow \infty$. Combining (8) with (9), for sufficiently large j we get

$$C_1 > C_0 \|D_{\varphi,u}^n - T\| \geq m_{z_j,n}(u, \varphi) > C_2,$$

which is a contradiction, and hence

$$\limsup_{|z| \rightarrow \infty} m_{z,n}(u, \varphi) \lesssim \|D_{\varphi,u}^n\|_e.$$

This completes the proof. \square

Combining Theorem 18 with Littlewood-Paley formulas for weighted Fock spaces $\mathcal{F}_{\alpha,\omega}^p$ (see [3, Theorem 1.1]), we can now provide the proof of Theorem 3. Clearly, only the upper estimate for essential norm of the operator $D_{\varphi,u}^n$ on weighted Fock spaces $\mathcal{F}_{\alpha,\omega}^p$ needs to be proven.

Proof of Theorem 3. Let k be a positive integer, and $V_{r_k}f(z) = f(r_k z)$ with $r_k = \frac{k}{k+1}$. Then V_{r_k} is a compact operator on $\mathcal{F}_{\alpha,\omega}^p$.

Let I denote the identity operator on $\mathcal{F}_{\alpha,\omega}^p$. Taking a number $R > 0$. For each $k \in \mathbb{N}$, we get

$$\begin{aligned} \|D_{\varphi,u}^n\|_e &\leq \|D_{\varphi,u}^n - D_{\varphi,u}^n \circ V_{r_k}\| = \sup_{\|f\|_{\mathcal{F}_{\alpha,\omega}^p} \leq 1} \|D_{\varphi,u}^n \circ (I - V_{r_k})f\| \\ &\leq \sup_{\|f\|_{\mathcal{F}_{\alpha,\omega}^p} \leq 1} \left(\int_{|z| \leq R} |D_{\varphi,u}^n \circ (I - V_{r_k})f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} \omega(z) dA(z) \right)^{\frac{1}{p}} \\ &\quad + \sup_{\|f\|_{\mathcal{F}_{\alpha,\omega}^p} \leq 1} \left(\int_{|z| > R} |D_{\varphi,u}^n \circ (I - V_{r_k})f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} \omega(z) dA(z) \right)^{\frac{1}{p}} \\ &:= J_{1,k} + J_{2,k}. \end{aligned}$$

On one hand, it follows from [3, Theorem 1.1] that

$$\begin{aligned} J_{2,k} &\lesssim \left(\sup_{|z| > R} \frac{m_{z,n}(u, \varphi)\omega(z)}{\omega(\varphi(z))} \right) \\ &\quad \cdot \sup_{\|f\|_{\mathcal{F}_{\alpha,\omega}^p} \leq 1} \left(\int_{|z| > R} \frac{|(((I - V_{r_k})f)^{(n)} \circ \varphi)(z)|^p e^{-\frac{p\alpha}{2}|\varphi(z)|^2}}{(1 + |\varphi(z)|)^{pn}} \omega(\varphi(z)) dA(z) \right)^{\frac{1}{p}} \\ &\lesssim |a|^{-\frac{2}{p}} \left(\sup_{|z| > R} \frac{m_{z,n}(u, \varphi)\omega(z)}{\omega(\varphi(z))} \right) \sup_{\|f\|_{\mathcal{F}_{\alpha,\omega}^p} \leq 1} \|(I - V_{r_k})f\| \\ &\lesssim |a|^{-\frac{2}{p}} (1 + \|V_{r_k}\|) \left(\sup_{|z| > R} \frac{m_{z,n}(u, \varphi)\omega(z)}{\omega(\varphi(z))} \right) \lesssim |a|^{-\frac{2}{p}} \sup_{|z| > R} \frac{m_{z,n}(u, \varphi)\omega(z)}{\omega(\varphi(z))}. \end{aligned}$$

On the other hand, for any $\delta \in (0, \alpha)$, we conclude from [3, Proposition 3.3] that

$$\mathcal{F}_{\alpha,\omega}^p \subset \mathcal{F}_{\alpha+\delta}^\infty.$$

This together with Lemma 5 yields

$$\begin{aligned}
J_{1,k} &\leq \left(\sup_{|z| \leq R} |u(z)| \right) \sup_{\|f\|_{\mathcal{F}_{\alpha,\omega}^p} \leq 1} \left(\int_{|z| \leq R} |(I - V_{r_k})f^{(n)}(\varphi(z))|^p e^{-\frac{p\alpha}{2}|z|^2} \omega(z) dA(z) \right)^{\frac{1}{p}} \\
&\leq \left(\sup_{|z| \leq R} |u(z)| \right) \sup_{\|f\|_{\mathcal{F}_{\alpha,\omega}^p} \leq 1} \sup_{|z| \leq R} |(I - V_{r_k})f^{(n)}(\varphi(z))| \left(\int_{|z| \leq R} e^{-\frac{p\alpha}{2}|z|^2} \omega(z) dA(z) \right)^{\frac{1}{p}} \\
&\leq C \left(\sup_{|z| \leq R} |u(z)| \right) \sup_{\|f\|_{\mathcal{F}_{\alpha+\delta}^\infty} \leq 1} \sup_{|z| \leq R} |(I - V_{r_k})f^{(n)}(\varphi(z))|.
\end{aligned}$$

For each $f(z) = \sum_{j=0}^{\infty} a_j z^j$ with $\|f\|_{\mathcal{F}_{\alpha+\delta}^\infty} \leq 1$. By an elementary computation, we obtain

$$|a_j| \leq \left(\frac{e(\alpha + \delta)}{j} \right)^{\frac{j}{2}}, \quad j \geq 1.$$

It follows that

$$\begin{aligned}
J_{1,k} &\lesssim \left(\sup_{|z| \leq R} |u(z)| \right) \sup_{\|f\|_{\mathcal{F}_{\alpha+\delta}^\infty} \leq 1} \sup_{|z| \leq R_\varphi} |(I - V_{r_k})f^{(n)}(z)| \\
&\leq \left(\sup_{|z| \leq R} |u(z)| \right) \\
&\quad \cdot \sup_{\|f\|_{\mathcal{F}_{\alpha+\delta}^\infty} \leq 1} \sup_{|z| \leq R_\varphi} \sum_{j=n+1}^{\infty} |a_j| j(j-1) \cdots (j-n+1) \left(1 - \left(\frac{k}{k+1} \right)^{j-n} \right) |z|^{j-n} \\
&\leq \left(\sup_{|z| \leq R} |u(z)| \right) \sup_{\|f\|_{\mathcal{F}_{\alpha+\delta}^\infty} \leq 1} \sup_{|z| \leq R_\varphi} \sum_{i=1}^{\infty} |a_{i+n}| \frac{(i+n)!}{i!} \left(1 - \left(\frac{k}{k+1} \right)^i \right) |z|^i \\
&\leq \frac{1}{k+1} \left(\sup_{|z| \leq R} |u(z)| \right) \sum_{i=1}^{\infty} \frac{(i+n)!}{(i-1)!} R_\varphi^i \left(\frac{e(\alpha + \delta)}{i+n} \right)^{\frac{i+n}{2}},
\end{aligned}$$

where $R_\varphi = \max_{|z| \leq R} |\varphi(z)|$, and the series in the last term is convergent. Hence,

$$\begin{aligned}
\|D_{\varphi,u}^n\|_e &\leq \limsup_{k \rightarrow \infty} \|D_{\varphi,u}^n - D_{\varphi,u}^n \circ V_{r_k}\| \leq \limsup_{k \rightarrow \infty} J_{1,k} + \limsup_{k \rightarrow \infty} J_{2,k} \\
&\lesssim |a|^{-\frac{2}{p}} \sup_{|z| > R} \frac{m_{z,n}(u, \varphi)\omega(z)}{\omega(\varphi(z))}.
\end{aligned}$$

Therefore, the desired consequence holds by letting $R \rightarrow \infty$. This finishes the proof. \square

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