

On numbers of the form $\theta[n^c] + \varphi$ with smooth indices

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Abstract—In this paper, we study two problems about numbers of the form $\theta[n^c] + \varphi$ with smooth indices.

First, let $1 < c < 3/2$ be a fixed real number. Let θ be an irrational number and φ a real number. Let $T > 5$ be a real number. Then there are infinitely many y -smooth numbers n such that

$$\|\theta[n^c] + \varphi\| < n^{-\frac{(3-2c)(1-1/T)+100\varepsilon}{27-1/T}},$$

where $y = (\log n)^T$.

Second, let $1 < c < 2$ be a fixed real number. Let θ be an irrational number and φ a real number. Then for all sufficiently large real x and arbitrary real $R \geq \max\{\frac{8+\tau}{c-\varepsilon}, \frac{9}{2-c-\varepsilon}\} + 0.124820$ we have

$$|\{n \in S(x, y) \mid \|\theta[n^c] + \varphi\| \text{ is an } R\text{-almost prime}\}| \gg \frac{x}{\log^2 x},$$

where $x^\varepsilon < y < x$ and $S(x, y)$ is the set of y -smooth numbers smaller than x .

Keywords: exponential sums, smooth numbers, Diophantine approximation, almost primes

I. INTRODUCTION

Recall that a positive integer n is called y -smooth, if its largest prime factor $P^+(n)$ is at most y . Smooth numbers are widely recognized as playing a pivotal role in analytic number theory, and the study of their distribution constitutes a core focus in relevant research. For illustrative examples pertaining to both smooth numbers and their distribution, interested readers are directed to references [1], [3], [10] and [18]. For $2 \leq y \leq x$, recall the notation

$$S(x, y) := \{n \leq x : P^+(n) \leq y\}, \quad \Psi(x, y) := |S(x, y)|.$$

The size of the parameter y with respect to x is vital in the study of smooth numbers. The smaller the value of y , the sparser the set $S(x, y)$ becomes; consequently, the corresponding problem grows increasingly difficult to tackle. If we write $u = \frac{\log x}{\log y}$ then for fixed u we have $\Psi(x, y) \sim \rho(u)x$ where $\rho(u)$ denotes the Dickman function defined by $\rho(u) = 1$ for $0 \leq u \leq 1$ and for $u \geq 1$ is defined as the unique continuous solution to the differential-difference equation $u\rho'(u) = -\rho(u-1)$. This asymptotic formula was first given by Dickman [5] for fixed values of u and as $x \rightarrow \infty$. When $y = (\log x)^\kappa$ for some fixed $\kappa > 1$, by (1.14) in the paper [10] we have

$$x^{1-1/\kappa-\varepsilon} \leq \Psi(x, (\log x)^\kappa) \leq x^{1-1/\kappa+\varepsilon} \quad (x \rightarrow \infty). \quad (1)$$

Because of this sparsity, many results about smooth numbers from this example above were until recently only known conditional on assumptions such as the GRH.

On the other hand, the following Diophantine inequality

$$\|\theta n + \varphi\| < n^{-\alpha} \quad (2)$$

have been studied by many scholars. Vinogradov [24] first considered this Diophantine approximation in prime variables with $\alpha = 1/5 - \varepsilon$. Later Harman [12] and Matomaki [17] improved the exponent to $\alpha = 7/22$ and $\alpha = 1/3 - \varepsilon$ respectively. Instead of considering primes, Baker [4] and Zhu [25] considered (2) with smooth numbers. Dimitrov [6] also considered the case for Piatetski-Shapiro primes. The following Diophantine inequality

$$\|\alpha p^2 + \beta\| < p^{-\theta} \quad (3)$$

is also studied by many scholars. The first result in this direction is due to Ghosh [8], who obtained $\theta = 1/8 - \varepsilon$, while the best result to date is due to Harman [12], with $\theta = 2/13 - \varepsilon$. Dimitrov and Lazarova [7] also considered (3) with Piatetski-Shapiro primes. Inspired by these, we consider the problem involving Piatetski-Shapiro sequences

with smooth indices, i.e. the sequence of the form $\{[n^c]\}_{n=1}^{\infty}$ with n a smooth number.

Theorem 1. *Let $1 < c < 3/2$ be a fixed real number, let θ be an irrational number, let φ a real number, and let $T > 5$ be a real number. There are infinitely many $n \in \mathbb{N}$ free of prime factors greater than $(\log n)^T$ such that*

$$\|\theta[n^c] + \varphi\| < n^{-\frac{(3-2c)(1-1/T)}{27-1/T} + 100\varepsilon}. \quad (4)$$

The Piatetski-Shapiro prime is a fundamental object in analytic number theory, named after the Soviet mathematician Israel Moiseevich Piatetski-Shapiro. It refers to primes of the form $[n^c]$, where $c > 1$ is a real. In 1953 Piatetski-Shapiro [19] showed that there are infinitely many Piatetski-Shapiro primes for $c \in (1, 12/11)$. It is worth emphasizing that the range $c \in (1, 12/11)$ was improved by Kolesnik [14], Heath-Brown [13], Kolesnik [15] and Liu and Rivat [16]. The best known result is due to Rivat and Sargos [20], which improved the range for c to $(2426/2817, 1)$. Inspired by the above research, we study the almost prime of the form $[\theta[n^c] + \varphi]$ with smooth indices. Whether there exist primes of the form $[\theta[n^c] + \varphi]$ with smooth indices is still an open problem.

Before stating Theorem 2, we give the definitions of the type of an irrational number and the R -almost primes. **Type of an irrational number**

For an irrational number θ , its type τ is defined as follows

$$\tau = \sup\{\rho \in \mathbb{R} : \liminf_{\substack{n \rightarrow \infty \\ n \in \mathbb{Z}^+}} n^\rho \|n\alpha\| = 0\}, \quad (5)$$

where $\|u\|$ denotes the distance of u from the nearest integer. By Dirichlet's approximation theorem, we have $\tau \geq 1$. If θ is of finite type τ , then θ^{-1} and $k\theta$ are also of finite type τ .

R-almost primes

For every integer $R \geq 1$, we say that a natural number is an R -almost prime if it has at most R prime factors, counted with multiplicity.

Theorem 2. *Let $1 < c < 2$ be a fixed real number. Let θ be an irrational number of finite type τ and φ a real number. Then for all sufficiently large real x and arbitrary real $R \geq \max\{\frac{8+\tau}{c-\varepsilon}, \frac{9}{2-c-\varepsilon}\} + 0.124820$ we have*

$$|\{n \in S(x, y) | [\theta[n^c] + \varphi] \text{ is an } R\text{-almost prime}\}| \gg \frac{x}{\log^2 x}, \quad (6)$$

where $x^\varepsilon < y < x$.

II. SOME NOTATIONS

Throughout this paper, we fix a real number c with $1 < c < 3/2$. Let $S(x, y)$ denote the set of y -smooth numbers which is less than x . And let $\Psi(x, y)$ denote the cardinal of the set $S(x, y)$. Let $S_a(x, y)$ denote the set of integers in $[x, ax)$ free of prime factors $> y$. We define $P^+(n)$ to be the largest prime factor of n . We shall frequently use ε with a abuse of notation to mean a small positive number, possibly a different one each time. Given a real number x , we write $e(x) = e^{2\pi ix}$. We use $[t]$, $\{t\}$ and $\|t\|$ to denote the integral part of t , the fractional part of t and the distance from t to the nearest integer, respectively. The notation $n \sim N$ means $N < n \leq 2N$ and $f(x) \ll g(x)$ means $f(x) = O(g(x))$.

III. SOME LEMMAS

Before proving two theorems, we shall list some lemmas which are necessary for establishing two theorems.

Lemma 1. *Let x_1, \dots, x_N be a real sequence with $\|x_n\| \geq M^{-1}$ ($1 \leq n \leq N$) where M is a positive integer. Then*

$$\sum_{h=1}^M \left| \sum_{n=1}^N e(hx_n) \right| \geq N/6.$$

Proof. For the proof of this lemma, see [2]. □

To treat the floor functions in the exponential sums, we need the following so-called 'Partition of Unity'.

Lemma 2. For $Z \in \mathbb{N}$ and $x \in \mathbb{R}$ there exist periodic functions $g_z(t)$ with period 1 ($0 \leq z \leq 2Z - 1$), such that

$$\begin{aligned} 0 < g_z(t) \leq 1, & \quad \text{if } |t - \frac{z}{2Z}| < \frac{1}{2Z}, \\ g_z(t) = 0, & \quad \text{if } \frac{1}{2Z} \leq |t - \frac{z}{2Z}| \leq \frac{1}{2}, \\ g_z(t) = \sum_{|n| \leq Z(\log x)^4} \beta_n^{(z)} e(nt) & + O(x^{-\log \log x}), \end{aligned}$$

where

$$|\beta_n^{(z)}| \leq \frac{1}{2Z}.$$

We also have

$$\sum_{z=0}^{2Z-1} g_z(t) = 1.$$

Proof. This is Lemma 2 in [21]. □

Lemma 3. Let $a(l)$ be a sequence of complex numbers, and let Q and L be positive reals. Then

$$\left| \sum_{L < l \leq 2L} a(l) \right|^2 \leq \left(1 + \frac{L}{Q}\right) \sum_{0 \leq q \leq Q} \left(1 - \frac{q}{Q}\right) \operatorname{Re} \left(\sum_{L \leq l \leq 2L-q} a(l+q) \overline{a(l)} \right).$$

Proof. This is Lemma 5 in [13]. □

Lemma 4. Let M be a positive real. Suppose that $M < M' < 2M$. Let f be a function with $f''(x)$ is continuous in $[M, M']$. If $\lambda \ll |f''(x)| \ll \lambda$ for $x \in [M, M']$. Then

$$\sum_{M < n < M'} e(f(n)) \ll M\lambda^{1/2} + \lambda^{-1/2}.$$

Proof. For the proof of this lemma, see [9]. □

We need the following approximation due to of Vaaler.

Lemma 5. Let x be a real, let $\psi(x) = x - [x] - 1/2$. Then for any $H \geq 1$ there exist reals a_h and b_h such that

$$\left| \psi(x) - \sum_{0 < |h| \leq H} a_h e(xh) \right| \leq \sum_{|h| \leq H} b_h e(xh),$$

with

$$a_h \ll \frac{1}{|h|}, b_h \ll \frac{1}{H}.$$

Proof. For the proof of this lemma, see [22]. □

In order to state next lemma, we introduce some notation. We say that an N -element set of integers A has a level of distribution D if for a given multiplicative function $f(d)$ we have

$$\sum_{d \leq D} \max_{\gcd(s,d)=1} \left| |\{a \in A, a \equiv s \pmod{d}\}| - \frac{f(d)}{d} N \right| \leq \frac{N}{\log^2 N}.$$

We define

$$\delta_R = 0.124820, \quad R \geq 5.$$

We have the following lemma.

Lemma 6. *Suppose A is an N -element set of positive integers with a level of distribution D and degree β in the sense that $a < D^\beta$ for any $a \in A$ hold with some real number $\beta < R - \delta_R$. Then*

$$|\{a \in A : a \text{ is an } R\text{-almost prime}\}| \gg \frac{N}{\log^2 N}.$$

Proof. For the proof of this lemma, see [11, Chapter 5, Proposition 1]. □

We also need the following lemma to detect the condition $d|[\theta[n^c] + \phi]$, which we call the division trick.

Lemma 7. *Let d and n be two nature numbers, let $\theta > 0$, $c > 0$ and ϕ be reals. Then $d|[\theta[n^c] + \phi]$ if and only if $E_{\theta,c,\phi,d}(n) := \left\lfloor \frac{\theta[n^c] + \phi}{d} \right\rfloor - \left\lfloor \frac{\theta[n^c] + \phi - 1}{d} \right\rfloor = 1$.*

Proof. The proof of this lemma is easy, we left it to readers. □

Lemma 8. *Suppose that $2 \leq y \leq R \leq n \leq x$, with $n \in S(x, y)$. Then there is a unique triple (p, u, v) satisfying*

- (i) $n = puv$,
- (ii) $p \leq y$,
- (iii) $R/p < v \leq R$ with $P^-(v) \geq p$ and $P^+(v) \leq y$,
- (iv) $u \leq x/pv$ with $P^+(u) \leq p$.

Proof. This is Lemma 10.1 in [23]. □

Lemma 9. *Let I be an interval. If f is continuously differentiable on I , f' is monotonic on I , and $\|f'\| \geq \lambda > 0$ on I then*

$$\sum_{n \in I} e(f(n)) \ll \lambda^{-1}.$$

Proof. This is Theorem 2.1 in [9]. □

Lemma 10. *If Z is large enough positive real then*

$$\int_{-1/2}^{1/2} \left| \sum_{r \in I} e(\alpha r) \right| d\alpha \ll \log Z,$$

where I is an interval with $I \subset [Z, 2Z]$.

Proof. We have

$$\begin{aligned} & \int_{-1/2}^{1/2} \left| \sum_{r \in I} e(\alpha r) \right| d\alpha \\ &= \int_{-1/Z}^{1/Z} \left| \sum_{r \in I} e(\alpha r) \right| d\alpha + \int_{1/Z}^{1/2} \left| \sum_{r \in I} e(\alpha r) \right| d\alpha + \int_{-1/2}^{-1/Z} \left| \sum_{r \in I} e(\alpha r) \right| d\alpha \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

By trivial estimation we have

$$I_1 \leq \int_{-1/Z}^{1/Z} Z d\alpha \ll 1. \tag{7}$$

By Lemma 9 we have

$$I_2 \leq \int_{1/Z}^{1/2} \alpha^{-1} d\alpha \ll \log Z. \tag{8}$$

Similarly

$$I_3 \ll \log Z. \quad (9)$$

Combining (7)-(9) we have

$$\int_{-1/2}^{1/2} \left| \sum_{r \in I} e(\alpha r) \right| d\alpha \ll \log Z.$$

This completes the proof of this lemma. \square

IV. PROOF OF THE THEOREM 1

By Dirichlet's approximation theorem, for any irrational number θ , there exists an integer q such that

$$\left| \theta - \frac{a}{q} \right| < \frac{1}{q^2}, \quad (a, q) = 1. \quad (10)$$

We may suppose that ε is sufficiently small, so that

$$\gamma := \frac{(3-2c)(1-1/T)}{27-1/T} - 100\varepsilon$$

is positive.

Define x by

$$x^{\frac{c+\gamma}{2}} = q. \quad (11)$$

Suppose that $n \in \mathbb{N}$ satisfies

$$\left\| \frac{a[n^c]}{q} + \varphi \right\| \leq x^{-\gamma}, \quad n \in S(x, y), \quad y = (\log x)^T. \quad (12)$$

This implies

$$\begin{aligned} \|\theta[n^c] + \varphi\| &\leq x^{-\gamma} + \left| \theta - \frac{a}{q} \right| \cdot x^c \\ &\leq x^{-\gamma} + \frac{x}{x^{1+\gamma}} < n^{-\frac{(3-2c)(1-1/T)}{27-1/T} + 100\varepsilon}, \end{aligned}$$

from (10)–(12). Hence it will suffice to show that (12) has a solution n .

Let $S = S_2(x^{\frac{1+\gamma}{2}}, y)$, $J = S_2(x^{\frac{1-\gamma}{2}}, y)$. By (1.1),

$$x^{\frac{1+\gamma}{2}(1-\frac{1}{T})-\frac{\varepsilon}{16}} \ll |S| \ll x^{\frac{1+\gamma}{2}(1-\frac{1}{T})+\frac{\varepsilon}{16}}, \quad (13)$$

$$x^{\frac{1-\gamma}{2}(1-\frac{1}{T})-\frac{\varepsilon}{16}} \ll |J| \ll x^{\frac{1-\gamma}{2}(1-\frac{1}{T})+\frac{\varepsilon}{16}}, \quad (14)$$

We will show that there are solutions of (12) with

$$n = uv, \quad u \in S, \quad v \in J.$$

Suppose there are no such solutions of (12). Let

$$S_h = \sum_{u \in S} \sum_{v \in J} e\left(\frac{ha[u^c v^c]}{q}\right).$$

Let $H = \lfloor x^\gamma \rfloor + 1$. We deduce from Lemma 1 and (13), (14) that

$$\sum_{h=1}^H |S_h| \gg x^{1-\frac{1}{T}-\frac{\varepsilon}{8}}. \quad (15)$$

Now we estimate the upper bounds of $\sum_{h=1}^H |S_h|$. First we have

$$\sum_{h=1}^H |S_h| \ll x^\gamma \max_{h \leq H} |S_h|. \quad (16)$$

By Lemma 2 with $Z = \lfloor x^{\gamma+\varepsilon} \rfloor + 1$, we have

$$\begin{aligned} S_h &= \sum_{u \leq x^{\frac{1+\gamma}{2}(1-1/T)}} a(u) \sum_{v \leq x^{\frac{1-\gamma}{2}(1-1/T)}} b(v) e(ha[u^c v^c]/q) \sum_{z=0}^{2Z-1} g_z(u^c v^c) \\ &= \sum_{z=0}^{2Z-1} \sum_{u \leq x^{\frac{1+\gamma}{2}(1-1/T)}} a(u) \sum_{v \leq x^{\frac{1-\gamma}{2}(1-1/T)}} b(v) e(ha[u^c v^c]/q) g_z(u^c v^c) \\ &= \sum_{z=0}^{2Z-1} \sum_{u \leq x^{\frac{1+\gamma}{2}(1-1/T)}} a(u) \sum_{v \leq x^{\frac{1-\gamma}{2}(1-1/T)}} b(v) (e(ha(u^c v^c - z/2Z)/q) \\ &\quad + O(H/Z)) g_z(u^c v^c) \\ &= \sum_{z=0}^{2Z-1} \sum_{u \leq x^{\frac{1+\gamma}{2}(1-1/T)}} a(u) \sum_{v \leq x^{\frac{1-\gamma}{2}(1-1/T)}} b(v) e(ha(u^c v^c - z/2Z)/q) \\ &\quad \times g_z(u^c v^c) + O(x^{1-1/T-\varepsilon}), \end{aligned}$$

where $a(u)$ and $b(v)$ are the indicator function of smooth number. By Lemma 2, we have

$$g_z(u^c v^c) = \sum_{|l| \leq Z \log^4 x} \beta_z^{(l)} e(lu^c v^c) + O(x^{-\log \log x}).$$

This implies

$$\begin{aligned} S_h &= \sum_{z=0}^{2Z-1} \sum_{|l| \leq Z \log^4 x} \beta_z^{(l)} \sum_{u \leq x^{\frac{1+\gamma}{2}(1-1/T)}} a(u) \sum_{v \leq x^{\frac{1-\gamma}{2}(1-1/T)}} b(v) e((ha/q + l)u^c v^c) \\ &\quad + O(x^{1-1/T-\varepsilon}) \\ &\ll x^{\gamma+\varepsilon} \left| \sum_{u \leq x^{\frac{1+\gamma}{2}(1-1/T)}} a(u) \sum_{v \leq x^{\frac{1-\gamma}{2}(1-1/T)}} b(v) e((ha/q + l)u^c v^c) \right| \\ &\quad + O(x^{1-1/T-\varepsilon}) \\ &=: x^{\gamma+\varepsilon} S' + O(x^{1-1/T-\varepsilon}). \end{aligned}$$

By Lemma 3 with $Q = \lfloor x^{4\gamma+8\varepsilon} \rfloor + 1$ (note that $Q \ll x^{\frac{1+\gamma}{2}(1-1/T) - \varepsilon}$), we have

$$\begin{aligned} S'^2 &\ll \frac{x^{2(1-1/T)}}{Q} + \frac{x^{1-1/T}}{Q} \sum_{1 \leq s \leq Q} \sum_{u \sim x^{\frac{(1+\gamma)(1-1/T)}{2}}} \\ &\quad \times \left| \sum_{v \sim x^{\frac{(1-\gamma)(1-1/T)}{2}}} e((ha/q + l)sv^{c-1}v^c) \right|. \end{aligned}$$

Note that $|ha/q + l| \geq 1/q$, and apply Lemma 4 to the innermost sum, we see the innermost sum is

$$\ll x^{\frac{1+\gamma}{2}(1-\frac{1}{T})} x^{\frac{\gamma+\varepsilon}{2}} Q^{1/2} u^{\frac{c-1}{2}} v^{\frac{c-2}{2}} + q^{1/2} u^{\frac{1-c}{2}} v^{\frac{2-c}{2}}.$$

It thus follows that

$$S' \ll x^{1-1/t-3\gamma-\varepsilon}.$$

Upon collecting the estimates above, we find that

$$\sum_{n=1}^H S_h \ll x^{1-1/t-\varepsilon}.$$

This leads to a contradiction. This completes the proof of the theorem.

V. PROOF OF THE THEOREM 2

Let $A = \{\lfloor \theta \lfloor n^c \rfloor + \phi \mid n \text{ is a } y\text{-smooth number and } n \leq x\}$. First let D be the level of the distribution of A . By Lemma 5 with $H = D \log^3 x$ and Lemma 7 we have

$$\begin{aligned} & \sum_{d \leq D} \max_{\gcd(s,d)=1} \left| |\{a \in A, a \equiv s \pmod{d}\}| - \frac{\Psi(x,y)}{d} \right| \\ &= \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} \left(\left\lfloor \frac{\theta \lfloor n^c \rfloor + \phi}{d} \right\rfloor - \left\lfloor \frac{\theta \lfloor n^c \rfloor + \phi - 1}{d} \right\rfloor \right) - \frac{\Psi(x,y)}{d} \right| \\ &= \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} \frac{1}{d} + O \left(\sum_{\substack{n \leq x \\ P^+(n) \leq y}} \psi \left(\frac{\theta \lfloor n^c \rfloor + \phi}{d} \right) \right) + O \left(\sum_{\substack{n \leq x \\ P^+(n) \leq y}} \psi \left(\frac{\theta \lfloor n^c \rfloor + \phi - 1}{d} \right) \right) - \frac{\Psi(x,y)}{d} \right| \\ &\ll \sum_{d \leq D} \left(\left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} \psi \left(\frac{\theta \lfloor n^c \rfloor + \phi}{d} \right) \right| + \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} \psi \left(\frac{\theta \lfloor n^c \rfloor + \phi - 1}{d} \right) \right| \right) \\ &\ll \sum_{d \leq D} \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e \left(\frac{\theta h \lfloor n^c \rfloor}{d} \right) \right| + O \left(\frac{\Psi(x,y)}{\log^3 x} \right) \\ &\ll \max_{1 \leq h \leq H} \log x \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e \left(\frac{\theta h \lfloor n^c \rfloor}{d} \right) \right| + O \left(\frac{\Psi(x,y)}{\log^3 x} \right). \end{aligned}$$

By Lemma 2 with $Z = D \log^5 x$ we have

$$\begin{aligned} & \sum_{d \leq D} \max_{\gcd(s,d)=1} \left| |\{a \in A, a \equiv s \pmod{d}\}| - \frac{\Psi(x,y)}{d} \right| \\ &\ll \max_{1 \leq h \leq H} \log x \sum_{z=0}^{2Z-1} \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e \left(\frac{\theta h \lfloor n^c \rfloor}{d} \right) \right| g_z(n^c) + O \left(\frac{\Psi(x,y)}{\log^3 x} \right) \end{aligned}$$

$$\begin{aligned}
&\ll \max_{1 \leq h \leq H} \log x \sum_{z=0}^{2Z-1} \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e\left(\frac{\theta hn^c}{d} - \frac{\theta z}{2dZ}\right) + O\left(\frac{h\theta}{dZ}\right) \right| g_z(n^c) + O\left(\frac{\Psi(x, y)}{\log^3 x}\right) \\
&\ll \max_{1 \leq h \leq H} \log x \sum_{z=0}^{2Z-1} \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e\left(\frac{\theta hn^c}{d}\right) \right| g_z(n^c) \\
&\quad + O\left(\log x \sum_{z=0}^{2Z-1} g_z(n^c) \sum_{d \leq D} \sum_{\substack{n \leq x \\ P^+(n) \leq y}} \frac{h\theta}{dZ}\right) \\
&\quad + O\left(\frac{\Psi(x, y)}{\log^3 x}\right) \\
&\ll \max_{1 \leq h \leq H} \log x \sum_{z=0}^{2Z-1} \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e\left(\frac{\theta hn^c}{d}\right) \right| g_z(n^c) + \frac{\Psi(x, y)}{\log^3 x} \\
&\ll \max_{1 \leq h \leq H} \log x \sum_{z=0}^{2Z-1} \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e\left(\frac{\theta hn^c}{d}\right) g_z(n^c) \right| + \frac{\Psi(x, y)}{\log^3 x} \\
&\ll \max_{1 \leq h \leq H} \log x \sum_{z=0}^{2Z-1} \sum_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e\left(\frac{\theta hn^c}{d}\right) \sum_{|r| \leq Z \log^4 x} \beta_r^{(z)} e(rn^c) \right| + \frac{\Psi(x, y)}{\log^3 x} \\
&\ll \max_{1 \leq h \leq H} \log x \sum_{z=0}^{2Z-1} \sum_{d \leq D} \sum_{|r| \leq Z \log^4 x} \beta_r^{(z)} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e\left(\left(\frac{\theta h}{d} + r\right) n^c\right) \right| + \frac{\Psi(x, y)}{\log^3 x} \\
&\ll \max_{1 \leq h \leq H} ZD \log^5 x \max_{d \leq D} \left| \sum_{\substack{n \leq x \\ P^+(n) \leq y}} e\left(\left(\frac{\theta h}{d} + r\right) n^c\right) \right| + \frac{\Psi(x, y)}{\log^3 x}
\end{aligned}$$

Now we estimate the innermost sum in the last expression. Let $s = \frac{\theta h}{d} + r$, recall that the type of θ is τ , the type of the irrational θ/d is also τ , then by the definition of the type of the irrational number, we have $\|\theta h/d\| > Ch^{-\tau-\varepsilon}$, where C is a constant. By Lemma 8 we have

$$\sum_{\substack{n \sim M \\ P^+(m) \leq y}} e(sn^c) = \sum_{p \leq y} \sum_{\substack{M_1/p < u \leq M_1 \\ P^-(u) \geq p \\ P^+(u) \leq y}} \sum_{\substack{v \sim M/(up) \\ P^+(v) \leq p}} e(s(puv)^c),$$

where $1 \leq M \leq x$ and $M_1 < M$ to be determine later.

Next, we divide the ranges of p, v and u into $\log^3 N$ dyadic intervals to get

$$\begin{aligned}
& \sum_{p \leq y} \sum_{\substack{M_1/p < u \leq M_1 \\ P^-(u) \geq p \\ P^+(u) \leq y}} \sum_{\substack{v \sim M/(up) \\ P^+(v) \leq p}} e(s(puv)^c) \\
& \ll \sum_{p \leq y} \sum_{\substack{M_1/p < u \leq M_1 \\ P^-(u) \geq p \\ P^+(u) \leq y}} \left| \sum_{\substack{v \sim M/(up) \\ P^+(v) \leq p}} e(s(puv)^c) \right| \\
& \ll \sum_{p \leq y} \sum_{M_1/p < u \leq M_1} \left| \sum_{\substack{v \sim M/(up) \\ P^+(v) \leq p}} e(s(puv)^c) \right| \\
& = \sum_{p \leq y} \sum_{M_1/p < u \leq M_1} \sum_{\substack{v \sim M/(up) \\ P^+(v) \leq p}} b(u, p) e(s(puv)^c) \\
& \leq \sum_{\substack{2 \leq P \leq y \\ P=2^k}} \sum_{\substack{M_1/2P \leq L \leq M_1 \\ L=2^r}} \sum_{\substack{M/4PL \leq K \leq 2M/PL \\ K=2^j}} |S'(P, K, L)|,
\end{aligned}$$

where

$$S' := S'(P, K, L) = \sum_{v \sim K} \sum_{\substack{p \sim P \\ P^+(v) \leq p}} \sum_{\substack{u \sim L \\ uv \sim M}} b(u, p) e(sp^c u^c v^c), \quad (17)$$

and $b(u, p) \ll 1$ be a complex number.

Here we note that

$$\begin{aligned}
S' &= \int_{-1/2}^{1/2} \sum_{v \sim K} \sum_{\substack{p \sim P \\ P^+(v) \leq p}} \sum_{u \sim L} e(\alpha up) b(u, p) e(sp^c u^c v^c) \sum_{f \sim M/v} e(-\alpha f) d\alpha \\
&= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v \sim K} \sum_{f \sim M/v} e(-\alpha f) \sum_{p \sim P} \sum_{u \sim L} e(\alpha up - \gamma p) b(u, p) e(sp^c u^c v^c) \sum_{\substack{w \sim P \\ w \geq P^+(v)}} e(\gamma w) d\alpha d\gamma \\
&= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v \sim K} \sum_{f \sim M/v} e(-\alpha f) \sum_{\substack{w \sim P \\ w \geq P^+(v)}} e(\gamma w) \sum_{p \sim P} \sum_{u \sim L} e(\alpha up - \gamma p) b(u, p) e(sp^c u^c v^c) d\alpha d\gamma.
\end{aligned}$$

So we get

$$|S'| = \left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v \sim K} a(v; \alpha, \gamma) \sum_{u \sim L} \sum_{p \sim P} b(u, p; \alpha, \gamma) e(sp^c u^c v^c) d\alpha d\gamma \right|,$$

where $a(v; \alpha, \gamma) = \sum_{f \sim M/v} e(-\alpha f) \sum_{\substack{w \sim P \\ w \geq P^+(v)}} e(\gamma w)$ and $b(u, p; \alpha, \gamma) = e(\alpha up - \gamma p)b(u, p)$.

Let $r = up$. Then we have

$$S' \ll \left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v \sim K} a(v; \alpha, \gamma) \sum_{r \sim PL} c(r; \alpha, \gamma) e(sr^c v^c) da d\gamma \right|,$$

where $c(r) = \tau(r)b(u, p; \alpha, \gamma)$.

If $(\alpha, \gamma) \in \Omega$, $\max_{v \sim K} |a(v; \alpha, \gamma)| \neq 0$ and $\max_{r \sim PL} |c(r; \alpha, \gamma)| \neq 0$, let

$$T_0(\alpha, \gamma) = \sum_{v \sim K} \frac{a(v; \alpha, \gamma)}{\max_{v \sim K} |a(v; \alpha, \gamma)|} \sum_{r \sim PL} \frac{c(r; \alpha, \gamma)}{\max_{r \sim PL} |c(r; \alpha, \gamma)|} e(sr^c v^c).$$

So we have

$$\begin{aligned} S' &\ll \left| \int \int_{(\alpha, \gamma) \in \Omega} \max_{v \sim K} |a(v; \alpha, \gamma)| \max_{r \sim PL} |c(r; \alpha, \gamma)| |T_0(\alpha, \gamma)| da d\gamma \right| \\ &= \int \int_{(\alpha, \gamma) \in \Omega} \max_{v \sim K} |a(v; \alpha, \gamma)| \max_{r \sim PL} |c(r; \alpha, \gamma)| f(h) T_0(\alpha, \gamma) da d\gamma, \end{aligned} \quad (18)$$

where $|f(h)| = 1$.

Now we estimate $T_0(\alpha, \gamma)$. By Lemma 3 with $Q = D^4 x^{2\varepsilon}$ we have

$$T_0(\alpha, \gamma)^2 \ll \frac{M^2}{Q} + \frac{M}{Q} \left(\sum_{q \leq Q} \sum_{v \sim K} \left| \sum_{r \sim PL} e(sr^c((v+q)^c - v^c)) \right| \right).$$

Choosing $K = D^4 x^{3\varepsilon}$ and $M_1 = \frac{M}{D^4 x^{3\varepsilon}}$. Utilizing Lemma 4 to the innermost sum in the last expression we get

$$T_0(\alpha, \gamma) \ll D^{-2} x^{1-\varepsilon} + D^{-2} x^{1/2-\varepsilon} (D^{4+\tau/2} x^{1-c/2+\varepsilon} + D^{9/2} x^{c/2})^{1/2}.$$

By formula (18) and Lemma 10 we have

$$S' \ll D^{-2} x^{1-\varepsilon} + D^{-2} x^{1/2-\varepsilon} (D^{4+\tau/2} x^{1-c/2+\varepsilon} + D^{9/2} x^{c/2})^{1/2}.$$

So we have

$$\sum_{\substack{n \sim M \\ P^+(n) \leq y}} e(sn^c) \ll D^{-2} x^{1-\varepsilon} + D^{-2} x^{1/2-\varepsilon} (D^{4+\tau/2} x^{1-c/2+\varepsilon} + D^{9/2} x^{c/2})^{1/2}.$$

Since

$$\begin{aligned} &\sum_{\substack{n \leq x \\ P^+(n) \leq y}} e\left(\left(\frac{\theta h}{d} + r\right) n^c\right) \\ &\ll \log x \max_{1 \leq M \leq x} \sum_{\substack{n \sim M \\ P^+(n) \leq y}} e(sn^c) \\ &\ll D^{-2} x^{1-\varepsilon} + D^{-2} x^{1/2-\varepsilon} (D^{4+\tau/2} x^{1-c/2+\varepsilon} + D^{9/2} x^{c/2})^{1/2}. \end{aligned}$$

So we have

$$\begin{aligned} & \sum_{d \leq D} \max_{\gcd(s,d)=1} \left| |\{a \in A, a \equiv s \pmod{d}\}| - \frac{x}{d} \right| \\ & \ll x^{1-\varepsilon} + x^{1/2-\varepsilon} (D^{4+\tau/2} x^{1-c/2+\varepsilon} + D^{9/2} x^{c/2})^{1/2} + \frac{\Psi(x, y)}{\log^3 x} \\ & \ll D^{2+\tau/4} x^{1-c/4} + D^{9/4} x^{1/2+c/4} + \frac{\Psi(x, y)}{\log^3 x}. \end{aligned}$$

So by the definition of the level of the distribution of the set A we have

$$D \leq \min\{x^{\frac{c-\varepsilon}{8+\tau}}, x^{\frac{2-c-\varepsilon}{9}}\}.$$

So by Lemma 6 when

$$R > \max\left\{\frac{8+\tau}{c-\varepsilon}, \frac{9}{2-c-\varepsilon}\right\} + \delta_R = \max\left\{\frac{8+\tau}{c-\varepsilon}, \frac{9}{2-c-\varepsilon}\right\} + 0.124820.$$

we have

$$|\{n \in S(x, y) | [\theta[n^c] + \varphi] \text{ is an } R\text{-almost prime}\}| \gg \frac{x}{\log^2 x},$$

where $x^\varepsilon < y < x$.

This completes the proof of this theorem.

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CONFLICT OF INTEREST

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