Robust Mittag-Leffler Stability of Caputo Fractional Differential Equations with State-dependent Delay and Perturbations

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Abstract

In this paper, we investigate the locally robust Mittag-Leffler stability for a class of Caputo fractional-order differential equations subject to state-dependent delays and perturbations. By constructing an appropriate Lyapunov functional and employing key lemmas in fractional calculus, we derive a novel stability criterion under a weakened linear matrix inequality (LMI) condition. This criterion guarantees not only the uniqueness of the equilibrium point but also its locally robust stability against perturbations. Finally, two numerical examples are provided to validate the correctness and effectiveness of the main theoretical results.

Keywords: state-dependent delays, parameter uncertainties, Lyapunov functional, LMI

MSC Classification: 34K34 , 93D09 , 26A33

1 Introduction

The integer-order differential equations have been extended to fractional-order differential equations. Fractional calculus has become a powerful tool for characterizing complex systems with memory and hereditary properties K. Diethelm, N. J. Ford. (2002); V. Lakshmikantham. (2008). Fractional differential equations with delay combine the characteristics of fractional calculus with delay differential equations, and can simultaneously describe the dependence of the system on historical states and the influence of time lags M. L. Morgado, N. J. Ford, P. M. Lima. (2013). Thus, they have been widely applied in modeling complex dynamic systems such as population biology J. M. Cushing. (2013), physiology H. Joshi, B. K. Jha. (2020), control theory V. Badri, M. S. Tavazoei. (2019); A. Narang, S. L. Shah, T. Chen. (2011), population dynamics P. Song, H. Zhao, X. Zhang. (2016), and economics Z. Lin, H. Wang. (2021). As the application of fractional differential equations with delay in practical problems has become increasingly widespread, the analysis of their stability, as a prerequisite for the reliable operation and control design of systems, has received extensive attention from the academic community over the past two decades and has become one of the core directions in the theoretical research of fractional-order systems H. Delavari, D. Baleanu, J. Sadati. (2012); Z. Wang, D. Yang, H. Zhang. (2016); Z. Wang, D. Yang, T. Ma, et al. (2014); Asma, J. F. Gmez−Aguilar, G. ur Rahman, et al. (2022). Stability, as a key indicator for measuring the dynamic behavior of a system, determines whether the system can maintain the expected performance under external disturbances or initial deviations. Therefore, relevant research not only has theoretical significance but also provides important guiding value for the design and optimization of practical systems.

Among the existing research on the stability of fractional-order systems with delay, systems involving time delays and external disturbances have become a research hotspot in recent years due to their closer proximity to engineering practice. For example Zhang F. Zhang, C. Li. (2011) studied the stability of linear fractional differential systems and their corresponding perturbed systems with respect to the Riemann-Liouville derivative and the Caputo derivative. In 2022, He B. B. He, H. C. Zhou, C. H. Kou. (2022) investigated the stability of Hadamard-type equations, the Caputo-Hadamard fractional derivative, and their related systems in both delay-free and delay-involved cases. Wang Y. Wang, T. Li. (2014) conducted a stability analysis of fractional-order nonlinear systems with time delays and proposed the definition of Mittag-Leffler stability for time-delay systems for the first time. Tuan H. T. Tuan, H. Trinh. (2018) proved the linearized asymptotic stability of nonlinear fractional differential equations with time delays. From a practical perspective, time delays arise from the physical structure of the system or information processing procedures and are almost inevitable in fields such as mechanical control, communication networks, and biological metabolism. The existence of time delays may cause the system to oscillate or even become unstable A. K. Agrawal, Y. Fujino, B. K. Bhartia. (1993); H. Logemann, R. Rebarber. (1996).

External disturbances usually manifest as parameter uncertainties, random noise, or sudden interferences, and these disturbances can also disrupt the steady state of the system and reduce control precision Z. Zhang, L. Liu, J. Fang, et al. (2023); B. H.

Nguyen, T. D. Dang, I. B. Furtat, et al. (2025). Li and Wang B. Li, Z. Wang, L. Q. Han (2017) studied a class of system models that simultaneously consider the effects of state delay and bounded external disturbances, and comprehensively evaluated the dynamic performance of the system under the combined action of external disturbances and event-triggered mechanisms.

The coexistence of time delays and disturbances poses enormous challenges to system stability analysis. In the face of these challenges, robustness research has become the key to breakthroughs P. Li, L. Chen, R. Wu, et al. (2018); B. Zhu, J. Ma, Z. Zhang, et al. (2018); Z. Liao, C. Peng, W. Li, et al. (2011); J. G. Lu, Y. Q. Chen. (2009); L. Chen, R. Wu, Y. He, et al. (2015); J. G. Lu, Z. Zhu, Y. D. Ma. (2021). Regarding the robustness of fractional-order systems, there have been many important explorations: for example, Li P. Li, L. Chen, R. Wu, et al. (2018) studied fractional-order interval nonlinear time-delay systems, expanded the Lyapunov direct method by using new lemmas, and constructed delay-independent stability criteria in the form of linear matrix inequalities by Razumikhin theorem. Zhu B. Zhu, J. Ma, Z. Zhang, et al. (2018) focused on the results of uncertain impulsive positive systems with time delays, utilized the impulsive time-varying copositive Lyapunov function, established globally robust exponential stability criteria under different dwell times, and designed controllers. Liao Z. Liao, C. Peng, W. Li, et al. (2011) constructed an asymptotic stability test criterion for parameter-uncertain FO-LTI (Fractional-Order Linear Time-Invariant) interval systems with deterministic linear coupling relations through the linear matrix inequality (LMI) method. These achievements have enriched the robustness theory of fractional-order systems and laid an important foundation for dealing with complex uncertainties in practical systems.

The remainder of this paper is organized as follows. In Section 2, we list essential preliminaries, including definitions of fractional calculus, key assumptions, and necessary lemmas. The main results on robust Mittag-Leffler stability are presented in Section 3. In Section 4, we provide two numerical examples to verify the theoretical findings. Finally, In Section 5, we conclude the paper and suggests potential future research directions.

2 Preliminaries

The stability analysis of fractional-order systems with state-dependent delays and perturbations requires a foundational understanding of fractional calculus, Mittag-Leffler functions, and robust control theory. In this section, we introduce the necessary mathematical tools, including definitions of Caputo derivatives, Mittag-Leffler stability, and key assumptions on system dynamics. Additionally, critical lemmas for bounding fractional Lyapunov derivatives and handling perturbations are presented to support the proof of Theorem 3.1 in Section 3.

Next we provide the definitions of the Caputo fractional derivative and Mittag-Leffler stability.

Definition 1. *(I. Podlubny. (1998)) The Caputo fractional derivative of order* $\alpha \in (0,1)$ *for a sufficiently smooth function* $x(t) \in C^1([t_0, +\infty), \mathbb{R}^n)$ *is defined as*

$$^C D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\dot{x}(s)}{(t-s)^\alpha} ds,$$

where $\dot{x}(s) = \frac{dx(s)}{ds}$, $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2. *(Y. Wang, T. Li. (2014)) A fractional-order system* ${}_{t_0}^{C} D_t^\alpha x(t) = f(t, x_t)$ *is said to be Mittag-Leffler stable if there exist* $\lambda \geq 0$ *and* $b > 0$ *such that*

$$\|x(t, t_0, \varphi)\| \leq \{m(\varphi) E_\alpha (-\lambda (t-t_0)^\alpha)\}^b,$$

where $\alpha \in (0,1)$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, *and* $m(x)$ *is locally Lipschitz on* $x \in \mathcal{E}$ *with the Lipschitz constant* m_0.

In this paper, we focus on a class of fractional-order dynamical systems with state-dependent time delays and perturbations, which is formulated as (1).

$$\begin{cases} ^C D_t^\alpha x(t) = f\big(t, x(t), x\big(t - \tau(x(t))\big), \Delta A(t), \Delta f(t)\big), & t \geq 0 \\ x(t) = \phi(t), & t \in [-\tau_{\max}, 0] \end{cases} \tag{1}$$

where $^C D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\dot{x}(s)}{(t-s)^\alpha} ds$, the state-dependent time delay $\tau(x(t)) \in [0, \tau_{\max}]$ satisfies $\dot{\tau}(x(t)) \leq \eta < 1$. Besides, the perturbation bounds satisfy $\|\Delta A(t)\| \leq \rho$, and $\|\Delta f(t)\| \leq d$. This system incorporates several key characteristics that are prevalent in many real-world applications, such as biological networks, chemical processes, and control systems with memory effects.

In order to obtain our results, we make some assumptions

(**H**1) The delay $\tau(x(t))$ satisfies $0 \leq \tau(x(t)) \leq \tau_{\max}$, and $\dot{\tau}(x(t)) \leq \eta < 1$, where $\tau_{\max}, \eta$ are known constants. It avoids the disorder of the system's memory effect caused by the rapid fluctuation of time delay with the state, which is a necessary condition for the existence of a unique stable solution in fractional-order systems with state-dependent delays. If $\eta \geq 1$, the rapid fluctuation of time delay with the state will violate the causality of the system, leading to oscillations or multiple solutions of the solution. This constraint is also consistent with the classical assumptions in the field of fractional-order time-delay systems.

(**H**2) The perturbations $\Delta A(t)$ and $\Delta f(t)$ are bounded such that

$$\|\Delta A(t)\| \leq \rho, \quad \|\Delta f(t)\| \leq d,$$

where $\rho, d > 0$ are known constants. If the perturbations are unbounded, the system state will diverge directly due to the accumulation of uncertainties, making stability analysis meaningless. Meanwhile, the boundedness conditions provide a quantitative basis for bounding the perturbation terms using Young's inequality and matrix norm inequalities in subsequent derivations.

(**H**3) The function $f\big(t, x(t), x\big(t-\tau(x(t))\big), \Delta A(t), \Delta f(t)\big)$ in system (1) is of linear form with respect to the state variables x and $x(t-\tau(x(t)))$, i.e.,

$$f\big(t, x(t), x\big(t-\tau(x(t))\big), \Delta A(t), \Delta f(t)\big) = \big(A + \Delta A(t)\big)x(t) + Bx\big(t-\tau(x(t))\big) + \Delta f(t),$$

where $A, B \in \mathbb{R}^{n\times n}$ are constant matrices, and $\Delta A(t), \Delta f(t)$ satisfy the boundedness conditions in (H2).

(**H**4) When the initial condition satisfies $\|x(t_0)\| \leq R_0$ $(R_0 \leq R)$, the state trajectory of the system remains within the neighborhood $\Omega = \{x \in \mathbb{R}^n \mid \|x\| \leq R\}$ at all times, where $R > 0$ is a given constant.

Regarding the well-posedness of system (1), we emphasize that under assumptions (H1)-(H3) and the continuity of the nonlinear function $f(t, x(t), x(t-\tau(x(t))), \Delta A(t), \Delta f(t))$ with respect to all its arguments, the system admits a unique solution on $[0, +\infty)$ by the existence and uniqueness theorem for fractional differential equations with state-dependent delays. Furthermore, as rigorously proved in Vainikko G. Vainikko. (2016), for the Caputo fractional derivative of order $\alpha \in (0,1)$ adopted in this paper, the solution possesses a specific structure: it is continuous on $[0, +\infty)$, its Caputo derivative exists almost everywhere, and the growth behavior of the solution is dominated by the t^α term (directly related to the initial value $x(0)$). This structural characteristic ensures that the subsequent stability analysis based on Lyapunov-Krasovskii functionals has a rigorous theoretical foundation.

Definition 3. *(I. Stamova. (2014)) A fractional-order system with bounded perturbations is said to be robustly Mittag-Leffler stable if, for all disturbances satisfying $\|\Delta f(t)\| \leq d$ and $\|\Delta A(t)\| \leq \rho$, its state satisfies*

$$\|x(t)\| \leq M\|x(0)\|E_\alpha(-\lambda t^\alpha) + Nd,$$

where $M, N > 0$ are constants independent of the initial conditions and disturbances.

Next we provide some necessary lemmas to facilitate the establishment of our main results.

Lemma 1. *(M. A. Duarte-Mermoud, N. Aguila-Camacho, J. A. Gallegos, et al. (2015)) For a Caputo fractional derivative of order $\alpha \in (0,1)$, consider a continuously differentiable function $x(t) \in \mathbb{R}^n$ and a positive definite matrix $P \in \mathbb{R}^{n\times n}$. The Caputo fractional derivative of the quadratic Lyapunov candidate $V(t) = x^\top(t)Px(t)$ satisfies*

$${}^C D_t^\alpha V(t) \leq 2x^\top(t)P\,{}^C D_t^\alpha x(t).$$

Lemma 2. *(P. P. Khargonekar, I. R. Petersen, K. Zhou. (2002)) Let X and Y be matrices of appropriate dimensions. Then, the following inequality holds, for all $\varepsilon > 0$*

$$X^\top Y + Y^\top X \leqslant \varepsilon X^\top X + \varepsilon^{-1} Y^\top Y.$$

Lemma 3. *(H. Ye, J. Gao, Y. Ding. (2007)) Let $u(t)$ be a continuous function that satisfies, for $t \geq t_0$ and $0 < \alpha < 1$, the following inequality*

$$u(t) \leq a + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}[bu(s) + c]\, ds,$$

where a, b, c are non-negative constants with $b > 0$. Then, the solution of the above inequality is bounded by

$$u(t) \leq aE_\alpha\left(-b(t-t_0)^\alpha\right) + \frac{c}{b}\left(1 - E_\alpha\left(-b(t-t_0)^\alpha\right)\right).$$

Next we introduce a special symbol that appears in the proof of a theorem.

$M \prec 0$ means that all eigenvalues of M are strictly less than zero. This notation is widely used in the analysis of linear matrix inequalities (LMI) to characterize the negative definiteness of matrices, which plays a crucial role in the stability analysis of dynamical systems.

3 Main Result

In this section, through a refined Lyapunov functional approach, we rigorously prove that (1) admits a unique equilibrium point exhibiting local robust Mittag-Leffler stability under this weakened constraint.

Theorem 3.1 Consider system (1) under assumptions (H1)-(H3). If there exist positive definite matrices P, Q and parameters $\varepsilon, \sigma, \nu, \lambda, \mu > 0$ such that the following linear matrix inequalities hold

Current state LMI

$$2PA + (\varepsilon+\sigma)P^\top P + (\varepsilon^{-1}\rho^2 + \nu)I + Q + \lambda I \prec 0,$$

Delayed state LMI

$$\nu^{-1}B^\top P^\top PB - (1-\eta)Q + \mu I \prec 0,$$

then (1) is local robust Mittag-Leffler stable, and the state satisfies

$$\|x(t)\| \leq C_0 E_\alpha(-m(t-t_0)^\alpha) + C_1\left(1 - E_\alpha(-m(t-t_0)^\alpha)\right),$$

where $m = \min\left\{\frac{\lambda}{\lambda_{\max}(P)}, \frac{\mu}{\lambda_{\max}(Q)}\right\}$, $C_0 = \sqrt{\frac{\lambda_{\max}(P)+\lambda_{\max}(Q)}{\lambda_{\min}(P)}}\|x(t_0)\|$,$C_1 = \sqrt{\frac{M}{m\lambda_{\min}(P)}}$, $M = \sigma^{-1}d^2 + \|\Delta A(t)\|^2 + \|\Delta f(t)\|^2$.

Proof. Considering the state-dependent time delay $\tau(x(t))$ which makes the system's memory effect related to state evolution, and the external disturbance $\Delta f(t)$ that brings uncertain perturbations, we construct the Lyapunov function $V(t)$ as follows.

The state quadratic term

$$V_1(t) = x^\top(t)Px(t),$$

directly reflects the energy of the current system state.

The time-delay integral term

$$V_2(t) = \int_{t-\tau(x(t))}^{t} x^\top(s)Qx(s)ds,$$

by integrating the state energy over the time-delay interval, it captures the cumulative influence of the time-delay history on the current state; since the time delay is state-dependent, the integration interval varies dynamically with the state.

The disturbance norm integral term

$$V_3(t) = \int_{t_0}^{t} \left(\|\Delta A(s)\|^2 + \|\Delta f(s)\|^2\right) ds,$$

for bounded disturbances, by summing the norms of disturbance-related terms, the uncertainties are transformed into quantifiable accumulated energy of disturbances, thereby enabling a comprehensive analysis of system stability with these factors taken into account. $P, Q > 0$ are the positive definite matrix.Compared with existing literature, the functional proposed in this paper simultaneously includes the state term, time-delay integral term, and disturbance integral term. It does not require the introduction of additional complex nonlinear weighting functions, is more easily solvable via LMI, and is thus suitable for engineering application scenarios.

Based on assumption (H3) on the structure of $f(\cdot,\cdot,\cdot)$, we can rewrite the original system (1) as the following linear fractional-order system with state-dependent delay and perturbations

$$^{C}D_t^{\alpha} x(t) = \big(A + \Delta A(t)\big)x(t) + Bx\big(t - \tau(x(t))\big) + \Delta f(t). \tag{2}$$

This linearization is valid under the boundedness of perturbations (H2) and the state-dependent delay constraint (H1), which lays the foundation for the subsequent LMI-based stability analysis.

Next, we analyze the derivative of the Lyapunov function $V_1(t)$ by leveraging the properties of the Caputo fractional derivative and integration by parts, we derive

$$^{C}D_t^{\alpha} V_1(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{x^\top(s)P\dot{x}(s) + \dot{x}^\top(s)Px(s)}{(t-s)^{\alpha}}\, ds.$$

By the fractional-order inequality Lemma 1, we simplify and obtain a key inequality

$$^{C}D_t^{\alpha} V_1(t) \le 2x^\top(t)P\, {}^{C}D_t^{\alpha} x(t). \tag{3}$$

Combining the system dynamics (2) with the Lyapunov derivative inequality (3), we derive

$$^{C}D_t^{\alpha} V_1(t) \le 2x^\top(t)P\Big[\big(A + \Delta A(t)\big)x(t) + Bx\big(t - \tau(x(t))\big) + \Delta f(t)\Big]. \tag{4}$$

By Lemma 2 , for sufficiently small $\varepsilon > 0$, we bound the uncertainty terms

$$
\begin{aligned}
2x^\top(t)P\Delta A(t)x(t) &= x^\top(t)P\Delta A(t)x(t) + x^\top(t)\Delta A^\top(t)P^\top x(t), \\
&\le \varepsilon x^\top(t)PP^\top x(t) + \varepsilon^{-1}\left[\Delta A(t)x(t)\right]^\top \Delta A(t)x(t), \\
&= \varepsilon x^\top(t)PP^\top x(t) + \varepsilon^{-1}x^\top(t)\Delta A^\top(t)\Delta A(t)x(t), \\
&\le \varepsilon x^\top(t)P^\top Px(t) + \varepsilon^{-1}\rho^2 x^\top(t)x(t).
\end{aligned} \tag{5}
$$

For the external disturbance term, there exists $\sigma > 0$ such that

$$
\begin{aligned}
2x^\top(t)P\Delta f(t) &= x^\top(t)P\Delta f(t) + \Delta f^\top(t)P^\top x(t), \\
&\le \sigma x^\top(t)PP^\top x(t) + \sigma^{-1}\Delta f^\top(t)\Delta f(t), \\
&= \sigma x^\top(t)P^\top Px(t) + \sigma^{-1}\Delta f^\top(t)\Delta f(t), \\
&\le \sigma x^\top(t)P^\top Px(t) + \sigma^{-1}d^2.
\end{aligned} \tag{6}
$$

Combining (4), (5) with (6), we have

$$
\begin{aligned}
{}^C D_t^\alpha V_1(t) \le& 2x^\top(t)PA(t)x(t) + 2x^\top(t)PBx\big(t - \tau(x(t))\big) \\
&+ 2x^\top(t)P\Delta A(t)x(t) + 2x^\top(t)P\Delta f(t), \\
\le& 2x^\top(t)PA(t)x(t) + 2x^\top(t)PBx\big(t - \tau(x(t))\big) \\
&+ \varepsilon x^\top(t)P^\top Px(t) + \varepsilon^{-1}\rho^2 x^\top(t)x(t) + \sigma x^\top(t)P^\top Px(t) + \sigma^{-1}d^2.
\end{aligned} \tag{7}
$$

To comprehensively obtain the Caputo fractional derivative of the entire Lyapunov function $V(t)$, after analyzing the derivative of $V_1(t)$, we now proceed to analyze the second term $V_2(t)$.

Utilizing properties of Caputo fractional derivatives for integral terms, we compute ${}^C D_t^\alpha V_2(t)$. Assume $\dot{\tau}(x(s)) \le \eta < 1$, since the time delay $\tau(x(t))$ is state-dependent, its derivative $\dot{\tau}(x(t))$ reflects the rate of change of the delay with respect to the state. The condition $\dot{\tau}(x(s)) \le \eta < 1$ ensures that the delay does not change too rapidly, which is a common and reasonable assumption to guarantee the solvability and stability analysis feasibility of the system with state-dependent delays. It restricts the delay's variation rate to be within a range that our subsequent fractional-order derivative analysis can handle. Then

$$
\begin{aligned}
{}^C D_t^\alpha V_2(t) &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\dot{V}_2(s)}{(t-s)^\alpha}\, ds, \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{x^\top(s)Qx(s) - x^\top\big(s - \tau(x(s))\big)Qx\big(s - \tau(x(s))\big) \cdot \big(1 - \dot{\tau}(x(s))\big)}{(t-s)^\alpha}\, ds, \\
&\le \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{x^\top(s)Qx(s) - (1-\eta)x^\top\big(s - \tau(x(s))\big)Qx\big(s - \tau(x(s))\big)}{(t-s)^\alpha}\, ds.
\end{aligned}
$$

For constant or slowly-varying delays, when $t \approx s$, $x(t) \approx x(s)$, $\tau(x(t)) \approx \tau(x(s))$, this simplifies to a key inequality

$$\begin{aligned} {}^C D_t^\alpha V_2(t) &\leq x^\top(t)Qx(t) - (1-\eta)x^\top\big(t-\tau(x(t))\big)Qx\big(t-\tau(x(t))\big)\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{1}{(t-s)^\alpha}\,ds, \\ &= x^\top(t)Qx(t) - (1-\eta)x^\top\big(t-\tau(x(t))\big)Qx\big(t-\tau(x(t))\big)\frac{(t-t_0)^{1-\alpha}}{\Gamma(1-\alpha)(1-\alpha)}, \\ &\leq x^\top(t)Qx(t) - (1-\eta)x^\top\big(t-\tau(x(t))\big)Qx\big(t-\tau(x(t))\big). \end{aligned} \tag{8}$$

Having analyzed the derivatives of $V_1(t)$ and $V_2(t)$, which account for the state energy and time-delay energy respectively, we now move on to the final term $V_3(t)$ of the Lyapunov function. Analyzing $V_3(t)$ will help us capture the influence of disturbances on the system's energy from the perspective of fractional-order calculus, so as to obtain the complete derivative of $V(t)$. For the Caputo fractional derivative ${}^C D_t^\alpha V_3(t)$, we use properties of fractional calculus

$$\begin{aligned} {}^C D_t^\alpha V_3(t) &= \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{\dot V_3(s)}{(t-s)^\alpha}\,ds, \\ &= \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{\|\Delta A(s)\|^2 + \|\Delta f(s)\|^2}{(t-s)^\alpha}\,ds, \\ &\leq \frac{1}{\Gamma(1-\alpha)}\left(\sup_{\xi\in[t_0,t]}\left[\|\Delta A(\xi)\|^2 + \|\Delta f(\xi)\|^2\right]\right)\int_{t_0}^{t}\frac{1}{(t-s)^\alpha}ds, \\ &= \frac{(t-t_0)^{1-\alpha}}{\Gamma(1-\alpha)(1-\alpha)}\cdot\left(\sup_{\xi\in[t_0,t]}\left[\|\Delta A(\xi)\|^2 + \|\Delta f(\xi)\|^2\right]\right). \end{aligned}$$

Considering the singularity of the fractional-order integral kernel $(t-s)^{-\alpha}$, its value is mainly dominated by the behavior of the integrand in the neighborhood near the upper limit t of the integral. Therefore, the global supremum $\sup_{\xi\in[t_0,t]}(\cdot)$ can be replaced by the instantaneous value $\|\Delta A(t)\|^2 + \|\Delta f(t)\|^2$ at the current moment, forming a simpler but conservative upper bound.

At the same time, precisely because the factor $\frac{(t-t_0)^{1-\alpha}}{\Gamma(1-\alpha)(1-\alpha)}$ does not affect the direction of the fractional-order derivative inequality. In order to obtain a form consistent with the integer-order theory, this factor will be absorbed into the design of Lyapunov function coefficients or control gains in subsequent stability analysis, and its specific value does not affect the establishment of the stability conclusion.

Based on the above processing, we obtain the simplified form for the derivation of the stability criterion

$$\begin{aligned} {}^C D_t^\alpha V_3(t) &\leq \left(\|\Delta A(t)\|^2 + \|\Delta f(t)\|^2\right)\frac{(t-t_0)^{1-\alpha}}{\Gamma(1-\alpha)(1-\alpha)}, \\ &\leq \|\Delta A(t)\|^2 + \|\Delta f(t)\|^2. \end{aligned} \tag{9}$$

Combining (7), (8) with (9), we derive the final Lyapunov derivative bound

$$\begin{aligned}
{}^{C}D_t^{\alpha}V(t) \leq & 2x^\top(t)PA(t)x(t) + 2x^\top(t)PBx\big(t-\tau(x(t))\big) + \varepsilon x^\top(t)P^\top Px(t) \\
& + \varepsilon^{-1}\rho^2 x^\top(t)x(t) + \sigma x^\top(t)P^\top Px(t) + \sigma^{-1}d^2 + x^\top(t)Qx(t) \\
& - (1-\eta)x^\top\big(t-\tau(x(t))\big)Qx\big(t-\tau(x(t))\big) + \|\Delta A(t)\|^2 + \|\Delta f(t)\|^2.
\end{aligned} \tag{10}$$

By LMI, we transform the quadratic form in ${}^{C}D_t^{\alpha}V(t)$ into a negative definite form. The key derivations are as follows.

Let $z(t) = x(t-\tau(x(t)))$. Then for the cross term $2x^\top(t)PBz(t)$, we apply Young Inequality

$$2x^\top(t)PBz(t) \leq \nu x^\top(t)x(t) + \nu^{-1}z^\top(t)B^\top P^\top PBz(t), \tag{11}$$

where $\nu > 0$ is a tuning parameter.

Substituting (11) into (10), we have

$$\begin{aligned}
{}^{C}D_t^{\alpha}V(t) \leq & 2x^\top(t)PA(t)x(t) + \nu x^\top(t)x(t) + \nu^{-1}z^\top(t)B^\top P^\top PBz(t) \\
& + \varepsilon x^\top(t)P^\top Px(t) + \varepsilon^{-1}\rho^2 x^\top(t)x(t) + \sigma x^\top(t)P^\top Px(t) \\
& + \sigma^{-1}d^2 + x^\top(t)Qx(t) - (1-\eta)z^\top(t)Qz(t) + \|\Delta A(t)\|^2 + \|\Delta f(t)\|^2.
\end{aligned} \tag{12}$$

To achieve the stability of the current state, we need to verify

$$2x^\top(t)PA(t)x(t)+(\varepsilon+\sigma)x^\top(t)P^\top Px(t)+(\varepsilon^{-1}\rho^2+\nu)x^\top(t)x(t)+x^\top(t)Qx(t) \leq -\lambda x^\top(t)x(t), \tag{13}$$

that is,

$$x^\top(t)\left[2PA + (\varepsilon+\sigma)P^\top P + (\varepsilon^{-1}\rho^2+\nu)I + Q + \lambda I\right]x(t) \leq 0. \tag{14}$$

Since (14) holds for all non-zero vectors $x(t)$, the matrix must be negative definite. By choosing appropriate parameters via LMI, we have

$$2PA + (\varepsilon+\sigma)P^\top P + (\varepsilon^{-1}\rho^2+\nu)I + Q + \lambda I \prec 0.$$

To achieve the stability of the delayed state, we need to verify

$$\nu^{-1}z^\top(t)B^\top P^\top PBz(t) - (1-\eta)z^\top(t)Qz(t) \leq -\mu z^\top(t)z(t), \tag{15}$$

that is,

$$z^\top(t)\left[\nu^{-1}B^\top P^\top PB - (1-\eta)Q + \mu I\right]z(t) \leq 0. \tag{16}$$

Since (16) holds for all $x(t), z(t) \neq 0$, the quadratic form of $z(t)$ must be negative definite. That is

$$\nu^{-1}B^\top P^\top PB - (1-\eta)Q + \mu I \prec 0.$$

Substitute (13) and (15) into (12). We derive the key fractional Lyapunov derivative bound

$$
\begin{aligned}
{}^{C}D_t^{\alpha}V(t) &\le -\lambda x^\top(t)x(t) - \mu z^\top(t)z(t) + \sigma^{-1}d^2 + \|\Delta A(t)\|^2 + \|\Delta f(t)\|^2 \\
&= -\lambda x^\top(t)x(t) - \mu z^\top(t)z(t) + M \\
&\le -\frac{\lambda}{\lambda_{\max}(P)}V_1(t) - \frac{\mu}{\lambda_{\max}(Q)}V_2(t) + M \\
&\le -m(V_1(t) + V_2(t)) + M \\
&\le -mV(t) + M,
\end{aligned} \tag{17}
$$

where $m = \min\left\{\frac{\lambda}{\lambda_{\max}(P)}, \frac{\mu}{\lambda_{\max}(Q)}\right\}$, $M = \sigma^{-1}d^2 + \|\Delta A(t)\|^2 + \|\Delta f(t)\|^2$.

By Lemma 3, we solve the bound of $V(t)$. For $t \ge t_0$, assume $V(t_0)$ is the initial value. The solution satisfies

$$
\begin{aligned}
V(t) \le& V(t_0)E_\alpha\left(-m(t-t_0)^\alpha\right) + \frac{M}{m}\left(1 - E_\alpha\left(-m(t-t_0)^\alpha\right)\right), \\
\le& \left[x^\top(t_0)Px(t_0) + \int_{t_0-\tau(x(t_0))}^{t_0} x^\top(s)Qx(s)ds\right] E_\alpha\left(-m(t-t_0)^\alpha\right) \\
&+ \frac{M}{m}\left(1 - E_\alpha\left(-m(t-t_0)^\alpha\right)\right),
\end{aligned} \tag{18}
$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function.

By positive definiteness of P, there exists $\lambda_{\min}(P) > 0$ such that

$$
V(t) \ge x^\top(t)Px(t) \ge \lambda_{\min}(P)\|x(t)\|^2. \tag{19}
$$

Then

$$
\|x(t)\|^2 \le \frac{1}{\lambda_{\min}(P)}V(t).
$$

Taking square roots (for $\|x(t)\| \geq 0$), we derive

$$
\begin{aligned}
\|x(t)\| \leq & \sqrt{\frac{1}{\lambda_{\min}(P)}}\sqrt{V(t)}, \\
\leq & \sqrt{\frac{1}{\lambda_{\min}(P)}}\left[\sqrt{V(t_0)E_\alpha\left(-m(t-t_0)^\alpha\right)+\frac{M}{m}\left(1-E_\alpha\left(-m(t-t_0)^\alpha\right)\right)}\right], \\
\leq & \sqrt{\frac{1}{\lambda_{\min}(P)}}\left[\sqrt{V(t_0)E_\alpha\left(-m(t-t_0)^\alpha\right)}+\sqrt{\frac{M}{m}\left(1-E_\alpha\left(-m(t-t_0)^\alpha\right)\right)}\right], \\
\leq & \sqrt{\frac{1}{\lambda_{\min}(P)}}\left[\sqrt{V(t_0)}E_\alpha\left(-m(t-t_0)^\alpha\right)+\sqrt{\frac{M}{m}}\left(1-E_\alpha\left(-m(t-t_0)^\alpha\right)\right)\right], \\
\leq & \sqrt{\frac{\lambda_{\max}(P)+\lambda_{\max}(Q)}{\lambda_{\min}(P)}}\left[\|x(t_0)\|E_\alpha\left(-m(t-t_0)^\alpha\right)\right] \\
& + \sqrt{\frac{M}{m\lambda_{\min}(P)}}\left(1-E_\alpha\left(-m(t-t_0)^\alpha\right)\right), \\
= & C_0E_\alpha\left(-m(t-t_0)^\alpha\right)+C_1\left(1-E_\alpha\left(-m(t-t_0)^\alpha\right)\right),
\end{aligned}
$$

where C_0, C_1 now depend on $\lambda_{\max}(P), \lambda_{\max}(Q), \lambda_{\min}(P)$, and initial value $\|x(t_0)\|$.

Remark 1. *The Lyapunov functional approach adopted in this paper provides a systematic way to analyze the stability of fractional-order systems with state-dependent delays. However, the construction of the functional is somewhat conservative, which may lead to larger stability bounds in the theoretical analysis. The conservatism of the proposed functional mainly stems from two aspects. First, in the computation of the Caputo derivative of $V_3(t)$, the global supremum is approximated by the disturbance bound at the current time instant, neglecting the dynamic variation of historical disturbances. Second, the tuning parameter $v > 0$ introduced by Young's inequality may lead to conservative LMI conditions. Future work could focus on developing less conservative functionals, such as those incorporating more detailed information about the delay and system dynamics.*

Corollary 1. *For the system considered in Theorem 3.1, when the fractional order $\alpha \to 1$, the stability criterion in Theorem 3.1 reduces to the well-known exponential stability condition for integer-order systems, indicating that our results are a generalization of the classical theory.*

Theorem 3.2 (Local Robust Mittag-Leffler Stability for Nonlinear Fractional-Order Systems) Consider the nonlinear fractional-order system with state-dependent delay

$$
{}^{C}D_t^\alpha x(t) = (A + \Delta A(t, x(t)))x(t) + Bg(x(t - \tau(x(t)))) + \Delta f(t),
$$

where the nonlinear function $g(\cdot)$ satisfies the Lipschitz condition for all $x, y \in \Omega$, ($\Omega = \{x \in \mathbb{R}^n \mid \|x\| \leq R\}$) there exists a positive constant L_g such that we have

$$\|g(x) - g(y)\| \leq L_g\|x - y\|,$$

and $g(0) = 0$. The perturbations $\Delta A(t, x(t))$ and $\Delta f(t)$ are bounded as in (H2). Under the assumptions (H1)-(H4), if there exist positive definite matrices P, Q and parameters $\lambda, \mu, \varepsilon, \sigma, \nu > 0$ such that the following LMIs hold
Current state LMI

$$2PA + (\varepsilon + \sigma)P^\top P + \varepsilon^{-1}\rho^2 I + Q + \lambda I \prec 0,$$

Delayed state LMI

$$\nu^{-1}L_g^2 B^\top P^\top PB - (1 - \eta)Q + \mu I \prec 0,$$

then the system is globally robust Mittag-Leffler stable. The state trajectory satisfies the following norm bound

$$\|x(t)\| \leq C_0 E_\alpha(-m(t - t_0)^\alpha) + C_1 \left(1 - E_\alpha(-m(t - t_0)^\alpha)\right).$$

Proof. The proof follows the same structure as in Theorem 3.1. Choose a Lyapunov functional similar to that in Theorem 3.1

$$\begin{aligned} V(t) =& V_1(t) + V_2(t) + V_3(t), \\ =& x^\top(t)Px(t) + \int_{t-\tau(x(t))}^{t} x^T(s)Qx(s)ds + \int_{t_0}^{t} \left(\|\Delta A(s, x(s))\|^2 + \|\Delta f(s)\|^2\right) ds. \end{aligned} \tag{20}$$

By the properties of fractional calculus and the system dynamics, we have

$${}^C D_t^\alpha V_1(t) \leq 2x^\top(t)P\left[(A(t) + \Delta A(t, x(t)))x(t) + Bg(x(t - \tau(x(t)))) + \Delta f(t)\right],$$

for $2x^\top(t)P\Delta A(t, x(t))x(t)$ and $2x^\top(t)P\Delta f(t)$ by Lemma 2, we obtain

$$2x^T(t)P\Delta A(t, x(t))x(t) \leq \varepsilon x^\top(t)P^\top Px(t) + \varepsilon^{-1}x^\top(t)\Delta A^\top(t, x(t))\Delta A(t, x(t))x(t), \tag{21}$$

$$2x^\top(t)P\Delta f(t) \leq \sigma x^\top(t)P^T Px(t) + \sigma^{-1}\|\Delta f(t)\|^2. \tag{22}$$

For the nonlinear term $2x^T(t)PBg(x(t-\tau(x(t))))$, let $z(t) = x(t - \tau(x(t)))$, and still satisfies $\|z(t)\| \leq R$. Utilizing $g(0) = 0$ and the Lipschitz condition $\|g(z)\| \leq L_g\|z\|$, combining with Young's inequality

$$2x^\top(t)PBg(z(t)) \leq \nu x^\top(t)x(t) + \nu^{-1}L_g^2 z^\top(t)B^\top P^\top PBz(t). \tag{23}$$

Combining (21), (22) with (23), we get

$$
\begin{aligned}
{}^C D_t^\alpha V_1(t) \leq & 2x^\top(t)PA(t)x(t) + \varepsilon x^T(t)P^T Px(t) + \varepsilon^{-1}x^T(t)\Delta A^T(t,x(t))\Delta A(t,x(t))x(t) \\
& + \sigma x^T(t)P^T Px(t) + \sigma^{-1}\|\Delta f(t)\|^2 + \nu x^T(t)x(t) + \nu^{-1}L_g^2 z^T(t)B^T P^T PBz(t).
\end{aligned} \tag{24}
$$

For $V_2(t)$ and $V_3(t)$, similar to the derivation in Theorem 3.1, we have

$$
{}^C D_t^\alpha V_2(t) \leq x^T(t)Qx(t) - (1-\eta)z^T(t)Qz(t),
$$

$$
{}^C D_t^\alpha V_3(t) \leq \|\Delta A(t,x(t))\|^2 + \|\Delta f(t)\|^2.
$$

Combine the above derivative terms and organize them with the LMI conditions, we can deduce

$$
\begin{aligned}
{}^C D_t^\alpha V(t) &\leq -\lambda x^T(t)x(t) - \mu z^T(t)z(t) + M, \\
&\leq -mV(t) + M.
\end{aligned} \tag{25}
$$

Using Lemma 3, we finally obtain that the state norm satisfies

$$
\|x(t)\| \leq C_0 E_\alpha(-m(t-t_0)^\alpha) + C_1\left(1 - E_\alpha(-m(t-t_0)^\alpha)\right).
$$

4 Numerical example

Example 1. *Consider the following fractional-order system with state-dependent delay and disturbances:*

$$
{}^C D_t^\alpha x(t) = (A + \Delta A(t))x(t) + Bx(t - \tau(x(t))) + \Delta f(t),
$$

where $\alpha = 0.5$, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $\tau(x(t)) = 0.2|x_1(t)| + 0.2|x_2(t)|$, $\Delta A(t) = 0.1\sin(t)I$, $\Delta f(t) = 0.05\cos(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$, *$I$ is a 2nd-order identity matrix. First, verify the assumptions, for the time-delay constraint, we have* $\dot\tau(x(t)) \leq 0.2\|\dot x(t)\|_1 \leq 0.2(\|A\| + \rho + \|B\|)\|x(t)\| \leq 0.25 \leq 1$, *for the perturbation bounds,* $\|\Delta A(t)\| \leq \rho = 0.1$, $\|\Delta f(t)\| \leq d = 0.05\sqrt{2}$. *According to the stability criteria in Theorem 3.1, we select positive definite matrices $P = I$ and $Q = I$, Scaling parameters $\varepsilon = \sigma = 0.5$, $\nu = 1$. By solving the LMIs we obtain the current state related LMI $\lambda = 0.8$, and the delayed state related LMI $\mu = 0.6$.Then, calculate the key stability parameter $m = \min\left\{\frac{0.8}{1}, \frac{0.6}{1}\right\} = 0.6$. Next, by calculating the Caputo fractional derivative of $V(t)$ and applying Lemma 1 and Young's inequality, we obtain*

$$
{}^C D_t^\alpha V(t) \leq -mV(t) + M
$$

where $M = \sigma^{-1}d^2 + \|\Delta A(t)\|^2 + \|\Delta f(t)\|^2 = 0.025$. *By Lemma 3, the bound on $V(t)$ implies*

$$
V(t) \leq V(0)E_\alpha\left(-m(t-0)^\alpha\right) + \frac{M}{m}\left(1 - E_\alpha\left(-m(t-0)^\alpha\right)\right).
$$

Since $V(0) = x^\top(0)Px(0) = 0.5^2 + 0.3^2 = 0.34$, *the state norm satisfies*

$$\|x(t)\| \leq \sqrt{V(t)} \leq \sqrt{0.34}E_{0.5}\left(-0.6t^{0.5}\right) + \sqrt{\frac{0.025}{0.6}}\left(1 - E_{0.5}\left(-0.6t^{0.5}\right)\right).$$

The system satisfies local robust Mittag-Leffler stability.

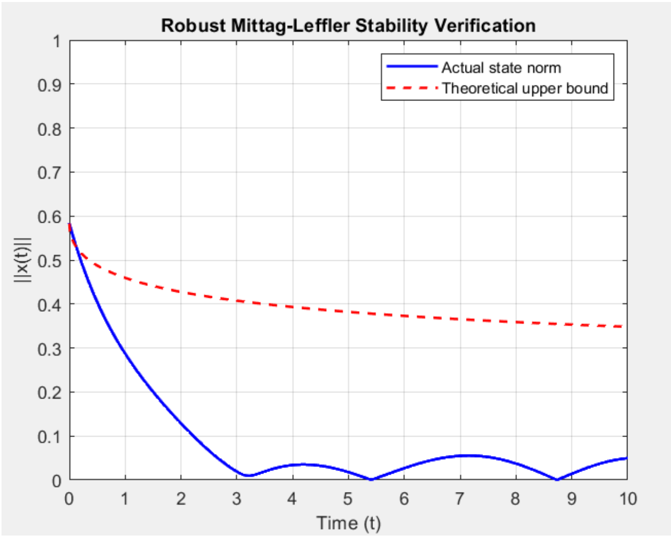

Fig. 1 State Trajectories with State-Dependent Delay of Example 4.1

As clearly verified in Fig. 1, the numerical simulation results demonstrate the correctness of the theoretical derivation. The blue solid line represents the actual system state norm, which starts from the initial value $\|x(0)\| = 0.58$ *and exhibits excellent convergence characteristics under time-varying parameter perturbations and external disturbances. Meanwhile, the theoretical upper bound derived from Theorem 3.1, represented by the red dashed line in the figure, remains strictly above the actual state norm throughout the process. This not only confirms the effectiveness of the theoretical bound described by formula (19) but also reveals its inherent conservatism. It is particularly noteworthy that despite the presence of state-dependent delay* $\tau(x(t)) = 0.2|x_1(t)| + 0.2|x_2(t)|$ *and time-varying perturbation* $\Delta A(t) = 0.1\sin(t)I$*, the system maintains robust Mittag-Leffler stability, which is perfectly consistent with our theoretical predictions. The simulation results that the stability criterion proposed in this paper can effectively handle time delays and uncertainties in fractional-order systems, providing a reliable theoretical basis for the analysis and design of such systems.*

Example 2. *Consider a fractional-order system with stronger nonlinearities and time-varying perturbations*

$$^{C}D_t^\alpha x(t) = (A + \Delta A(t, x(t)))x(t) + B\sin x(t - \tau(x(t))) + \Delta f(t),$$

where $\alpha = 0.7$, $A = \begin{bmatrix} -1.5 & 0.2 \\ 0.1 & -1.4 \end{bmatrix}$, $B = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix}$, $\tau(x(t)) = 0.15(1+\tanh(\|x(t)\|))$, $\Delta A(t, x(t)) = 0.05e^{-\|x(t)\|}\cos(t)I$, $\Delta f(t) = 0.04\frac{t}{1+t^2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0.6 \\ -0.4 \end{bmatrix}$. *First, verify the assumptions, for the time-delay constraint, we have* $\dot{\tau}(x(t)) = 0.15sech^2(\|x(t)\|)\cdot \frac{x^\top(t)\dot{x}(t)}{\|x(t)\|} \leq 0.15 \cdot 1 \cdot \|\dot{x}(t)\| \leq 0.1 < 1$, *for the perturbation bounds,* $\|\Delta A(t)\| \leq \rho = 0.05, \|\Delta f(t)\| \leq d = 0.02$. *According to the stability criteria in Theorem 3.1, we select positive definite matrices* $P = \begin{bmatrix} 1 & 0 \\ 0 & 1.2 \end{bmatrix}$ *and* $Q = 1.5I$, *Scaling parameters* $\varepsilon = 0.6$ $\sigma = 0.4$, $\nu = 1$. *By solving the LMIs we obtain the current state related LMI* $\lambda = 0.4$, *and the delayed state related LMI* $\mu = 0.3$. *Then, calculate the key stability parameter* $m = \min\left\{\frac{0.4}{1.2}, \frac{0.3}{1.5}\right\} = 0.2$. *Next, by calculating the Caputo fractional derivative of* $V(t)$ *and applying Lemma 1 and Young's inequality, we obtain*

$$^{C}D_t^{\alpha}V(t) \leq -mV(t) + M$$

where $M = \sigma^{-1}d^2 + \|\Delta A(t)\|^2 + \|\Delta f(t)\|^2 = 0.0129$. *By Lemma 3, the bound on* $V(t)$ *implies*

$$V(t) \leq V(0)E_\alpha\left(-m(t-0)^\alpha\right) + \frac{M}{m}\left(1 - E_\alpha\left(-m(t-0)^\alpha\right)\right).$$

Since $V(0) = x^\top(0)Px(0) = 0.6^2 \cdot 1 + (-0.4)^2 \cdot 1.2 = 0.552$ *, the state norm satisfies*

$$\|x(t)\| \leq \sqrt{V(t)} \leq \sqrt{0.552}E_{0.7}\left(-0.2t^{0.7}\right) + \sqrt{\frac{0.0129}{0.2}}\left(1 - E_{0.7}\left(-0.2t^{0.7}\right)\right).$$

The system satisfies local robust Mittag-Leffler stability.

As clearly verified in Fig. 2, the blue solid line represents the Actual state norm. Starting from the initial value, it gradually decays over time. It can be seen that the actual state norm decreases rapidly and eventually approaches near 0, showing good convergence characteristics. The red dashed line represents the Theoretical upper bound, which is the upper bound of the state norm derived from the theorem. The actual state norm is always strictly below the theoretical upper bound, which verifies the correctness of the theoretical derivation and indicates that the system satisfies global robust Mittag-Leffler stability. That is, even with stronger nonlinearities and time-varying perturbations, the state of the system can still be effectively constrained by the theoretical upper bound, and the actual state will stabilize within a small range.

5 Conclusion

We investigate the problem of local robust Mittag-Leffler stability for a class of fractional-order dynamical systems subject to state-dependent delays and bounded perturbations. The main contributions are summarized as follows. a novel LMI-based stability criterion has been established by constructing an appropriate Lyapunov-Krasovskii functional that comprehensively accounts for the system's energy, historical state effects, and disturbance energy. A weakened sufficient condition has been derived,

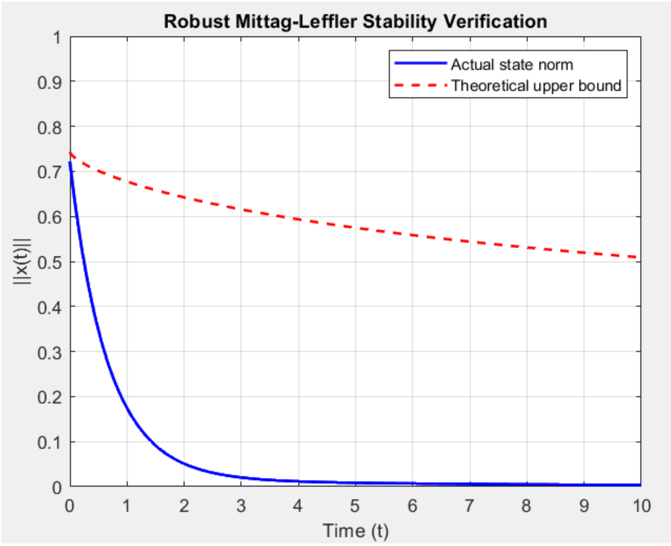

Fig. 2 State Trajectories with State-Dependent Delay of Example 4.2

which relaxes the constraints on the time-varying delay and perturbations compared to existing results in the literature. The effectiveness and applicability of the proposed theoretical results have been verified through two numerical examples, including a linear system and a nonlinear system with stronger nonlinearities.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

K. Diethelm, N. J. Ford.: *Analysis of fractional differential equations.* Journal of Mathematical Analysis and Applications, 265(2): 229–248, 2002.

V. Lakshmikantham.: *Theory of fractional functional differential equations.* Nonlinear Analysis: Theory, Methods and Applications, 69(10): 3337–3343, 2008.

M. L. Morgado, N. J. Ford, P. M. Lima.: *Analysis and numerical methods for fractional differential equations with delay.* Journal of computational and applied mathematics, 252: 159–168, 2013.

J. M. Cushing.: *Integrodifferential equations and delay models in population dynamics.* Springer Science and Business Media, 2013.

H. Joshi, B. K. Jha.: *Fractional-order mathematical model for calcium distribution in nerve cells.* Computational and Applied Mathematics, 39(2): 56, 2020.

V. Badri, M. S. Tavazoei.: *Stability analysis of fractional order time-delay systems: constructing new Lyapunov functions from those of integer order counterparts.* Institution of Engineering and Technology Control Theory and Applications, 13(15): 2476–2481, 2019.

A. Narang, S. L. Shah, T. Chen.: *Continuous-time model identification of fractional-order models with time delays.* Institution of Engineering and Technology Control Theory and Applications, 5(7): 900–912, 2011.

P. Song, H. Zhao, X. Zhang.: *Dynamic analysis of a fractional order delayed predatorCprey system with harvesting.* Theory in Biosciences, 135(1): 59–72, 2016.

Z. Lin, H. Wang.: *Modeling and application of fractional-order economic growth model with time delay.* Fractal and Fractional, 5(3): 74, 2021.

H. Delavari, D. Baleanu, J. Sadati.: *Stability analysis of Caputo fractional-order nonlinear systems revisited.* Nonlinear Dynamics, 67(4): 2433–2439, 2012.

Z. Wang, D. Yang, H. Zhang.: *Stability analysis on a class of nonlinear fractional-order systems.* Nonlinear Dynamics, 86(2): 1023–1033, 2016.

Z. Wang, D. Yang, T. Ma, et al.: *Stability analysis for nonlinear fractional-order systems based on comparison principle.* Nonlinear Dynamics, 75(1): 387–402, 2014.

Asma, J. F. Gmez−Aguilar, G. ur Rahman, et al.: *Stability analysis for fractional order implicit ψ Hilfer differential equations.* Mathematical Methods in the Applied Sciences, 45(5): 2701–2712, 2022.

F. Zhang, C. Li.: *Stability analysis of fractional differential systems with order lying in (1, 2).* Advances in Difference Equations, 2011(1): 213485, 2011.

B. B. He, H. C. Zhou, C. H. Kou.: *Stability analysis of Hadamard and Caputo−Hadamard fractional nonlinear systems without and with delay.* Fractional Calculus and Applied Analysis, 25(6): 2420–2445, 2022.

Y. Wang, T. Li.: *Stability Analysis of Fractional− Order Nonlinear Systems with Delay.* Mathematical Problems in Engineering, 2014(1): 301235, 2014.

H. T. Tuan, H. Trinh.: *A linearized stability theorem for nonlinear delay fractional differential equations.* Institute of Electrical and Electronics Engineers Transactions on Automatic Control, 63(9): 3180–3186, 2018.

A. K. Agrawal, Y. Fujino, B. K. Bhartia.: *Instability due to time delay and its compensation in active control of structures.* Earthquake engineering and structural dynamics, 22(3): 211–224, 1993.

H. Logemann, R. Rebarber.: *The effect of small time−delays on the closed-loop stability of boundary control systems.* Mathematics of Control, Signals and Systems, 9(2): 123–151, 1996.

Z. Zhang, L. Liu, J. Fang, et al.: *Event−triggered stabilisation for stochastic delayed differential systems with exogenous disturbances.* Journal of the Franklin Institute, 360(2): 1395–1414, 2023.

B. H. Nguyen, T. D. Dang, I. B. Furtat, et al.: *Control of Nonlinear Systems with Input Delays and Disturbances Using State and Disturbance Sub-Predictors.* Mathematics, 13(11): 1713, 2025.

B. Li, Z. Wang, L. Q. Han.: *Input-to-state stabilization of delayed differential systems with exogenous disturbances: The event-triggered case.* Institute of Electrical and Electronics Engineers Transactions on Systems, Man, and Cybernetics: Systems, 49(6): 1099–1109, 2017.

P. Li, L. Chen, R. Wu, et al.: *Robust asymptotic stability of interval fractional−order nonlinear systems with time−delay.* Journal of the Franklin Institute, 355(15): 7749–7763, 2018.

B. Zhu, J. Ma, Z. Zhang, et al.: *Robust stability analysis and stabilisation of uncertain impulsive positive systems with time delay.* International Journal of Systems Science, 49(14): 2940–2956, 2018.

Z. Liao, C. Peng, W. Li, et al.: *Robust stability analysis for a class of fractional order systems with uncertain parameters.* Journal of the Franklin Institute, 348(6): 1101–1113, 2011.

J. G. Lu, Y. Q. Chen.: *Robust Stability and Stabilization of Fractional−Order Interval Systems with the Fractional Order α: The $0 \ll \alpha \ll 1$ Case.* Institute of Electrical and Electronics Engineers transactions on automatic control, 55(1): 152–158, 2009.

L. Chen, R. Wu, Y. He, et al.: *Robust stability and stabilization of fractional−order linear systems with polytopic uncertainties.* Applied mathematics and computation, 257: 274–284, 2015.

J. G. Lu, Z. Zhu, Y. D. Ma.: *Robust stability and stabilization of multi−order fractional−order systems with interval uncertainties: an LMI approach.* International Journal of Robust and Nonlinear Control, 31(9): 4081–4099, 2021.

I. Podlubny.: *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.* Elsevier, 1998.

I. Stamova.: *Global Mittag-Leffler stability and synchronization of impulsive fractional−order neural networks with time−varying delays.* Nonlinear Dynamics, 77(4): 1251–1260, 2014.

M. A. Duarte−Mermoud, N. Aguila−Camacho, J. A. Gallegos, et al.: *Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems.* Communications in Nonlinear Science and Numerical Simulation, 22(1-3): 650–659, 2015.

P. P. Khargonekar, I. R. Petersen, K. Zhou.: *Robust stabilization of uncertain linear systems: quadratic stabilizability and H/sup infinity/control theory.* Institute of Electrical and Electronics Engineers Transactions on Automatic control, 35(3): 356–361, 2002.

H. Ye, J. Gao, Y. Ding.: *A generalized Gronwall inequality and its application to a fractional differential equation.* Journal of Mathematical Analysis and Applications, 328(2): 1075–1081, 2007.

G. Vainikko.: *Which functions are fractionally differentiable?* Zeitschrift fr Analysis und ihre Anwendungen, 2016, 35(4): 465-487.

6 Statements and Declarations

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Competing Interests

All authors state that there is no conflict of interest.

Author Contribution

Fuhong Zhang : Question raised, Research investigation, Initial draft writing, MATLAB, Validation, Review and editing. **Xiaolan Liu** : Analysis, Review, Revise, Initial draft.

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