Spectral properties of bounded operator matrices on semi-Hilbertian spaces and their applications

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Abstract

Let $\boldsymbol{A}$ and $\boldsymbol{D}$ be positive operators on the Hilbert spaces $\boldsymbol{\mathcal{X}_1}$ and $\boldsymbol{\mathcal{X}_2}$, respectively, and define $\mathbb{A} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} \end{pmatrix}$. Furthermore, let $\mathbb{T} = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{K} \\ \boldsymbol{L} & \boldsymbol{N} \end{pmatrix}$ denote the $\mathbb{A}$-bounded operator matrices. The results are derived by analyzing the entries of diagonal operator matrices ($\boldsymbol{K} = \boldsymbol{L} = \boldsymbol{0}$), off-diagonal operator matrices ($\boldsymbol{M} = \boldsymbol{N} = \boldsymbol{0}$), and upper triangular operator matrices ($\boldsymbol{L} = \boldsymbol{0}$) to characterize the spectral properties of the operator matrices $\mathbb{T}$ on semi-Hilbertian spaces. Firstly, for diagonal operator matrices, the spectral properties of $\mathbb{T}$ are determined by the spectral properties of its diagonal entries. Secondly, for off-diagonal operator matrices, the spectral structure is more intricate than that of diagonal matrices and is closely associated with the product of the off-diagonal entries. Thirdly, for upper triangular operator matrices, we establish the spectral distribution of $\mathbb{T}$ and provide the necessary and sufficient conditions under which the spectra of $\mathbb{T}$ coincide with the union of the spectra of its diagonal entries. Finally, an example of spectral analysis of the pseudo-Hermitian operator matrices is presented to demonstrate the conclusions.

Keywords: Positive operator, Semi-inner product, Spectrum, Operator matrix

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1 Introduction

The study of the spectra of self-adjoint operators has been a central topic of operator theory [1–4], with applications spanning numerous branches of physics, particularly in quantum mechanics [5–7]. Typically, the state space is modeled as a Hilbert space

$(\mathcal{X}, \langle\cdot,\cdot\rangle)$, in which physical observable measurements are represented by self-adjoint operators, where self-adjointness is defined with respect to the chosen inner product. However, most linear operators are not self-adjoint under the standard inner product. As a result, pseudo-Hermitian quantum mechanics introduces more general criteria[8–11] by defining a new inner product. This new inner product provides a framework in which operators corresponding to physical observable measurements satisfy the revised self-adjointness criteria, ensuring both operator symmetry and real spectra.

Throughout this paper, $\mathcal{B}(\mathcal{X})$ denote the set of all bounded linear operators on $\mathcal{X}$. For any $T \in \mathcal{B}(\mathcal{X})$, the symbols $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\overline{\mathcal{R}(T)}$, $\mathcal{N}(T)$, and T^* represent the domain, range, closure of the range, null space, and adjoint of T, respectively. For $T, S \in \mathcal{B}(\mathcal{X})$, if

$$TST = T, \quad STS = S, \quad ST = P_{\overline{\mathcal{R}(T)}}, \quad TS = P_{\overline{\mathcal{R}(T)}}|_{\mathcal{R}(T)\oplus\mathcal{R}(T)^\perp},$$

where $P_{\overline{\mathcal{R}(T)}}$ is the orthogonal projection on $\overline{\mathcal{R}(T)}$, then S is referred to as the Moore-Penrose inverse of T, denoted by $T^\dagger$, i.e., $S = T^\dagger$[12]. Let $\mathcal{B}(\mathcal{X})^+ = \{A \in \mathcal{B}(\mathcal{X}) : \langle Ax, x\rangle \geq 0, \forall x \in \mathcal{X}\}$. If $A \in \mathcal{B}(\mathcal{X})^+$, then the positive operator A induces a semi-inner product defined as $\langle x, y\rangle_A := \langle Ax, y\rangle$, which generates the seminorm $\|x\|_A = \sqrt{\langle Ax, x\rangle}$, for all $x \in \mathcal{X}$. Consequently, $(\mathcal{X}, \langle\cdot,\cdot\rangle_A)$ is termed a semi-Hilbertian space. For $T \in \mathcal{B}(\mathcal{X})$, if there exists $S \in \mathcal{B}(\mathcal{X})$ such that $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$, then the operator equation $AS = T^*A$ admits a solution S, referred to as the A-adjoint of T. This solution can be derived using the Douglas range inclusion theorem[13]. The set of all operators possessing A-adjoints is given by

$$\mathcal{B}_A(\mathcal{X}) = \{T \in \mathcal{B}(\mathcal{X}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}.$$

If the equation $AS = T^*A$ has a unique solution, this solution is denoted by $T^\sharp$, where $T^\sharp = A^\dagger T^* A$. Applying the Douglas range inclusion theorem again, the set of all operators with $A^{1/2}$-adjoints is

$$\begin{aligned}\mathcal{B}_{A^{1/2}}(\mathcal{X}) &= \{T \in \mathcal{B}(\mathcal{X}) : \mathcal{R}(T^*A^{1/2}) \subseteq \mathcal{R}(A^{1/2})\}\\ &= \{T \in \mathcal{B}(\mathcal{X}) : \exists\ c > 0, \text{ such that } \|Tx\|_A \leq c\|x\|_A, \forall x \in \mathcal{X}\}.\end{aligned}$$

If $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$, T is called an A-bounded operator, and it satisfies $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$. It is worth noting that $\mathcal{B}_A(\mathcal{X}) \subset \mathcal{B}_{A^{1/2}}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X})$[25].

In recent years, the study of operators on semi-Hilbertian spaces has gained increasing attention and achieved significant progress. Notable advancements include researches on A-numerical radius [14–18], A-spectral theory [19–24], the correspondence between operators in $\mathcal{B}_{A^{1/2}}(\mathcal{X})$ and classes of operators in other mathematical spaces [17, 24, 25]. Operator matrices play a crucial role in system theory, nonlinear analysis, and various mathematical and physical problems [26]. Furthermore, any operator $T \in \mathcal{B}(\mathcal{X})$ can be expressed as an operator matrix through space decomposition, and its properties can often be characterized by the entries of block operator matrices (see [27–29] for more details). Despite these developments, the $\mathbb{A}$-spectral properties of block operator matrices remain largely unexplored in the existing literature.

Specifically, analyzing operator matrices within this framework is crucial. They model coupled subsystems or systems with internal structures in pseudo-Hermitian settings. Understanding the $\mathbb{A}$-spectra of block operator matrices, for instance, provides insights into the stability and dynamical behavior of the composite system.

The primary objective of this paper is to describe the spectral properties of operator matrices on semi-Hilbertian product spaces. Section 2 provides foundational concepts and defines the fine A-spectra of $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$, including the A-residual, A-continuous, and A-defect spectra. Section 3 focuses on the $\mathbb{A}$-spectral analysis of structured operator matrices. Subsection 3.1 studies the $\mathbb{A}$-spectra of diagonal operator matrices, which serves as a foundation. Subsection 3.2 extends the analysis to off-diagonal operator matrices. Subsection 3.3 investigates a more general case of upper triangular operator matrices. Finally, Subsection 3.4 presents an application of the $\mathbb{A}$-spectral results for off-diagonal operator matrices in pseudo-Hermitian quantum mechanics. This methodology establishes a robust framework for addressing the $\mathbb{A}$-spectral problem of operator matrices.

2 Preliminary results and notations

This section presents fundamental notations, definitions, and results utilized in the subsequent sections. For $A \in \mathcal{B}(\mathcal{X})^+$, the Moore-Penrose inverse of A and the corresponding orthogonal projection onto $\overline{\mathcal{R}(A)}$ play a crucial role in semi-Hilbertian spaces generated by A. To simplify notation, we denote $P_{\overline{\mathcal{R}(A)}} = A^\dagger A$ as P. An operator $T \in \mathcal{B}_A(\mathcal{X})$ is termed A-unitary if $T^\sharp T = TT^\sharp = P$.

The study in Ref.[19] develops the spectral analysis of operators on the subspace $\mathcal{B}_{A^{1/2}}(\mathcal{X})$ as follows.

Definition 2.1 *An operator $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$ is said to be invertible in $\mathcal{B}_{A^{1/2}}(\mathcal{X})$ if there exists an operator $S \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$ such that*

$$ATS = AST = A.$$

The operator S is referred to as the A-inverse of T.

Remark 2.2 *Since $A^\dagger A = P = (A^{1/2})^\dagger A^{1/2}$, and $AP = A$, $A^{1/2}P = PA^{1/2} = A^{1/2}$, equivalent descriptions of the A-invertible operator are provided as follows:*

(1) $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$ has an A-inverse $S \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$ if and only if $A^{1/2}TS = A^{1/2}ST = A^{1/2}$.

(2) $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$ has an A-inverse $S \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$ if and only if $PTS = PST = P$.

Definition 2.3 *Let $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$. The operator AT is defined as injective if for all $x \in \overline{\mathcal{R}(A)}$, $ATx = 0$ implies $x = 0$. Furthermore, AT is defined as surjective if for all $y \in \overline{\mathcal{R}(A)}$, there exists $x \in \mathcal{X}$ such that $ATx = y$.*

Definition 2.4 *Let $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$. The A-resolvent set and the A-spectra of T are respectively defined as*

$$\rho_A(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible in } \mathcal{B}_{A^{1/2}}(\mathcal{X})\},$$

and $\sigma_A(T) = \mathbb{C} \setminus \rho_A(T)$, *i.e.,*

$$\sigma_A(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } \mathcal{B}_{A^{1/2}}(\mathcal{X})\}.$$

Additionally, the following sets are introduced:

(1) *The A-point spectrum of T is denoted by* $\sigma_{A_p}(T) = \{\lambda \in \mathbb{C} : A(T - \lambda I)$ *is not injective*$\}$[17];
(2) *The A-residual spectrum of T is denoted by* $\sigma_{A_r}(T) = \{\lambda \in \mathbb{C} : A(T - \lambda I)$ *is injective, but* $\overline{\mathcal{R}(A(T-\lambda I))} \neq \overline{\mathcal{R}(A)}\}$;
(3) *The A-continuous spectrum of T is denoted by* $\sigma_{A_c}(T) = \{\lambda \in \mathbb{C} : A(T - \lambda I)$ *is injective,* $\overline{\mathcal{R}(A(T-\lambda I))} = \overline{\mathcal{R}(A)}$, *but* $\mathcal{R}(A(T-\lambda I)) \neq \overline{\mathcal{R}(A)}\}$.

Remark 2.5 *If* $T \in \mathcal{B}_{A^{1/2}}(X)$ *and* $\lambda \in \rho_A(T)$, *then there exists an analytic function* $R_\lambda(T) \in \mathcal{B}_{A^{1/2}}(X)$ *such that* $A(T - \lambda I)R_\lambda(T) = AR_\lambda(T)(T - \lambda I) = A$. *According to Remark 2.2, the A-point spectrum of T can also be expressed as*

$$\begin{aligned}\sigma_{A_p}(T) &= \{\lambda \in \mathbb{C} : A^{1/2}(T - \lambda I) \text{ is not injective}\}\\ &= \{\lambda \in \mathbb{C} : P(T - \lambda I) \text{ is not injective}\}.\end{aligned}$$

In general, $\mathcal{R}(A(T-\lambda I)) \subset \overline{\mathcal{R}(A)}$, *and if* $\overline{\mathcal{R}(A(T-\lambda I))} = \overline{\mathcal{R}(A)}$, *we say that* $\mathcal{R}(A(T-\lambda I))$ *is dense in* $\overline{\mathcal{R}(A)}$. *Moreover,* $\sigma_A(T) = \sigma_{A_p}(T) \cup \sigma_{A_r}(T) \cup \sigma_{A_c}(T)$.

Remark 2.6 *Let* $T \in \mathcal{B}_{A^{1/2}}(X)$. *The 1-A-point spectrum, the 1-A-residual spectrum and the A-defect spectrum of T are defined by*

$$\begin{aligned}\sigma_{A_{p,1}}(T) &= \{\lambda \in \sigma_{A_p}(T) : \mathcal{R}(A(T-\lambda I)) = \overline{\mathcal{R}(A)}\},\\ \sigma_{A_{r,1}}(T) &= \{\lambda \in \sigma_{A_r}(T) : \mathcal{R}(A(T-\lambda I)) = \overline{\mathcal{R}(A(T-\lambda I))}\}.\\ \sigma_{A_\delta}(T) &= \{\lambda \in \mathbb{C} : \mathcal{R}(A(T-\lambda I)) \neq \overline{\mathcal{R}(A)}\};\end{aligned}$$

Lemma 2.7 [19] *If* T_1 *and* T_2 *are invertible in* $\mathcal{B}_{A^{1/2}}(X)$, *then so is* T_1T_2. *Furthermore, if* $T_1T_2 = T_2T_1$ *in* $\mathcal{B}_{A^{1/2}}(X)$, *then* T_1T_2 *is invertible in* $\mathcal{B}_{A^{1/2}}(X)$ *if and only if* T_1 *and* T_2 *are invertible in* $\mathcal{B}_{A^{1/2}}(X)$.

Previously, we rigorously defined the A-residual spectrum and A-continuous spectrum based on the properties of injectivity and surjectivity of the operator $A(T-\lambda I)$. The subsequent theorem elucidates the precise relationship between the A-spectral points of the operator T and those of its adjoint T^*.

Lemma 2.8 *Let* $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$. *Then*

$$\mathcal{N}(PT) = \mathcal{R}(PT^*)^\perp, \ \overline{\mathcal{R}(PT)} = \mathcal{N}(PT^*)^\perp.$$

Proof Since $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$, it follows that $T^* \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$. Consequently, $\mathcal{N}(A)$ is an invariant subspace of both T and T^*. Therefore, we have $PTP = PT$ and $PT^*P = PT^*$.

If $x \in \mathcal{N}(PT)$, this implies $PTx = 0$. For all $y \in \mathcal{D}(PT^*)$, we obtain

$$\langle PTPx, y\rangle = \langle x, PT^*Py\rangle = \langle x, PT^*y\rangle = 0,$$

which indicates that $x \in \mathcal{R}(PT^*)^\perp$.

Conversely, if $x \in \mathcal{R}(PT^*)^\perp$, then for all $y \in \mathcal{D}(PT^*)$, we have $\langle x, PT^*y\rangle = 0$. By the definition of adjoint operators, this yields

$$\langle x, PT^*y\rangle = \langle x, PT^*Py\rangle = \langle PTPx, y\rangle = 0,$$

implying $PTx = 0$, i.e., $x \in \mathcal{N}(PT)$.

Since $\overline{\mathcal{R}(PT^*)} = (\mathcal{R}(PT^*)^\perp)^\perp = \mathcal{N}(PT)^\perp$, it follows that

$$\overline{\mathcal{R}(PT)} = \mathcal{N}(PT^*)^\perp.$$

□

Lemma 2.9 *Let $T \in \mathcal{B}_{A^{1/2}}(\mathcal{X})$.*

(1) If $\lambda \in \sigma_{A_p}(T)$, then $\overline{\lambda} \in \sigma_{A_p}(T^*) \cup \sigma_{A_r}(T^*)$.
(2) If $\lambda \in \sigma_{A_r}(T)$, then $\overline{\lambda} \in \sigma_{A_p}(T^*)$.

Proof (1) If $\lambda \in \sigma_{A_p}(T)$, there exists $x_0 \neq 0$ such that $P(T-\lambda I)x_0 = 0$. If $P(T^* - \overline{\lambda}I)$ is not injective, then $\overline{\lambda} \in \sigma_{A_p}(T^*)$. Otherwise, if $P(T^* - \overline{\lambda}I)$ is injective, for all $y \in \overline{\mathcal{R}(A(T^* - \overline{\lambda}I))}$, we have

$$\begin{aligned}\langle P(T-\lambda I)x_0, y\rangle &= \langle P(T-\lambda I)Px_0, y\rangle \\ &= \langle x_0, P(T^* - \overline{\lambda}I)Py\rangle \\ &= \langle x_0, P(T^* - \overline{\lambda}I)y\rangle = 0.\end{aligned}$$

This implies that $\overline{\mathcal{R}(P(T^* - \overline{\lambda}I))} \neq \overline{\mathcal{R}(A)}$, hence $\lambda \in \sigma_{A_r}(T^*)$.

(2) If $\lambda \in \sigma_{A_r}(T)$, then $\overline{\mathcal{R}(P(T-\lambda I))}^\perp \neq \overline{\mathcal{R}(A)}^\perp = \{0\}$. Since

$$\overline{\mathcal{R}(P(T-\lambda I))}^\perp = \mathcal{N}(P(T^* - \overline{\lambda}I)),$$

it follows that $\lambda \in \sigma_{A_p}(T^*)$. □

We next investigate the spectral properties of operator matrices on semi-Hilbertian product spaces. Let A and D be positive operators, and let $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Clearly, $\mathbb{A} \in \mathcal{B}(\mathcal{X}_1 \oplus \mathcal{X}_2)^+$, and it induces the semi-inner product $\langle x, y\rangle_{\mathbb{A}} = \langle \mathbb{A}x, y\rangle = \langle x_1, y_1\rangle_A + \langle x_2, y_2\rangle_D$, for all $x = (x_1, x_2) \in \mathcal{X}_1 \oplus \mathcal{X}_2$ and $y = (y_1, y_2) \in \mathcal{X}_1 \oplus \mathcal{X}_2$.

3 Spectra of $\mathbb{A}$-bounded operator matrices

Operator matrices with special structures arise frequently in applications, and understanding their $\mathbb{A}$-spectra is a fundamental problem. This section is divided into three subsections to analyze the $\mathbb{A}$-spectra of diagonal operator matrices, off-diagonal operator matrices, and upper triangular operator matrices. The final part demonstrates how the results for off-diagonal matrices can be applied to a problem in pseudo-Hermitian quantum mechanics.

3.1 $\mathbb{A}$-spectra of diagonal operator matrices

Let $\mathbb{T} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$ be an $\mathbb{A}$-bounded diagonal operator matrix, where $M \in \mathcal{B}_{A^{1/2}}(\mathcal{X}_1)$ and $N \in \mathcal{B}_{D^{1/2}}(\mathcal{X}_2)$. Diagonal operator matrices are the most elementary structured operator matrices, and their spectral properties provide a baseline for understanding more complicated structures. The structure of the spectral points is described below.

Theorem 3.1 *Let* $\mathbb{T} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \in \mathcal{B}_{\mathbb{A}^{1/2}}(\mathcal{X}_1 \oplus \mathcal{X}_2)$. *Then*

(1) $\sigma_{\mathbb{A}_p}(\mathbb{T}) = \sigma_{A_p}(M) \cup \sigma_{D_p}(N)$;
(2) $\sigma_{\mathbb{A}_\delta}(\mathbb{T}) = \sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N)$;
(3) $\sigma_{\mathbb{A}}(\mathbb{T}) = \sigma_A(M) \cup \sigma_D(N)$;
(4) $\sigma_{\mathbb{A}_r}(\mathbb{T}) = (\sigma_{A_r}(M) \setminus \sigma_{D_p}(N)) \cup (\sigma_{D_r}(N) \setminus \sigma_{A_p}(M))$;
(5) $\sigma_{\mathbb{A}_c}(\mathbb{T}) = (\sigma_{A_c}(M) \cap \sigma_{D_c}(N)) \cup (\sigma_{A_c}(M) \cap \rho_D(N)) \cup (\sigma_{D_c}(N) \cap \rho_A(M))$.

Proof The proofs of (1) and (2) are straightforward.

(3) Suppose $\lambda \in \rho_A(M) \cap \rho_D(N)$. There exist $R_\lambda(M) \in \mathcal{B}_{A^{1/2}}(\mathcal{X}_1)$ and $R_\lambda(N) \in \mathcal{B}_{D^{1/2}}(\mathcal{X}_2)$ such that

$$A(M - \lambda I)R_\lambda(M) = AR_\lambda(M)(M - \lambda I) = A$$

and

$$D(N - \lambda I)R_\lambda(N) = DR_\lambda(N)(N - \lambda I) = D.$$

Define $R_\lambda(\mathbb{T}) = \begin{pmatrix} R_\lambda(M) & 0 \\ 0 & R_\lambda(N) \end{pmatrix}$. Then

$$\mathbb{A}(\lambda I - \mathbb{T})R_\lambda(\mathbb{T}) = \mathbb{A}R_\lambda(\mathbb{T})(\lambda I - \mathbb{T}) = \mathbb{A},$$

and $R_\lambda(\mathbb{T}) \in \mathcal{B}_{\mathbb{A}^{1/2}}(\mathcal{X}_1 \oplus \mathcal{X}_2)$, implying $\lambda \in \rho_{\mathbb{A}}(\mathbb{T})$. The converse can be proved similarly.

(4) If $\lambda \in \sigma_{\mathbb{A}_r}(\mathbb{T})$ and $\mathbb{A}(\mathbb{T} - \lambda I)$ is injective, then $A(M - \lambda I)$ and $D(N - \lambda I)$ are also injective, implying that $\lambda \notin \sigma_{A_p}(M) \cup \sigma_{D_p}(N)$. Since $\mathcal{R}(\mathbb{A}(\mathbb{T} - \lambda I))$ is not dense, this further implies that either $\mathcal{R}(A(M - \lambda I))$ or $\mathcal{R}(D(N - \lambda I))$ is not dense. If $\mathcal{R}(A(M - \lambda I))$ is not dense, then $\lambda \in \sigma_{A_r}(M) \setminus \sigma_{D_p}(N)$. If $\mathcal{R}(D(N - \lambda I))$ is not dense, then $\lambda \in \sigma_{D_r}(N) \setminus \sigma_{A_p}(M)$. Thus,

$$\sigma_{\mathbb{A}_r}(\mathbb{T}) \subset (\sigma_{A_r}(M) \setminus \sigma_{D_p}(N)) \cup (\sigma_{D_r}(N) \setminus \sigma_{A_p}(M)).$$

On the other hand, if $\lambda \in \sigma_{A_r}(M) \setminus \sigma_{D_p}(N)$, it indicates that $A(M - \lambda I)$ and $D(N - \lambda I)$ are injective and $\mathcal{R}(A(M - \lambda I))$ is not dense, so $\lambda \in \sigma_{\mathbb{A}_r}(\mathbb{T})$. Similarly, if $\lambda \in \sigma_{D_r}(N) \setminus \sigma_{A_p}(M)$, then $\lambda \in \sigma_{\mathbb{A}_r}(\mathbb{T})$.

(5) If $\lambda \in \sigma_{\mathbb{A}_c}(\mathbb{T})$, $\mathbb{A}(\mathbb{T} - \lambda I)$ is injective, and $\mathcal{R}(\mathbb{A}(\mathbb{T} - \lambda I))$ is dense, then $A(M - \lambda I)$ and $D(N - \lambda I)$ are injective, and their ranges are both dense. Since $A(M - \lambda I)$ is injective and its range is dense, $\lambda \in \sigma_{A_c}(M) \cap \rho_A(M)$. Similarly, $\lambda \in \sigma_{D_c}(N) \cap \rho_D(N)$. Combining this with $\lambda \notin \rho_A(M) \cap \rho_D(N)$, we obtain:

$$\begin{aligned} \lambda &\in (\sigma_{A_c}(M) \cup (\rho_A(M) \setminus \rho_D(N))) \cap (\sigma_{D_c}(N) \cup (\rho_D(N) \setminus \rho_A(M))) \\ &= (\sigma_{A_c}(M) \cap \sigma_{D_c}(N)) \cup (\sigma_{A_c}(M) \cap \rho_D(N)) \cup (\sigma_{D_c}(N) \cap \rho_A(M)). \end{aligned}$$

The reverse proof follows a similar argument. □

3.2 $\mathbb{A}$-spectra of off-diagonal operator matrices

Let $P = A^\dagger A$, $Q = D^\dagger D$, and $\mathbb{P} = \mathbb{A}^\dagger \mathbb{A} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$ denote the orthogonal projection onto $\overline{\mathcal{R}(\mathbb{A})} = \overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(D)}$. The $\mathbb{A}$-bounded off-diagonal operator matrices are denoted by $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$. It is worth emphasizing that $\mathbb{T} \in \mathcal{B}_{\mathbb{A}^{1/2}}(\mathcal{X}_1 \oplus \mathcal{X}_2)$ implies $\mathbb{PTP} = \mathbb{PT}$, which is equivalent to $PKQ = PK$ and $QLP = QL$. Here we analyze the relationship between the $\mathbb{A}$-spectra of off-diagonal operator matrices and the spectra of the product of its subdiagonal elements.

The following theorem asserts that the $\mathbb{A}$-spectral structure of $\mathbb{T}$ and $-\mathbb{T}$ is identical.

Theorem 3.2 *Let $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$ be an $\mathbb{A}$-bounded off-diagonal operator matrix. Then the following statements hold:*

(1) $\sigma_{\mathbb{A}}(\mathbb{T}) = \sigma_{\mathbb{A}}(-\mathbb{T})$;
(2) $\sigma_{\mathbb{A}_p}(\mathbb{T}) = \sigma_{\mathbb{A}_p}(-\mathbb{T})$;
(3) $\sigma_{\mathbb{A}_r}(\mathbb{T}) = \sigma_{\mathbb{A}_r}(-\mathbb{T})$.

Proof (1) Since $\mathbb{P}(\mathbb{T} - \lambda I) = \mathbb{P}(\mathbb{PT} - \lambda I)$, it follows that $\sigma_{\mathbb{A}}(\mathbb{PT}) = \sigma_{\mathbb{A}}(\mathbb{T})$. Let $\mathbb{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \in \mathcal{B}_{\mathbb{A}}(\mathcal{X}_1 \oplus \mathcal{X}_2)$, then $\mathbb{J}^\sharp = \begin{pmatrix} P & 0 \\ 0 & -Q \end{pmatrix}$, and thus $\mathbb{J}\mathbb{J}^\sharp = \mathbb{J}^\sharp \mathbb{J} = \mathbb{P}$, which implies that $\mathbb{J}$ is $\mathbb{A}$-unitary. Observing that

$$\mathbb{J}^\sharp \mathbb{T} \mathbb{J} = \begin{pmatrix} 0 & -PK \\ -QL & 0 \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} 0 & -K \\ -L & 0 \end{pmatrix} = -\mathbb{PT},$$

by applying [19, Theorem 5.14], we conclude that

$$\sigma_{\mathbb{A}}(\mathbb{T}) = \sigma_{\mathbb{A}}(\mathbb{J}^\sharp \mathbb{T} \mathbb{J}) = \sigma_{\mathbb{A}}(-\mathbb{PT}) = \sigma_{\mathbb{A}}(-\mathbb{T}).$$

(2) Suppose $\lambda \in \sigma_{\mathbb{A}_p}(\mathbb{T})$, then there exists $z \neq \mathbf{0}$ such that $\mathbb{P}(\mathbb{T} - \lambda I)z = \mathbf{0}$. Since

$$\begin{aligned} \mathbb{P}(\mathbb{T} - \lambda I)z = \mathbf{0} &\Longleftrightarrow \mathbb{P}(-\mathbb{J}^\sharp \mathbb{T} \mathbb{J} - \lambda I)z = \mathbf{0} \Longleftrightarrow \mathbb{J}^\sharp(-\mathbb{T} - \lambda I)\mathbb{J}z = \mathbf{0} \\ &\Longleftrightarrow \mathbb{J}\mathbb{J}^\sharp(-\mathbb{T} - \lambda I)\mathbb{J}z = \mathbf{0} \Longleftrightarrow \mathbb{P}(-\mathbb{T} - \lambda I)\mathbb{J}z = \mathbf{0}, \end{aligned}$$

together with $\mathbb{J}z \neq \mathbf{0}$, we deduce that $\lambda \in \sigma_{\mathbb{A}_p}(-\mathbb{T})$.

(3) Observe that $\mathcal{N}(\mathbb{A})$ is an invariant space of $\mathbb{T}$ and $\mathbb{J}^\sharp(\mathbb{T} - \lambda I)\mathbb{J} = \mathbb{P}(-\mathbb{T} - \lambda I)$, then

$$\overline{\mathcal{R}(\mathbb{A}(-\mathbb{T} - \lambda I))} = \overline{\mathcal{R}(\mathbb{P}(-\mathbb{T} - \lambda I))} = \overline{\mathcal{R}(\mathbb{J}^\sharp(\mathbb{T} - \lambda I)\mathbb{J})}.$$

For all $y \in \mathcal{R}(\mathbb{P}(\mathbb{T} - \lambda I))$, there exists $x \in \mathcal{X}_1 \oplus \mathcal{X}_2$ such that $y = \mathbb{P}(\mathbb{T} - \lambda I)x = \mathbb{P}(\mathbb{T} - \lambda I)\mathbb{P}x$. Let $z = \mathbb{J}^\sharp x$, then

$$\mathbb{J}^\sharp(\mathbb{T} - \lambda I)\mathbb{J}z = \mathbb{J}^\sharp \mathbb{P}(\mathbb{T} - \lambda I)\mathbb{P}x = \mathbb{J}^\sharp y \in \mathcal{R}(\mathbb{J}^\sharp(\mathbb{T} - \lambda I)\mathbb{J}),$$

i.e., $\mathbb{J}^\sharp \mathcal{R}(\mathbb{P}(\mathbb{T} - \lambda I)) \subset \mathcal{R}(\mathbb{J}^\sharp(\mathbb{T} - \lambda I)\mathbb{J})$. For all $w \in \mathcal{R}(\mathbb{J}^\sharp(\mathbb{T} - \lambda I)\mathbb{J})$, there exists $z \in \mathcal{X}_1 \oplus \mathcal{X}_2$ such that $\mathbb{J}z = \mathbb{P}x$ and

$$w = \mathbb{J}^\sharp(\mathbb{T} - \lambda I)\mathbb{J}z = \mathbb{J}^\sharp \mathbb{P}(\mathbb{T} - \lambda I)\mathbb{P}x = \mathbb{J}^\sharp \mathbb{P}(\mathbb{T} - \lambda I)x,$$

i.e., $\mathcal{R}(\mathbb{J}^\sharp(\mathbb{T}-\lambda I)\mathbb{J}) \subset \mathbb{J}^\sharp\mathcal{R}(\mathbb{P}(\mathbb{T}-\lambda I))$, hence

$$\overline{\mathcal{R}(\mathbb{J}^\sharp(\mathbb{T}-\lambda I)\mathbb{J})} = \mathbb{J}^\sharp\overline{\mathcal{R}(\mathbb{P}(\mathbb{T}-\lambda I))}.$$

As $\mathbb{J}^\sharp$ is $\mathbb{A}$-unitary, it follows that

$$\begin{aligned}\overline{\mathcal{R}(\mathbb{A}(-\mathbb{T}-\lambda I))} &= \overline{\mathcal{R}(\mathbb{J}^\sharp(\mathbb{T}-\lambda I)\mathbb{J})} = \mathbb{J}^\sharp\overline{\mathcal{R}(\mathbb{P}(\mathbb{T}-\lambda I))} \\ &= \overline{\mathcal{R}(\mathbb{P}(\mathbb{T}-\lambda I))} = \overline{\mathcal{R}(\mathbb{A}(\mathbb{T}-\lambda I))} \neq \overline{\mathcal{R}(\mathbb{A})}.\end{aligned}$$

Together with the fact that $\mathbb{P}(\mathbb{T}+\lambda I)$ is injective if and only if $\mathbb{P}(\mathbb{T}-\lambda I)$ is injective, we conclude that $\sigma_{\mathbb{A}_r}(\mathbb{T}) = \sigma_{\mathbb{A}_r}(-\mathbb{T})$. □

For $\mathbb{A}$-bounded off-diagonal operator matrices, the spectral structure is closely linked to the product operators KL and LK.

Theorem 3.3 *Let* $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$ *be an* $\mathbb{A}$*-bounded off-diagonal operator matrix. Then*

$$\sigma_{\mathbb{A}_p}(\mathbb{T}) = \{\lambda \in \mathbb{C} : \lambda^2 \in \sigma_{A_p}(KL) \cup \sigma_{D_p}(LK)\}.$$

Proof For $\lambda \in \sigma_{\mathbb{A}_p}(\mathbb{T})$, there exists a nonzero vector z such that $\mathbb{P}(\mathbb{T}-\lambda I)z = \mathbf{0}$. Consequently,

$$\mathbb{P}(\mathbb{T}^2-\lambda^2 I)z = \mathbb{P}(\mathbb{T}+\lambda I)\mathbb{P}(\mathbb{T}-\lambda I)z = \mathbf{0},$$

indicating that $\lambda^2 \in \sigma_{\mathbb{A}_p}(\mathbb{T}^2)$. Using the equation

$$\begin{aligned}\mathbb{P}(\mathbb{T}^2-\lambda^2 I) &= \mathbb{P}(\mathbb{T}+\lambda I)\mathbb{P}(\mathbb{T}-\lambda I) \\ &= \mathbb{P}(\mathbb{T}-\lambda I)\mathbb{P}(\mathbb{T}+\lambda I) \\ &= \begin{pmatrix} P(KL-\lambda^2 I) & 0 \\ 0 & Q(LK-\lambda^2 I) \end{pmatrix},\end{aligned} \tag{3.1}$$

and combining Theorem 3.1, we deduce that $\lambda^2 \in \sigma_{A_p}(KL)$ or $\lambda^2 \in \sigma_{D_p}(LK)$. Thus,

$$\sigma_{\mathbb{A}_p}(\mathbb{T}) \subset \{\lambda \in \mathbb{C} : \lambda^2 \in \sigma_{A_p}(KL) \cup \sigma_{D_p}(LK)\}.$$

Since $\sigma_{\mathbb{A}_p}(\mathbb{T}) = \sigma_{\mathbb{A}_p}(-\mathbb{T})$, it follows that

$$\sigma_{\mathbb{A}_p}(\mathbb{T}) \cup \sigma_{\mathbb{A}_p}(-\mathbb{T}) \subset \{\lambda \in \mathbb{C} : \lambda^2 \in \sigma_{A_p}(KL) \cup \sigma_{D_p}(LK)\}.$$

Conversely, if $\lambda^2 \in \sigma_{A_p}(KL)$, there exists a nonzero vector x such that $P(KL-\lambda^2)x = 0$. Let $z = \begin{pmatrix} x \\ 0 \end{pmatrix} \neq \mathbf{0}$. Then

$$\mathbb{P}(\mathbb{T}^2-\lambda^2 I)z = \begin{pmatrix} P(KL-\lambda^2 I) & 0 \\ 0 & Q(LK-\lambda^2 I) \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \mathbf{0},$$

implying that $\mathbb{P}(\mathbb{T}^2-\lambda^2 I)$ is not injective. Hence, there exists a nonzero vector u such that $\mathbb{P}(\mathbb{T}^2-\lambda^2 I)u = 0$. If $\mathbb{P}(\mathbb{T}-\lambda I)$ is injective, Eq. (3.1) implies that $\mathbb{P}(\mathbb{T}+\lambda I)u \in \mathcal{N}(\mathbb{P}(\mathbb{T}-\lambda I))$, and thus $\mathbb{P}(\mathbb{T}+\lambda I)u = 0$, i.e., $\lambda \in \sigma_{\mathbb{A}_p}(-\mathbb{T})$. Similarly, if $\mathbb{P}(\mathbb{T}+\lambda I)$ is injective, then $\lambda \in \sigma_{\mathbb{A}_p}(\mathbb{T})$. By analogous reasoning, if $\lambda^2 \in \sigma_{D_p}(LK)$, it follows that $\lambda \in \sigma_{\mathbb{A}_p}(\mathbb{T}) \cup \sigma_{\mathbb{A}_p}(-\mathbb{T}) = \sigma_{\mathbb{A}_p}(\mathbb{T})$. Therefore, $\sigma_{\mathbb{A}_p}(\mathbb{T}) = \{\lambda \in \mathbb{C} : \lambda^2 \in \sigma_{A_p}(KL) \cup \sigma_{D_p}(LK)\}$. □

Theorem 3.4 *Let* $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$ *be an* $\mathbb{A}$*-bounded off-diagonal operator matrix. If* $\mathbb{A}\mathbb{T} = \mathbb{T}^*\mathbb{A}$, *then*

$$\sigma_{\mathbb{A}_p}(\mathbb{T}) \subset \mathbb{R}.$$

Proof For $\lambda \in \sigma_{\mathbb{A}_p}(\mathbb{T})$, according to Theorem 3.3, we have $\lambda^2 \in \sigma_{A_p}(KL) \cup \sigma_{D_p}(LK)$. If $\lambda^2 \in \sigma_{A_p}(KL)$, then there exists $x(\neq 0) \in \overline{\mathcal{R}(A)}$ such that

$$A(KL - \lambda^2 I)x = 0. \tag{3.2}$$

$\mathbb{A}\mathbb{T} = \mathbb{T}^*\mathbb{A}$ implies that

$$AK = L^*D \text{ and } DL = K^*A. \tag{3.3}$$

Substituting Eq.(3.3) into Eq.(3.2), we obtain $L^*K^*Ax = \lambda^2 Ax$. Taking the conjugates on both sides, then

$$AKLx = \overline{\lambda}^2 Ax.$$

Compare with Eq.(3.2) and $Ax \neq 0$, it follows that $\lambda^2 = \overline{\lambda}^2$, i.e., $\lambda^2 \in \mathbb{R}$. For $x(\neq 0) \in \overline{\mathcal{R}(A)}$ and D is positive, we have

$$\lambda^2 \langle Ax, x\rangle = \langle AKLx, x\rangle = \langle L^*DLx, x\rangle \geq 0.$$

Since $\langle Ax, x\rangle > 0$, then $\lambda^2 \geq 0$. If $\lambda^2 \in \sigma_{D_p}(LK)$, by an analogous argument, we also obtain $\lambda^2 \geq 0$. Consequently, $\lambda \in \mathbb{R}$. □

Theorem 3.5 *Let* $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$ *be an* $\mathbb{A}$*-bounded off-diagonal operator matrix and suppose that* $\mathcal{R}(\mathbb{A})$ *is closed. Then*

$$\sigma_{\mathbb{A}}(\mathbb{T}) = \{\lambda \in \mathbb{C} : \lambda^2 \in \sigma_A(KL) \cup \sigma_D(LK)\}.$$

Proof To prove $\sigma_{\mathbb{A}}(\mathbb{T}) = \{\lambda \in \mathbb{C} : \lambda^2 \in \sigma_A(KL) \cup \sigma_D(LK)\}$, it suffices to show that $\rho_{\mathbb{A}}(\mathbb{T}) = \{\lambda \in \mathbb{C} : \lambda^2 \in \rho_A(KL) \cap \rho_D(LK)\}$.

If $0 \in \rho_{\mathbb{A}}(\mathbb{T})$, then $0 \in \rho_{\mathbb{A}}(\mathbb{T}^2)$, where $\mathbb{T}^2 = \begin{pmatrix} KL & 0 \\ 0 & LK \end{pmatrix}$. By Theorem 3.1, $0 \in \rho_A(KL) \cap \rho_D(LK)$.

For $0 \neq \lambda \in \rho_{\mathbb{A}}(\mathbb{T})$, this implies that $A(KL - \lambda^2 I)$ is both injective and surjective. If it is not injective, there would exist a nonzero vector x_0 such that $A(KL - \lambda^2 I)x_0 = 0$. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ \frac{1}{\lambda} Lx_0 \end{pmatrix}.$$

Then $\mathbb{A}(\mathbb{T} - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$, contradicting $\lambda \in \rho_{\mathbb{A}}(\mathbb{T})$. Hence, $A(KL - \lambda^2 I)$ is injective. Similarly, we can prove that $A(KL - \lambda^2 I)$ is surjective. Given $\lambda \in \rho_{\mathbb{A}}(\mathbb{T})$, for all $\begin{pmatrix} f & g \end{pmatrix}^{\mathrm{T}} \in \overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(D)}$, the equations

$$\begin{cases} -\lambda Ax + AKy = f \\ DLx - \lambda Dy = g \end{cases} \implies \begin{cases} -\lambda Px + PKy = A^\dagger f \\ QLx - \lambda Qy = D^\dagger g \end{cases}$$

have a solution. In particular, if $g=0$, then $Qy=\frac{1}{\lambda}QLx$. Substituting this and $PKQ=PK$ into the first equation yields

$$-\lambda Px+PKy=-\lambda Px+PKQy=\frac{1}{\lambda}P(KL-\lambda^2I)x=A^\dagger f.$$

Since $\mathcal{R}(A)$ is closed, it follows that $A(KL-\lambda^2I)x=\lambda AA^\dagger f=\lambda f$. The arbitrariness of f implies that $A(KL-\lambda^2I)$ is surjective, hence $\lambda^2\in\rho_A(KL)$. By analogous reasoning, $\lambda^2\in\rho_D(LK)$. Therefore, $\rho_{\mathbb{A}}(\mathbb{T})\subset\{\lambda\in\mathbb{C}:\lambda^2\in\rho_A(KL)\cap\rho_D(LK)\}$.

Conversely, if $\lambda^2\in\rho_A(KL)\cap\rho_D(LK)$, then $\lambda^2\in\rho_{\mathbb{A}}(\mathbb{T}^2)$. By applying Eq. (3.1) in conjunction with Lemma 2.7, we deduce that $\lambda\in\rho_{\mathbb{A}}(\mathbb{T})\cap\rho_{\mathbb{A}}(-\mathbb{T})$. According to Theorem 3.2, $\rho_{\mathbb{A}}(\mathbb{T})=\rho_{\mathbb{A}}(-\mathbb{T})$, and therefore $\rho_{\mathbb{A}}(\mathbb{T})=\{\lambda\in\mathbb{C}:\lambda^2\in\rho_A(KL)\cup\rho_D(LK)\}$. □

Next, we establish the relationship between the point spectra of KL and LK. To achieve this, we require some auxiliary results. The semi-inner product $\langle x,y\rangle_A$ induces an inner product $[\cdot,\cdot]$ on the quotient space $\mathcal{X}/\mathcal{N}(A)$ defined by $[\overline{x},\overline{y}]=\langle x,y\rangle_A$, for all $\overline{x},\overline{y}\in\mathcal{X}/\mathcal{N}(A)$. The completion of $(\mathcal{X}/\mathcal{N}(A),[\cdot,\cdot])$ is isometrically isomorphic to $\mathcal{R}(A^{1/2})$ with a new inner product

$$(A^{1/2}x,A^{1/2}y)=\langle Px,Py\rangle, \tag{3.4}$$

for all $x,y\in\mathcal{X}$, where P denotes the orthogonal projection on $\overline{\mathcal{R}(A)}$.

In what follows, the Hilbert space $(\mathcal{R}(A^{1/2}),(\cdot,\cdot))$ will be denoted by $\boldsymbol{\mathcal{R}}(A^{1/2})$. Define $Z_A:\mathcal{X}\to\boldsymbol{\mathcal{R}}(A^{1/2})$ by $Z_Ax=Ax$ for any $x\in\mathcal{X}$. Furthermore, there exists a uniquely determined operator $\widetilde{T}\in\mathcal{B}(\boldsymbol{\mathcal{R}}(A^{1/2}))$ such that $Z_AT=\widetilde{T}Z_A$ for any $T\in\mathcal{B}_{A^{1/2}}(\mathcal{X})$ (see [25]). Additionally, the following properties hold:

Proposition 3.6 [17] *Let $T\in\mathcal{B}_{A^{1/2}}(\mathcal{X})$ and let $\widetilde{T}\in\mathcal{B}(\boldsymbol{\mathcal{R}}(A^{1/2}))$ denote the operator corresponding to T. Then $\sigma(\widetilde{T})\subset\sigma_A(T)$ and $\|\widetilde{T}\|_{\boldsymbol{\mathcal{R}}(A^{1/2})}=\|T\|_A$.*

Proposition 3.7 [30] *Let $A\in\mathcal{B}(\mathcal{X}_1)^+$ and $D\in\mathcal{B}(\mathcal{X}_2)^+$. For any operators $L\in\mathcal{B}(\mathcal{X}_1,\mathcal{X}_2)$ and $K\in\mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)$ satisfying $DL=K^*A$, there exist uniquely determined operators $\widetilde{L}\in\mathcal{B}(\boldsymbol{\mathcal{R}}(A^{1/2}),\boldsymbol{\mathcal{R}}(D^{1/2}))$ and $\widetilde{K}\in\mathcal{B}(\boldsymbol{\mathcal{R}}(D^{1/2}),\boldsymbol{\mathcal{R}}(A^{1/2}))$ such that $Z_DL=\widetilde{L}Z_A$, $Z_AK=\widetilde{K}Z_D$, and $\widetilde{L}^*=\widetilde{K}$.*

In the following theorem, we establish the relationship between $\sigma_{A_p}(KL)$ and $\sigma_{D_p}(LK)$.

Theorem 3.8 *Let $\mathbb{T}=\begin{pmatrix}0&K\\L&0\end{pmatrix}$ be an $\mathbb{A}$-bounded off-diagonal operator matrix satisfying $DL=K^*A$. Then*

$$\sigma_{A_p}(KL)\setminus\{0\}=\sigma_{D_p}(LK)\setminus\{0\}.$$

Proof For $\mu\in\sigma_{A_p}(KL)\setminus\{0\}$, there exists a nonzero vector x_0 such that

$$AKLx_0=\mu Ax_0. \tag{3.5}$$

According to Proposition 3.7, there exist uniquely determined operators $\widetilde{L} \in \mathcal{B}(\boldsymbol{\mathcal{R}}(A^{1/2}), \boldsymbol{\mathcal{R}}(D^{1/2}))$ and $\widetilde{K} \in \mathcal{B}(\boldsymbol{\mathcal{R}}(D^{1/2}), \boldsymbol{\mathcal{R}}(A^{1/2}))$ such that $AKx = \widetilde{K}Dx$, $DLx = \widetilde{L}Ax$, for all $x \in \mathcal{X}$. Then $AKLx_0 = \widetilde{K}\widetilde{L}Ax_0$, substituting it into Eq. (3.5) and multiplying by $\widetilde{L}$, we have

$$\widetilde{L}\widetilde{K}\widetilde{L}Ax_0 = \mu(\widetilde{L}Ax_0),$$

it follows that $DLKLx_0 = \mu DLx_0$, i.e., $D(LK - \mu)Lx_0 = 0$ with $Lx_0 \neq 0$. Therefore, $\mu \in \sigma_{D_p}(LK) \setminus \{0\}$. □

By combining Theorem 3.3 and Theorem 3.8, we obtain the following corollary.

Corollary 3.9 *Let* $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$ *be an* $\mathbb{A}$*-bounded off-diagonal operator matrix satisfying* $DL = K^*A$. *Then*

$$\begin{aligned} \sigma_{\mathbb{A}_p}(\mathbb{T}) \setminus \{0\} &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \in \sigma_{A_p}(KL)\} \\ &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \in \sigma_{D_p}(LK)\}. \end{aligned}$$

The following example demonstrates the effectiveness of Theorem 3.4 and Corollary 3.9.

Example 3.10 *Let* $\mathcal{X} = \ell^2(\mathbb{N})$, *where* $\ell^2(\mathbb{N})$ *denotes the space of square-summable complex sequences. For any* $x = (x_1, x_2, \ldots, x_k, \ldots) \in \ell^2(\mathbb{N})$, *consider the operators* A, D, K, L *defined by*

$$\begin{aligned} Ax_k &= \frac{1}{k}x_k, \\ Dx_k &= \frac{k}{k+1}x_k, \\ Kx_k &= \begin{cases} 0, & k = 1 \\ x_{k-1}, & k \geq 2 \end{cases}, \\ Lx_k &= \frac{1}{k}x_{k+1}. \end{aligned}$$

We have the following results:

(1) $A > 0$ *and* $D > 0$. *Hence, the operator matrix* $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ *induces a semi-Hilbert space structure on* $\mathcal{X}_{\mathbb{A}} = \mathcal{X}_A \oplus \mathcal{X}_D$, *equipped with the semi-inner product*

$$\langle x, y \rangle_{\mathbb{A}} = \langle \mathbb{A}x, y \rangle, \; x, y \in \mathcal{X}_{\mathbb{A}},$$

where $\langle \cdot, \cdot \rangle$ *denotes the standard inner product on* $\ell^2(\mathbb{N})$;

(2) The off-diagonal operator matrix

$$\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$$

is $\mathbb{A}$*-bounded, and*

$$\begin{aligned} \sigma_{\mathbb{A}_p}(\mathbb{T}) \setminus \{0\} &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \in \sigma_{A_p}(KL)\} \\ &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \in \sigma_{D_p}(LK)\} \\ &= \{\pm\frac{1}{\sqrt{k}}, k = 1, 2, 3, \ldots\}. \end{aligned}$$

Proof It is straightforward to verify (1). Moreover, a direct computation yields

$$\mathbb{T}^* = \begin{pmatrix} 0 & L^* \\ K^* & 0 \end{pmatrix},$$

where $K^* x_k = x_{k+1}$, and

$$L^* x_k = \begin{cases} 0, & k = 1 \\ \dfrac{1}{k-1} x_{k-1}, & k \geq 2, \end{cases}$$

for any $x = (x_1, x_2, \dots, x_k, \dots) \in \ell^2(\mathbb{N})$.

Clearly, $\mathbb{T}$ is non-selfadjoint, and we have the identities $AK = L^* D = \begin{cases} 0, & k = 1 \\ \frac{1}{k} x_{k-1}, & k \geq 2 \end{cases}$ and $DL = K^* A = \dfrac{1}{k+1} x_{k+1}$. It follows that

$$\mathbb{A}\mathbb{T} = \mathbb{T}^* \mathbb{A}.$$

Furthermore,

$$A(KL - \mu I) x_k = \frac{1}{k}(\alpha_k - \mu) x_k,$$

where $\alpha_k = \begin{cases} 0, & k = 1 \\ \dfrac{1}{k-1}, & k \geq 2 \end{cases}$. It is not injective if and only if there exists k such that $\mu = \alpha_k$, i.e.,

$$\mu \in \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}.$$

Moreover,

$$D(LK - \nu I) x_k = \frac{k}{k+1} \left(\frac{1}{k} - \nu \right) x_k.$$

It is not injective if and only if there exists k such that $\nu = \dfrac{1}{k}$, i.e.,

$$\nu \in \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}.$$

Consequently, the A-point spectrum of KL is given by

$$\sigma_{A_p}(KL) = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\},$$

and the D-point spectrum of LK is given by

$$\sigma_{D_p}(LK) = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}.$$

Applying Corollary 3.9, we conclude that

$$\begin{aligned} \sigma_{\mathbb{A}_p}(\mathbb{T}) \setminus \{0\} &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \in \sigma_{A_p}(KL)\} \\ &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \in \sigma_{D_p}(LK)\} \\ &= \{\pm \frac{1}{\sqrt{k}}, k = 1, 2, 3, \dots\}. \end{aligned}$$

□

Based on Lemma 2.9 and Theorem 3.3, we proceed to investigate the relationship between the remaining $\mathbb{A}$-spectral points and the spectral points of the product operators of off-diagonal entries.

Theorem 3.11 *Let* $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$ *be an* $\mathbb{A}$*-bounded off-diagonal operator matrix satisfying* $DL = K^*A$*. Then*

$$\begin{aligned}\sigma_{\mathbb{A}_r}(\mathbb{T}) \setminus \{0\} &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \notin \sigma_{A_p}(KL), \overline{\lambda}^2 \in \sigma_{D_p}(K^*L^*)\} \\ &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \notin \sigma_{D_p}(LK), \overline{\lambda}^2 \in \sigma_{A_p}(L^*K^*)\}.\end{aligned}$$

Proof It suffices to prove that $\sigma_{\mathbb{A}_r}(\mathbb{T})\setminus\{0\} = \{\lambda \in \mathbb{C}\setminus\{0\} : \lambda^2 \notin \sigma_{A_p}(KL), \overline{\lambda}^2 \in \sigma_{D_p}(K^*L^*)\}$. The proof for the second equation is entirely analogous.

If $\lambda \in \sigma_{\mathbb{A}_r}(\mathbb{T}) \setminus \{0\}$, by Lemma 2.9, we have $\overline{\lambda} \in \sigma_{\mathbb{A}_p}(\mathbb{T}^*) \setminus \{0\}$. Applying Theorem 3.3, it follows that

$$\overline{\lambda}^2 \in \sigma_{A_p}(L^*K^*) \cup \sigma_{D_p}(K^*L^*).$$

Since $\sigma_{A_p}(L^*K^*) \setminus \{0\} = \sigma_{D_p}(K^*L^*) \setminus \{0\}$, we conclude that $\overline{\lambda}^2 \in \sigma_{D_p}(K^*L^*)$. Moreover, since $\lambda \notin \sigma_{\mathbb{A}_p}(\mathbb{T})$, it implies that $\lambda^2 \notin \sigma_{A_p}(KL)$. Thus,

$$\sigma_{\mathbb{A}_r}(\mathbb{T}) \setminus \{0\} \subset \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \notin \sigma_{A_p}(KL), \overline{\lambda}^2 \in \sigma_{D_p}(K^*L^*)\}.$$

Conversely, if $\lambda \in \{\lambda \in \mathbb{C}\setminus\{0\} : \lambda^2 \notin \sigma_{A_p}(KL), \overline{\lambda}^2 \in \sigma_{D_p}(K^*L^*)\}$, then $\overline{\lambda}^2 \in \sigma_{D_p}(K^*L^*)$ and Lemma 2.9 implies that $\lambda^2 \in \sigma_{D_p}(LK) \cup \sigma_{D_r}(LK)$. Since $\lambda^2 \notin \sigma_{A_p}(KL)$ and $\sigma_{A_p}(KL) \setminus \{0\} = \sigma_{D_p}(LK) \setminus \{0\}$, it follows that $\lambda^2 \in \sigma_{D_r}(LK)$. Using Eq. (3.1), we deduce that $\lambda \in \sigma_{\mathbb{A}_r}(\mathbb{T}) \setminus \{0\} \cup \sigma_{\mathbb{A}_r}(-\mathbb{T}) \setminus \{0\}$. By Theorem 3.2(3) and the above argument, we obtain

$$\sigma_{\mathbb{A}_r}(\mathbb{T}) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \notin \sigma_{A_p}(KL), \overline{\lambda}^2 \in \sigma_{D_p}(K^*L^*)\}.$$

□

Theorem 3.12 *Let* $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$ *be an* $\mathbb{A}$*-bounded off-diagonal operator matrix satisfying* $DL = K^*A$*. Then*

$$\begin{aligned}\sigma_{\mathbb{A}_c}(\mathbb{T}) \setminus \{0\} &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \notin \sigma_{A_p}(KL), \overline{\lambda}^2 \notin \sigma_{A_p}(L^*K^*), \mathcal{R}(A(KL - \lambda^2 I)) \neq \overline{\mathcal{R}(A)}\} \\ &= \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \notin \sigma_{D_p}(LK), \overline{\lambda}^2 \notin \sigma_{D_p}(K^*L^*), \mathcal{R}(D(LK - \lambda^2 I)) \neq \overline{\mathcal{R}(D)}\}.\end{aligned}$$

Proof It suffices to prove that $\sigma_{\mathbb{A}_c}(\mathbb{T}) \setminus \{0\} = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \notin \sigma_{A_p}(KL), \overline{\lambda}^2 \notin \sigma_{A_p}(L^*K^*), \mathcal{R}(A(KL - \lambda^2 I)) \neq \overline{\mathcal{R}(A)}\}$. The proof for the second equation is entirely analogous.

If $\lambda \in \sigma_{\mathbb{A}_c}(\mathbb{T}) \setminus \{0\}$, then $\lambda \notin \sigma_{\mathbb{A}_p}(\mathbb{T})$ and $\lambda \notin \sigma_{\mathbb{A}_r}(\mathbb{T})$. Hence, $\lambda^2 \notin \sigma_{A_p}(KL)$ and $\overline{\lambda}^2 \notin \sigma_{A_p}(L^*K^*)$. Together with $\lambda \notin \rho_{\mathbb{A}}(\mathbb{T})$, it implies that $\mathcal{R}(A(KL - \lambda^2 I)) \neq \overline{\mathcal{R}(A)}$. Therefore, $\sigma_{\mathbb{A}_c}(\mathbb{T}) \setminus \{0\} \subset \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^2 \notin \sigma_{A_p}(KL), \overline{\lambda}^2 \notin \sigma_{A_p}(L^*K^*), \mathcal{R}(A(KL - \lambda^2 I)) \neq \overline{\mathcal{R}(A)}\}$.

Conversely, if $\lambda \neq 0$, $\lambda^2 \notin \sigma_{A_p}(KL)$ implies that $\lambda \notin \sigma_{\mathbb{A}_p}(\mathbb{T})$. Moreover, $\overline{\lambda}^2 \notin \sigma_{A_p}(L^*K^*)$ indicates that $\lambda \notin \sigma_{\mathbb{A}_r}(\mathbb{T})$, and $\mathcal{R}(A(KL - \lambda^2 I)) \neq \overline{\mathcal{R}(A)}$ shows that $\lambda \notin \rho_{\mathbb{A}}(\mathbb{T})$. Consequently, $\lambda \in \sigma_{\mathbb{A}_c}(\mathbb{T})$. □

3.3 $\mathbb{A}$-spectra of upper triangular operator matrices

Let $\mathbb{T} = \begin{pmatrix} M & K \\ 0 & N \end{pmatrix}$ be an $\mathbb{A}$-bounded upper triangular operator matrix. Upper-triangular operator matrices are the most general form among the three types considered. Their analysis unifies and extends the previous results. The following results demonstrate that the distribution of spectral points of upper triangular matrices can be controlled by the spectral points of the principal diagonal entries.

Theorem 3.13 *Let $\mathbb{T} = \begin{pmatrix} M & K \\ 0 & N \end{pmatrix}$ be an $\mathbb{A}$-bounded upper triangular operator matrix. Then:*

(1) $(\sigma_A(M) \setminus \sigma_D(N)) \cup (\sigma_D(N) \setminus \sigma_A(M)) \subset \sigma_{\mathbb{A}}(\mathbb{T}) \subset \sigma_A(M) \cup \sigma_D(N)$*;*
(2) $\sigma_{A_p}(M) \subset \sigma_{\mathbb{A}_p}(\mathbb{T}) \subset \sigma_{A_p}(M) \cup \sigma_{D_p}(N)$*;*
(3) $\sigma_{D_\delta}(N) \subset \sigma_{\mathbb{A}_\delta}(\mathbb{T}) \subset \sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N)$*;*
(4) $(\sigma_A(M) \setminus \sigma_{A_p}(M)) \cap \sigma_{D_r}(N) \subset \sigma_{\mathbb{A}_r}(\mathbb{T}) \subset \sigma_{A_r}(M) \cup \sigma_{D_p}(N) \cup \sigma_{D_r}(N)$*.*

Proof (1) Consider the equation

$$\mathbb{T} - \lambda I = \begin{pmatrix} I & 0 \\ 0 & N - \lambda I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I \end{pmatrix} \begin{pmatrix} M - \lambda I & 0 \\ 0 & I \end{pmatrix}. \tag{3.6}$$

Since the second term is always $\mathbb{A}$-invertible, using Lemma 2.7, we obtain $\sigma_{\mathbb{A}}(\mathbb{T}) \subset \sigma_A(M) \cup \sigma_D(N)$.

If $\lambda \in \sigma_A(M) \setminus \sigma_D(N)$ and $\lambda \in \rho_{\mathbb{A}}(\mathbb{T})$, from Eq. (3.6), it follows that $\lambda \in \rho_A(M)$, which contradicts $\lambda \in \sigma_A(M)$. Therefore, $\lambda \in \sigma_{\mathbb{A}}(\mathbb{T})$. Similarly, $\sigma_D(N) \setminus \sigma_A(M) \subset \sigma_{\mathbb{A}}(\mathbb{T})$. In conclusion,

$$(\sigma_A(M) \setminus \sigma_D(N)) \cup (\sigma_D(N) \setminus \sigma_A(M)) \subset \sigma_{\mathbb{A}}(\mathbb{T}) \subset \sigma_A(M) \cup \sigma_D(N).$$

(2) If $\lambda \in \sigma_{A_p}(M)$, there exists a nonzero vector x_0 such that $A(M - \lambda I)x_0 = 0$. Let $u = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$, where $u \neq \mathbf{0}$. Then

$$\mathbb{A}(\mathbb{T} - \lambda I)u = \mathbf{0},$$

which implies that $\sigma_{A_p}(M) \subset \sigma_{\mathbb{A}_p}(\mathbb{T})$.

If $\lambda \in \sigma_{\mathbb{A}_p}(\mathbb{T})$, there exist elements x, y, not all zero, such that

$$\mathbb{A}(\mathbb{T} - \lambda I) \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}, \quad \text{i.e.,} \quad \begin{cases} A(M - \lambda I)x + AKy & = 0, \\ D(N - \lambda I)y & = 0. \end{cases}$$

If $y \neq 0$, then $\lambda \in \sigma_{D_p}(N)$. If $y = 0$, then $x \neq 0$ and $A(M - \lambda I)x = 0$, which implies $\lambda \in \sigma_{A_p}(M)$. Therefore, $\lambda \in \sigma_{A_p}(M) \cup \sigma_{D_p}(N)$.

(3) It is evident that $\sigma_{D_\delta}(N) \subset \sigma_{\mathbb{A}_\delta}(\mathbb{T})$. If $\lambda \in \sigma_{\mathbb{A}_\delta}(\mathbb{T})$, then $\mathcal{R}(\mathbb{A}(\mathbb{T} - \lambda I)) \neq \overline{\mathcal{R}(\mathbb{A})}$, which implies

$$\mathcal{R}(A(M - \lambda I)) + \mathcal{R}(AK|_{\mathcal{D}(D(N-\lambda I))}) \neq \overline{\mathcal{R}(A)} \quad \text{or} \quad \mathcal{R}(D(N - \lambda I)) \neq \overline{\mathcal{R}(D)}.$$

If $\mathcal{R}(A(M - \lambda I)) + \mathcal{R}(AK|_{\mathcal{D}(D(N-\lambda I))}) \neq \overline{\mathcal{R}(A)}$ and $\lambda \notin \sigma_{A_\delta}(M)$, then $\mathcal{R}(A(M - \lambda I)) = \overline{\mathcal{R}(A)}$ and $\mathcal{R}(A(M - \lambda I)) + \mathcal{R}(AK|_{\mathcal{D}(D(N-\lambda I))}) = \overline{\mathcal{R}(A)}$, leading to a contradiction. Hence, $\lambda \in \sigma_{A_\delta}(M)$. If $\mathcal{R}(D(N - \lambda I)) \neq \overline{\mathcal{R}(D)}$, this shows that $\lambda \in \sigma_{D_\delta}(N)$. Therefore,

$$\sigma_{\mathbb{A}_\delta}(\mathbb{T}) \subset \sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N).$$

(4) If $\lambda \in (\sigma_A(M) \setminus \sigma_{A_p}(M)) \cap \sigma_{D_r}(N)$, this indicates that $A(M-\lambda I)$ and $D(N-\lambda I)$ are injective. Setting $\mathbb{A}(\mathbb{T}-\lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$, which yields that $A(M-\lambda I)x + AKy = 0$ and $D(N-\lambda I)y = 0$. As $D(N-\lambda I)$ is injective, then $y=0$. Substituting it into the first equation and combining $A(M-\lambda I)$ is injective, we obtain $x = 0$. Hence, $\mathbb{A}(\mathbb{T}-\lambda I)$ is injective. Because $\mathcal{R}(D(LK-\lambda I)) \neq \overline{\mathcal{R}(D)}$, it shows that $\lambda \in \sigma_{\mathbb{A}_r}(\mathbb{T})$, then $\overline{\lambda} \in \sigma_{\mathbb{A}_p}(\mathbb{T}^*)$. Consider the equation

$$\mathbb{P}(\mathbb{T}^* - \overline{\lambda} I) = \begin{pmatrix} P(M^* - \overline{\lambda} I) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ QK^* & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q(N^* - \overline{\lambda} I) \end{pmatrix},$$

we have $\overline{\lambda} \in \sigma_{A_p}(M^*) \cup \sigma_{D_p}(N^*)$. Lemma 2.9 leads up to

$$\lambda \in \sigma_{A_p}(M) \cup \sigma_{A_r}(M) \cup \sigma_{D_p}(N) \cup \sigma_{D_r}(N).$$

Since $\lambda \notin \sigma_{\mathbb{A}_p}(\mathbb{T})$ and together with $\sigma_{A_p}(M) \subset \sigma_{\mathbb{A}_p}(\mathbb{T})$, thus $\lambda \notin \sigma_{A_p}(M)$. Hence

$$(\sigma_A(M) \setminus \sigma_{A_p}(M)) \cap \sigma_{D_r}(N) \subset \sigma_{\mathbb{A}_r}(\mathbb{T}) \subset \sigma_{A_r}(M) \cup \sigma_{D_p}(N) \cup \sigma_{D_r}(N).$$

□

From the discussion above, the set $\sigma_A(M) \cup \sigma_D(N)$, $\sigma_{A_p}(M) \cup \sigma_{D_p}(N)$, $\sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N)$ are larger than, respectively, $\sigma_{\mathbb{A}}(\mathbb{T})$, $\sigma_{\mathbb{A}_p}(\mathbb{T})$, $\sigma_{\mathbb{A}_\delta}(\mathbb{T})$. In the following, we obtain the spectral equalities between $\mathbb{T}$ and the diagonal entries of $\mathbb{T}$ for those three cases.

Theorem 3.14 *Let* $\mathbb{T} = \begin{pmatrix} M & K \\ 0 & N \end{pmatrix}$ *be an* $\mathbb{A}$*-bounded upper triangular operator matrix. Then*

(1) $\sigma_A(M) \cup \sigma_D(N) = \sigma_{\mathbb{A}}(\mathbb{T}) \cup G$, *where* $G \subset \sigma_{D_{p,1}}(N) \cap \sigma_{A_{r,1}}(M)$;
(2) $\sigma_{A_p}(M) \cup \sigma_{D_p}(N) = \sigma_{\mathbb{A}_p}(\mathbb{T}) \cup G$, *where* $G \subset \sigma_{D_p}(N) \cap \sigma_{A_\delta}(M)$;
(3) $\sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N) = \sigma_{\mathbb{A}_\delta}(\mathbb{T}) \cup G$, *where* $G \subset \sigma_{D_{p,1}}(N) \cap \sigma_{A_\delta}(M)$.

Proof (1) Since

$$(\sigma_A(M) \setminus \sigma_D(N)) \cup (\sigma_D(N) \setminus \sigma_A(M)) = (\sigma_A(M) \cup \sigma_D(N)) \setminus (\sigma_A(M) \cap \sigma_D(N))$$

and

$$(\sigma_A(M) \setminus \sigma_D(N)) \cup (\sigma_D(N) \setminus \sigma_A(M)) \subset \sigma_{\mathbb{A}}(\mathbb{T}) \subset \sigma_A(M) \cup \sigma_D(N),$$

we have $\lambda \in \sigma_A(M) \cap \sigma_D(N)$, while $\lambda \in \sigma_A(M) \cup \sigma_D(N)$ but $\lambda \notin \sigma_{\mathbb{A}}(\mathbb{T})$. By using Proposition 3.6, there exists a unique $\widetilde{\mathbb{T}} \in \mathcal{B}(\boldsymbol{\mathcal{R}}(A^{1/2}) \oplus \boldsymbol{\mathcal{R}}(D^{1/2}))$ such that $\mathbb{A}\mathbb{T} = \widetilde{\mathbb{T}}\mathbb{A}$ and $\lambda \in \rho(\widetilde{\mathbb{T}})$. Then $\widetilde{\mathbb{T}} - \lambda I$ is bounded below and for all $z = \begin{pmatrix} x & y \end{pmatrix}^{\mathrm{T}} \in \mathcal{X}_1 \oplus \mathcal{X}_2$,

$$\|(\widetilde{\mathbb{T}} - \lambda I)\mathbb{A}z\|_{\boldsymbol{\mathcal{R}}(A^{1/2}) \oplus \boldsymbol{\mathcal{R}}(D^{1/2})} \geq c\|\mathbb{A}z\|_{\boldsymbol{\mathcal{R}}(A^{1/2}) \oplus \boldsymbol{\mathcal{R}}(D^{1/2})}.$$

Taking $y = 0$, then

$$\|(\widetilde{M} - \lambda\widetilde{I})Ax\|_{\boldsymbol{\mathcal{R}}(A^{1/2})} \geq c\|Ax\|_{\boldsymbol{\mathcal{R}}(A^{1/2})}, \text{ for } x \in \mathcal{X}_1.$$

This indicates that $\widetilde{M} - \lambda\widetilde{I}$ is bounded below in $\boldsymbol{\mathcal{R}}(A^{1/2})$, together with Proposition 3.6, we obtain $\|(M-\lambda I)x\|_A \geq c\|x\|_A$, which indicates that $\lambda \in \sigma_{A_{r,1}}(M)$. Because $\mathbb{A}(\mathbb{T}-\lambda I)$ is surjective, then $D(N-\lambda I)$ is surjective and $\lambda \in \sigma_{D_{p,1}}(N)$.

(2) We just want to proof that $\sigma_{A_p}(M) \cup \sigma_{D_p}(N) \subset \sigma_{\mathbb{A}_p}(\mathbb{T}) \cup G$. If $\lambda \in \sigma_{A_p}(M) \cup \sigma_{D_p}(N)$ and $\lambda \notin \sigma_{\mathbb{A}_p}(\mathbb{T})$, we have $\lambda \in \sigma_{D_p}(N) \setminus \sigma_{A_p}(M)$. We claim that $\lambda \in \sigma_A(M)$. Otherwise,

if $\lambda \in \rho_A(M)$, according to $\lambda \in \sigma_{D_p}(N)$, and hence $\lambda \in \sigma_{\mathbb{A}_p}(\mathbb{T})$, which leads up to the contradiction. By combining $\lambda \notin \sigma_{A_p}(M)$, then $\lambda \in \sigma_{A_\delta}(M)$, this is $G \subset \sigma_{D_p}(N) \cup \sigma_{A_\delta}(M)$.

(3) From Theorem 3.13, we have

$$\sigma_{\mathbb{A}_\delta}(\mathbb{T}) \cup (\sigma_{D_{p,1}}(N) \cap \sigma_{A_\delta}(M)) \subset \sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N).$$

Conversely, if $\lambda \in (\sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N)) \setminus \sigma_{\mathbb{A}_\delta}(\mathbb{T})$, again using Theorem 3.13, it follows that $\lambda \in \sigma_{A_\delta}(M) \setminus \sigma_{D_\delta}(N)$, and hence $\lambda \in \sigma_{D_{p,1}}(N) \cup \rho_D(N)$.

If $\lambda \in \rho_D(N)$, there exists $R_\lambda(N)$ such that

$$Q(N - \lambda I)R_\lambda(N) = QR_\lambda(N)(N - \lambda I) = Q.$$

Using the equation

$$\mathbb{P}(\mathbb{T} - \lambda I) = \begin{pmatrix} I & PKQR_\lambda(N) \\ 0 & I \end{pmatrix} \begin{pmatrix} P(M - \lambda I) & 0 \\ 0 & Q(N - \lambda I) \end{pmatrix},$$

together with $\lambda \notin \sigma_{\mathbb{A}_\delta}(\mathbb{T})$, we deduce that $\mathcal{R}(A(M - \lambda I)) = \overline{\mathcal{R}(A)}$, which contradicts $\lambda \in \sigma_{A_\delta}(M)$. Therefore, $\lambda \in \sigma_{D_{p,1}}(N)$. Hence,

$$(\sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N)) \setminus \sigma_{\mathbb{A}_\delta}(\mathbb{T}) \subset \sigma_{D_{p,1}}(N) \cap \sigma_{A_\delta}(M).$$

□

Corollary 3.15 *Let* $\mathbb{T} = \begin{pmatrix} M & K \\ 0 & N \end{pmatrix}$ *be an* $\mathbb{A}$*-bounded upper triangular operator matrix. Then:*

(1) $\sigma_A(M) \cup \sigma_D(N) = \sigma_{\mathbb{A}}(\mathbb{T}) \Longleftrightarrow \sigma_{D_{p,1}}(N) \cap \sigma_{A_{r,1}}(M) = \emptyset$;
(2) $\sigma_{A_p}(M) \cup \sigma_{D_p}(N) = \sigma_{\mathbb{A}_p}(\mathbb{T}) \Longleftrightarrow \sigma_{D_p}(N) \cap \sigma_{A_\delta}(M) = \emptyset$.
(3) $\sigma_{A_\delta}(M) \cup \sigma_{D_\delta}(N) = \sigma_{\mathbb{A}_\delta}(\mathbb{T}) \Longleftrightarrow \sigma_{D_{p,1}}(N) \cap \sigma_{A_\delta}(M) = \emptyset$.

3.4 Example: Spectral analysis of a class of pseudo-Hermitian operator matrices

In this subsection, we present an example to investigate eigenvalue distributions under $\mathcal{PT}$-symmetric constraints. Mathematically, $\mathcal{PT}$-symmetry is characterized by pseudo-Hermiticity. Here, the operator T satisfies $A^{-1}T^*A = T$. The operator A is positive and self-adjoint, and is known as the metric operator. Building on this, the presence of pseudo-Hermiticity implies that non-Hermitian systems can be related to Hermitian systems via a similarity transformation, thereby preserving the real spectral characteristic. For a more detailed mathematical foundation of pseudo-Hermitian systems, please refer to Ref. [8]. Next, we show how the results of Subsection 3.2 apply to a specific model in pseudo-Hermitian quantum mechanics. Consider two non-reciprocal subsystems within a semi-Hilbert space defined as $\mathcal{X} = \mathcal{X}_a \oplus \mathcal{X}_b$.

Modified inner product: In the framework of $\mathcal{PT}$-symmetric systems, the generators of the subspaces $\mathcal{X}_a$ and $\mathcal{X}_b$ are defined by the metric operators $A = \omega_1 a^* a$ and $D = \omega_2 b^* b$, respectively. Here, a, a^* and b, b^* represent the standard bosonic creation and annihilation operators, which satisfy the commutation relations $[a, a^*] = 1$ and $[b, b^*] = 1$. The terms a^*a and b^*b act as particle-number operators, with eigenvalues n_a and n_b, respectively.

The parameters $\omega_1, \omega_2 > 0$ represent the eigenfrequencies of the generators. The eigenvalues of A and D are non-negative real numbers $n_a\omega_1$ and $n_b\omega_2$ (where n_a

and n_b are non-negative integers). The self-adjointness of A and D arises from their symmetric structures (e.g., a^*a being a Hermitian operator), and the non-negativity of A and D are reflected by their eigenvalues.

The semi-Hilbert space $\mathcal{X} = \mathcal{X}_a \oplus \mathcal{X}_b$ is generated by $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, equipped with the semi-inner product $\langle \psi, \phi \rangle_{\mathbb{A}} = \langle \mathbb{A}\psi, \phi \rangle$, for all $\psi, \phi \in \mathcal{X}$. The non-negativity of $\mathbb{A}$ is directly inherited from the non-negativity of A and D.

Pseudo-Hermitian modification of the Hamiltonian: The properties of the system are determined by the dynamical operator, which is given by

$$\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}.$$

The interaction terms $K : \mathcal{X}_b \to \mathcal{X}_a$ and $L : \mathcal{X}_a \to \mathcal{X}_b$ are defined as $K = \kappa_1 a^* b$ and $L = \kappa_2 b^* a$, where κ_1, κ_2 represent the coupling strength. Since

$$\mathbb{T}^* = \begin{pmatrix} 0 & \overline{\kappa_2} a^* b \\ \overline{\kappa_1} b^* a & 0 \end{pmatrix},$$

it follows that $\mathbb{T}^* \neq \mathbb{T}$ in general. Under the modified inner product, we assume that $\mathbb{A}\mathbb{T} = \mathbb{T}^*\mathbb{A}$, the pseudo-Hermitian condition requires $\kappa_2(n_a, n_b) = \frac{\omega_1}{\omega_2} \overline{\kappa_1}(n_a, n_b) \frac{n_a+1}{n_b+1}$. We observe that $\frac{n_a}{n_b} \approx c$ (a constant) when $n_a \gg 1, n_b \gg 1$, this simplifies the parameters to constants, i.e., $\kappa_2 \approx c\,\overline{\kappa_1} \frac{\omega_1}{\omega_2}$. Thus, by introducing the metric operator $\mathbb{A}$, the system satisfies the pseudo-Hermiticity condition

$$\mathbb{A}\mathbb{T} = \mathbb{T}^*\mathbb{A}.$$

For the interaction terms, their pseudo-Hermitian relationship becomes

$$DL = K^* A \text{ and } AK = L^* D.$$

Spectral decomposition and operator structure analysis: For the generating operator $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, where A and D denote operators on two respective subsystems, and the dynamical operator $\mathbb{T} = \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix}$, where K and L represent coupling operators between subsystems. By invoking Theorem 3.3, we obtain

$$\sigma_{\mathbb{A}p}(\mathbb{T}) = \lambda \in \mathbb{C} : \lambda^2 \in \sigma A_p(KL) \cup \sigma_{D_p}(LK).$$

Furthermore, leveraging the pseudo-Hermitian symmetric parameter constraint $\kappa_2 \approx c, \overline{\kappa_1} \dfrac{\omega_1}{\omega_2}$, we derive

$$\sigma_{A_p}(KL) = c|\kappa_1|^2, \frac{\omega_1}{\omega_2}, n_a(n_b + 1)$$

and

$$\sigma_{D_p}(LK) = c|\kappa_1|^2, \frac{\omega_1}{\omega_2}, (n_a + 1)n_b.$$

These results indicate that the eigenvalues of $\mathbb{T}$ are real numbers, which verifies Theorem 3.4 and aligns with the results under the unbroken $\mathcal{PT}$-symmetric phase. Additionally, Theorem 3.8 establishes that

$$\sigma_{D_p}(LK) \setminus \{0\} = \sigma_{A_p}(KL) \setminus \{0\}.$$

This finding corroborates that the excitation numbers of the two subsystems are equal, i.e., $n_a = n_b$. Under the explicit constraint $n_a = n_b$, for $\lambda = 0$, the corresponding eigenstates represent steady configurations where energy transfer between subsystems is prohibited.

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