Second Order Monotone Finite-Difference Schemes for Time-Fractional Quasilinear Parabolic Equations

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Abstract

This study presents a monotone finite-difference scheme designed to approximate the initial-boundary value problem for a time-fractional quasilinear parabolic equation. The maximum principle for the corresponding differential problem is established, and two-sided estimates of the numerical solution are derived directly from the input data, without any assumptions on their signs. *A priori* estimates in the C-norm are also obtained for both the exact and the approximate solutions. In particular, these estimates are in perfect agreement with the corresponding ones for the exact solution of the original differential problem, indicating that the proposed difference scheme preserves the qualitative properties of the continuous problem. The convergence of the scheme in the grid L_2-norm is proven using the energy inequality method combined with a Gronwall-type inequality. In addition, a convergence criterion is employed to evaluate the stability of the proposed scheme.

Keywords: Monotone finite-difference scheme, two-sides estimates, maximum principle, minimum principle, time-fractional quasilinear parabolic equation, convergence, stability.

1 Introduction

The maximum principle is a fundamental concept in the theory of partial differential equations (PDEs), traditionally applied to linear elliptic and parabolic equations. This principle provides crucial insights into the behavior of solutions, particularly in terms of their extrema, which can be useful for proving uniqueness, existence, and stability of solutions. In recent years, the extension of the maximum principle to fractional partial differential equations (FPDEs) has gained significant attention due to their increasing applications in various fields, such as physics, finance, and biology.

Fractional derivatives, characterized by non-integer orders, introduce memory and non-local effects into the equations, making them more suitable for modeling complex phenomena, such as anomalous diffusion and long-range interactions. The study of FPDEs is a rapidly growing area of research [3, 5, 24, 25, 30–33, 52, 53, 58], and developing a maximum principle for these equations is essential for understanding the qualitative behavior of their solutions. Proofs of various versions of the maximum principle for time-space FPDEs and their applications can be found in many works [2,7,10,26,27,37–39].

In the theory of finite-difference schemes (FDSs), the maximum principle plays an important role in ensuring that the finite-difference solution is consistent with the input data and converges when studying the uniform norm. Monotonicity is well-known for being a property of finite-difference methods that satisfy the grid maximum principle [56]. The well-conditioning of the algebraic equation systems resulting from monotone schemes has made them very important in computational practice [17]. Monotone schemes also enable numerical approximations that avoid oscillations, even for solutions that are not smooth (including those with discontinuities) [16]. Many studies focus on the development and analysis of monotone FDSs for integer linear PDEs in mathematical physics, considering various boundary conditions, as discussed in books [56, 57] and articles [20, 40, 43, 50]. For nonlinear cases, notable studies on monotone FDSs include articles [15, 16, 19, 21, 42, 44, 46–49]. Regarding fractional derivatives, Jiang's work [22] demonstrates the monotone properties of numerical solutions for the time-fractional Fokker–Planck equation, specifically with respect to initial values and truncation errors. Based on this, he proved the stability and convergence of the solution under the discrete

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L_1-norm. In a later study [23], Jiang and colleagues introduced a monotone finite volume method for time-fractional Fokker-Planck equations and proved its unconditional stability. They also showed that the convergence rate of this method is first-order in space, which can improve to second-order if the spatial grid is sufficiently fine. A key feature of their method is that it preserves the monotonicity of physical variables like density and concentration. Additionally, in [11], Li and Wang's work on complete monotonicity-preserving schemes [35] is extended, showing that these schemes satisfy the discrete comparison principle for time-fractional differential equations with variable coefficients. Furthermore, the works of Shkhanukov et al. on monotone difference schemes for FPDEs [4, 8, 9, 29] are worth noting.

The nonstandard grid maximum principle serves as a key tool for deriving two-sided bounds on the approximate solutions of FDSs. Nevertheless, as shown in [13], the sharpness of these estimates in the discrete setting is generally inferior to that of their counterparts in the continuous differential framework [28]. Faragó and collaborators extended this line of research by establishing analogous estimates for the finite element method, particularly in the context of linear problems and nonlinear problems with bounded nonlinearities (see, e.g., [12, 14]). Traditionally, the application of the grid maximum principle relies on the assumption that the input data maintain a constant sign. However, more refined two-sided estimates that do not impose this restriction have been developed and employed in [45, 51] to construct a generalized FDS in canonical form. To the best of our knowledge, no such two-sided estimates have previously been established for difference schemes associated with time-fractional quasilinear parabolic equations (TFQPEs). This work aims to fill that gap by introducing a theoretical framework for such estimates. Remarkably, the resulting bounds for the approximate solution are shown to be fully consistent with those for the exact solution.

The mathematical foundation of difference schemes is built upon three fundamental concepts: consistency, stability, and convergence. These notions are closely related and are formally unified by the Filippov–Ryaben'kii theorem [6, 55], commonly referred to in the literature as the Lax equivalence theorem [54]. This classical result states that, for a well-posed linear initial-boundary value problem (IBVP), any consistent FDS is convergent if and only if it is stable. In this framework, consistency ensures that the discrete scheme faithfully approximates the continuous differential problem. However, the situation becomes more intricate in the nonlinear setting. In particular, convergence alone is not sufficient to guarantee stability, as highlighted in [18]. To address this, the work in [41] generalizes the Lax equivalence theorem to encompass abstract nonlinear difference problems, where the underlying operators act in finite-dimensional Banach spaces. Within this extended theory, the convergence criterion remains valid, but only for numerical methods that exhibit unconditional stability.

The main contributions of the present study are as follows. This paper focuses on a monotone FDS that approximates the IBVP for a TFQPE. Interestingly, this scheme preserves the qualitative properties of the original differential problem. The maximum principle for the differential problem is established, and a two-sided estimate of the difference solution is derived using the input data, with no assumptions made about their signs. An a priori estimate in the uniform norm for the approximate solution is found to coincide exactly with the corresponding estimate for the exact solution. Moreover, it provides a sharper bound compared to the results obtained in the linear case, as presented in [30–32]. Using the energy inequality method and a Gronwall-type inequality, the convergence of the difference scheme in the grid L_2-norm is proven. Based on this, the convergence criterion in [41] is applied to assess the stability of the scheme.

The paper is organized as follows. Section 2 introduces the IBVP for a TFQPE and the maximum principle for its classical solution. Section 3 presents the nonstandard maximum principle for the FDS in the canonical form and its important corollary. Section 4 focuses on constructing a second order monotone FDS for the given problem, proving the two-sided estimates of the approximate solution, and obtaining the a priori estimate based on the input data. Section 5 and Section 6 present the evaluation of stability and convergence of the proposed scheme in the discrete L_2-norm.

2 Statement of the problem and maximum principle

In the domain $\bar{Q}_T = \{(x,t) : x \in \bar{\Omega},\, t \in [0,T]\}$, $\bar{\Omega} = \{x : 0 \le x \le l\} = \Omega \cup \{0,l\}$ we study numerical solutions of the IBVP for a TFQPE with bounded nonlinearity [33]

$$\mathcal{D}_t^\alpha u = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{1}$$

with the initial and boundary conditions, given by

$$\begin{gathered} u(x,0) = u_0(x), \quad x \in \bar{\Omega}, \\ u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad t > 0. \end{gathered} \tag{2}$$

We assume that the nonlinear diffusion coefficient $k(u)$ satisfies $0 < k_1 \le k(u) \le k_2$, $\forall\, u \in \mathbb{R}$, where k_1, k_2 are positive constants. Moreover, $k \in C^2(\mathbb{R})$ and k is globally Lipschitz continuous, i.e., there exists a constant $L > 0$ such that

$$|k(u_1) - k(u_2)| \le L|u_1 - u_2|, \quad \forall\, u_1, u_2 \in \mathbb{R}. \tag{3}$$

The Caputo fractional derivative $\mathcal{D}_t^\alpha$ is defined as [53]

$$\mathcal{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}(t-s)^{-\alpha} ds, \quad 0 < \alpha < 1. \tag{4}$$

Here $\Gamma(\cdot)$ denotes the usual gamma function.

In the paper, we assume that there exits a unique solution of problem (1)–(2). Notation $W_t^1((0,T))$ is defined as the space of functions $g \in C^1((0,T])$ such that $g' \in L((0,T))$, where $L((0,T))$ denotes the set of Lebesgue integrable functions on $(0,T)$.

Lemma 1 ([1,38]). *Let a function $g(t) \in W_t^1((0,T)) \cap C([0,T])$ attain its maximum (minimum) over the interval $[0,T]$ at the point $t = t_0 \in (0,T]$. Then the Caputo fractional derivative of the function $g(t)$ is non-negative (non-positive) at the point t_0 for any α, $0 < \alpha < 1$*

$$\mathcal{D}_t^\alpha g(t_0) \ge 0 \quad (\mathcal{D}_t^\alpha g(t_0) \le 0), \quad 0 < \alpha < 1. \tag{5}$$

The maximum principle for the TFQPE (1) is given by the following theorem.

Theorem 1. *Let a function $u \in CW_T(\Omega) := C(\bar{Q}_T) \cap W_T^1((0,T)) \cap C_x^2(\Omega)$ be a classical solution of the TFQPE (1) in the domain Q_T and $f(x,t) \le 0$, $(x,t) \in Q_T$. Then either $u(x,t) \le 0$, $(x,t) \in \bar{Q}_T$ or the function u attains its positive maximum on the bottom or two-side parts $\partial Q_T = \big(\bar{\Omega} \times \{0\}\big) \cup (\{0,l\} \times [0,T])$ of the boundary of the domain Q_T, i.e.*

$$u(x,t) \le \max\left\{0, \max_{(x,t)\in\partial Q_T}\{u_0(x), \mu_1(t), \mu_2(t)\}\right\}, \quad \forall (x,t) \in \bar{Q}_T. \tag{6}$$

Proof. This theorem is proved using the technique in [38]. Assume that the conclusion of the theorem is false. In this case, there exists (x_0, t_0), where $x_0 \in \Omega$ and $0 < t_0 \le T$, such that

$$u(x_0,t_0) > \max\left\{0, \max_{(x,t)\in\partial Q_T}\{u_0(x), \mu_1(t), \mu_2(t)\}\right\} = m_0 \ge 0.$$

We now define the quantity $\varepsilon := u(x_0,t_0) - m_0 > 0$ and introduce the auxiliary function

$$w(x,t) := u(x,t) + \frac{\varepsilon(T-t)}{2T}, \quad (x,t) \in \bar{Q}_T.$$

Based on the definition of the function w and the assumptions of the theorem, we can conclude that w satisfies the following properties

$$w(x,t) \le u(x,t) + \frac{\varepsilon}{2}, \quad (x,t) \in \bar{Q}_T,$$

$$w(x_0,t_0) \geq u(x_0,t_0) = \varepsilon + m_0 \geq \varepsilon + u(x,t) \geq \varepsilon + w(x,t) - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + w(x,t), \quad (x,t) \in \partial Q_T.$$

The final property implies that the function w cannot attain its maximum value on the portion ∂Q_T of the boundary of the domain Q_T. Suppose that the global maximum of w over the closure $\bar{Q}_T$ is achieved at the point (x_1, t_1). Then it must hold that $x_1 \in \Omega$, $0 < t_1 \leq T$, and

$$w(x_1,t_1) \geq w(x_0,t_0) \geq \varepsilon + m_0 \geq \varepsilon.$$

By applying Lemma 1, together with the necessary conditions for a function to attain its maximum within an open domain Ω, we arrive at the following relations

$$\mathcal{D}_t^\alpha w(t_1) \geq 0, \quad \frac{\partial w}{\partial x}(x_1,t_1) = 0, \quad \frac{\partial^2 w}{\partial x^2}(x_1,t_1) \leq 0. \tag{7}$$

According to the construction of the function w, the function u is subject to the following relation

$$u(x,t) = w(x,t) - \frac{\varepsilon(T-t)}{2T}, \quad (x,t) \in \bar{Q}_T.$$

The well-known formula

$$\mathcal{D}_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} t^{\beta-\alpha}, \quad 0 < \alpha \leq 1, \quad \beta > 0$$

for the Caputo fractional derivative leads to

$$\mathcal{D}_t^\alpha u = \mathcal{D}_t^\alpha w + \frac{\varepsilon t^{1-\alpha}}{2T\Gamma(2-\alpha)}. \tag{8}$$

Applying formulas (7)–(8), we obtain a sequence of equalities and inequalities evaluated at the point (x_1, t_1)

$$\begin{aligned}
&\mathcal{D}_t^\alpha u(x_1,t_1) - \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)(x_1,t_1) - f(x_1,t_1) \\
&= \mathcal{D}_t^\alpha u(x_1,t_1) - k(u)\frac{\partial^2 u}{\partial x^2}(x_1,t_1) - k'(u)\left(\frac{\partial u(x_1,t_1)}{\partial x}\right)^2 - f(x_1,t_1) \\
&= \mathcal{D}_t^\alpha w(x_1,t_1) + \frac{\varepsilon t_1^{1-\alpha}}{2T\Gamma(2-\alpha)} - k(u)\frac{\partial^2 w}{\partial x^2}(x_1,t_1) - k'(u)\left(\frac{\partial w(x_1,t_1)}{\partial x}\right)^2 - f(x_1,t_1) \\
&\geq \frac{\varepsilon t_1^{1-\alpha}}{2T\Gamma(2-\alpha)} > 0,
\end{aligned}$$

which contradicts the theorem's assumption that the function u is a solution of equation (1). This contradiction implies that the initial assumption made at the beginning of the proof is invalid, and hence, the theorem is established. □

By similar reasoning as above, the minimum principle can be obtained.

Theorem 2. *Let a function $u \in CW_T(\Omega)$ be a classical solution of the TFQPE (1) in the domain Q_T and $f(x,t) \geq 0$, $(x,t) \in Q_T$. Then either $u(x,t) \geq 0$, $(x,t) \in \bar{Q}_T$ or the function u attains its negative minimum on the bottom or two-side parts ∂Q_T of the boundary of the domain Q_T, i.e.*

$$u(x,t) \geq \min\left\{0, \min_{(x,t)\in\partial Q_T}\{u_0(x), \mu_1(t), \mu_2(t)\}\right\}, \quad \forall (x,t) \in \bar{Q}_T. \tag{9}$$

Theorem 3. *Let u be a classical solution of the problem (1)–(2). Then the following estimate in C-norm holds valid*

$$\|u\|_{C(\bar{Q}_T)} \leq \max\{M_0, M_1, M_2\} + \frac{T^\alpha}{\Gamma(1+\alpha)} M, \tag{10}$$

where

$$M = \|f\|_{C(\bar{Q}_T)} = \max_{(x,t)\in\bar{Q}_T} |f(x,t)|, \quad M_0 = \|u_0\|_{C(\bar{\Omega})} = \max_{x\in\bar{\Omega}} |u_0(x)|,$$

$$M_1 = \|\mu_1\|_{C(0,T)} = \max_{t\in[0,T]} |\mu_1(t)|, \quad M_2 = \|\mu_2\|_{C(0,T)} = \max_{t\in[0,T]} |\mu_2(t)|.$$

Proof. To prove the theorem, we first introduce an auxiliary function w

$$w(x,t) := u(x,t) - \frac{M}{\Gamma(1+\alpha)} t^\alpha, \quad (x,t) \in \bar{Q}_T.$$

Obviously, the function w is a classical solution of the following problem

$$D_t^\alpha w = \frac{\partial}{\partial x}\left(k\left(w + \frac{Mt^\alpha}{\Gamma(1+\alpha)}\right)\frac{\partial w}{\partial x}\right) + f(x,t) - M, \quad (x,t) \in \Omega \times (0,T],$$

$$w(x,0) = u_0(x), \quad x \in \bar{\Omega},$$

$$w(0,t) = \mu_1(t) - \frac{Mt^\alpha}{\Gamma(1+\alpha)}, \quad w(l,t) = \mu_2(t) - \frac{Mt^\alpha}{\Gamma(1+\alpha)}, \quad t > 0.$$

Since $f(x,t) - M \leq 0$, $(x,t) \in \bar{Q}_T$, then the maximum principle applied to the classical solution w leads to the estimate

$$w(x,t) \leq \max\{M_0, M_1, M_2\}, \quad (x,t) \in \bar{Q}_T.$$

For the function u, we get

$$u(x,t) = w(x,t) + \frac{M}{\Gamma(1+\alpha)} t^\alpha \leq \max\{M_0, M_1, M_2\} + \frac{T^\alpha}{\Gamma(1+\alpha)} M, \quad (x,t) \in \bar{Q}_T. \tag{11}$$

The minimum principle from Theorem 2 applied to the auxiliary function

$$w(x,t) := u(x,t) + \frac{M}{\Gamma(1+\alpha)} t^\alpha, \quad (x,t) \in \bar{Q}_T$$

leads to the estimate

$$u(x,t) \geq -\max\{M_0, M_1, M_2\} - \frac{T^\alpha}{\Gamma(1+\alpha)} M, \quad (x,t) \in \bar{Q}_T,$$

that together with the estimate (11) finishes the proof of the theorem. □

3 The nonstandard maximum principle for difference schemes with variable sign input data

Let in the n-dimensional Euclidian space a finite number of points of the grid Ω_h is given. To each point $x \in \Omega_h$ we associate one and only one stencil $\mathcal{M}(x)$ - a subset of Ω_h, containing this point. The set $\mathcal{M}'(x) = \mathcal{M}(x) \setminus x$ is called *neighborhood* of the point x. Let the functions $A(x)$, $B(x,\xi)$, $F(x)$ be given at $x \in \Omega_h$, $\xi \in \Omega_h$ and they take real values. Next, to each point $x \in \Omega_h$ corresponds one and only one equation of the form [56]

$$A(x)\, y(x) = \sum_{\xi \in \mathcal{M}'(x)} B(x,\xi)\, y(\xi) + F(x), \quad x \in \Omega_h, \tag{12}$$

which is called the *canonical form* of the FDS. According to [56], the point x is called a *boundary grid node*, if Dirichlet condition is posed

$$y(x) = \mu(x), \quad x \in \gamma, \tag{13}$$

where γ is the set of the boundary nodes, $\bar{\Omega}_h = \Omega_h \cup \gamma$. We will assume the fulfilment of the usual positivity conditions for the coefficients

$$A(x) > 0, \quad B(x,\xi) > 0 \quad \text{for all} \quad \xi \in \mathcal{M}'(x), \tag{14}$$

$$D(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x,\xi) > 0. \tag{15}$$

Lemma 2 ([45]). *Suppose that the positivity conditions for coefficients* (14)–(15) *are fulfilled. Then the maximum and minimum values of the solution of the FDS* (12) *belong to the range of the input data*

$$\min_{x\in\Omega_h}\frac{F(x)}{D(x)}\le y(x)\le\max_{x\in\Omega_h}\frac{F(x)}{D(x)},\quad x\in\Omega_h. \tag{16}$$

Corollary 1 ([56]). *Assume that conditions of the Lemma 2 are satisfied. Then in the grid analog of the C-norm, the solution of finite-difference problem* (12) *satisfies the estimate*

$$\|y\|_C=\max_{x\in\Omega_h}|y(x)|\le\left\|\frac{F}{D}\right\|_C. \tag{17}$$

Remark 1. *The estimates* (16), (17) *are at the interior nodes $x\in\Omega_h$. If the boundary nodes are also considered, i.e. $x\in\bar\Omega_h$, then instead of* (16), (17) *for the solution of difference problem* (12)–(13) *the following estimates hold*

$$\min\left\{\min_{x\in\gamma}\mu(x),\min_{x\in\Omega_h}\frac{F(x)}{D(x)}\right\}\le y(x)\le\max\left\{\max_{x\in\gamma}\mu(x),\max_{x\in\Omega_h}\frac{F(x)}{D(x)}\right\},\quad x\in\bar\Omega_h,$$

$$\|y\|_{\bar C}\le\max\left\{\|\mu\|_{C_\gamma},\left\|\frac{F}{D}\right\|_C\right\},$$

where

$$\|v\|_{C_\gamma}=\max_{x\in\gamma}|v(x)|,\quad \|v\|_{\bar C}=\max_{x\in\bar\Omega_h}|v(x)|.$$

4 Finite-difference scheme

In the domain $\bar Q_T$ we define the usual uniform grid in space and time

$$\bar\omega=\bar\omega_h\times\bar\omega_\tau,\quad \bar\omega_h=\left\{x_i=ih,\ i=\overline{0,N},\ hN=l\right\},\quad \bar\omega_h=\omega_h\cup\{x_0=0,\ x_N=l\},$$

$$\bar\omega_\tau=\left\{t_n=n\tau,\ n=\overline{0,N_0},\ \tau N_0=T\right\},\quad \bar\omega_\tau=\omega_\tau\cup\{t_0=0\}.$$

A widespread difference approximation of fractional derivative (4) is the so-called $L1$ method [36, 52, 59] which is defined as follows

$$\mathcal{D}_t^\alpha u(x_i,t_{n+1})=\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{n}b_j\frac{u(x_i,t_{n+1-j})-u(x_i,t_{n-j})}{\tau^\alpha}+O\left(\tau^{2-\alpha}\right), \tag{18}$$

where

$$b_j=(j+1)^{1-\alpha}-j^{1-\alpha},\quad j=0,1,\ldots,n.$$

It is direct to check that

$$\begin{gathered}b_j>0,\quad j=0,1,\ldots,n,\\ 1=b_0>b_1>\ldots>b_n,\quad b_n\xrightarrow{n\to\infty}0,\\ b_n+\sum_{j=1}^{n}(b_{j-1}-b_j)=1,\quad n\ge1.\end{gathered} \tag{19}$$

Let us define the discrete fractional differential operator Δ_t^α as a difference analog of the Caputo derivative of the order α $(0<\alpha<1)$

$$\Delta_t^\alpha u=\Delta_t^\alpha u(x_i,t_{n+1})=\frac{1}{\tau^\alpha\Gamma(2-\alpha)}\sum_{j=0}^{n}b_j\left(u(x_i,t_{n+1-j})-u(x_i,t_{n-j})\right).$$

Then from (18) we have

$$\mathcal{D}_t^\alpha u(x_i,t_{n+1})=\Delta_t^\alpha u(x_i,t_{n+1})+O\left(\tau^{2-\alpha}\right). \tag{20}$$

For the numerical solution $y(x,t) \approx u(x,t)$ of the problem (1)–(2) we construct the FDS

$$\Delta_t^\alpha y = (a(y)\hat{y}_{\bar{x}})_x + \hat{\varphi}, \tag{21}$$

$$y(x,0) = u_0(x), \quad x \in \bar{\omega}_h, \tag{22}$$

$$y_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}). \tag{23}$$

The functional

$$a(y) = 0.5(k(y_{i-1}) + k(y_i)), \tag{24}$$

is as usually chosen from the second-order approximation condition for the elliptic operator [56]

$$(a(u)\hat{u}_{\bar{x}})_x - \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) = O(h^2 + \tau).$$

Here and further we use the usual notations of the theory of FDSs [56]

$$y = y_i^n = y(x_i, t_n), \quad y_t = (\hat{y} - y)/\tau, \quad \hat{y} = y_i^{n+1},$$

$$v_{\bar{x}} = (v_i - v_{i-1})/h, \quad v_x = (v_{i+1} - v_i)/h, \quad \hat{\varphi} = f(x_i, t_{n+1}).$$

It is easy to show that the difference scheme (21)–(23) has the second order of approximation with respect to the spatial variable and the first order with respect to the time variable.

This difference scheme (21)–(23) is written in the canonical form (12)

$$A_i^n y_{i-1}^{n+1} - C_i^n y_i^{n+1} + B_i^n y_{i+1}^{n+1} = -F_i^n, \quad i = 1, 2, \ldots, N-1, \tag{25}$$

$$y_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}), \tag{26}$$

with coefficients defined as follows

$$A_i^n = \frac{\alpha_0}{h^2} a\left(y_i^n\right), \quad B_i^n = \frac{\alpha_0}{h^2} a\left(y_i^n\right), \quad \alpha_0 = \tau^\alpha \Gamma(2-\alpha), \tag{27}$$

$$C_i^n = 1 + A_i^n + B_i^n, \quad D_i^n = C_i^n - A_i^n - B_i^n = 1,$$

$$F_i^0 = y_i^0 + \alpha_0 \varphi_i^1, \quad F_i^n = b_n y_i^0 + \sum_{j=1}^{n} \left(b_{j-1} - b_j\right) y_i^{n-j+1} + \alpha_0 \varphi_i^{n+1}, \quad n \geq 1.$$

The scheme (25)–(26) is monotone if the positivity conditions of the coefficients (14)–(15) are satisfied [56], i.e.

$$A_i^n > 0, \quad B_i^n > 0, \quad D_i^n = C_i^n - A_i^n - B_i^n > 0.$$

From (24) it follows that $a(y) > 0$ for all y and thus, the coefficients A_i^n, B_i^n in (27) are positive, i.e. the scheme (21)–(23) is unconditionally monotone.

To get two-sided estimates for the solution of the difference scheme (21)–(23), we first present two useful lemmas.

Lemma 3 ([34])**.** *Let $\{p_n\}$ be a sequence defined by*

$$p_0 = 1, \quad p_n = \sum_{j=1}^{n} (b_{j-1} - b_j) p_{n-j}, \quad n \geq 1. \tag{28}$$

Then it holds that

$$\Gamma(2-\alpha) \sum_{j=1}^{n} p_{n-j} \leq \frac{n^\alpha}{\Gamma(1+\alpha)}. \tag{29}$$

Lemma 4. *The solution of the difference scheme* (21)–(23) *satisfies*

$$y_i^{n+1} \geq \min_{1\leq i\leq N-1} u_{0i} + \alpha_0 \sum_{k=1}^{n+1} p_{n-k+1} \min_{1\leq i\leq N-1} f_i^k, \tag{30}$$

$$y_i^{n+1} \leq \max_{1\leq i\leq N-1} u_{0i} + \alpha_0 \sum_{k=1}^{n+1} p_{n-k+1} \max_{1\leq i\leq N-1} f_i^k \tag{31}$$

for $i = 1, 2, \ldots, N-1$, $n = 0, 1, \ldots, N_0 - 1$, *where* p_n *is defined by* (28).

Proof. By the lemma 2, based on estimate (16), for arbitrary $t = t_n \in \omega_\tau$ and for all $i = 1, \ldots, N-1$ we have

$$\min_{1\leq i\leq N-1} F_i^n \leq y_i^{n+1} \leq \max_{1\leq i\leq N-1} F_i^n.$$

From that we can deduce

$$y_i^{n+1} \geq b_n \min_{1\leq i\leq N-1} y_i^0 + \sum_{j=1}^{n} (b_{j-1} - b_j) \min_{1\leq i\leq N-1} y_i^{n-j+1} + \alpha_0 \min_{1\leq i\leq N-1} f_i^{n+1}, \tag{32}$$

$$y_i^{n+1} \leq b_n \max_{1\leq i\leq N-1} y_i^0 + \sum_{j=1}^{n} (b_{j-1} - b_j) \max_{1\leq i\leq N-1} y_i^{n-j+1} + \alpha_0 \max_{1\leq i\leq N-1} f_i^{n+1}. \tag{33}$$

We prove inequality (30) below. The proof of (31) is similar. For $n = 0$, estimate (32) yields

$$y_i^1 \geq \min_{1\leq i\leq N-1} y_i^0 + \alpha_0 \min_{1\leq i\leq N-1} f_i^1 = \min_{1\leq i\leq N-1} u_{0i} + \alpha_0 p_0 \min_{1\leq i\leq N-1} f_i^1.$$

Thus, (30) is valid for $n = 0$. Assume that, for an arbitrary n, inequality (30) is also true, i.e.

$$y_i^n \geq \min_{1\leq i\leq N-1} u_{0i} + \alpha_0 \sum_{k=1}^{n} p_{n-k} \min_{1\leq i\leq N-1} f_i^1. \tag{34}$$

Using (32) again, from (28), (34) we get

$$\begin{aligned}
y_i^{n+1} &\geq b_n \min_{1\leq i\leq N-1} u_{0i} + \alpha_0 \min_{1\leq i\leq N-1} f_i^{n+1} + \sum_{j=1}^{n} (b_{j-1} - b_j) \left(\min_{1\leq i\leq N-1} u_{0i} + \alpha_0 \sum_{k=1}^{n-j+1} p_{n-j-k+1} \min_{1\leq i\leq N-1} f_i^k \right) \\
&= \min_{1\leq i\leq N-1} u_{0i} + \alpha_0 \min_{1\leq i\leq N-1} f_i^{n+1} + \alpha_0 \sum_{j=1}^{n} (b_{j-1} - b_j) \left(\sum_{k=1}^{n-j+1} p_{n-j-k+1} \min_{1\leq i\leq N-1} f_i^k \right) \\
&= \min_{1\leq i\leq N-1} u_{0i} + \alpha_0 \min_{1\leq i\leq N-1} f_i^{n+1} + \alpha_0 \sum_{k=1}^{n} \min_{1\leq i\leq N-1} f_i^k \left(\sum_{j=1}^{n-k+1} (b_{j-1} - b_j) p_{n-j-k+1} \right) \\
&= \min_{1\leq i\leq N-1} u_{0i} + \alpha_0 \sum_{k=1}^{n+1} p_{n-k+1} \min_{1\leq i\leq N-1} f_i^k.
\end{aligned}$$

So the estimate (30) is correct for $n + 1$. By the principle of induction, the lemma is proved. □

With the help of lemmas 3, 4, we obtain two-sided estimates for the solution of the difference scheme (21)–(23) through the input data without the assumption that the input data are of definite sign.

Theorem 4. *The FDS* (21)–(23) *is unconditionally monotone (without constraints on the steps τ and h) and its solution satisfies the following two-sided estimates*

$$m_1 \le y_i^{n+1} \le m_2, \quad i = \overline{0, N}, \quad n = \overline{0, N_0 - 1}, \tag{35}$$

where m_1, m_2 are constants defined as follow

$$m_1 = \min_{x \in \bar{\Omega}, t \in [0,T]} \{u_0(x), \mu_1(t), \mu_2(t)\} + \frac{T^\alpha}{\Gamma(1+\alpha)} \min\left\{0, \min_{(x,t)\in \bar{Q}_T} f(x,t)\right\},$$

$$m_2 = \max_{x \in \bar{\Omega}, t \in [0,T]} \{u_0(x), \mu_1(t), \mu_2(t)\} + \frac{T^\alpha}{\Gamma(1+\alpha)} \max\left\{0, \max_{(x,t)\in \bar{Q}_T} f(x,t)\right\}.$$

Proof. If the minimum of the solution $y(x,t)$ of the difference problem (21)–(23) is reached at the boundary nodes, then we get

$$y_i^{n+1} \ge \min\{\mu_1^{n+1}, \mu_2^{n+1}\} \ge \min_{t\in[0,T]} \{\mu_1(t), \mu_2(t)\}. \tag{36}$$

On the other hand, assuming that the minimum of the solution $y(x,t)$ occurs at internal grid nodes, inequality (29) together with (30) implies that

$$\begin{aligned} y_i^{n+1} &\ge \min_{x\in\bar{\Omega}_h} u_0(x) + \left(\min_{(x,t)\in\bar{Q}_T} f(x,t)\right) \tau^\alpha \Gamma(2-\alpha) \sum_{k=1}^{n+1} p_{n-k+1} \\ &\ge \min_{x\in\bar{\Omega}_h} u_0(x) + \frac{(n+1)^\alpha \tau^\alpha}{\Gamma(1+\alpha)} \min\left\{0, \min_{(x,t)\in\bar{Q}_T} f(x,t)\right\} \\ &\ge \min_{x\in\bar{\Omega}_h} u_0(x) + \frac{T^\alpha}{\Gamma(1+\alpha)} \min\left\{0, \min_{(x,t)\in\bar{Q}_T} f(x,t)\right\}. \end{aligned} \tag{37}$$

From (36), (37) we obtain the left inequality (35). The right inequality (35) is derived by similar reasoning. Thus the theorem is proved. □

Remark 2. *The two-sided estimates obtained in* (35) *are completely consistent with the maximum-minimum principles* (6), (9) *for the exact solution of the differential problem* (1)–(4)*, and in this sense, one can say that the FDSs inherit the properties of the differential problem.*

Based on the corollary of the difference maximum principle, we in a standard way obtain the important a priori estimate in the strong C-norm. We first present the following supporting lemma.

Lemma 5. *The solution of the difference scheme* (21)–(23) *satisfies*

$$\|y^{n+1}\|_C \le \|u_0\|_C + \alpha_0 \sum_{k=1}^{n+1} p_{n-k+1} \|f^k\|_C, \tag{38}$$

where

$$\|v\|_C = \max_{x\in\omega_h} |v(x)|, \quad \|v\|_{C_\gamma} = \max_{x\in\gamma} |v(x)|, \quad \|v\|_{\bar{C}} = \max_{x\in\bar{\omega}_h} |v(x)|.$$

Proof. Based on corollary 1, from (17) we conclude that

$$\|y^{n+1}\|_C \le \|b_n y^0 + \sum_{j=1}^{n} (b_{j-1} - b_j) y^{n-j+1} + \alpha_0 \varphi^{n+1}\|_C,$$

which leads to

$$\|y^{n+1}\|_C \le b_n \|u_0\|_C + \sum_{j=1}^{n} (b_{j-1} - b_j) \|y^{n-j+1}\|_C + \alpha_0 \|f^{n+1}\|_C. \tag{39}$$

For $n = 0$ we have

$$\|y^1\|_C \le \|u_0\|_C + \alpha_0 p_0 \|f^1\|_C.$$

This means that (38) is true for $n = 0$. Suppose that, the estimate (38) is also true for an arbitrary n ($n = 1, 2, \dots, N_0 - 1$), i.e.

$$\|y^n\|_C \le \|u_0\|_C + \alpha_0 \sum_{k=1}^{n} p_{n-k} \|f^k\|_C. \tag{40}$$

Now we need to prove that (38) is true for $n+1$. Indeed, from (39), (40) it follows that

$$\begin{aligned}
\|y^{n+1}\|_C &\le b_n\|u_0\|_C + \alpha_0\|f^{n+1}\|_C + \sum_{j=1}^{n}(b_{j-1} - b_j)\left(\|u_0\|_C + \alpha_0 \sum_{k=1}^{n-j+1} p_{n-j-k+1}\|f^k\|_C\right) \\
&= \|u_0\|_C + \alpha_0\|f^{n+1}\|_C + \alpha_0 \sum_{j=1}^{n}(b_{j-1} - b_j)\left(\sum_{k=1}^{n-j+1} p_{n-j-k+1}\|f^k\|_C\right) \\
&= \|u_0\|_C + \alpha_0\|f^{n+1}\|_C + \alpha_0 \sum_{k=1}^{n}\|f^k\|_C\left(\sum_{j=1}^{n-k+1}(b_{j-1} - b_j)p_{n-j-k+1}\right) \\
&= \|u_0\|_C + \alpha_0 \sum_{k=1}^{n+1} p_{n-k+1}\|f^k\|_C.
\end{aligned}$$

By the principle of induction, the lemma is proved. □

Theorem 5. *For the solution of the difference scheme* (21)–(23) *for any* $t_n \in \omega_\tau$ *the following a priori estimate is valid*

$$\|y^{n+1}\|_{\bar{C}} \le \max\left\{\|u_0\|_{\bar{C}}, \max_{t\in[0,T]}\{|\mu_1(t)|, |\mu_2(t)|\}\right\} + \frac{T^\alpha}{\Gamma(1+\alpha)} \max_{t\in[0,T]}\|f(t)\|_{\bar{C}}. \tag{41}$$

Proof. Obviously, for the boundary grid nodes we have

$$\|y^{n+1}\|_{C_\gamma} \le \max\left\{|\mu_1(t_{n+1})|, |\mu_2(t_{n+1})|\right\} \le \max_{t\in[0,T]}\left\{|\mu_1(t)|, |\mu_2(t)|\right\}. \tag{42}$$

On the other hand, for the internal grid nodes, using estimate (38) of lemma 5 in conjunction with inequality (29) of lemma 3, we obtain the following series of inequalities

$$\begin{aligned}
\|y^{n+1}\|_C &\le \|u_0\|_{\bar{C}} + \left(\max_{t\in[0,T]}\|f(t)\|_{\bar{C}}\right)\tau^\alpha\Gamma(2-\alpha)\sum_{k=1}^{n+1} p_{n-k+1} \\
&\le \|u_0\|_{\bar{C}} + \frac{(n+1)^\alpha\tau^\alpha}{\Gamma(1+\alpha)} \max_{t\in[0,T]}\|f(t)\|_{\bar{C}} \\
&\le \|u_0\|_{\bar{C}} + \frac{T^\alpha}{\Gamma(1+\alpha)} \max_{t\in[0,T]}\|f(t)\|_{\bar{C}}.
\end{aligned} \tag{43}$$

From (42) and (43), the required relation (41) follows. This completes the proof. □

Remark 3. *Inequality* (41) *provides an* a priori *estimate for the solution of the difference method, derived from the original problem's input data. It is fully consistent with the corresponding estimate* (10) *for the solution of the differential problem. Notably, this estimate is more accurate than those obtained for the linear case, as presented in the works of A.V. Lapin and colleagues* [30–32].

However, due to the nonlinearity of the original problem, classical stability of the difference scheme cannot be inferred directly from estimate (41). *Using a convergence criterion, we will discuss the stability of the difference scheme* (21)–(23) *in the next section, based on the results in* [41].

5 Convergence in the grid L_2-norm

In this section, we additionally assume that the exact solution of problem (1)–(4) is sufficiently smooth, namely, $u(x,t) \in C^{4,2}(Q_T)$. We use the energy inequality method to obtain estimates of the error and convergence results in the discrete L_2-norm. Defining the approximation error $\hat{\psi}$ in the interior nodes as

$$\hat{\psi} = -\Delta_t^\alpha u + (a(u)\hat{u}_{\bar{x}})_x + \hat{\varphi},$$

the grid-function error $z = y - u$ is the solution of the discrete problem

$$\Delta_t^\alpha z = (a(y)\hat{y}_{\bar{x}})_x - (a(u)\hat{u}_{\bar{x}})_x + \hat{\psi}, \tag{44}$$

with initial and boundary conditions

$$z\,(x,0) = 0, \quad x \in \bar{\omega}_h, \tag{45}$$

$$z\,(0,t) = z\,(l,t) = 0, \quad t \in \omega_\tau. \tag{46}$$

It is easy to see that $\hat{\psi} = O\left(h^2 + \tau\right)$ in all the nodes. We now define the following inner products and the corresponding norms

$$(u,v) = \sum_{x\in\omega_h} hu(x)v(x), \quad \|u\| = \sqrt{(u,u)},$$

$$(u,v] = \sum_{x\in\omega_h^+} hu(x)v(x), \quad ||u]| = \sqrt{(u,u]},$$

where $\omega_h^+ = \omega_h \cup \{l\}$. The following results will be useful to prove the convergence of the scheme. With this purpose, we will use the formula of summation by parts [56], and Gronwall type inequality [34].

Lemma 6 (Summation by parts). *For any grid functions u, v defined in ω_h vanishing on the boundary points $x \in \{0,l\}$, the following identity holds:*

$$(u_x, v) = -(u, v_{\bar{x}}]. \tag{47}$$

Lemma 7 ([3]). *For every function $v(t)$ defined on the grid $\bar{\omega}_\tau$ one has the inequality*

$$v^{n+1}\Delta_t^\alpha y \geq \frac{1}{2}\Delta_t^\alpha (y)^2. \tag{48}$$

Lemma 8 (Gronwall type inequality). *Suppose that the nonnegative sequences $\{\omega^n\}$, $\{g^n\}$, $n = 0,1,2,\ldots$ satisfy*

$$\Delta_t^\alpha \omega^n \leq \lambda_1\omega^n + \lambda_2\omega^{n-1} + g^n, \quad n \geq 1,$$

where $\lambda_1, \lambda_2 = const \geq 0$. Then, there exists a positive constant τ^* such that, when $\tau \leq \tau^*$,

$$\omega^n \leq 2\left(\omega^0 + \frac{t_n^\alpha}{\Gamma(1+\alpha)}\max_{0\leq j\leq n} g^j\right) E_\alpha\left(2\lambda t_n^\alpha\right), \quad 1 \leq n \leq N_0, \tag{49}$$

where $E_\alpha(v) = \sum\limits_{k=0}^{\infty} \dfrac{v^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function and $\lambda = \lambda_1 + \lambda_2/(2-2^{1-\alpha})$.

The following assertion holds.

Theorem 6. *The solution of the FDS (21)–(23) converges to the exact solution of differential problem (1)–(4), and the following estimate of the method accuracy holds*

$$\|\hat{z}\| \leq C\left(h^2 + \tau\right), \tag{50}$$

C being a positive constant independent of the discretization parameters.

Proof. Taking the inner product of Eq. (44) by $\hat{z}$, we obtain

$$(\Delta_t^\alpha z, \hat{z}) = (\hat{z}, (a(y)\hat{y}_{\bar{x}})_x - (a(u)\hat{u}_{\bar{x}})_x) + (\hat{z}, \hat{\psi}). \tag{51}$$

Taking into account the fact that

$$(a(y)\hat{y}_{\bar{x}})_x - (a(u)\hat{u}_{\bar{x}})_x = (a(y)\hat{z}_{\bar{x}})_x + ((a(y) - a(u))\,\hat{u}_{\bar{x}})_x,$$

and applying the summation by parts formula (47) to the first term on the right-hand side in (51), we obtain

$$RHS := (\hat{z}, (a(y)\hat{y}_{\bar{x}})_x - (a(u)\hat{u}_{\bar{x}})_x) = (\hat{z}, (a(y)\hat{z}_{\bar{x}})_x) + (\hat{z}, ((a(y) - a(u))\,\hat{u}_{\bar{x}})_x)$$
$$= -\,(\hat{z}_{\bar{x}}, a(y)\hat{z}_{\bar{x}}] - (\hat{z}_{\bar{x}}, (a(y) - a(u))\,\hat{u}_{\bar{x}}] = -\left((\hat{z}_{\bar{x}})^2, a(y)\right] - (\hat{z}_{\bar{x}}, (a(y) - a(u))\,\hat{u}_{\bar{x}}].$$

Because the coefficient $a(y) \geq k_1 > 0$, $\forall y \in \mathbb{R}$ the following evaluation is satisfied

$$\left((\hat{z}_{\bar{x}})^2, a(y)\right] \geq k_1||\hat{z}_{\bar{x}}]|^2.$$

Therefore,

$$-\left((\hat{z}_{\bar{x}})^2, a(y)\right] \leq -k_1||\hat{z}_{\bar{x}}]|^2.$$

By assumption (3), we obtain $|a(y) - a(u)| \leq L|z|_{(0.5)}$, where $|z_i|_{(0.5)} = 0.5\,(|z_i| + |z_{i-1}|)$. From this it follows that

$$-\,(\hat{z}_{\bar{x}}, (a(y) - a(u))\,\hat{u}_{\bar{x}}] \leq L\left(|\hat{z}_{\bar{x}}|\,, |z_i|_{(0.5)}\,|\hat{u}_{\bar{x}}|\right].$$

The solution of problem (1)–(4) is sufficiently smooth, and hence

$$|\hat{u}_{\bar{x},i}| \leq \frac{1}{h}\int\limits_{x_{i-1}}^{x_i} \left|\frac{\partial \hat{u}}{\partial x}\right| dx \leq c.$$

Then, applying the ε-inequality, we obtain the estimate

$$-\,(\hat{z}_{\bar{x}}, (a(y) - a(u))\,\hat{u}_{\bar{x}}] \leq \varepsilon_1 Lc||\hat{z}_{\bar{x}}]|^2 + \frac{Lc}{4\varepsilon_1}\|z\|^2.$$

Therefore, for the first term on the right-hand side in (51), we have

$$RHS \leq -(k_1 - \varepsilon_1 Lc)||\hat{z}_{\bar{x}}]|^2 + \frac{Lc}{4\varepsilon_1}\|z\|^2, \tag{52}$$

and the second term satisfies the following estimate

$$(\hat{z}, \hat{\psi}) \leq \frac{\|\hat{z}\|^2 + \|\hat{\psi}\|^2}{2}. \tag{53}$$

On the other hand, using the inequality (48) of the lemma 7, we get the estimate for the left-hand side of the equality (51) as follows

$$(\Delta_t^\alpha z, \hat{z}) \geq \frac{1}{2}\Delta_t^\alpha \|z\|^2. \tag{54}$$

Thus, from the results (51)–(54) we derive the estimate

$$\frac{1}{2}\Delta_t^\alpha \|z\|^2 + (k_1 - \varepsilon_1 Lc)||\hat{z}_{\bar{x}}]|^2 \leq \frac{Lc}{4\varepsilon_1}\|z\|^2 + \frac{1}{2}\|\hat{z}\|^2 + \frac{1}{2}\|\hat{\psi}\|^2.$$

We take the value ε_1 sufficiently small, namely, such that the inequality $k_1 - \varepsilon_1 Lc \geq 0$ is satisfied. Then we arrive at the final estimate

$$\Delta_t^\alpha \|z\|^2 \leq \|\hat{z}\|^2 + \frac{Lc}{2\varepsilon_1}\|z\|^2 + \|\hat{\psi}\|^2 \leq \|\hat{z}\|^2 + \frac{Lc}{2\varepsilon_1}\|z\|^2 + C_1(h^2 + \tau)^2.$$

Applying the finite-difference analog (49) of the Gronwall lemma to the last inequality, we obtain the desired estimate. The proof of the theorem is complete. □

6 Stability of difference scheme

The criterion for the convergence of nonlinear difference schemes proved in [41] allows one to prove the unconditional stability of difference methods for which convergence has already been proved.

In the sequel, we assume that problem (1)–(4) is Hadamard well-posed, i.e., the following conditions are satisfied

(a) There exits a unique solution $u(x,t) \in C^{4,2}(Q_T)$ for all input data;

(b) The solution is stable in the L_2-norm for all $u, \tilde{u} \in C^{4,2}(Q_T)$ with respect to small disturbances in the initial data

$$\max_{0\le t\le T} \|\tilde{u} - u\|_{L_2(\bar{\Omega})} \le c_0 \|\tilde{u}_0 - u_0\|_{L_2(\bar{\Omega})},$$

where $c_0 = const > 0$, $\tilde{u}$ is the solution of problem (1)–(4) with the perturbed initial condition $\tilde{u}_0$.

Obviously, a similar estimate holds for the perturbed difference scheme

$$\|\tilde{y} - \tilde{u}\| \le C\left(h^2 + \tau\right). \tag{55}$$

Based on the above, from (50), (55) we conclude that

$$\begin{aligned}\max_{t\in\bar{\omega}_\tau} \|\tilde{y} - y\| &\le \max_{t\in\bar{\omega}_\tau} \|y - u\| + \max_{t\in\bar{\omega}_\tau} \|\tilde{y} - \tilde{u}\| + \max_{t\in\bar{\omega}_\tau} \|\tilde{u} - u\| \\ &\le 2C\left(h^2 + \tau\right) + \max_{0\le t\le T} \|\tilde{u} - u\|.\end{aligned}$$

Since we do not perturb the boundary conditions, we see that

$$|R(u)| = \left| \|u\| - \|u\|_{L_2(\bar{\Omega})} \right| \le c_1 h, \quad c_1 = const > 0.$$

Obviously, for sufficiently small $h \le h_0$ and $\tau \le \tau_0$ satisfying the inequality

$$2C\left(h^2 + \tau\right) \le c_2 \|\tilde{u}_0 - u_0\|_{L_2(\bar{\Omega})}, \quad c_2 = const > 0,$$

the difference scheme (21)–(23) is unconditionally stable in the L_2-norm with respect to the initial data, and one has the inequality

$$\max_{t\in\bar{\omega}_\tau} \|\tilde{y}(t) - y(t)\| \le c_3 \|\tilde{u}_0 - u_0\|_{L_2(\bar{\Omega})}, \quad c_3 = const > 0.$$

7 Conclusions

In conclusion, this paper presents a monotone FDS for approximating the IBVP of a TFQPE. The maximum principle for the differential problem has been established, and a two-sided estimate of the solution has been derived directly from the input data, without making any assumptions about their signs. Notably, the nonstandard maximum principle for the difference solution is fully consistent with the maximum principle of the original differential problem, meaning that the proposed scheme preserves the properties of the original equation. The convergence of the scheme in the grid L_2-norm has been rigorously proven using the energy inequality method and a Gronwall-type inequality. Additionally, the convergence criterion from [41] was successfully applied to assess the scheme's stability. These findings lay a strong foundation for the further development and analysis of numerical methods for nonlinear FPDEs with unbounded nonlinearity.

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