Zeros of the independent domination polynomial of the cozero divisor graphs of ring $\mathbb{Z}_n$

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Abstract

In this paper, we investigate the zeros of the independent domination polynomial $D_i(\Gamma'(\mathbb{Z}_n), z)$ of the cozero divisor graph $\Gamma'(\mathbb{Z}_n)$. For biprime case, the nonzero zeros of $D_i(\Gamma'(\mathbb{Z}_n), z)$ lie on the unit circle and the set of limits of zeros is exactly $\{0\} \cup \{z \in \mathbb{C} : |z| = 1\}$, while in the triprime case all nonzero zeros lie in the fixed annulus $\{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 2\}$, and limit of nonzero zeros lie is the unit circle. For $n = p^{n_1}q$, with primes $p < q$, we characterize precisely when $D_i(\Gamma'(\mathbb{Z}_n), z)$ is unimodal and log-concave, it is both unimodal and log-concave if and only if $(p, q) = (2, 3)$, while for all other (p, q) it is never unimodal, and it is log-concave if and only if a simple arithmetic condition on (p, q, n_1) holds. In all cases, we show that the nonzero zeros reside in the annulus $\{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 2\}$, and that 0 is always a limit of zeros, which leads to a natural open problem on describing the limit set of nonzero zeros for the family $\{D_i(\Gamma'(\mathbb{Z}_{p^{n_1}q}), z)\}_{n_1 \geq 1}$.

Keywords: Cozero divisor graphs, independent domination polynomial; unimodal; log-concave; zeros; limit of zeros

AMS subject classification: 05C25, 05C31, 68R05, 05C69, 26C10, 68Q25.

1 Introduction

The interplay between commutative algebra and graph theory has been an active area of research for several decades. A classical construction associates to a commutative ring R,

various graphs that encode the ideal-theoretic or multiplicative relations among its elements, such as the zero-divisor graph, cozero divisor graph, and comaximal graph. These constructions have proved useful both for translating ring-theoretic questions into combinatorial language, and for using algebraic tools to study structural and spectral properties of graphs (see, for example, [1–4, 8–10, 23, 25]).

In this paper, we focus on the cozero divisor graph $\Gamma'(R)$ of a commutative ring R, introduced, and studied in [1–4]. The cozero divisor graph $\Gamma'(R)$ is the simple graph whose vertex set consists of the nonzero nonunit elements of R, where two distinct vertices x, y are adjacent if and only if $x \notin Ry$ and $y \notin Rx$, with Rx denoting the principal ideal generated by x. The cozero divisor graph has been investigated from many different viewpoints, including chromatic and clique numbers, planarity and outerplanarity, graph-theoretic characterisations of classes of rings, and metric graph invariants such as metric dimension and strong metric dimension [1–4, 8–10, 23, 25]. A particularly fruitful testing ground is provided by the finite quotient rings $\mathbb{Z}_n$, where n is a positive integer, see, for instance, [13, 32] for related work on zero-divisor and comaximal graphs of $\mathbb{Z}_n$.

On the graph-theoretic side, domination-type invariants form a central topic of modern graph theory [15, 17]. Given a finite simple graph G, a set $S \subseteq V(G)$ is a dominating set, if every vertex of G is either in S or adjacent to a vertex of S, and S is an independent dominating set if it is both dominating and independent, which is a NP-hard/complete problem [19]. The *independent domination polynomial* of G is defined as

$$D_i(G, z) = \sum_{k \geq 0} d_k z^k,$$

where d_k is the number of independent dominating sets of cardinality k. This graph polynomial was introduced and studied in [7, 14], and has since been investigated for various graph families, and under several operations such as corona products and joins [5–7, 14, 15, 18, 21]. In parallel, independent domination polynomials of graphs arising from commutative rings have attracted attention recently, see in particular [16, 29, 30] for results on zero-divisor and cozero divisor graphs. The independent domination polynomial encodes refined structural information about the distribution of independent dominating sets, and its coefficients and zeros carry subtle combinatorial and analytic data [22, 24]. Questions about unimodality and log-concavity of its coefficient sequence are closely related to classical themes in algebraic and enumerative combinatorics, while the location and limiting behaviour of its zeros connect to the rich theory of graph polynomials and their root distributions [20, 31].

In this work, we continue and extend the study initiated in [16, 26–29] of the independent domination polynomial of the cozero divisor graph of $\mathbb{Z}_n$. For n with prescribed prime factorisation, we investigate in detail the arithmetic and analytic properties of the polynomial $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$. More precisely, our main contributions can be summarised as follows: For

$n = p_1p_2p_3$ with primes $p_1 < p_2 < p_3$, with Cauchy-type root bounds for monic polynomials with bounded coefficients, we prove that every nonzero zeros of $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ lies in the closed annulus $\big\{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 2\big\}$, and that there are no positive real zeros (see Theorem 2.3). Also, we show that any nonzero limit of zeros of $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ lie on the unit circle (Theorem 2.4). For $n = p^{n_1}q$ with primes $p < q$ and $n_1 \geq 1$, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ was computed in [29] as

$$D_i\big(\Gamma'(\mathbb{Z}_n), z\big) = z^{p^{n_1-1}(q-1)} + \sum_{i=1}^{n_1-1} z^{(q-1)\left(p^{n_1-1}-p^{i-1}\right)+p^i-1} + z^{p^{n_1}-1}.$$

We carry out a detailed analysis of the exponent pattern, and show that the coefficient sequence is unimodal and log-concave if and only if $(p, q) = (2, 3)$. For $(p, q) \neq (2, 3)$, we give a complete characterisation of log-concavity in terms of explicit differences of its exponents (Theorem 3.2). Furthermore, Theorem 3.3 shows that all nonzero zeros of the polynomial lie in the annulus $\frac{1}{2} \leq |z| \leq 2$.

The main motivation of our work is that the cozero divisor graph $\Gamma'(\mathbb{Z}_n)$ is determined in a very precise way by the arithmetic of the finite ring $\mathbb{Z}_n$, especially by the prime factorization of n, the structure of principal ideals, and divisibility among nonzero nonunits. Therefore, the independent domination polynomial $D_i(\Gamma'(\mathbb{Z}_n), z)$ should be viewed as a compact invariant encoding ring-theoretic information in combinatorial form. In particular, its coefficients reflect how arithmetic constraints govern independent dominating sets, while its zeros reveal global analytic patterns forced by the multiplicative structure of $\mathbb{Z}_n$. Our results show that properties such as unimodality, log-concavity, and root distribution can distinguish special families such as $n = p^{n_1}q$, where the exceptional case $(p, q) = (2, 3)$ is singled out. Thus, the paper gives not only new results on graph polynomials, but also a new method for detecting and comparing structural features of finite rings through domination-based invariants. Moreover, the asymptotic behavior of zeros raises new ring-theoretic questions: for instance: How does the factorization of n control the zero distribution of $D_i(\Gamma'(\mathbb{Z}_n), z)$? which, in our view, provides further motivation for the study.

The rest of the paper is organised as follows: Section 2 is devoted to the cases $n = p_1p_2$ and $n = p_1p_2p_3$, where we obtain sharp bounds on its zeros, and describe the corresponding set of limit of zeros of the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$. In Section 3, we turn to the family $n = p^{n_1}q$, prove the unimodality and log-concavity for the cozero graph $\Gamma'(\mathbb{Z}_n)$, and establish uniform root bounds for the associated polynomials. We conclude the article with a brief discussion of future work.

2 Zeros and limits of independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$

In this section, we discuss the zeros and their bounds for the independent domination polynomial of $D_i(\Gamma'(\mathbb{Z}_n), z)$ for $n \in \{p_1p_2, p_1p_2p_3\}$.

The independent domination polynomial (see [29]) of $\Gamma'(\mathbb{Z}_{p_1p_2})$ is $D_i(\Gamma'(\mathbb{Z}_{p_1p_2}), z) = z^{p_1-1} + z^{p_2-1}$. The following proposition discusses its zeros.

Proposition 2.1. *Let $n = p_1p_2$ with primes $p_1 < p_2$ and $D_i(\Gamma'(\mathbb{Z}_n), z)$ be its independent domination polynomial. Then the zeros of $D_i(\Gamma'(\mathbb{Z}_n), z)$ are $z = 0$ with multiplicity $p_1 - 1$, and the $(p_2 - p_1)$-th roots of -1, that is, $z = \exp\left(\frac{(2k+1)\pi i}{p_2-p_1}\right)$, for $k = 0, 1, \ldots, p_2 - p_1 - 1$ lying on the unit circle.*

Proof. The proof is straightforward. □

For the family of (complex) polynomials $\{p_n(z)\}$, with $n \in \mathbb{N}$, a complex number $z \in \mathbb{C}$ is said to be limit of zeros of $\{p_n(z)\}$ if either $p_n(z) = 0$, for all sufficiently large $n \in \mathbb{N}$ or z is the limit point (limit curve) of the set $Z\{(p_n(z))\}$, where $Z\{(p_n(z))\}$ is the collection of the zeros of $p_n(z)$, see [11, 12].

The following result gives the limit of zeros of $D_i(\Gamma'(\mathbb{Z}_{p_1p_2}), z)$.

Proposition 2.2. *For the family of polynomials $\mathcal{F} = \{D_i(\Gamma'(\mathbb{Z}_{p_1p_2}), z) : p_1, p_2$ primes, $p_1 < p_2\}$, with its set of zeros $Z(D_i(\Gamma'(\mathbb{Z}_{p_1p_2}), z))$. The set of all limits of zeros of $\mathcal{F}$ is $\{0\} \cup \{z \in \mathbb{C} : |z| = 1\}$.*

Proof. For fixed primes $p_1 < p_2$, we have

$$D_{p_1,p_2}(z) = D_i(\Gamma'(\mathbb{Z}_{p_1p_2}), z) = z^{p_1-1} + z^{p_2-1} = z^{p_1-1}\big(1 + z^{p_2-p_1}\big).$$

Thus, its set of zeros is

$$Z(D_{p_1,p_2}) = \{0\} \ \cup \ \{\zeta \in \mathbb{C} : \zeta^{p_2-p_1} = -1\}.$$

For every pair of primes $p_1 < p_2$, $z = 0$ is a root of $D_{p_1,p_2}(z)$ with multiplicity $p_1 - 1 \geq 1$. Hence, by definition, $z = 0$ is a limit of zeros of the family $\mathcal{F}$. If $z \neq 0$ and $D_{p_1,p_2}(z) = 0$, then $1 + z^k = 0$ with $k = p_2 - p_1 \in \mathbb{N}$. Thus, $z^k = -1 = e^{i\pi}$, and we get

$$z = \exp\left(i\,\frac{\pi + 2\pi m}{k}\right), \qquad m = 0, 1, \ldots, k-1.$$

In particular, every nonzero zero of D_{p_1,p_2} lies on the unit circle, and thus any limit of a sequence of nonzero zeros is again on the unit circle. If $z_n \in Z(D_{p_{1,n},p_{2,n}}) \setminus \{0\}$ and $z_n \to z$, then

$$|z| = \lim_{n\to\infty} |z_n| = \lim_{n\to\infty} 1 = 1.$$

So, every nonzero limit of zeros of $\mathcal{F}$ belongs to $\{z \in \mathbb{C} : |z| = 1\}$.

Now, we will show that every point on the unit circle has a limit of zeros. Let $z_0 = e^{i\theta}$ with $\theta \in \mathbb{R}$, so that $|z_0| = 1$. We show that z_0 is a limit point of zeros of $\mathcal{F}$. With fixed $p_1 = 2$, and let p_2 range over the primes greater than 2. Then $k = p_2 - p_1 = p_2 - 2$ takes infinitely many values, and $k \to \infty$ as $p_2 \to \infty$. For each such k, the nonzero zeros of D_{2,p_2} are the k-th roots of -1, namely

$$\zeta_{k,m} = \exp\left(i\,\frac{\pi + 2\pi m}{k}\right), \qquad m = 0, 1, \ldots, k-1.$$

For fixed k, choose $m = m(k)$, such that $\left|\theta - \frac{\pi+2\pi m}{k}\right| \leq \frac{\pi}{k}$, which is always possible by choosing m as the integer closest to $\frac{k\theta-\pi}{2\pi}$. Thus, we have

$$\begin{aligned}\left|z_0 - \zeta_{k,m(k)}\right| = \left|e^{i\theta} - \exp\left(i\,\frac{\pi + 2\pi m(k)}{k}\right)\right| &\leq 2\left|\sin\frac{\theta - (\pi + 2\pi m(k))/k}{2}\right| \\ &\leq |\theta - \tfrac{\pi+2\pi m(k))}{k}| \leq \frac{\pi}{k}.\end{aligned}$$

As $p_2 \to \infty$, it follows that $k = p_2 - 2 \to \infty$, thus $\frac{\pi}{k} \to 0$ and $\zeta_{k,m(k)} \xrightarrow[k\to\infty]{} z_0$. By definition, each $\zeta_{k,m(k)}$ is a root of $D_{2,p_2}(z)$ for a specific prime p_2. So, z_0 is a limit point of $\bigcup_{p_1<p_2} Z(D_{p_1,p_2})$. Since $z_0 = e^{i\theta}$ was arbitrary on the unit circle, we have shown that every point of $\{z \in \mathbb{C} : |z| = 1\}$ is a limit of zeros of the family $\mathcal{F}$. Hence, the full set of limits of zeros of the family $\mathcal{F}$ is $\{0\} \cup \{z \in \mathbb{C} : |z| = 1\}$. □

For $n = p_1p_2p_3$ with primes $p_1 < p_2 < p_3$, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is given as (see [29, Theorem 2.1])

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_2p_3-p_3} + z^{p_2p_3-p_2} + z^{p_1p_3-p_3} + z^{p_1p_3-p_1} + z^{p_1p_2-p_1} + z^{p_1p_2-p_2}. \tag{2.1}$$

Next result gives the bounds for the nonzero zeros of $D_i(\Gamma'(\mathbb{Z}_n), z)$, when n is product of three primes.

Theorem 2.3. *Let $n = p_1p_2p_3$ with primes $p_1 < p_2 < p_3$ and $D_i(\Gamma'(\mathbb{Z}_n), z)$ be the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$. Then, 0 is the zero of $D_i(\Gamma'(\mathbb{Z}_n), z)$ with multiplicity $p_2(p_1-1)$, and every nonzero zeros in the closed annulus $\left\{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 2\right\}$. Moreover, the polynomial $D_i(\Gamma'(\mathbb{Z}_n), z)$ has no positive real zeros.*

Proof. We write the independent domination polynomial (2.1) in compact form as

$$D(z) = D_i(\Gamma'(\mathbb{Z}_n), z) = z^{a_1} + z^{a_2} + z^{a_3} + z^{a_4} + z^{a_5} + z^{a_6},$$

where

$$\begin{aligned} a_1 &= p_2p_3 - p_3 = p_3(p_2 - 1), \quad a_2 = p_2p_3 - p_2 = p_2(p_3 - 1), \\ a_3 &= p_1p_3 - p_3 = p_3(p_1 - 1), \quad a_4 = p_1p_3 - p_1 = p_1(p_3 - 1), \\ a_5 &= p_1p_2 - p_1 = p_1(p_2 - 1), \quad a_6 = p_1p_2 - p_2 = p_2(p_1 - 1). \end{aligned}$$

Since $p_1 < p_2 < p_3$, we have $a_6 = p_2(p_1-1) < a_5 = p_1(p_2-1) < a_3 = p_3(p_1-1)$, and similarly $a_6 < a_4 < a_1 < a_2$. Thus, $m = \min\{a_1, \ldots, a_6\} = a_6 = p_2(p_1-1)$. Hence, the polynomial $D(z)$ can be written as $D(z) = z^m Q(z)$, where $Q(z)$ is a polynomial with $Q(0) = 1$. This shows that $z = 0$ is a zero of $D(z)$ of multiplicity $m = p_2(p_1-1)$. Now, we write $Q(z)$ as

$$Q(z) = z^{a_1-m} + z^{a_2-m} + z^{a_3-m} + z^{a_4-m} + z^{a_5-m} + 1.$$

The exponents of $Q(z)$ are $b_j = a_j - m, (1 \le j \le 5)$, which are positive integers, and among them the largest is $b_{\max} = a_2 - m = \big(p_2p_3 - p_2\big) - \big(p_1p_2 - p_2\big) = p_2(p_3 - p_1) > 0$. Thus, $Q(z)$ can be written in the form

$$Q(z) = z^d + \sum_{k=0}^{d-1} c_k z^k,$$

where $d = b_{\max} = p_2(p_3 - p_1)$, $c_0 = 1$ and each $c_k \in \{0, 1\}$, since the coefficient 1 appears exactly at the exponents $\{b_1, \ldots, b_5, 0\}$ and is 0 elsewhere. In particular, $Q(z)$ is a monic polynomial with $|c_k| \le 1$ for all $0 \le k \le d-1$. The nonzero zeros of $D(z)$ are exactly the zeros of $Q(z)$, since z^m contributes only the zero at $z = 0$. Let ζ be any zero of $Q(z)$. Since Q is monic polynomial of degree d, so the Cauchy's root bound can be applied to

$$Q(z) = z^d + c_{d-1}z^{d-1} + \cdots + c_0,$$

which gives us

$$|\zeta| \le 1 + \max_{0\le k\le d-1} |c_k| \le 1 + 1 = 2.$$

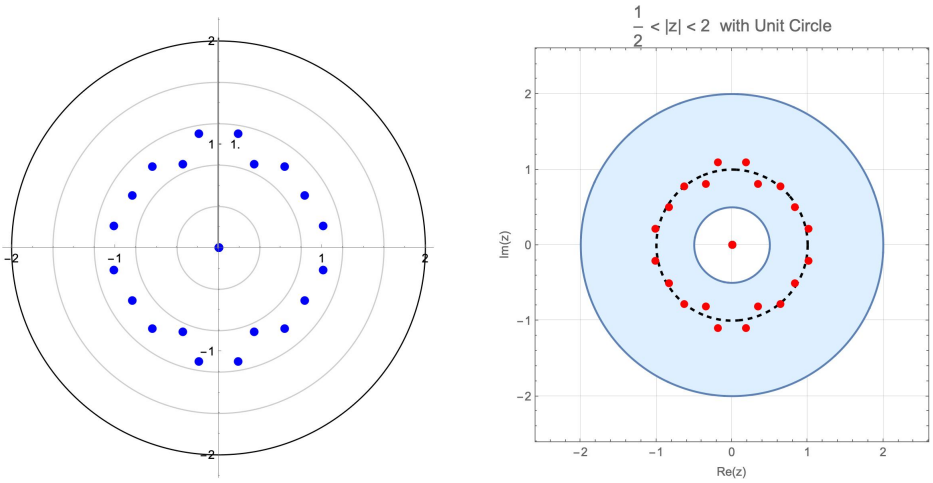
Thus, every zero of $Q(z)$, and hence every nonzero zero of $D(z)$ satisfies $|\zeta| \le 2$. To establish the lower bound, we examine the reciprocal polynomial

$$\widetilde{Q}(z) = z^d Q\left(\frac{1}{z}\right) = 1 + c_{d-1}z + \cdots + c_0 z^d.$$

This is a monic as the coefficient of z^d is $c_0 = 1$, and all its coefficients satisfy $|\cdot| \le 1$. The zeros of $\widetilde{Q}$ are precisely the reciprocals $\frac{1}{\zeta}$ of the zeros ζ of Q. Once more, utilizing Cauchy's inequality on $\widetilde{Q}$, we obtain

$$\left|\frac{1}{\zeta}\right| \le 1 + \max_{0\le k\le d-1} |c_k| \le 2.$$

Consequently, it follows that $|\zeta| \ge \frac{1}{2}$. By integrating these two inequalities, we deduce that any nonzero zero ζ of $D(z)$ fulfills the condition $\frac{1}{2} \le |\zeta| \le 2$. Lastly, we observe that all coefficients of $D(z)$ are nonnegative. Therefore, for $x > 0$, it follows that $D(x) > 0$. So, D has no positive real roots. Any real zeros, if they exist and are distinct from 0, must be on the negative real axis and satisfy $\frac{1}{2} \le |z| \le 2$. □



(a) Zeros of $D_i(\Gamma'(\mathbb{Z}_{105}), x)$ (b) Zeros of $D_i(\Gamma'(\mathbb{Z}_{105}), x)$.

Figure 1: Representation of the zeros of $D_i(\Gamma'(\mathbb{Z}_{105}), x)$.

The polar plot of the zeros of $D_i(\Gamma'(\mathbb{Z}_{105}), x) = x^{10} + x^{12} + x^{14} + x^{18} + x^{28} + x^{30}$ is shown in (a) Figure 1. Also, Figure (b) 1 shows the zeros (red points) of the polynomial $D_i(\Gamma'(\mathbb{Z}_{105}), x)$ lying in the annular region $\frac{1}{2} < |z| < 2$ (light blue), thereby illustrate Theorem 2.3. The unit circle $|z| = 1$ (black dashed) is included for reference.

The following result discuss the limit of the zeros of the polynomial $D_i(\Gamma'(\mathbb{Z}_{p_1p_2p_3}), z)$.

Theorem 2.4. *Let $n = p_1p_2p_3$, where $p_1 < p_2 < p_3$ are prime numbers, and let $D_n(z)$ denote its independent domination polynomial as defined in equation (2.1). Let L denote the set of limits of zeros of family $\{D_{p_1p_2p_3}(z) : p_1 < p_2 < p_3$ are primes$\}$. Then $0 \in L$ and $L \setminus \{0\} \subseteq \{z \in \mathbb{C} : |z| = 1\}$.*

Proof. Writing the exponents of $D_n(z)$ in compact form as

$$e_1 = p_2p_3 - p_3 = p_3(p_2 - 1), e_2 = p_2p_3 - p_2 = p_2(p_3 - 1), e_3 = p_1p_3 - p_3 = p_3(p_1 - 1),$$
$$e_4 = p_1p_3 - p_1 = p_1(p_3 - 1), e_5 = p_1p_2 - p_1 = p_1(p_2 - 1), e_6 = p_1p_2 - p_2 = p_2(p_1 - 1).$$

Since $p_1 < p_2 < p_3$, so it is easy to see that $e_{\min} = \min\{e_1, \ldots, e_6\} = e_6 = p_1p_2 - p_2$. Thus, $D_n(z) = z^{e_{\min}}Q_n(z)$, where

$$Q_n(z) = z^{e_1 - e_{\min}} + z^{e_2 - e_{\min}} + z^{e_3 - e_{\min}} + z^{e_4 - e_{\min}} + z^{e_5 - e_{\min}} + 1.$$

In particular, Q_n has constant term 1, so $z = 0$ is a zero of D_n of multiplicity $e_{\min} \geq 1$ for every n. Hence, by the definition of limit of zeros, 0 is a limit of zeros of the family $\{D_n\}$, so $0 \in L$. Also, by Theorem 2.3, any nonzero zero of $D_n(z)$ satisfies $\frac{1}{2} \leq |z| \leq 2$. Let $Z(D_n)$ be the set of zeros of $D_n(z)$. Consequently, the union $\bigcup_n (Z(D_n) \setminus \{0\})$ is contained in a compact

set, so by Bolzano–Weierstrass, it has at least one accumulation point in $\mathbb{C}$. In order to show that no nonzero limit zeros of $D_n(z)$ lie strictly inside the unit disk, we fix any $0 < \rho < 1$, and consider the closed disk $\overline{B(0,\rho)} = \{z : |z| \leq \rho\}$. We show that there exists N such that, for all $n \geq N$, D_n has no nonzero zeros in $\overline{B(0,\rho)}$. This implies that no nonzero limit of zeros lies with modulus $< \rho$. Since, $\rho < 1$ was arbitrary, so it follows that any nonzero limit of zeros must satisfy $|z| \geq 1$. Identifying the exponents of $Q_n(z)$ more explicitly, and considering their differences

$$e_1 - e_{\min} = p_3(p_2-1) - p_2(p_1-1), e_2 - e_{\min} = p_2(p_3-p_1), e_3 - e_{\min} = (p_1-1)(p_3-p_2),$$
$$e_4 - e_{\min} = p_1(p_3-1) - p_2(p_1-1), e_5 - e_{\min} = p_2 - p_1.$$

Thus, we see that $e_5 - e_{\min} = p_2 - p_1$ depends only on p_1, p_2, while the other four differences grow without bound as $p_3 \to \infty$. We write $Q_n(z)$ in a convenient way as $Q_n(z) = \big(1 + z^{m_n}\big) + R_n(z)$, where $m_n = p_2 - p_1 \geq 1$ and

$$R_n(z) = z^{e_1 - e_{\min}} + z^{e_2 - e_{\min}} + z^{e_3 - e_{\min}} + z^{e_4 - e_{\min}}.$$

Let z be such that $|z| \leq \rho < 1$. Then $|R_n(z)| \leq 4\rho^{\tau_n}$, where

$$\tau_n = \min\{e_1 - e_{\min}, e_2 - e_{\min}, e_3 - e_{\min}, e_4 - e_{\min}\}.$$

As n ranges over $p_1p_2p_3$ with $p_1 < p_2 < p_3$, we have $p_3 \to \infty$ as $n \to \infty$, and from the formulas above it follows that $\tau_n \to \infty$ with n. Therefore, we have

$$\sup_{|z|\leq\rho} |R_n(z)| \xrightarrow[n\to\infty]{} 0.$$

Conversely, for all n, and all $|z| \leq \rho$, it follows that

$$|1 + z^{m_n}| \geq 1 - |z|^{m_n} \geq 1 - \rho^{m_n} \geq 1 - \rho = \delta > 0,$$

since $m_n \geq 1$ and $\rho < 1$. Consequently, we derive

$$\inf_n \inf_{|z|\leq\rho} |1 + z^{m_n}| \geq \delta > 0.$$

Select N sufficiently large, such that $\sup_{|z|\leq\rho} |R_n(z)| < \frac{\delta}{2}$ for all $n \geq N$. For any $n \geq N$, and all $|z| \leq \rho$, it follows that

$$|Q_n(z) - (1 + z^{m_n})| = |R_n(z)| < \frac{\delta}{2} \quad \text{and} \quad |1 + z^{m_n}| \geq \delta.$$

According to Rouché's theorem, on the circle $|z| = \rho$, the functions Q_n and $1 + z^{m_n}$ possess an equivalent number of zeros inside the region $|z| < \rho$. The zeros of $1 + z^{m_n}$ correspond

precisely to the m_n-th roots of -1, all of which reside on the unit circle $|z| = 1$. Consequently, $1 + z^{m_n}$ possesses no zeros within the region $|z| < \rho$. Therefore, Q_n has no zeros in $|z| < \rho$, for all $n \geq N$. Since $D_n(z) = z^{e_{\min}} Q_n(z)$ and $e_{\min} \geq 1$, the only possible zero of D_n in $|z| < \rho$, for $n \geq N$ is $z = 0$. This shows that no nonzero limit of zeros can satisfy $|z| < \rho$. As $\rho \in (0, 1)$ was arbitrary, we conclude that every nonzero limit of zeros $w \in L$ satisfies $|w| \geq 1$.

A completely analogous argument using the reciprocal polynomials shows that no limit of zeros can have modulus strictly larger than 1. More precisely, let $E_n = \max\{e_1, \ldots, e_6\}$ be the largest exponent of D_n, and consider the polynomial

$$\widetilde{D}_n(z) = z^{E_n} D_n(1/z).$$

Its zeros are exactly the reciprocals of the zeros of D_n. A direct computation shows that $\widetilde{D}_n$ has the same form as D_n, with exponents shifted, and in particular it admits a factorisation

$$\widetilde{D}_n(z) = z^{e'_{\min}} \widetilde{Q}_n(z),$$

where $\widetilde{Q}_n(z)$ has constant term 1, and five further monomials with coefficients 1.

Repeating the argument of Q_n with new polynomial $\widetilde{Q}_n$, it follows that, for any fixed $\rho < 1$, there is N such that for all $n \geq N$, $\widetilde{Q}_n$ has no zeros in $|z| < \rho$, that is, D_n has no nonzero zeros with $|z| > 1/\rho$. Since $\rho < 1$ is arbitrary, it follows that any nonzero limit of zeros $w \in L$ must satisfy $|w| \leq 1$. Combining these two facts, we obtain that any nonzero limit of zeros $w \in L$ satisfies $|w| \geq 1$ and $|w| \leq 1$, and hence $|w| = 1$. Thus, it follows that

$$0 \in L \quad \text{and} \quad L \setminus \{0\} \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

□

3 Unimodal, log-concave and zeros of $D_i\big(\Gamma'(\mathbb{Z}_{p^{n_1}q}), z\big)$

Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with nonnegative coefficients, that is, $a_k \geq 0$ for all k. We say that $f(x)$ is *unimodal* if the coefficient sequence $(a_0, a_1, \ldots, a_n)$ is unimodal, that is, there exists an index m with $0 \leq m \leq n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_m \quad \text{and} \quad a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

The index m is called a *mode* of the polynomial. We say that $f(x)$ is *log-concave* if its coefficient sequence $(a_0, a_1, \ldots, a_n)$ is log-concave, that is, it satisfies

$$a_k^2 \geq a_{k-1}\, a_{k+1} \qquad \text{for all } k \text{ with } 1 \leq k \leq n-1.$$

A central reason for studying unimodal and log-concave polynomials is that their coefficient sequences behave in a remarkably regular way, and occur naturally throughout combinatorics and probability. If a polynomial $f(x)$ has nonnegative coefficients, and no internal zeros (that is, the set $\{k : a_k > 0\}$ is an interval of integers), then log-concavity $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq n-1$ already forces unimodality of $(a_0, \ldots, a_n)$ [31]. So, log-concave polynomials form a particularly well-behaved subclass of unimodal polynomials. A key structural motivation comes from the fact that if f has only real (nonpositive) zeros and nonnegative coefficients, then Newton's inequalities imply that the coefficient sequence is log-concave. Thus real-rootedness of a polynomial often serves as a powerful sufficient condition for log-concavity, and hence unimodality, for more see [20, 22, 31].

The following theorem from [29] gives the independent domination polynomial of the cozero divisor graph of $\mathbb{Z}_{p^{n_1}q}$

Theorem 3.1. [29] Let $n = p^{n_1}q$, where $p < q$ are primes, and let $\Gamma'(\mathbb{Z}_n)$ be the cozero divisor graph of the ring $\mathbb{Z}_n$. Then the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is

$$D_i\big(\Gamma'(\mathbb{Z}_n), z\big) = z^{p^{n_1-1}(q-1)} + \sum_{i=1}^{n_1-1} z^{(q-1)\left(p^{n_1-1}-p^{i-1}\right)+p^i-1} + z^{p^{n_1}-1}. \tag{3.2}$$

The following result gives the unimodal and log-concave properties of the independent domination polynomial of $\Gamma'(\mathbb{Z}_{p^{n_1}q})$, where $p < q$ are primes and $n_1 \in \mathbb{Z}$ is a positive integer.

Theorem 3.2. *Let $n = p^{n_1}q$ with primes $p < q$, positive integer $n_1 \geq 1$, and let $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ be its independent domination polynomial given as*

$$D_i\big(\Gamma'(\mathbb{Z}_n), z\big) = z^{s_0} + \sum_{i=1}^{n_1-1} z^{s_i} + z^{s_{n_1}}$$

where $s_0 = p^{n_1-1}(q-1), s_i = (q-1)\big(p^{n_1-1} - p^{i-1}\big) + p^i - 1$ $(1 \leq i \leq n_1 - 1)$, and $s_{n_1} = p^{n_1} - 1$. Then $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ is unimodal and log-concave if and only if $(p, q) = (2, 3)$, and if $(p, q) \neq (2, 3)$, then $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ is log-concave if and only if there is no index j with $1 \leq j \leq n_1$ such that $s_{j-1} - s_j = 2$. Equivalently, if $q - p = 2$ or if $(p, q) = (2, 5)$ with $n_1 \geq 2$, then $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ is not log-concave, and in all other cases $(p, q) \neq (2, 3)$, the polynomial $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ is log-concave but not unimodal.

Proof. With notations as in statement, we analyze the coefficients $s_0, s_1, \ldots, s_{n_1}$. For $i = 1$, we have

$$s_0 - s_1 = p^{n_1-1}(q-1) - \big((q-1)(p^{n_1-1} - 1) + p - 1\big) = q - p.$$

For $2 \leq i \leq n_1 - 1$, we have

$$s_{i-1} - s_i = (q-1)\big(p^{n_1-1} - p^{i-2}\big) + p^{i-1} - 1 - \Big((q-1)\big(p^{n_1-1} - p^{i-1}\big) + p^i - 1\Big)$$

$$= (q-1)\big(p^{i-1}-p^{i-2}\big) + p^{i-1}(1-p) = p^{i-2}(p-1)(q-p-1).$$

Finally, we have

$$s_{n_1-1} - s_{n_1} = (q-1)\big(p^{n_1-1}-p^{n_1-2}\big) + p^{n_1-1} - 1 - (p^{n_1}-1) = p^{n_1-2}(p-1)(q-p-1).$$

Given that $p < q$, it follows that $q-p > 0$ and $q-p-1 \geq 0$. Consequently, $s_0 > s_1$ and $s_{i-1} \geq s_i$, for $2 \leq i \leq n_1$. Furthermore, $s_{i-1} = s_i$ can only be true when $q-p-1 = 0$, which implies that $q = p+1$. The only primes differing by 1 are $p = 2, q = 3$, so, if $(p,q) = (2,3)$, then $s_0 > s_1 = s_2 = \cdots = s_{n_1}$, and if $(p,q) \neq (2,3)$, then $q-p-1 > 0$, hence if follows that $s_0 > s_1 > \cdots > s_{n_1}$. For $(p,q) = (2,3)$, we have $D_i\big(\Gamma'(\mathbb{Z}_{2^{n_1}\cdot 3}), z\big) = z^{2^{n_1}} + n_1 z^{2^{n_1}-1}$, which is unimodal and log-concave. Writing the independent domination polynomial as $D_i\big(\Gamma'(\mathbb{Z}_n), z\big) = \sum_{k\geq 0} c_k z^k$. For $(p,q) \neq (2,3)$, the exponents $s_0, \ldots, s_{n_1}$ are pairwise distinct, so $c_{s_j} = 1$, for $0 \leq j \leq n_1$, and $c_k = 0$ for all other k. We recall that a sequence $(a_0, a_1, \ldots, a_N)$ of nonnegative reals is unimodal if there exists an index m, such that

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_N.$$

We first note the following simple fact, if there exist indices $j < k < \ell$ with $a_j > 0$, $a_k = 0$ and $a_\ell > 0$, then $(a_0, \ldots, a_N)$ is not unimodal. Suppose (a_i) is unimodal with mode m. If $m \geq k$, then $a_0 \leq \cdots \leq a_m$, in particular $a_j \leq a_k = 0$, which contradicts $a_j > 0$. If $m < k$, then $a_m \geq \cdots \geq a_N$, so $a_\ell \leq a_k = 0$, contradicting $a_\ell > 0$. Consequently, if the coefficient sequence $(c_0, c_1, \ldots)$ of D_i is unimodal, then the support is the set of indices where the coefficients are nonzero, and hence

$$\operatorname{supp}(D_i) = \{k \geq 0 : c_k > 0\}$$

must be an interval of consecutive integers. Assume that $(p,q) \neq (2,3)$. Then, $s_0 > s_1 > \cdots > s_{n_1}$ and $s_0 - s_1 = q - p \geq 2$. Consequently, the support $\operatorname{supp}(D_i) = \{s_0, s_1, \ldots, s_{n_1}\}$ demonstrates a minimum gap of 2 between s_0 and s_1, signifying that it does not form an interval of consecutive integers. Hence, the coefficient sequence cannot be unimodal.

A sequence $(a_0, \ldots, a_N)$ is log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$, for $1 \leq i \leq N-1$. For $(p,q) \neq (2,3)$, we have $c_k \in \{0,1\}$, and for $(p,q) = (2,3)$, we have $c_k \in \{0,1,n_1\}$, with exactly one index k such that $c_k > 1$. For such sequences the only possible way to violate log-concavity is through a pattern $(c_{i-1}, c_i, c_{i+1}) = (1,0,1)$, since then $c_i^2 = 0 < 1 \cdot 1 = c_{i-1}c_{i+1}$. Therefore, D_i is log-concave if and only if $s_{j-1} - s_j \neq 2$, for all $1 \leq j \leq n_1$. Next, we solve $s_{j-1} - s_j = 2$ for j. If $j = 1$, then $s_0 - s_1 = q - p$. Thus $s_0 - s_1 = 2$ if and only if $q - p = 2$, that is, p and q are twin primes with $p \geq 3$. For $j \geq 2$, it follows that $s_{j-1} - s_j = p^{j-2}(p-1)(q-p-1)$. If $s_{j-1} - s_j = 2$, then p^{j-2}, $p-1$, and $q-p-1$ are positive integers, thus, a factorization of 2 into three positive integers is required. There are just two possibilities: $2 \cdot 1 \cdot 1$ or $1 \cdot 2 \cdot 1$.

So, either $p-1=2$ or $p-1=1$. For $p=2$, it follows that $p-1=1$, transforming the equation into $2^{j-2}(q-3)=2$. Thus, this leads to two scenarios: $2^{j-2}=1$ and $q-3=2$, or $2^{j-2}=2$ and $q-3=1$. The second situation yields $q=4$, which is not a prime number. In the initial scenario, we obtain $j=2$ and $q=5$, resulting in the triplet $(p,q,j)=(2,5,2)$. Given that $p=3$, it follows that $p-1=2$. Consequently, we have $2\cdot 3^{j-2}(q-4)=2$, leading to $3^{j-2}(q-4)=1$. Thus, the values of j and q must be 2 and 5, respectively. So, $(p,q,j)=(3,5,2)$. If $p\geq 5$, then $p-1\geq 4$, therefore $p^{j-2}(p-1)(q-p-1)\geq 4$, which cannot equal 2. Therefore, $s_{j-1}-s_j=2$ holds exactly if $j=1$ and $q-p=2$, that is, twin primes with $p\geq 3$, so $(p,q,j)=(2,5,2)$, and $(p,q,j)=(3,5,2)$, which are already included in $q-p=2$. That completes the proof. □

For $(p,q)=(2,3)$, we have $D_i\big(\Gamma'(\mathbb{Z}_{p^{n_1}q}),z\big)=z^{2^{n_1}-1}(z+n_1)$, and its zeros are $z=0$ with multiplicity $2^{n_1}-1$, and $z=-n_1$. Next, for remaining values, the following result gives the bounds for the zeros of the $D_i\big(\Gamma'(\mathbb{Z}_{p^{n_1}q}),z\big)$.

Theorem 3.3. *Let $n=p^{n_1}q$, where $p<q$ are prime numbers, and let $D_i(\Gamma'(\mathbb{Z}_n),z)$ denote its independent domination polynomial as specified in equation (3.2). Then $z=0$ is a zero of $D_i\big(\Gamma'(\mathbb{Z}_n),z\big)$ with multiplicity $m_0=p^{n_1}-1$, and all other zeros $\zeta\neq 0$ of $D_i\big(\Gamma'(\mathbb{Z}_n),z\big)$ lie in the closed annulus $\frac{1}{2}\leq|\zeta|\leq 2$.*

Proof. Let $D_i\big(\Gamma'(\mathbb{Z}_n),z\big)=\sum_{j=0}^{n_1} z^{s_j}$ be the given independent domination polynomial with $s_0=p^{n_1-1}(q-1), s_{n_1}=p^{n_1}-1$, and $s_i=(q-1)\big(p^{n_1-1}-p^{i-1}\big)+p^i-1$, for $1\leq i\leq n_1-1$. Factoring out $z^{s_{n_1}}$, we get $D_i\big(\Gamma'(\mathbb{Z}_n),z\big)=z^{s_{n_1}}Q(z)$, where

$$Q(z)=1+z^{s_0-s_{n_1}}+\sum_{i=1}^{n_1-1} z^{s_i-s_{n_1}}.$$

Since $Q(0)=1\neq 0$, it follows that $z=0$ is a zero of $D_i\big(\Gamma'(\mathbb{Z}_n),z\big)$ of multiplicity exactly $s_{n_1}=p^{n_1}-1$. The nonzero zeros of D_i are precisely the zeros of $Q(z)$. With degree $d=\deg Q=s_0-s_{n_1}$, and writing $Q(z)=\sum_{k=0}^{d} a_k z^k$. Since, $a_0=1, a_d=1$, and $0\leq a_k\leq 1$ for all k. So, by Cauchy's estimate for a polynomial

$$P(z)=a_d z^d+a_{d-1}z^{d-1}+\cdots+a_0,$$

all zeros satisfy

$$|z|\leq 1+\max_{0\leq k\leq d-1}\left|\frac{a_k}{a_d}\right|.$$

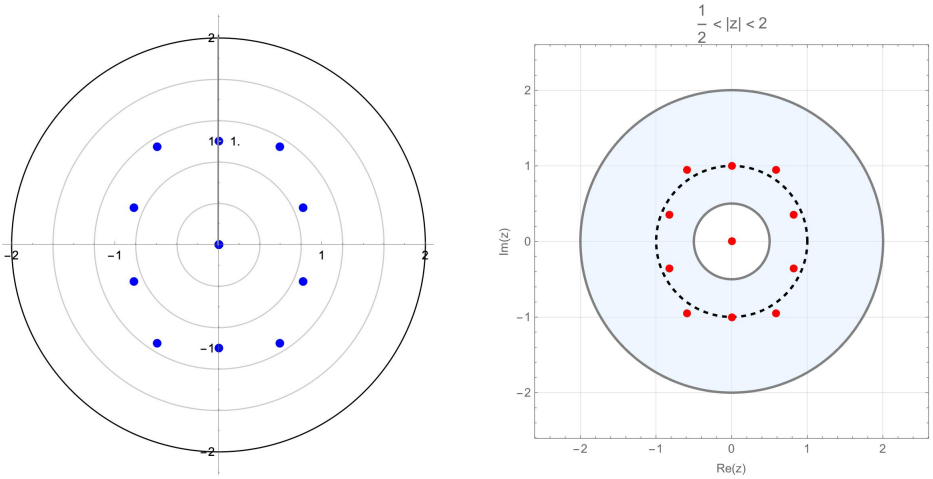
Here $a_d=1$, and $|a_k|\leq 1$, for all k, so we have

$$|z|\leq 1+\max_{0\leq k\leq d-1}|a_k|\leq 1+1=2.$$

Thus, every zero of Q satisfies $|z| \leq 2$. Again apply the corresponding Cauchy bound to $Q(z) = a_0 + a_1 z + \cdots + a_d z^d$, with $a_0 \neq 0$. A standard lower bound states that every zero z with $z \neq 0$ satisfies

$$|z| \geq \frac{1}{1 + \max\limits_{1 \leq k \leq d} \left|\frac{a_k}{a_0}\right|}.$$

With $a_0 = 1$ and $|a_k| \leq 1$ for all k, we obtain $\max\limits_{1 \leq k \leq d} \left|\frac{a_k}{a_0}\right| \leq 1$, and hence $|z| \geq \frac{1}{1+1} = \frac{1}{2}$. Thus, every nonzero zero ζ of $Q(z)$, and hence of $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ satisfies $\frac{1}{2} \leq |\zeta| \leq 2$. □



(a) Zeros of $D_i(\Gamma'(\mathbb{Z}_{135}), x)$ (b) Zeros of $D_i(\Gamma'(\mathbb{Z}_{135}), x)$.

Figure 2: Representation of the zeros of $D_i(\Gamma'(\mathbb{Z}_{135}), x)$.

Figure (a) 2 shows the zeros of the independent domination polynomial $D_i\big(\Gamma'(\mathbb{Z}_{3^3 5}), z\big) = z^{26} + z^{32} + z^{34} + z^{36}$ in the complex plane, displayed in polar coordinates. Apart from the multiple zero at the origin, all other zeros cluster near the unit circle, visually confirming that the nonzero zeros are confined to a narrow annulus around $|z| = 1$. Figure (b) 2 depicts the zeros of the independent domination polynomial $D_i\big(\Gamma'(\mathbb{Z}_{3^3 5}), z\big)$ inside the annulus $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$, shaded in light blue, as confirmed by Theorem 3.3. The inner circle $|z| = \frac{1}{2}$, the unit circle $|z| = 1$ (dashed), and the outer circle $|z| = 2$ are drawn for reference, and the red points illustrate that all nonzero zeros indeed lie inside this annulus.

Since, $z = 0$ is a zero of $D_i\big(\Gamma'(\mathbb{Z}_{p^{n_1} q}), z\big)$ of multiplicity $p^{n_1} - 1$ for every $n_1 \geq 1$. By the definition of limit of zeros, it follows immediately that $z = 0$ is a limit of zeros of $D_i\big(\Gamma'(\mathbb{Z}^{p^{n_1} q}), z\big)$. Assume that $p = 2$, $q = 3$, then

$$D_i\big(\Gamma'(\mathbb{Z}_{2^{n_1} 3}), z\big) = z^{2^{n_1}} + \sum_{i=1}^{n_1} z^{2^{n_1}-1} = z^{2^{n_1}-1}(z + n_1).$$

Thus, the zeros of $D_i\big(\Gamma'(\mathbb{Z}_{2^{n_1} 3}), z\big)$ are $z = 0$ with multiplicity $2^{n_1} - 1$, and simple zero

$z = -n_1$. As $n_1 \to \infty$ the nonzero root $-n_1$ tends to infinity, and has no finite accumulation point. Therefore, among finite complex numbers, only $z = 0$ is a limit of zeros of $D_i\big(\Gamma'(\mathbb{Z}_{2^{n_1}3}), z\big)$. For other non-zero zeros of $D_i\big(\Gamma'(\mathbb{Z}_{p^{n_1}q}), z\big)$, we have the following problem.

Problem 1. *Find the limit of the nonzero zeros of* $D_i\big(\Gamma'(\mathbb{Z}_{p^{n_1}q}), z\big)$.

Conclusion

For $n = p_1p_2$, the polynomial $D_i\big(\Gamma'(\mathbb{Z}_{p_1p_2}), z\big)$ has the nonzero zeros on the unit circle (Propositions 2.1, and 2.2). For $n = p_1p_2p_3$, the nonzero zeros of $D_i\big(\Gamma'(\mathbb{Z}_n), z\big)$ lie in an annulus (Theorem 2.3), and have all nonzero limit points on the unit circle (Theorem 2.4). For mixed prime powers $n = p^{n_1}q$, we characterised unimodality, log-concavity (Theorem 3.2) and zero distribution, and show that all nonzero zeros lie in $\{z : \frac{1}{2} \leq |z| \leq 2\}$ (Theorem 3.3). These results suggest further problems, including describing nonzero limit sets and extending the theory to wider classes of cozero divisor graphs of rings $\mathbb{Z}_n$, for larger values of n.

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