2-TOEPLITZ OPERATORS ON NEWTON SPACES

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Abstract. This paper initiates an exploration into the realm of 2-Toeplitz operators acting on Newton spaces. We establish explicit matrix representations for both the 2-Toeplitz operator itself and its adjoint. Furthermore, we delve into an analysis of their operator norm behavior, specifically characterizing the conditions under which these operators exhibit contractivity or expansivity when associated with analytic and harmonic symbols.

1. **Introduction**

Let $\mathbb{N}, \mathbb{N}_0$, $\mathbb{Z}^-$ and $\mathbb{R}^+$ denote the sets of all natural numbers, non-negative integers, negative integers, and positive real numbers, respectively. Additionally, $\lceil . \rceil$ represents the ceiling function and $\lfloor . \rfloor$ represents the floor function on the set of real numbers. For $n \in \mathbb{N}_0$, let $N_n(z)$ be the nth Newton polynomial defined by the coefficients in the following expansion,

$$(1-w)^z = \sum_{n=0}^{\infty} N_n(z) w^n$$

where $|w| < 1$ and z is any complex number. Alternatively, the nth Newton polynomial $N_n(z)$ is also defined by

$$N_n(z) = \frac{(-z)_n}{n!} = (-1)^n \binom{z}{n}, \text{ where } \binom{z}{n} = \begin{cases} \frac{z(z-1)(z-2)\cdots(z-(n-1))}{n!} & \text{if } n \geqslant 1 \\ 1 & \text{if } n = 0. \end{cases}$$

Let μ be a probability measure on $\mathbb{C}$ with finite moments, that is, $\int_{\mathbb{C}} |z|^n \, \mathrm{d}\mu(z) < \infty$ for all $n \in \mathbb{N}$. Let $\beta(x)$ be the discrete measure on real line with unit masses at $\left\{-\frac{1}{2} + \frac{n}{2} : n \in \mathbb{N}_0\right\}$, and let $\mathbb{P} = \left\{z \in \mathbb{C} : \mathrm{Re}(z) > -\frac{1}{2}\right\}$ and $\overline{\mathbb{P}} = \{z \in \mathbb{C} : \mathrm{Re}(z) \geqslant -\frac{1}{2}\}$. Define the measure μ on $\mathbb{C}$ by $d\mu(x+iy) = \frac{1}{2\pi} \frac{|\Gamma(x+iy+1)|^2}{\Gamma(2x+2)} \, dy \, d\beta(x)$. Then the Newton space $N^2(\mathbb{P})$ can be seen as the closure of the set of all Newton polynomials in $L^2(\mu)$, where $L^2(\mu)$ is the Lebesgue space of complex-valued measurable functions on $\mathbb{C}$ such that $\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 d\mu(z) < \infty$. Newton space is a Hilbert space, and the set of Newton polynomials $\{N_n(z)\}_{n=0}^{\infty}$ forms an orthonormal basis for $N^2(\mathbb{P})$ (see [2, 8, 9]). Therefore,

$$N^2(\mathbb{P}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n N_n(z) : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

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The reproducing kernel for $N^2(\mathbb{P})$ at $\lambda \in \mathbb{P}$ is given by

$$K(\lambda, z) = \frac{\Gamma(z+\overline{\lambda}+1)}{\Gamma(z+1)\Gamma(\overline{\lambda}+1)}, \quad \text{for all } z \in \mathbb{P}.$$

An inner product on $N^2(\mathbb{P})$ is defined as $\langle g, h\rangle = \int_{\mathbb{C}} g(z)\overline{h(z)}\mathrm{d}\mu(z)$ for all $g, h \in N^2(\mathbb{P})$. Let P be the orthogonal projection from $L^2(\mu)$ onto $N^2(\mathbb{P})$ given by

$$Pf(z) = \int_{\mathbb{C}} f(w)K(w,z)d\mu(w).$$

Let $L^\infty(\mu)$ be the space of all essentially bounded measurable functions with respect to the measure μ on $\mathbb{C}$ and $H^\infty(\mathbb{P})$ be the space of bounded analytic functions on $\mathbb{P}$. For $\phi \in L^\infty(\mu)$, the Toeplitz operator T_ϕ on $N^2(\mathbb{P})$ is defined by

$$T_\phi(f) = P(\phi f) \text{ for } f \in N^2(\mathbb{P}).$$

Linde [7] introduced Toeplitz operators T_ϕ on Newton space and investigated their properties. These operators are important due to their wide-ranging applications. Specifically, within Newton space (see [7]), Linde showed that for $\phi \in L^\infty(\mu)$:

(i) $T_{\alpha\phi+\beta\psi} = \alpha T_\phi + \beta T_\psi$;
(ii) $T_\phi^* = T_\phi$;
(iii) $T_\phi = 0$ if and only if $\phi = 0$ almost everywhere;
(iv) $T_\phi T_\psi = T_{\phi\psi}$ and $T_{\overline{\psi}} T_\phi = T_{\overline{\psi}\phi}$ for $\psi \in H^\infty(\mathbb{P})$.

Toeplitz operators on Newton space have been extensively studied [4–7], as have composition operators on $N^2(\mathbb{P})$ [3,7,8]. Han [2] characterized complex symmetric composition operators on $N^2(\mathbb{P})$ under a given conjugation. Gupta and Bhola [1] subsequently introduced 2-Toeplitz operators on the Hardy-Hilbert space, defined by matrices with alternating constant diagonals. These operators generalize multiplication and Toeplitz operators and inherit many of their properties, providing broader utility. Building on this progress, we define 2-Toeplitz operators on Newton space and investigate their contractivity and expansivity. We derive necessary and sufficient conditions for these properties to hold for both analytic and harmonic symbols.

2. 2-Toeplitz operator on Newton space

We begin this section by introducing the definition of 2-Toeplitz operators acting on the Newton space. Following this, we obtain the matrix representation of these operators under the harmonic orthonormal basis and explore its relationship with the adjoint matrix.

Let $\phi, \xi \in L^\infty(\mu)$ have series representations $\phi(z) = \sum_{n=0}^\infty \phi_n N_n(z) + \sum_{n=1}^\infty \phi_{-n}\overline{N_n(z)}$ and $\xi(z) = \sum_{n=0}^\infty \xi_n N_n(z) + \sum_{n=1}^\infty \xi_{-n}\overline{N_n(z)}$. Now, we define the functions $\phi_\xi, \xi_\phi : \overline{\mathbb{P}} \to \mathbb{C}$ as $\phi_\xi(z) = \sum_{n=0}^\infty \phi_n N_n(z) + \sum_{n=1}^\infty \phi_{-2n}\overline{N_{2n}(z)} + \sum_{n=1}^\infty \xi_{-(2n-1)}\overline{N_{2n-1}(z)}$ and $\xi_\phi(z) = \sum_{n=0}^\infty \xi_n N_n(z) + \sum_{n=1}^\infty \xi_{-2n}\overline{N_{2n}(z)} + \sum_{n=1}^\infty \phi_{-(2n-1)}\overline{N_{2n-1}(z)}$. Suppose $D_\infty = \Big\{(\phi,\xi) \in L^\infty(\mu) \times L^\infty(\mu) \,|\, \phi(z) = \sum_{n=0}^\infty \phi_n N_n(z) + \sum_{n=1}^\infty \phi_{-n}\overline{N_n(z)},$ and $\xi(z) = \sum_{n=0}^\infty \xi_n N_n(z) + \sum_{n=1}^\infty \xi_{-n}\overline{N_n(z)}$ and $\phi_\xi, \xi_\phi \in L^\infty(\mu)\Big\}$. For $f(z) = \sum_{n=0}^\infty a_n N_n(z) + \sum_{n=1}^\infty a_{-n}\overline{N_n(z)}$ in $L^2(\mu)$, we define the functions $f^-(z)$ and $f^+(z)$ as $f^-(z) = \sum_{n=1}^\infty a_{-(2n-1)}\overline{N_{2n-1}(z)} + \sum_{n=0}^\infty a_{2n+1}N_{2n+1}(z)$ and $f^+(z) = \sum_{n=1}^\infty a_{-2n}\overline{N_{2n}(z)} + \sum_{n=0}^\infty a_{2n}N_{2n}(z)$.

Definition 2.1. Let $(\phi,\xi)\in D_\infty$. The 2-Laurent operator with symbol (ϕ,ξ) is defined as the operator $L_{\phi,\xi}(f): L^2(\mu)\to L^2(\mu)$ such that

$$L_{\phi,\xi}(f)=L_{\phi_\xi}f^+ + L_{\xi_\phi}f^- \quad \text{for all } f\in L^2(\mu),$$

where L_{ϕ_ξ} and L_{ξ_ϕ} are classical Laurent operators.

Definition 2.2. Let $(\phi,\xi)\in D_\infty$. The 2-Toeplitz operator with a symbol (ϕ,ξ) is defined as the operator $T_{\phi,\xi}: N^2(\mathbb{P})\to N^2(\mathbb{P})$ such that

$$T_{\phi,\xi}(f)=PL_{\phi,\xi}(f) \quad \text{for all } f\in N^2(\mathbb{P}).$$

The subsequent Proposition demonstrates that 2-Toeplitz operators are bounded linear operators on Newton space.

Proposition 2.3. *The 2-Toeplitz operators are bounded linear operators on Newton space.*

Proof. Let $(\phi,\xi),(\psi,\eta)\in D_\infty$, $f\in N^2(\mathbb{P})$ and α be any scalar. Then by using the definition of 2-Toeplitz operator on Newton space, we have

$$\begin{aligned}
T_{\alpha(\phi,\xi)+(\psi,\eta)}f(z) =& T_{(\alpha\phi+\psi,\alpha\xi+\eta)}f(z)\\
=&P\left((\alpha\phi+\psi)_{(\alpha\xi+\eta)}f^+(z)+(\alpha\xi+\eta)_{(\alpha\phi+\psi)}\,f^-(z)\right)\\
=&P\Bigg(\sum_{i=0}^{\infty}(\alpha\phi_i+\psi_i)\,N_i(z)f^+(z)+\sum_{i=1}^{\infty}(\alpha\phi_{-2i}+\psi_{-2i})\,\overline{N_{2i}(z)}f^+(z)\\
&+\sum_{i=1}^{\infty}\left(\alpha\xi_{-(2i-1)}+\eta_{-(2i-1)}\right)\overline{N_{2i-1}(z)}f^+(z)+\sum_{i=0}^{\infty}(\alpha\xi_i+\eta_i)\,N_i(z)f^-(z)\\
&+\sum_{i=1}^{\infty}(\alpha\xi_{-2i}+\eta_{-2i})\,\overline{N_{2i}(z)}f^-(z)+\sum_{i=1}^{\infty}\left(\alpha\phi_{-(2i-1)}+\psi_{-(2i-1)}\right)\overline{N_{2i-1}(z)}f^-(z)\Bigg)\\
=&\alpha P\Bigg(\sum_{i=0}^{\infty}\phi_iN_i(z)f^+(z)+\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}(z)}f^+(z)+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}(z)}f^+(z)\\
&+\sum_{i=1}^{\infty}\xi_iN_i(z)f^-(z)+\sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}(z)}f^-(z)+\sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}(z)}f^-(z)\Bigg)\\
&+P\Bigg(\sum_{i=0}^{\infty}\psi_iN_i(z)f^+(z)+\sum_{i=1}^{\infty}\psi_{-2i}\overline{N_{2i}(z)}f^+(z)+\sum_{i=1}^{\infty}\eta_{-(2i-1)}\overline{N_{2i-1}(z)}f^+(z)\\
&+\sum_{i=0}^{\infty}\eta_i\overline{N_i(z)}f^-(z)+\sum_{i=1}^{\infty}\eta_{-2i}\overline{N_{2i}(z)}f^-(z)+\sum_{i=1}^{\infty}\psi_{-(2i-1)}\overline{N_{2i-1}(z)}f^-(z)\Bigg)\\
=&\alpha T_{\phi,\xi}(f)+T_{(\psi,\eta)}(f) \text{ for all } f\in N^2(\mathbb{P})
\end{aligned}$$

and

$$\begin{aligned}
\|T_{\phi,\xi}f\|=\left\|P\left(\phi_\xi f^+ + \xi_\phi f^-\right)\right\| \leqslant \left\|\phi_\xi f^+\right\|+\left\|\xi_\phi f^-\right\| \leqslant \|\phi_\xi\|_\infty\|f\|+\|\xi_\phi\|_\infty\|f\|\\
\leqslant 2\left(\|\phi\|_\infty+\|\xi\|_\infty\right)\|f\|.
\end{aligned}$$

This gives $\|T_{\phi,\xi}\|\leqslant 2\left(\|\phi\|_\infty+\|\xi\|_\infty\right)$. Hence, the 2-Toeplitz operators are bounded linear operators on Newton space. □

Remark 2.4. By using Proposition 2.3, we can easily show that the operator $T_{\phi,\xi}$ is closed in $N^2(\mathbb{P})$.

The following lemmas are useful for subsequent results.

Lemma 2.5. *[6] For any $m, n \geqslant 0$, the following hold:*

$$N_m(z)N_n(z) = \sum_{j=\max\{m,n\}}^{(m+n)} b_j(m,n)N_j(z)$$

where

$$\begin{pmatrix} b_m(m,n) \\ b_{m+1}(m,n) \\ b_{m+2}(m,n) \\ \vdots \\ b_{m+n}(m,n) \end{pmatrix} = \mathcal{N} \begin{pmatrix} N_m(m)N_n(m) \\ N_m(m+1)N_n(m+1) \\ N_m(m+2)N_n(m+2) \\ \vdots \\ N_m(m+n)N_n(m+n) \end{pmatrix}$$

and

$$\mathcal{N} = \begin{pmatrix} N_m(m) & 0 & 0 & 0 & \cdots & 0 \\ N_m(m+1) & N_{m+1}(m+1) & 0 & 0 & \cdots & 0 \\ N_m(m+2) & N_{m+1}(m+2) & N_{m+2}(m+2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N_m(m+n) & N_{m+1}(m+n) & N_{m+2}(m+n) & N_{m+3}(m+n) & \cdots & N_{m+n}(m+n) \end{pmatrix}$$

for $b_j(m,n) \in \mathbb{R}$.

Lemma 2.6. *[5] For any nonnegative integers m, k with $m \geqslant k$*

$$b_{m+j}(m,k) = (-1)^{k+j}\binom{k}{j}\binom{m+j}{k}$$

where $j = 0, 1, \ldots, k$.

Lemma 2.7. *[5, 6] For any nonnegative integers m, n, the following hold:*

(i) $\|N_m(z)N_n(z)\|^2 = \left\|\overline{N}_m(z)N_n(z)\right\|^2 = \displaystyle\sum_{j=\max\{m,n\}}^{(m+n)} |b_j(m,n)|^2$;

(ii) $\left\|N_m(z)\displaystyle\sum_{i=0}^{\infty} c_i N_i(z)\right\|^2 = \left\|\overline{N}_m(z)\displaystyle\sum_{i=0}^{\infty} c_i N_i(z)\right\|^2 = \displaystyle\sum_{i=m}^{\infty}\left|\sum_{j=i}^{m+i} c_{j-m} b_i(m, j-m)\right|^2$;

(iii) $\left\|P\left(\overline{N}_m(z)\displaystyle\sum_{i=0}^{\infty} c_i N_i(z)\right)\right\|^2 = \displaystyle\sum_{i=0}^{m-1}\left|\sum_{j=0}^{i} c_{m+j} b_{m+j}(m,i)\right|^2 + \sum_{i=m}^{\infty}\left|\sum_{j=i}^{m+i} c_j b_j(m,i)\right|^2$.

(iv) $P\left(\overline{N_n}N_m\right) = \begin{cases} \sum_{j=0}^{n} b_m(n, m-n+j)N_{m-n+j}(z) & \textit{if } m \geq n; \\ 0 & \textit{if } m < n, \end{cases}$

where for each i, $c_i \in \mathbb{C}$ and $b_m(n, m-n+j)$ is the solution of the matrix equation as in Lemma 2.5 for $0 \leqslant j \leqslant n$.

Consider the following notation: $\delta_{j,k,i}(m) = \displaystyle\sum_{j=\max\{k+m,i\}}^{k+m+i} c_j b_j\,(i, k+m)$ and $\Delta_{i,k,j}(m,n) = c_{2k-2i+m} b_{2k+n}(2k-2i+m, j)$ for all $c_i \in \mathbb{C}$.

Lemma 2.8. *The orthogonal projection P on Newton space satisfies the following:*

$$\left\| P\left(\sum_{k=1}^{\infty} a_k\overline{N_k}(z)\sum_{i=0}^{\infty} c_{2i}N_{2i}(z)+\sum_{k=1}^{\infty} d_k\overline{N_k}(z)\sum_{i=0}^{\infty} c_{2i+1}N_{2i+1}(z)\right)\right\|^2$$

$$=\sum_{k=0}^{\infty}\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,k,i}(0)\right|^2=\sum_{k=0}^{\infty}\left\{\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,2k,i}(0)\right|^2+\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,2k,i}(1)\right|^2\right\}$$

where $^{j}p_i=\begin{cases} a_i & \text{if } j \text{ is even;}\\ d_i & \text{if } j \text{ is odd.}\end{cases}$

Proof. On using Lemma 2.7, we compute

$$\left\| P\left(\sum_{k=1}^{\infty} a_k\overline{N_k}(z)\sum_{i=0}^{\infty} c_{2i}N_{2i}(z)+\sum_{k=1}^{\infty} d_k\overline{N}_k(z)\sum_{i=0}^{\infty} c_{2i+1}N_{2i+1}(z)\right)\right\|^2$$

$$=\Bigg\|\sum_{k=1}^{\infty} a_k\sum_{i=\lceil\frac{k}{2}\rceil}^{\infty} c_{2i}\sum_{j=0}^{k} b_{2i}(k,2i-k+j)N_{2i-k+j}(z)+$$

$$\sum_{k=1}^{\infty} d_k\sum_{i=\lfloor\frac{k}{2}\rfloor}^{\infty} c_{2i+1}\sum_{j=0}^{k} b_{2i+1}(k,2i+1-k+j)N_{2i+1-k+j}(z)\Bigg\|^2$$

$$\begin{aligned}&=\|(d_1c_1b_1(1,0)+a_2c_2b_2(2,0)+d_3c_3b_3(3,0)+\cdots)N_0(z)+(d_1c_1b_1(1,1)\\&\quad+a_1c_2b_2(1,1)+a_2c_2b_2(2,1)+\cdots)\,N_1(z)+(a_1c_2b_2(1,2)+d_1c_3b_3(1,2)+\cdots)\,N_2(z)\\&\quad+(d_1c_3b_3(1,3)+a_1c_4b_4(1,3)+d_2c_3b_3(2,3)+\cdots)\,N_3(z)+\cdots\ \|^2\end{aligned}$$

$$=\left\|\sum_{k=0}^{\infty}\left(\sum_{i=1}^{\infty} {}^{j}p_i\sum_{j=\max\{k,i\}}^{k+i} c_jb_j(i,k)\right)N_k(z)\right\|^2=\sum_{k=0}^{\infty}\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,k,i}(0)\right|$$

$$=\sum_{k=0}^{\infty}\left\{\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,2k,i}(0)\right|^2+\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,2k,i}(1)\right|^2\right\}.$$

□

Lemmas 2.7 and 2.8 imply the following.

Lemma 2.9. *For any non-negative integer m, the orthogonal projection P on Newton space satisfies the following:*

$$\left\| P\left(aN_m(z)\sum_{i=0}^{\infty} c_{2i}N_{2i}(z)+dN_m(z)\sum_{i=0}^{\infty} c_{2i+1}N_{2i+1}(z)\right)\right\|^2=\sum_{t=m}^{\infty}\left|\sum_{i=t-m}^{t}\lambda_ic_ib_t(i,m)\right|^2,$$

where $\lambda_i=\begin{cases} a & \text{if } i \text{ is even;}\\ d & \text{if } i \text{ is odd.}\end{cases}$

Lemma 2.10. *The orthogonal projection P on Newton space satisfies the following:*

$$\Bigg\| P\Bigg(\sum_{k=1}^{\infty} a_{2k}\overline{N_{2k}}(z)\sum_{i=0}^{\infty} c_{2i}N_{2i}(z)+\sum_{k=1}^{\infty} d_{2k-1}\overline{N_{2k-1}}(z)\sum_{i=0}^{\infty} c_{2i}N_{2i}(z)$$

$$+\sum_{k=1}^{\infty} d_{2k}\overline{N_{2k}}(z)\sum_{i=0}^{\infty} c_{2i+1}N_{2i+1}(z)+\sum_{k=1}^{\infty} a_{2k-1}\overline{N_{2k-1}}(z)\sum_{i=0}^{\infty} c_{2i+1}N_{2i+1}(z)\Bigg\|^2$$

$$=\sum_{k=0}^{\infty}\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,k,i}(0)\right|^2=\sum_{k=0}^{\infty}\left\{\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,2k,i}(0)\right|^2+\left|\sum_{i=1}^{\infty} {}^{j}p_i\delta_{j,2k,i}(1)\right|^2\right\}$$

where $^{j}p_i=\begin{cases} a_i & \textit{if } i \textit{ is even and } j \textit{ is even or } i \textit{ is odd and } j \textit{ is odd;}\\ d_i & \textit{if } i \textit{ is even and } j \textit{ is odd or } i \textit{ is odd and } j \textit{ is even.}\end{cases}$

Using Lemma 2.7, we examine the matrix representation of $T_{\phi,\xi}$ on Newton space.

Theorem 2.11. *Let* $(\phi,\xi)\in D_\infty$ *such that* $\phi_\xi(z)=\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}+\sum_{i=0}^{\infty}\phi_iN_i$ *and* $\xi_\phi(z)=\sum_{i=0}^{\infty}\xi_iN_i+\sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}+\sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}$ *be two harmonic symbols; then the matrix of the operator* $T_{\phi,\xi}$ *with respect to orthonormal basis* $\{N_n\}_{n\geqslant 0}$ *is given by*

$$[T_{\phi,\xi}]=\begin{pmatrix}
\phi_0b_0(0,0) & \phi_{-1}b_1(1,0) & \phi_{-2}b_2(2,0) & \phi_{-3}b_3(3,0) & \cdots\\
\phi_1b_1(1,0) & \sum_{i=0}^{1}\xi_ib_1(i,1) & \xi_{-1}b_2(1,1)+\phi_{-2}b_2(2,1) & \xi_{-2}b_3(2,1)+\phi_{-3}b_3(3,1) & \cdots\\
\phi_2b_2(2,0) & \sum_{i=1}^{2}\xi_ib_2(i,1) & \sum_{i=0}^{2}\phi_ib_2(i,2) & \xi_{-2}b_3(2,2)+\sum_{i=1}^{2}\phi_{-(2i-1)}b_3(2i-1,2) & \cdots\\
\phi_3b_3(3,0) & \sum_{i=2}^{3}\xi_ib_3(i,1) & \sum_{i=1}^{3}\phi_ib_3(i,2) & \sum_{i=0}^{3}\xi_ib_3(i,3) & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

where $b_i(\cdot,\cdot)\in\mathbb{R}$ *is denoted from Lemma 2.5.*

Proof. Let $\phi_\xi(z)=\sum_{i=0}^{\infty}\phi_iN_i+\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}$ and $\xi_\phi(z)=\sum_{i=0}^{\infty}\xi_iN_i+\sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}+\sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}$ be two harmonic symbols; the (m,n)th entry of the matrix $T_{\phi,\xi}$ with respect to orthonormal basis $\gamma=\{N_n\}_{n\geqslant 0}$ of $N^2(\mathbb{P})$ is given by,

$$\begin{aligned}\langle T_{\phi,\xi}N_n,N_m\rangle &=\left\langle P\left(\phi_\xi N_n^+ +\xi_\phi N_n^-\right),N_m\right\rangle\\ &=\left\langle P\left(\phi_\xi N_n^+\right),N_m\right\rangle+\left\langle P\left(\xi_\phi N_n^-\right),N_m\right\rangle. \end{aligned}\tag{2.1}$$

Consider the following two cases:

Case (i): If n is even. Then by using equation (2.1) and Lemma 2.7, we compute

$$\begin{aligned}\langle T_{\phi,\xi}N_n,N_m\rangle &=\left\langle P\left(\phi_\xi N_n^+\right),N_m\right\rangle\\ &=\left\langle P\left(\sum_{i=0}^{\infty}\phi_iN_iN_n+\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}N_n+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}N_n\right),N_m\right\rangle\\ &=\left\langle\sum_{i=0}^{\infty}\phi_i\sum_{j=\max\{i,n\}}^{i+n}b_j\left(i,n\right)N_j,N_m\right\rangle+\left\langle\sum_{i=1}^{\frac{n}{2}}\phi_{-2i}\sum_{j=0}^{2i}b_n(2i,n-2i+j)N_{n-2i+j}\right.\end{aligned}$$

$$+\sum_{i=1}^{\frac{n}{2}}\xi_{-(2i-1)}\sum_{j=0}^{2i-1}b_n(2i-1,n-(2i-1)+j)N_{n-(2i-1)+j},N_m\Big\rangle. \tag{2.2}$$

For $n \leqslant m$, by equation (2.2), we get

$$\begin{aligned}\langle T_{\phi,\xi}N_n,N_m\rangle &= \Big\langle\sum_{i=0}^{\infty}\phi_i\sum_{j=\max\{i,n\}}^{i+n}b_j(i,n)N_j,N_m\Big\rangle = \Big\langle\sum_{i=0}^{m-n-1}\phi_i\sum_{j=\max\{i,n\}}^{i+n}b_j(i,n)N_j\\ &\quad+\sum_{i=m-n}^{m}\phi_i\sum_{j=\max\{i,n\}}^{i+n}b_j(i,n)N_j+\sum_{i=m+1}^{\infty}\phi_i\sum_{j=\max\{i,n\}}^{i+n}b_j(i,n)N_j,N_m\Big\rangle\\ &=\Big\langle\sum_{i=m-n}^{m}\phi_i b_m(i,n)N_m,N_m\Big\rangle=\sum_{i=m-n}^{m}\phi_i b_m(i,n).\end{aligned}$$

For $n > m$, using Lemma 2.7 from equation (2.2), we get

$$\begin{aligned}\langle T_{\phi,\xi}N_n,N_m\rangle &= \Big\langle\sum_{i=1}^{\frac{n}{2}}\phi_{-2i}\sum_{j=0}^{2i}b_n(2i,n-2i+j)N_{n-2i+j}\\ &\quad+\sum_{i=1}^{\frac{n}{2}}\xi_{-(2i-1)}\sum_{j=0}^{2i-1}b_n(2i-1,n-(2i-1)+j)N_{n-(2i-1)+j},N_m\Big\rangle.\end{aligned} \tag{2.3}$$

Consider the following cases:
Case (a): If m is odd, then by using equation (2.3) we have

$$\begin{aligned}\langle T_{\phi,\xi},N_n,N_m\rangle &= \Big\langle\sum_{i=1}^{\frac{n-m-1}{2}}\phi_{-2i}\sum_{j=0}^{2i}b_n(2i,n-2i+j)N_{n-2i+j}\\ &\quad+\sum_{i=\frac{n-m+1}{2}}^{\frac{n}{2}}\phi_{-2i}\sum_{j=0}^{2i}b_n(2i,n-2i+j)N_{n-2i+j}\\ &\quad+\sum_{i=1}^{\frac{n-m-1}{2}}\xi_{-(2i-1)}\sum_{j=0}^{2i-1}b_n(2i-1,n-(2i-1)+j)N_{n-(2i-1)+j}\\ &\quad+\sum_{i=\frac{n-m+1}{2}}^{\frac{n}{2}}\xi_{-(2i-1)}\sum_{j=0}^{2i-1}b_n(2i-1,n-(2i-1)+j)N_{n-(2i-1)+j},N_m\Big\rangle\\ &=\Big\langle\sum_{i=\frac{n-m+1}{2}}^{\frac{n}{2}}\phi_{-2i}b_n(2i,m)N_m+\sum_{i=\frac{n-m+1}{2}}^{\frac{n}{2}}\xi_{-(2i-1)}b_n(2i-1,m)N_m,N_m\Big\rangle\\ &=\sum_{i=\frac{n-m+1}{2}}^{\frac{n}{2}}\phi_{-2i}b_n(2i,m)+\sum_{i=\frac{n-m+1}{2}}^{\frac{n}{2}}\xi_{-(2i-1)}b_n(2i-1,m).\end{aligned}$$

Case (b): If m is even, then similarly by using equation (2.3), we have

$$\langle T_{\phi,\xi}N_n, N_m\rangle = \sum_{i=\frac{n-m}{2}}^{\frac{n}{2}} \phi_{-2i} b_n(2i,m) + \sum_{i=\frac{n-m+2}{2}}^{\frac{n}{2}} \xi_{-(2i-1)} b_n(2i-1,m).$$

Case (ii): If n is odd, then by equation (2.1) and Lemma 2.7, we have

$$\begin{aligned}
\langle T_{\phi,\xi}N_n, N_m\rangle &= \left\langle P\left(\xi_\phi N_n^-\right), N_m\right\rangle \\
&= \left\langle P\left(\sum_{i=0}^{\infty}\xi_i N_i N_n + \sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}N_n + \sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}N_n\right), N_m\right\rangle \\
&= \left\langle \sum_{i=0}^{\infty}\xi_i \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n)N_j, N_m\right\rangle + \left\langle \sum_{i=1}^{\frac{n-1}{2}}\xi_{-2i}\sum_{j=0}^{2i} b_n(2i, n-2i+j)N_{n-2i+j}\right. \\
&\quad \left. + \sum_{i=1}^{\frac{n+1}{2}}\phi_{-(2i-1)}\sum_{j=0}^{2i-1} b_n(2i-1, n-(2i-1)+j)N_{n-(2i-1)+j}, N_m\right\rangle .
\end{aligned} \tag{2.4}$$

For $n \leqslant m$, equation (2.4) yields $\langle T_{\phi,\xi}N_n, N_m\rangle = \sum\limits_{i=m-n}^{m} \xi_i b_m(i,n)$. For $n > m$, it yields $\langle T_{\phi,\xi}N_n, N_m\rangle = \sum\limits_{i=\frac{n-m}{2}}^{\frac{n-1}{2}} \xi_{-2i} b_n(2i,m) + \sum\limits_{i=\frac{n-m+2}{2}}^{\frac{n+1}{2}} \phi_{-(2i-1)} b_n(2i-1,m)$ if m is odd and $\langle T_{\phi,\xi}N_n, N_m\rangle = \sum\limits_{i=\frac{n-m+1}{2}}^{\frac{n-1}{2}} \xi_{-2i} b_n(2i,m) + \sum\limits_{i=\frac{n-m+1}{2}}^{\frac{n+1}{2}} \phi_{-(2i-1)} b_n(2i-1,m)$ if m is even. Hence, for $n \leqslant m$, the (m,n)th entry is given by

$$\langle T_{\phi,\xi}N_n, N_m\rangle = \begin{cases} \sum\limits_{i=m-n}^{m} \phi_i b_m(i,n) & \text{if } n \text{ is even;} \\ \sum\limits_{i=m-n}^{m} \xi_i b_m(i,n) & \text{if } n \text{ is odd} \end{cases}$$

and for $n > m$, the (m,n)th entry is given by

$$\langle T_{\phi,\xi}N_n, N_m\rangle = \begin{cases} \sum\limits_{i=\frac{n-m+1}{2}}^{\frac{n}{2}} \phi_{-2i} b_n(2i,m) + \sum\limits_{i=\frac{n-m+1}{2}}^{\frac{n}{2}} \xi_{-(2i-1)} b_n(2i-1,m) & \text{if } n \text{ is even and } m \text{ is odd;} \\ \sum\limits_{i=\frac{n-m}{2}}^{\frac{n}{2}} \phi_{-2i} b_n(2i,m) + \sum\limits_{i=\frac{n-m+2}{2}}^{\frac{n}{2}} \xi_{-(2i-1)} b_n(2i-1,m) & \text{if } n \text{ is even and } m \text{ is even;} \\ \sum\limits_{i=\frac{n-m}{2}}^{\frac{n-1}{2}} \xi_{-2i} b_n(2i,m) + \sum\limits_{i=\frac{n-m+2}{2}}^{\frac{n+1}{2}} \phi_{-(2i-1)} b_n(2i-1,m) & \text{if } n \text{ is odd and } m \text{ is odd;} \\ \sum\limits_{i=\frac{n-m+1}{2}}^{\frac{n-1}{2}} \xi_{-2i} b_n(2i,m) + \sum\limits_{i=\frac{n-m+1}{2}}^{\frac{n+1}{2}} \phi_{-(2i-1)} b_n(2i-1,m) & \text{if } n \text{ is odd and } m \text{ is even} \end{cases}$$

where m and n are non-negative integers and the matrix representation of $T_{\phi,\xi}$ with respect to the orthonormal basis $\{N_n\}_{n\geqslant 0}$ is given by

$$\begin{pmatrix}
\phi_0 b_0(0,0) & \phi_{-1}b_1(1,0) & \phi_{-2}b_2(2,0) & \phi_{-3}b_3(3,0) & \cdots \\
\phi_1 b_1(1,0) & \sum\limits_{i=0}^{1}\xi_i b_1(i,1) & \xi_{-1}b_2(1,1)+\phi_{-2}b_2(2,1) & \xi_{-2}b_3(2,1)+\phi_{-3}b_3(3,1) & \cdots \\
\phi_2 b_2(2,0) & \sum\limits_{i=1}^{2}\xi_i b_2(i,1) & \sum\limits_{i=0}^{2}\phi_i b_2(i,2) & \xi_{-2}b_3(2,2)+\sum\limits_{i=1}^{2}\phi_{-(2i-1)}b_3(2i-1,2) & \cdots \\
\phi_3 b_3(3,0) & \sum\limits_{i=2}^{3}\xi_i b_3(i,1) & \sum\limits_{i=1}^{3}\phi_i b_3(i,2) & \sum\limits_{i=0}^{3}\xi_i b_3(i,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

□

Using Lemma 2.7, we examine the matrix representation of the adjoint of the operator $T_{\phi,\xi}$ on Newton space.

Theorem 2.12. *Let $(\phi,\xi)\in D_\infty$ be such that for each fixed non-negative integer m, ${}^m k_{\phi,\xi}=\sum_{i=1}^{\infty}\overline{{}^m a_{-i}}N_i+\sum_{i=0}^{\infty}{}^m\lambda_i\overline{N_i}\in L^\infty(\mu)$ where ${}^m\lambda_i=\begin{cases}\overline{\phi_i} & \text{if } m \text{ is even;}\\ \overline{\xi_i} & \text{if } m \text{ is odd}\end{cases}$ and ${}^m a_i=\begin{cases}\phi_i & \text{if } i \text{ is even and } m \text{ is even or } i \text{ is odd and } m \text{ is odd;}\\ \xi_i & \text{if } i \text{ is odd and } m \text{ is even or } i \text{ is even and } m \text{ is odd.}\end{cases}$ Then, the m-th row of the matrix of $T_{{}^m k_{\phi,\xi}}$ is the m-th row of the adjoint of the matrix of $T_{\phi,\xi}$ with respect to the orthonormal basis $\{N_n\}_{n\geqslant 0}$.*

Proof. Suppose ${}^m k_{\phi,\xi}=\sum_{i=1}^{\infty}\overline{{}^m a_{-i}}N_i+\sum_{i=0}^{\infty}{}^m\lambda_i\overline{N_i}$, where ${}^m\lambda_i=\begin{cases}\overline{\phi_i} & \text{if } m \text{ is even;}\\ \overline{\xi_i} & \text{if } m \text{ is odd}\end{cases}$ and ${}^m a_i=\begin{cases}\phi_i & \text{if } i \text{ is even and } m \text{ is even or } i \text{ is odd and } m \text{ is odd;}\\ \xi_i & \text{if } i \text{ is odd and } m \text{ is even or } i \text{ is even and } m \text{ is odd}\end{cases}$ be the harmonic symbols; the (m,n)th entry of the matrix of $T_{{}^m k_{\phi,\xi}}$ with respect to the orthonormal basis $\gamma=\{N_n\}_{n\geqslant 0}$ of $N^2(\mathbb{P})$ is

$$\begin{aligned}
\left\langle T_{{}^m k_{\phi,\xi}}N_n, N_m\right\rangle &= \left\langle P\left({}^m k_{\phi,\xi}N_n\right), N_m\right\rangle \\
&= \left\langle P\left(\sum_{i=1}^{\infty}\overline{{}^m a_{-i}}N_iN_n+\sum_{i=0}^{\infty}{}^m\lambda_i\overline{N_i}N_n\right), N_m\right\rangle \\
&= \left\langle \sum_{i=0}^{n}{}^m\lambda_i\sum_{j=0}^{i}b_n(i,n-i+j)N_{n-i+j}, N_m\right\rangle \\
&\quad + \left\langle \sum_{i=1}^{\infty}\overline{{}^m a_{-i}}\sum_{j=\max\{i,n\}}^{i+n}b_j(i,n)N_j, N_m\right\rangle. \qquad (2.5)
\end{aligned}$$

Consider the following two cases:
Case (i): For $n\geqslant m$. Then by using equation (2.5), we have

$$\begin{aligned}
\left\langle T_{{}^m k_{\phi,\xi}}N_n, N_m\right\rangle &= \left\langle \sum_{i=0}^{n}{}^m\lambda_i\sum_{j=0}^{i}b_n(i,n-i+j)N_{n-i+j}, N_m\right\rangle \\
&= \left\langle \sum_{i=0}^{n-m-1}{}^m\lambda_i\sum_{j=0}^{i}b_n(i,n-i+j)+\sum_{i=n-m}^{n}{}^m\lambda_i\sum_{j=0}^{i}b_n(i,n-i+j), N_m\right\rangle
\end{aligned}$$

$$= \left\langle \sum_{i=n-m}^{n} {}^{m}\lambda_i b_n(i,m) N_m, N_m \right\rangle$$

$$= \sum_{i=n-m}^{n} {}^{m}\lambda_i b_n(i,m), \text{ where } {}^{m}\lambda_i = \begin{cases} \overline{\phi_i} & \text{if } m \text{ is even;} \\ \overline{\xi_i} & \text{if } m \text{ is odd.} \end{cases}$$

Case (ii): For $n < m$ from equation (2.5), we obtain

$$\left\langle T_{{}^{m}k_{\phi,\xi}} N_n, N_m \right\rangle = \left\langle \sum_{i=1}^{m-n-1} \overline{{}^{m}a_{-i}} \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j + \sum_{i=m-n}^{m} \overline{{}^{m}a_{-i}} \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j \right.$$
$$\left. + \sum_{i=m+1}^{\infty} \overline{{}^{m}a_{-i}} \sum_{j=\max\{i,n\}}^{i+n} b_j(i,n) N_j, N_m \right\rangle = \sum_{i=m-n}^{m} \overline{{}^{m}a_{-i}} b_m(i,n).$$

Thus, the (m,n)th entry of $T_{{}^{m}k_{\phi,\xi}}$ is given by

$$\left\langle T_{{}^{m}k_{\phi,\xi}} N_n, N_m \right\rangle = \begin{cases} \displaystyle\sum_{i=n-m}^{n} {}^{m}\lambda_i b_n(i,m) & \text{if } n \geqslant m; \\ \displaystyle\sum_{i=m-n}^{m} \overline{{}^{m}a_{-i}} b_m(i,n) & \text{if } n < m \end{cases}$$

where m and n are non-negative integers and the matrix representation of $T_{{}^{m}k_{\phi,\xi}}$ with respect to the orthonormal basis $\{N_n\}_{n\geqslant 0}$ is given by

$$\begin{pmatrix} \overline{\phi_0} b_0(0,0) & \overline{\phi_1} b_1(1,0) & \overline{\phi_2} b_2(2,0) & \overline{\phi_3} b_3(3,0) & \cdots \\ \overline{\phi_{-1}} b_1(1,0) & \displaystyle\sum_{i=0}^{1} \overline{\xi_i} b_1(i,1) & \displaystyle\sum_{i=1}^{2} \overline{\xi_i} b_2(i,1) & \displaystyle\sum_{i=2}^{3} \overline{\xi_i} b_3(i,1) & \cdots \\ \overline{\phi_{-2}} b_2(2,0) & \overline{\xi_{-1}} b_2(1,1) + \overline{\phi_{-2}} b_2(2,1) & \displaystyle\sum_{i=0}^{2} \overline{\phi_i} b_2(i,2) & \displaystyle\sum_{i=1}^{3} \overline{\phi_i} b_3(i,2) & \cdots \\ \overline{\phi_{-3}} b_3(3,0) & \overline{\xi_{-2}} b_3(2,1) + \overline{\phi_{-3}} b_3(3,1) & \overline{\xi_{-2}} b_3(2,2) + \displaystyle\sum_{i=1}^{2} \overline{\phi_{-(2i-1)}} b_3(2i-1,2) & \displaystyle\sum_{i=0}^{3} \overline{\xi_i} b_3(i,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Therefore, the matrix representation of the adjoint operator $T^*_{\phi,\xi}$ on the Newton space can be described row-wise. Specifically, for each non-negative integer m, the m-th row of $T^*_{\phi,\xi}$ coincides with the m-th row of $T_{{}^{m}k_{\phi,\xi}}$, where

$${}^{m}k_{\phi,\xi} = \sum_{i=1}^{\infty} \overline{{}^{m}a_{-i}}\, N_i + \sum_{i=0}^{\infty} {}^{m}\lambda_i\, \overline{N_i},$$

with ${}^{m}a_i = \begin{cases} \phi_i & \text{if } i \text{ is even and } m \text{ is even or } i \text{ is odd and } m \text{ is odd;} \\ \xi_i & \text{if } i \text{ is odd and } m \text{ is even or } i \text{ is even and } m \text{ is odd} \end{cases}$ and ${}^{m}\lambda_i = \begin{cases} \overline{\phi_i}, & \text{if } m \text{ is even,} \\ \overline{\xi_i}, & \text{if } m \text{ is odd.} \end{cases}$ Thus, in general, the operator $T_{\phi,\xi}$ is not self adjoint. $\square$

The following examples show that, in general the operator $T_{\phi,\xi}$ is neither normal nor isometric on $N^2(\mathbb{P})$.

Example 2.13. Let $\phi_\xi(z) = \frac{1}{4}N_2(z)$, $\xi_\phi(z) = 0$, and $f(z) = \displaystyle\sum_{i=0}^{\infty} \frac{1}{2^i} N_i(z)$. Then ${}^{m}k_{\phi,\xi}(z) = \frac{1}{4}\overline{N_2(z)}$ for even m and ${}^{m}k_{\phi,\xi}(z) = 0$ for odd m. On using Lemma 2.5, we compute that

$$\begin{aligned}T_{\phi,\xi}f(z) =&P\left(\phi_\xi f^+(z)+\xi_\phi f^-(z)\right)\\ =&P\left(\frac{1}{4}N_2(z)\sum_{i=0}^{\infty}\frac{1}{2^{2i}}N_{2i}(z)\right)=\frac{1}{4}\sum_{i=0}^{\infty}\frac{1}{2^{2i}}\sum_{k=\max\{2i,2\}}^{2i+2}b_k(2i,2)N_k(z)\\ =&\frac{1}{4}\sum_{k=1}^{\infty}\frac{5k(2k-1)}{4^k}N_{2k}(z)-\frac{1}{4}\sum_{k=1}^{\infty}\frac{2k(2k+1)}{4^k}N_{2k+1}(z),\end{aligned}$$

which further implies that

$$\begin{aligned}\|T_{\phi,\xi}f(z)\|^2 =&\frac{1}{16}\left\{\sum_{k=1}^{\infty}\left|\frac{5k(2k-1)}{4^k}\right|^2+\sum_{k=1}^{\infty}\left|\frac{2k(2k+1)}{4^k}\right|^2\right\}\\ =&\frac{176263}{253125}.\end{aligned}$$

For $m\in\mathbb{N}_0$, Theorem 2.12 gives that the m-th row of the matrix of the adjoint operator $T^*_{\phi,\xi}$ coincides with the m-th row of $T_{{}^m k_{\phi,\xi}}$. Therefore, by Parseval's identity, we have

$$\begin{aligned}\left\|T^*_{\phi,\xi}f(z)\right\|^2 &= \sum_{m=0}^{\infty}|\langle T^*_{\phi,\xi}f(z),N_m\rangle|^2=\sum_{m=0}^{\infty}|\langle T_{{}^m k_{\phi,\xi}}f(z),N_m\rangle|^2\\ &=\sum_{m=0}^{\infty}|\langle T_{{}^{2m} k_{\phi,\xi}}f(z),N_{2m}\rangle|^2+\sum_{m=0}^{\infty}|\langle T_{{}^{2m+1} k_{\phi,\xi}}f(z),N_{2m+1}\rangle|^2. \quad (2.6)\end{aligned}$$

But, for all odd m, ${}^m k_{\phi,\xi}=0$ and hence equation (2.6) implies that

$$\left\|T^*_{\phi,\xi}f(z)\right\|^2=\sum_{m=0}^{\infty}|\langle T_{{}^{2m} k_{\phi,\xi}}f(z),N_{2m}\rangle|^2.$$

Now, on using Lemma 2.7, we have

$$\begin{aligned}T_{{}^{2m} k_{\phi,\xi}}f(z) &= P\left({}^{2m}k_{\phi,\xi}f(z)\right)=P\left(\frac{1}{4}\overline{N_2(z)}\sum_{i=0}^{\infty}\frac{1}{2^i}N_i(z)\right)=\frac{1}{4}P\left(\overline{N_2(z)}\sum_{i=0}^{\infty}\frac{1}{2^i}N_i(z)\right)\\ &=\sum_{i=0}^{1}\left\{\sum_{j=0}^{i}\frac{1}{2^{4+j}}b_{2+j}(2,i)\right\}N_i(z)+\sum_{i=2}^{\infty}\left\{\sum_{j=i}^{2+i}\frac{1}{2^{j+2}}b_j(2,i)\right\}N_i(z),\end{aligned}$$

which implies that

$$\begin{aligned}\left\|T^*_{\phi,\xi}f(z)\right\|^2 =& \sum_{m=0}^{\infty}\left|\left\langle\sum_{i=0}^{1}\left\{\sum_{j=0}^{i}\frac{1}{2^{4+j}}b_{2+j}(2,i)\right\}N_i(z)\right.\right.\\ &\left.\left.+\sum_{i=2}^{\infty}\left\{\sum_{j=i}^{2+i}\frac{1}{2^{j+2}}b_j(2,i)\right\}N_i(z),N_{2m}(z)\right\rangle\right|^2\\ =&\frac{1}{2^8}+\sum_{i=1}^{\infty}\left|\sum_{j=2i}^{2+2i}\frac{1}{2^{j+2}}b_j(2,2i)\right|^2=\frac{1247}{253125}.\end{aligned}$$

Thus, $\left\|T^*_{\phi,\xi}f(z)\right\|^2<\|T_{\phi,\xi}f(z)\|^2$. Hence, the operator $T_{\phi,\xi}$ is not normal.

Example 2.14. Let $\phi_\xi(z) = N_2(z), \xi_\phi(z) = 2N_2(z)$ and $f(z) = \sum\limits_{i=0}^{\infty} \frac{1}{2^i} N_i$, then $\|f(z)\|^2 = 2$. On using Lemma 2.5, we compute

$$\begin{aligned}
T_{\phi,\xi} f(z) =& P\left(\phi_\xi f^+(z) + \xi_\phi f^-(z)\right) \\
=& P\left(N_2(z) \sum_{i=0}^{\infty} \frac{1}{2^{2i}} N_{2i}(z)\right) + P\left(2N_2(z) \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} N_{2i+1}(z)\right) \\
=& \sum_{i=0}^{\infty} \frac{1}{2^{2i}} \sum_{j=\max\{2i,2\}}^{2i+2} b_j(2i,2) N_j(z) \\
& \qquad + 2\sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} \sum_{j=\max\{2i+1,2\}}^{2i+3} b_j(2i+1,2) N_j(z) \\
=& \sum_{k=1}^{\infty} \left\{ \frac{k(2k-1)}{2^{2k-2}} + \frac{k(2k-1)}{2^{2k}} - \frac{4k(2k-1)}{2^{2k-1}} \right\} N_{2k}(z) + \\
& \sum_{k=1}^{\infty} \left\{ \frac{2k(2k+1)}{2^{2k+1}} + \frac{2k(2k+1)}{2^{2k-1}} - \frac{2k(2k+1)}{2^{2k}} \right\} N_{2k+1}(z).
\end{aligned}$$

This implies, $\|T_{\phi,\xi} f(z)\|^2 = \frac{343264}{28125}$ and $\|f(z)\|^2 = 2$. Thus, the operator $T_{\phi,\xi}$ is not isometric.

3. Expansivity and contractivity of 2-Toeplitz operator on $N^2(\mathbb{P})$

In this section, we investigate the expansivity and contractivity of 2-Toeplitz operators on the Newton space. Let us first recall the definitions of these properties for a bounded linear operator T on a Hilbert space: T is contractive if $T^*T \leqslant I$, expansive if $T^*T \geqslant I$, and isometric if $T^*T = I$. In the following theorem, we characterize the contractivity and expansivity of 2-Toeplitz operators $T_{\phi,\xi}$ with analytic symbols on $N^2(\mathbb{P})$.

Theorem 3.1. *Let $(\phi, \xi) \in D_\infty$ such that $\phi_\xi(z) = \sum\limits_{j=0}^{m} a_j N_j(z)$ and $\xi_\phi(z) = \sum\limits_{j=0}^{m} d_j N_j(z)$ for some $m \in \mathbb{N}$ and $a_i, d_i \in \mathbb{C}$. Then the following holds:*

(i) *$T_{\phi,\xi}$ is contractive if and only if*

$$\begin{aligned}
\sum_{k=0}^{\infty} \Bigg\{ & \left| \sum_{j=0}^{\min\{m,2k\}} a_j \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \Delta_{i,k,j}(0,0) + \sum_{j=1}^{\min\{m,2k\}} d_j \sum_{i=1}^{\lceil \frac{j}{2} \rceil} \Delta_{i,k,j}(1,0) \right|^2 \\
& + \left| \sum_{j=1}^{\min\{m,2k+1\}} a_j \sum_{i=1}^{\lceil \frac{j}{2} \rceil} \Delta_{i,k,j}(2,1) + \sum_{j=0}^{\min\{m,2k+1\}} d_j \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \Delta_{i,k,j}(1,1) \right|^2 \Bigg\} \leqslant \sum_{i=0}^{\infty} |c_i|^2 .
\end{aligned}$$

(ii) *$T_{\phi,\xi}$ is expansive if and only if*

$$\sum_{k=0}^{\infty}\left\{\left|\sum_{j=0}^{\min\{m,2k\}} a_j \sum_{i=0}^{\left\lfloor \frac{j}{2}\right\rfloor}\Delta_{i,k,j}(0,0)+\sum_{j=1}^{\min\{m,2k\}} d_j \sum_{i=1}^{\left\lceil \frac{j}{2}\right\rceil}\Delta_{i,k,j}(1,0)\right|^2\right.$$
$$\left.+\left|\sum_{j=1}^{\min\{m,2k+1\}} a_j \sum_{i=1}^{\left\lceil \frac{j}{2}\right\rceil}\Delta_{i,k,j}(2,1)+\sum_{j=0}^{\min\{m,2k+1\}} d_j \sum_{i=0}^{\left\lfloor \frac{j}{2}\right\rfloor}\Delta_{i,k,j}(1,1)\right|^2\right\}\geqslant\sum_{i=0}^{\infty}|c_i|^2$$

for any $c_i \in \mathbb{C}$.

Proof. Let $f(z)=\sum_{i=0}^{\infty}c_iN_i(z)\in N^2(\mathbb{P})$. Then $\|f(z)\|^2=\sum_{i=0}^{\infty}|c_i|^2$ and by using Lemma 2.5, we have

$$\begin{aligned}
T_{\phi,\xi}f(z) =& P\left(\phi_\xi f^+(z)+\xi_\phi f^-(z)\right)\\
=& P\left(\sum_{j=0}^{m}a_jN_j(z)\sum_{i=0}^{\infty}c_{2i}N_{2i}(z)\right)+P\left(\sum_{j=0}^{m}d_jN_j(z)\sum_{i=0}^{\infty}c_{2i+1}N_{2i+1}(z)\right)\\
=& P\left(\sum_{j=0}^{m}\sum_{i=0}^{\infty}a_jc_{2i}N_j(z)N_{2i}(z)\right)+P\left(\sum_{j=0}^{m}\sum_{i=0}^{\infty}d_jc_{2i+1}N_j(z)N_{2i+1}(z)\right)\\
=& P\left(\sum_{j=0}^{m}\sum_{i=0}^{\infty}a_jc_{2i}\sum_{k=\max\{2i,j\}}^{2i+j}b_k(2,j)N_k(z)\right)\\
&+P\left(\sum_{j=0}^{m}\sum_{i=0}^{\infty}d_jc_{2i+1}\sum_{f=\max\{2i+1,j\}}^{2i+1+j}b_f(2i+1,j)N_f(z)\right)\\
=& \sum_{i=0}^{\infty}a_0c_{2i}\sum_{k=2i}^{2i}b_k(2i,0)N_k(z)+\sum_{i=0}^{\infty}d_0c_{2i+1}\sum_{f=2i+1}^{2i+1}b_f(2i+1,0)N_f(z)\\
&+\sum_{i=0}^{\infty}a_1c_{2i}\sum_{k=\max\{2i,1\}}^{2i+1}b_k(2i,1)N_k(z)+\sum_{i=0}^{\infty}d_1c_{2i+1}\sum_{f=\max\{2i+1,1\}}^{2i+2}b_f(2i+1,1)N_f(z)\\
&+\sum_{i=0}^{\infty}a_2c_{2i}\sum_{k=\max\{2i,2\}}^{2i+2}b_k(2i,2)N_k(z)+\sum_{i=0}^{\infty}d_2c_{2i+1}\sum_{f=\max\{2i+1,2\}}^{2i+3}b_f(2i+1,2)N_f(z)\\
&+\ldots+\sum_{i=0}^{\infty}a_mc_{2i}\sum_{k=\max\{2i,m\}}^{2i+m}b_k(2i,m)N_k(z)\\
&+\sum_{i=0}^{\infty}d_mc_{2i+1}\sum_{f=\max\{2i+1,m\}}^{2i+1+m}b_f(2i+1,m)N_f(z)\\
=& a_0c_0b_0(0,0)N_0(z)+\{d_0c_1b_1(1,0)+a_1c_0b_1(0,1)+d_1c_1b_1(1,1)\}N_1(z)\\
&+\{a_0c_2b_2(2,0)+d_1c_1b_2(1,1)+a_1c_2b_2(2,1)+a_2c_0b_2(0,2)+d_2c_1b_2(1,2)
\end{aligned}$$

$$+\ a_2c_2b_2(2,2)\}\, N_2(z) + \cdots .$$

Thus,

$$\begin{aligned} T_{\phi,\xi}f(z) =& \sum_{k=0}^{\infty}\left\{\sum_{j=0}^{\min\{m,2k\}} a_j \sum_{i=0}^{\lfloor \frac{j}{2}\rfloor} \Delta_{i,k,j}(0,0) + \sum_{j=1}^{\min\{m,2k\}} d_j \sum_{i=1}^{\lceil \frac{j}{2}\rceil} \Delta_{i,k,j}(1,0)\right\} N_{2k}(z) + \\ & \sum_{k=0}^{\infty}\left\{\sum_{j=1}^{\min\{m,2k+1\}} a_j \sum_{i=1}^{\lceil \frac{j}{2}\rceil} \Delta_{i,k,j}(2,1) + \sum_{j=0}^{\min\{m,2k+1\}} d_j \sum_{i=0}^{\lfloor \frac{j}{2}\rfloor} \Delta_{i,k,j}(1,1)\right\} N_{2k+1}(z). \end{aligned}$$

Therefore, the operator $T_{\phi,\xi}$ is contractive on $N^2(\mathbb{P})$ if and only if

$$\begin{aligned} \sum_{k=0}^{\infty}\Bigg\{&\left|\sum_{j=0}^{\min\{m,2k\}} a_j \sum_{i=0}^{\lfloor \frac{j}{2}\rfloor} \Delta_{i,k,j}(0,0) + \sum_{j=1}^{\min\{m,2k\}} d_j \sum_{i=1}^{\lceil \frac{j}{2}\rceil} \Delta_{i,k,j}(1,0)\right|^2 \\ &+ \left|\sum_{j=1}^{\min\{m,2k+1\}} a_j \sum_{i=1}^{\lceil \frac{j}{2}\rceil} \Delta_{i,k,j}(2,1) + \sum_{j=0}^{\min\{m,2k+1\}} d_j \sum_{i=0}^{\lfloor \frac{j}{2}\rfloor} \Delta_{i,k,j}(1,1)\right|^2\Bigg\} \leqslant \sum_{i=0}^{\infty} |c_i|^2 . \end{aligned}$$

Analogously, we establish the result for expansivity. This proves the result. □

Example 3.2. (i) Let $\phi_\xi(z) = -1,\ \xi_\phi(z) = -\frac{1}{2} + \frac{1}{2}N_1(z)$ and $f(z) = \frac{1}{2} + \frac{1}{2}N_1(z) + \frac{1}{2}N_2(z)$. Then $\|f\|^2 = \frac{3}{4}$ and

$$\begin{aligned} \sum_{k=0}^{\infty}\Bigg\{&\left|\sum_{j=0}^{\min\{m,2k\}} a_j \sum_{i=0}^{\lfloor \frac{j}{2}\rfloor} \Delta_{i,k,j}(0,0) + \sum_{j=1}^{\min\{m,2k\}} d_j \sum_{i=1}^{\lceil \frac{j}{2}\rceil} \Delta_{i,k,j}(1,0)\right|^2 \\ &+ \left|\sum_{j=1}^{\min\{m,2k+1\}} a_j \sum_{i=1}^{\lceil \frac{j}{2}\rceil} \Delta_{i,k,j}(2,1) + \sum_{j=0}^{\min\{m,2k+1\}} d_j \sum_{i=0}^{\lfloor \frac{j}{2}\rfloor} \Delta_{i,k,j}(1,1)\right|^2\Bigg\} \\ =& |a_0c_0|^2 + |d_0c_1 + d_1c_1b_1(1,1)|^2 + |a_0c_1 + d_1c_1b_2(1,1)|^2 \\ =& \frac{1}{2} < \frac{3}{4}. \end{aligned}$$

Thus, the operator $T_{\phi,\xi}$ is contractive.

(ii) Let $\phi_\xi(z) = \frac{1}{2}N_1(z),\ \xi_\phi(z) = 1$ and $f(z) = 1 + \frac{1}{2}N_1(z) + \frac{1}{2}N_2(z)$. Then $\|f\|^2 = \frac{3}{2}$ and

$$\begin{aligned} \sum_{k=0}^{\infty}\Bigg\{&\left|\sum_{j=0}^{\min\{m,2k\}} a_j \sum_{i=0}^{\lfloor \frac{j}{2}\rfloor} \Delta_{i,k,j}(0,0) + \sum_{j=1}^{\min\{m,2k\}} d_j \sum_{i=1}^{\lceil \frac{j}{2}\rceil} \Delta_{i,k,j}(1,0)\right|^2 \\ &+ \left|\sum_{j=1}^{\min\{m,2k+1\}} a_j \sum_{i=1}^{\lceil \frac{j}{2}\rceil} \Delta_{i,k,j}(2,1) + \sum_{j=0}^{\min\{m,2k+1\}} d_j \sum_{i=0}^{\lfloor \frac{j}{2}\rfloor} \Delta_{i,k,j}(1,1)\right|^2\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= |d_0c_1b_1(1,0) + a_1c_0b_1(0,1)|^2 + |a_1c_2b_2(1,2)|^2 + |a_1c_2b_3(2,1)|^2 \\
&= \frac{29}{16} > \frac{3}{2}.
\end{aligned}
$$

Thus, the operator $T_{\phi,\xi}$ is expansive.

Corollary 3.3. Let $(\phi, \xi) \in D_\infty$ such that $\phi_\xi(z) = \sum_{j=0}^{m} a_j N_j(z), \xi_\phi(z) = \sum_{j=0}^{m} d_j N_j(z)$ for some $m \in \mathbb{N}$ and $a_i, d_i \in \mathbb{C}$. Then the following holds:

(i) If $T_{\phi,\xi}$ is contractive, then

$$\left(\frac{1}{|c_0|^2 + |c_1|^2}\right)\left\{|c_0a_0|^2 + \sum_{i=1}^{m} |c_0a_i + ic_1(d_{i-1} - d_i)|^2 + |(m+1)c_1d_m|^2\right\} \leqslant 1.$$

(ii) If $T_{\phi,\xi}$ is expansive, then

$$\left(\frac{1}{|c_0|^2 + |c_1|^2}\right)\left\{|c_0a_0|^2 + \sum_{i=1}^{m} |c_0a_i + ic_1(d_{i-1} - d_i)|^2 + |(m+1)c_1d_m|^2\right\} \geqslant 1$$

for any $c_0, c_1 \in \mathbb{C}$.

Proof. Let $f(z) = c_0N_0(z) + c_1N_1(z)$, then $\|f(z)\|^2 = |c_0|^2 + |c_1|^2$ and by utilizing Lemma 2.5, we obtain

$$
\begin{aligned}
T_{\phi,\xi}f(z) &= P\left(\phi_\xi f^+(z) + \xi_\phi f^-(z)\right) \\
&= P\left(\sum_{j=0}^{m} a_jN_j(z)c_0 + \sum_{j=0}^{m} d_jN_j(z)c_1N_1(z)\right) \\
&= c_0\sum_{j=0}^{m} a_jN_j(z) + c_1\sum_{j=0}^{m} d_j \sum_{k=\max\{1,j\}}^{j+1} b_k(j,1)N_k(z) \\
&= c_0a_0 + \sum_{i=1}^{m} \{c_0a_i + ic_1(d_{i-1} - d_i)\} N_i(z) + (m+1)c_1d_mN_{m+1}(z).
\end{aligned}
$$

Thus if $T_{\phi,\xi}$ contractive, then we have

$$|c_0a_0|^2 + \sum_{i=1}^{m} |c_0a_i + ic_1(d_{i-1} - d_i)|^2 + |(m+1)c_1d_m|^2 \leqslant |c_0|^2 + |c_1|^2.$$

This implies,

$$\left(\frac{1}{|c_0|^2 + |c_1|^2}\right)\left\{|c_0a_0|^2 + \sum_{i=1}^{m} |c_0a_i + ic_1(d_{i-1} - d_i)|^2 + |(m+1)c_1d_m|^2\right\} \leqslant 1.$$

Similarly, we conclude the result for expansivity. This completes the proof. □

Remark 3.4. In general, the converse of Corollary 3.3 does not hold, which can be seen with the help of the functions ϕ_ξ, ξ_ϕ, and f discussed in Example 3.2.

We use the following lemma to prove Theorem 3.6.

Lemma 3.5. *[5] Let $m \in \mathbb{N}$. For any $k = 0, 1, 2, \ldots, m$, let*

$$S_m(m,k) := |b_m(m,k)|^2 + |b_{m+1}(m,k)|^2 + \cdots + |b_{m+k}(m,k)|^2$$

and for any $k = m+1, m+2, \ldots$, let

$$S_k(m,k) := |b_k(m,k)|^2 + |b_{k+1}(m,k)|^2 + \cdots + |b_{k+m}(m,k)|^2 .$$

Then, the sequences $\{S_m(m,k)\}_{k=0}^{m}$ and $\{S_k(m,k)\}_{k=m+1}^{\infty}$ are increasing.

By employing Lemma 3.5, we discuss the special instance of Theorem 3.1.

Theorem 3.6. *Let $(\phi, \xi) \in D_\infty$ such that $\phi_\xi(z) = aN_m(z)$ and $\xi_\phi(z) = dN_m(z)$ for some $m \in \mathbb{N}$ and non-zero $a, d \in \mathbb{C}$, then the following holds:*

(a) (i) *$T_{\phi,\xi}$ is expansive if and only if*

$$\sum_{t=m}^{\infty} \left| \sum_{i=t-m}^{t} \lambda_i c_i b_t(i,m) \right|^2 \geqslant \sum_{i=0}^{\infty} |c_i|^2 .$$

(ii) *Suppose $T_{\phi,\xi}$ is expansive, then $|a| \geqslant 1$ if i is even and $|d| \geqslant 1$ if i is odd.*

(b) (i) *$T_{\phi,\xi}$ is contractive if and only if*

$$\sum_{t=m}^{\infty} \left| \sum_{i=t-m}^{t} \lambda_i c_i b_t(i,m) \right|^2 \leqslant \sum_{i=0}^{\infty} |c_i|^2 .$$

(ii) *Suppose $T_{\phi,\xi}$ is contractive then $\{S_m(m,k)\}_{k=0}^{m}$ is bounded by g, where $g = \max\left\{\frac{1}{|a|^2}, \frac{1}{|d|^2}\right\}$.*

for any $c_i \in \mathbb{C}, i \in \mathbb{N}_0$ and where $\lambda_i = \begin{cases} a & \text{if } i \text{ is even;} \\ d & \text{if } i \text{ is odd.} \end{cases}$

Proof. Let $f(z) = \sum_{i=0}^{\infty} c_i N_i(z) \in N^2(\mathbb{P})$. Then, we have $\|f(z)\|^2 = \sum_{i=0}^{\infty} |c_i|^2$. Using Lemma 2.9, we compute

$$\begin{aligned} \|T_{\phi,\xi} f(z)\|^2 &= \| P\left(\phi_\xi f^+(z) + \xi_\phi f^-(z)\right) \|^2 \\ &= \left\| P\left(aN_m(z) \sum_{i=0}^{\infty} c_{2i} N_{2i}(z) + dN_m(z) \sum_{i=0}^{\infty} c_{2i+1} N_{2i+1}(z) \right) \right\|^2 \\ &= \sum_{t=m}^{\infty} \left| \sum_{i=t-m}^{t} \lambda_i c_i b_t(i,m) \right|^2 , \text{ where } \lambda_i = \begin{cases} a & \text{if } i \text{ is even;} \\ d & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

Thus $T_{\phi,\xi}$ is expansive on $N^2(\mathbb{P})$ if and only if $\|T_{\phi,\xi}(f)\|^2 \geqslant \|f\|^2$ or, equivalently,

$$\sum_{t=m}^{\infty} \left| \sum_{i=t-m}^{t} \lambda_i c_i b_t(i,m) \right|^2 \geqslant \sum_{i=0}^{\infty} |c_i|^2 .$$

Similarly, $T_{\phi,\xi}$ is contractive on $N^2(\mathbb{P})$ if and only if

$$\sum_{t=m}^{\infty} \left| \sum_{i=t-m}^{t} \lambda_i c_i b_t(i,m) \right|^2 \leqslant \sum_{i=0}^{\infty} |c_i|^2 .$$

Using Proposition 2.11 of [5], we can conclude that $|a| \geqslant 1$ if i is even, and $|d| \geqslant 1$ if i is odd. Moreover, on applying Theorem 2.15 from [5], we deduce that the sequence $\{S_m(m,k)\}_{k=0}^{m}$ is bounded by $\frac{1}{|a|^2}$ for even i and by $\frac{1}{|d|^2}$ for odd i. ☐

The following example shows that the converse of Theorem 3.6 does not hold.

Example 3.7. (a) Let $m=1, a=1, d=1.001, c_n=\left(\frac{1}{2}\right)^n$. From Theorem 3.6, we have

$$\begin{aligned}\sum_{t=1}^{\infty}\left|\sum_{i=t-1}^{t}\lambda_i c_i b_t(i,1)\right|^2 &= \sum_{t=1}^{\infty} t^2\left(\lambda_t c_t-\lambda_{t-1}c_{t-1}\right)^2\\ &= \sum_{t=1}^{\infty} t^2\lambda_t^2 c_t^2+\sum_{t=1}^{\infty}\lambda_{t-1}^2 c_{t-1}^2 t^2-2\sum_{t=1}^{\infty}\lambda_t\lambda_{t-1}c_t c_{t-1}t^2.\end{aligned}$$

Here, $\sum_{t=1}^{\infty} t^2\lambda_t^2c_t^2=\sum_{t=1}^{\infty}(2t)^2\lambda_{2t}^2c_{2t}^2+\sum_{t=1}^{\infty}(2t-1)^2\lambda_{2t-1}^2c_{2t-1}^2=\frac{1088}{3375}+\frac{21}{50}=\frac{5011}{6750}$. Similarly, we get $\sum_{t=1}^{\infty}\lambda_{t-1}^2c_{t-1}^2t^2=\frac{593}{200}$ and $\sum_{t=1}^{\infty}\lambda_t\lambda_{t-1}t^2c_tc_{t-1}=\frac{37}{25}$. Thus, it follows that $\sum_{t=1}^{\infty}\left|\sum_{i=t-1}^{t}\lambda_i c_i b_t(i,1)\right|^2\approx\frac{20012}{27000}$ and $\sum_{i=0}^{\infty}|c_i|^2=\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{2i}=\frac{4}{3}$. Hence, the operator $T_{\phi,\xi}$ is not expansive.

(b) Let $m=1, a=1, d=-\frac{1}{3}, c_n=\left(\frac{1}{2}\right)^n$ and $\{S_m(m,k)\}_{k=0}^{m}$ is bounded by 9. Using Theorem 3.6, we have,

$$\begin{aligned}\sum_{t=1}^{\infty}\left|\sum_{i=t-1}^{t}\lambda_i c_i b_t(i,1)\right|^2 &=\sum_{t=1}^{\infty} t^2\lambda_t^2 c_t^2+\sum_{t=1}^{\infty}\lambda_{t-1}^2 c_{t-1}^2 t^2-2\sum_{t=1}^{\infty}\lambda_t\lambda_{t-1}t^2c_{t-1}c_t\\ &=\frac{37}{100}+\frac{91}{50}+\frac{99}{100}>\frac{4}{3}=\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{2i}=\sum_{i=0}^{\infty}|c_i|^2.\end{aligned}$$

Hence, the operator $T_{\phi,\xi}$ is not contractive.

Corollary 3.8. Let $\phi_\xi(z)=aN_2(z)$ and $\xi_\phi(z)=dN_2(z)$ for non-zero $a,d\in\mathbb{C}$, then the following holds:

(i) $T_{\phi,\xi}$ is contractive if and only if

$$\begin{aligned}&\sum_{k=0}^{\infty}(k+1)^2\Big\{(2k+1)^2\left|a\left(c_{2k}+c_{2k+2}\right)-2dc_{2k+1}\right|^2\\ &+(2k+3)^2\left|d\left(c_{2k+1}+c_{2k+3}\right)-2ac_{2k+2}\right|^2\Big\}\leqslant\sum_{i=0}^{\infty}|c_i|^2.\end{aligned}$$

(ii) $T_{\phi,\xi}$ is expansive if and only if

$$\begin{aligned}&\sum_{k=0}^{\infty}(k+1)^2\Big\{(2k+1)^2\left|a\left(c_{2k}+c_{2k+2}\right)-2dc_{2k+1}\right|^2\\ &+(2k+3)^2\left|d\left(c_{2k+1}+c_{2k+3}\right)-2ac_{2k+2}\right|^2\Big\}\geqslant\sum_{i=0}^{\infty}|c_i|^2\end{aligned}$$

for any $c_i\in\mathbb{C}$ and $i\in\mathbb{N}_0$.

Proof. From Theorem 3.6, we have $\|T_{\phi,\xi}f(z)\|^2 = \sum_{t=2}^{\infty}\left|\sum_{i=t-2}^{t}\lambda_i c_i b_t(2,i)\right|^2$. Using Lemma 2.6, we compute

$$\begin{aligned}\left|\sum_{i=t-2}^{t}\lambda_i c_i b_t(2,i)\right|^2 &= |\lambda_{t-2}c_{t-2}b_t(2,t-2)+\lambda_{t-1}c_{t-1}b_t(2,t-1)+\lambda_t c_t b_t(2,t)|^2\\ &= \left|\lambda_{t-2}c_{t-2}\binom{t}{t-2}+\lambda_{t-1}c_{t-1}(-2)\binom{t}{t-2}+\lambda_t c_t\binom{t}{t-2}\right|^2\\ &= \left|\binom{t}{t-2}\right|^2 |\lambda_{t-2}c_{t-2}-2\lambda_{t-1}c_{t-1}+\lambda_t c_t|^2.\end{aligned}$$

On substituting $t-2=k$, we get

$$\begin{aligned}\sum_{t=2}^{\infty}\left|\sum_{i=t-2}^{t}\lambda_i c_i b_t(2,i)\right|^2 &= \sum_{k=0}^{\infty}\left|\binom{k+2}{k}\right|^2 |\lambda_k c_k - 2c_{k+1}\lambda_{k+1}+\lambda_{k+2}c_{k+2}|^2\\ &= \sum_{k=0}^{\infty}\left\{\left|\binom{2k+2}{2k}\right|^2 |\lambda_{2k}c_{2k}-2\lambda_{2k+1}c_{2k+1}+\lambda_{2k+2}c_{2k+2}|^2\right.\\ &\quad \left.+\left|\binom{2k+3}{2k+1}\right|^2 |\lambda_{2k+1}c_{2k+1}-2\lambda_{2k+2}c_{2k+2}+\lambda_{2k+3}c_{2k+3}|^2\right\}\\ &\quad \sum_{k=0}^{\infty}(k+1)^2\left\{(2k+1)^2|\lambda_{2k}c_{2k}-2\lambda_{2k+1}c_{2k+1}+\lambda_{2k+2}c_{2k+2}|^2\right.\\ &\quad \left.+(2k+3)^2|\lambda_{2k+1}c_{2k+1}-2\lambda_{2k+2}c_{2k+2}+\lambda_{2k+3}c_{2k+3}|^2\right\}\\ &= \sum_{k=0}^{\infty}(k+1)^2\left\{(2k+1)^2|a(c_{2k}+c_{2k+2})-2dc_{2k+1}|^2\right.\\ &\quad \left.+(2k+3)^2|d(c_{2k+1}+c_{2k+3})-2ac_{2k+2}|^2\right\}.\end{aligned}$$

Thus $T_{\phi,\xi}$ is contractive on $N^2(\mathbb{P})$ if and only if

$$\begin{aligned}&\sum_{k=0}^{\infty}(k+1)^2\left\{(2k+1)^2|a(c_{2k}+c_{2k+2})-2dc_{2k+1}|^2\right.\\ &\left.+(2k+3)^2|d(c_{2k+1}+c_{2k+3})-2ac_{2k+2}|^2\right\} \leqslant \sum_{i=0}^{\infty}|c_i|^2.\end{aligned}$$

Similarly, we get the condition for expansivity. □

Remark 3.9. If $\phi_\xi = a\overline{N_m}(z)$ and $\xi_\phi = d\overline{N_n}(z)$ for Some $m,n\in\mathbb{N}$ and $a,d\in\mathbb{C}$, then by using Theorem 2.19 and Corollary 2.21 of [5], we directly say that $T_{\phi,\xi}$ is never contractive and never expansive.

The following theorem addresses the necessary conditions for expansivity of 2-Toeplitz operators for harmonic symbols.

Theorem 3.10. *Let* $(\phi,\xi)\in D_\infty$ *such that* $\phi_\xi(z)=\sum_{i=0}^{\infty}\phi_i N_i(z)+\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}(z)+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}(z)$ *and* $\xi_\phi(z)=\sum_{i=0}^{\infty}\xi_i N_i(z)+\sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}(z)+\sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}(z)$. *If* $T_{\phi,\xi}$ *is expansive, then*

$$\sum_{i=2}^{\infty}|\phi_i+i\xi_{i-1}-i\xi_i|^2\geqslant 2-|\phi_0+\phi_{-1}|^2-|\phi_1+\xi_0-\xi_1-\phi_{-1}|^2.$$

Proof. Let $f(z)=N_0(z)+N_1(z)\in N^2(\mathbb{P})$. Then $\|f(z)\|^2=2$ and by using Lemma 2.5 and Lemma 2.7, we obtain

$$\begin{aligned}T_{\phi,\xi}f(z)=&P\left(\phi_\xi N_0(z)+\xi_\phi N_1(z)\right)\\ =&P\left(\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}(z)+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}(z)+\sum_{i=0}^{\infty}\phi_i N_i(z)\right.\\ &\left.+\sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}(z)N_1(z)+\sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}(z)N_1(z)+\sum_{i=0}^{\infty}\xi_i N_i(z)N_1(z)\right)\\ =&\sum_{i=0}^{\infty}\phi_i N_i(z)+\phi_{-1}\sum_{j=0}^{1}b_1(1,j)N_j(z)+\sum_{i=0}^{\infty}\xi_i\sum_{j=\max\{i,1\}}^{i+1}b_j(i,1)N_j(z).\end{aligned}$$

Using Lemma 2.6, we compute

$$T_{\phi,\xi}f(z)=(\phi_0+\phi_{-1})+(\phi_1+\xi_0-\xi_1-\phi_{-1})N_1(z)+\sum_{i=2}^{\infty}(\phi_i+i\xi_{i-1}-i\xi_i)N_i(z).$$

Thus, if $T_{\phi,\xi}$ is expansive, then we have

$$|\phi_0+\phi_{-1}|^2+|\phi_1+\xi_0-\xi_1-\phi_{-1}|^2+\sum_{i=2}^{\infty}|\phi_i+i\xi_{i-1}-i\xi_i|^2\geqslant 2,$$

which implies that

$$\sum_{i=2}^{\infty}|\phi_i+i\xi_{i-1}-i\xi_i|^2\geqslant 2-|\phi_0+\phi_{-1}|^2-|\phi_1+\xi_0-\xi_1-\phi_{-1}|^2.$$

This completes the proof. ☐

The subsequent theorem outlines the necessary condition under which the adjoint of 2-Toeplitz operator exhibits expansivity for harmonic symbols.

Theorem 3.11. *Let* $(\phi,\xi)\in D_\infty$ *such that* $\phi_\xi(z)=\sum_{i=0}^{\infty}\phi_i N_i(z)+\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}(z)+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}(z)$ *and* $\xi_\phi(z)=\sum_{i=0}^{\infty}\xi_i N_i(z)+\sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}(z)+\sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}(z)$. *If* $T^*_{\phi,\xi}$ *is expansive, then*

$$\sum_{m=2}^{\infty}|(1-m)\phi_{-m}+m\xi_{-m+1}|^2\geqslant 2-|\phi_0+\phi_1|^2-|\xi_0-\xi_1|^2.$$

Proof. Let $f(z) = N_0(z) + N_1(z) \in N^2(\mathbb{P})$, then $\|f(z)\|^2 = 2$. For $m \in \mathbb{N}_0$, Theorem 2.12 gives that the m-th row of the matrix of the adjoint operator $T^*_{\phi,\xi}$ coincides with the m-th row of $T_{{}^mk_{\phi,\xi}}$. On using Lemma 2.7, we obtain

$$\begin{aligned}
T_{{}^mk_{\phi,\xi}} f(z) =& P\left({}^mk_{\phi,\xi} N_0(z) + {}^mk_{\phi,\xi} N_1(z)\right) \qquad (3.1)\\
=& P\Bigg(\sum_{i=1}^{\infty} \overline{{}^ma_{-i}} N_i(z) + \sum_{i=0}^{\infty} {}^m\lambda_i \overline{N_i(z)} + \sum_{i=1}^{\infty} \overline{{}^ma_{-i}} N_i(z) N_1(z) \\
&\qquad + \sum_{i=0}^{\infty} {}^m\lambda_i \overline{N_i(z)} N_1(z)\Bigg)\\
=& \sum_{i=1}^{\infty} \overline{{}^ma_{-i}} N_i(z) + \sum_{i=1}^{\infty} \overline{{}^ma_{-i}} \sum_{j=\max\{i,1\}}^{i+1} b_j(1,i) N_j(z) \\
&\qquad + {}^m\lambda_1 N_0(z) + ({}^m\lambda_0 - {}^m\lambda_1) N_1(z)\\
=& {}^m\lambda_0 + {}^m\lambda_1 + ({}^m\lambda_0 - {}^m\lambda_1) N_1(z) + \sum_{i=2}^{\infty} \left(\overline{{}^ma_{-i}} + i\overline{{}^ma_{-i+1}} - i\overline{{}^ma_{-i}}\right) N_i(z). \qquad (3.2)
\end{aligned}$$

Further, from Parseval's identity and equation (3.2), we have

$$\begin{aligned}
\left\|T^*_{\phi,\xi} f(z)\right\|^2 =& \sum_{m=0}^{\infty} \left|\langle T^*_{\phi,\xi} f(z), N_m(z)\rangle\right|^2 = \sum_{m=0}^{\infty} \left|\left\langle T_{m_{k_{\phi,\xi}}} f(z), N_m(z)\right\rangle\right|^2\\
=& \sum_{m=0}^{\infty} \Bigg|\Bigg\langle ({}^m\lambda_0 +^m \lambda_1) + ({}^m\lambda_0 - {}^m\lambda_1) N_1(z) + \sum_{i=2}^{\infty} \big(\overline{{}^ma_{-i}} + i\overline{{}^ma_{-i+1}} \\
&\qquad - i\overline{{}^ma_{-i}}\big) N_i(z), N_m(z)\Bigg\rangle\Bigg|^2\\
=& \left|{}^0\lambda_1 + {}^0\lambda_1\right|^2 + \left|{}^1\lambda_0 - {}^1\lambda_1\right|^2 + \sum_{m=2}^{\infty} \left|\overline{{}^ma_{-m}} + m\overline{{}^ma_{-m+1}} - m\overline{{}^ma_{-m}}\right|^2\\
=& |\phi_0 + \phi_1|^2 + |\xi_0 - \xi_1|^2 + \sum_{m=2}^{\infty} |(1-m)\phi_{-m} + m\xi_{-m+1}|^2.
\end{aligned}$$

Thus, if $T^*_{\phi,\xi}$ is expansive, then we have

$$|\phi_0 + \phi_1|^2 + |\xi_0 - \xi_1|^2 + \sum_{m=2}^{\infty} |(1-m)\phi_{-m} + m\xi_{-m+1}|^2 \geqslant 2,$$

which implies that

$$\sum_{m=2}^{\infty} |(1-m)\phi_{-m} + m\xi_{-m+1}|^2 \geqslant 2 - |\phi_0 + \phi_1|^2 - |\xi_0 - \xi_1|^2.$$

This completes the proof. □

The following theorem establishes the necessary and sufficient conditions for the expansivity and contractivity of 2-Toeplitz operators with harmonic symbols.

Theorem 3.12. *Let* $(\phi,\xi)\in D_\infty$ *such that* $\phi_\xi(z)=\sum\limits_{i=0}^{\infty}\phi_iN_i(z)+\sum\limits_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}(z)+\sum\limits_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}(z)$ *and* $\xi_\phi(z)=\sum\limits_{i=0}^{\infty}\xi_iN_i(z)+\sum\limits_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}(z)+\sum\limits_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}(z)$. *Then the following holds:*

(i) $T_{\phi,\xi}$ *is contractive if and only if*

$$\sum_{k=0}^{\infty}\left\{\left|\sum_{j=0}^{2k}\phi_j\sum_{i=0}^{\lfloor\frac{j}{2}\rfloor}\Delta_{i,k,j}(0,0)+\sum_{j=1}^{2k}\xi_j\sum_{i=1}^{\lceil\frac{j}{2}\rceil}\Delta_{i,k,j}(1,0)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(0)\right|^2\right.$$
$$\left.+\left|\sum_{j=1}^{2k+1}\phi_j\sum_{i=1}^{\lceil\frac{j}{2}\rceil}\Delta_{i,k,j}(2,1)+\sum_{j=0}^{2k+1}\xi_j\sum_{i=0}^{\lfloor\frac{j}{2}\rfloor}\Delta_{i,k,j}(1,1)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(1)\right|^2\right\}\leqslant\sum_{i=0}^{\infty}|c_i|^2.$$

(ii) $T_{\phi,\xi}$ *is expansive if and only if*

$$\sum_{k=0}^{\infty}\left\{\left|\sum_{j=0}^{2k}\phi_j\sum_{i=0}^{\lfloor\frac{j}{2}\rfloor}\Delta_{i,k,j}(0,0)+\sum_{j=1}^{2k}\xi_j\sum_{i=1}^{\lceil\frac{j}{2}\rceil}\Delta_{i,k,j}(1,0)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(0)\right|^2\right.$$
$$\left.+\left|\sum_{j=1}^{2k+1}\phi_j\sum_{i=1}^{\lceil\frac{j}{2}\rceil}\Delta_{i,k,j}(2,1)+\sum_{j=0}^{2k+1}\xi_j\sum_{i=0}^{\lfloor\frac{j}{2}\rfloor}\Delta_{i,k,j}(1,1)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(1)\right|^2\right\}\geqslant\sum_{i=0}^{\infty}|c_i|^2$$

where $c_i\in\mathbb{C}$ *and* ${}^{j}\nu_i=\begin{cases}\phi_i & \textit{if } i \textit{ is even and } j \textit{ is even or } i \textit{ is odd and } j \textit{ is odd;}\\ \xi_i & \textit{if } i \textit{ is even and } j \textit{ is odd or } i \textit{ is odd and } j \textit{ is even.}\end{cases}$

Proof. Let $f(z)=\sum\limits_{i=0}^{\infty}c_iN_i(z)\in N^2(\mathbb{P})$. Then we have $\|f(z)\|^2=\sum\limits_{i=0}^{\infty}|c_i|^2$ and by using Theorem 3.1 and Lemma 2.10, we compute

$$\begin{aligned}\|T_{\phi,\xi}f(z)\|^2=&\|P\left(\phi_\xi f^+(z)+\xi_\phi f^-(z)\right)\|^2\\
=&\left\|P\left(\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}(z)\sum_{k=0}^{\infty}c_{2k}N_{2k}(z)+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}(z)\sum_{k=0}^{\infty}c_{2k}N_{2k}(z)+\right.\right.\\
&\left.\sum_{i=0}^{\infty}\phi_iN_i(z)\sum_{k=0}^{\infty}c_{2k}N_{2k}(z)\right)+P\left(\sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}(z)\sum_{k=0}^{\infty}c_{2k}N_{2k+1}(z)+\right.\\
&\left.\left.\sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}(z)\sum_{k=0}^{\infty}c_{2k+1}N_{2k+1}(z)+\sum_{i=0}^{\infty}\xi_iN_i(z)\sum_{k=0}^{\infty}c_{2k+1}N_{2k+1}(z)\right)\right\|^2\\
=&\left\|P\left(\sum_{i=0}^{\infty}\phi_iN_i(z)\sum_{k=0}^{\infty}c_{2k}N_{2k}(z)+\sum_{i=0}^{\infty}\xi_iN_i(z)\sum_{k=0}^{\infty}c_{2k+1}(z)N_{2k+1}(z)\right)+\right.\\
&P\left(\sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}(z)\sum_{k=0}^{\infty}c_{2k}N_{2k}(z)+\sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}(z)\sum_{k=0}^{\infty}c_{2k}N_{2k}(z)+\right.\end{aligned}$$

$$\begin{aligned}&\left.\sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}(z)\sum_{k=0}^{\infty}c_{2k+1}N_{2k+1}(z)+\sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}(z)\sum_{k=0}^{\infty}c_{2k+1}N_{2k+1}(z)\right)\Bigg\|^2\\ =&\sum_{k=0}^{\infty}\left\{\left|\sum_{j=0}^{2k}\phi_j\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor}\Delta_{i,k,j}(0,0)+\sum_{j=1}^{2k}\xi_j\sum_{i=1}^{\lceil \frac{j}{2}\rceil}\Delta_{i,k,j}(1,0)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(0)\right|^2\right.\\ &\left.+\left|\sum_{j=1}^{2k+1}\phi_j\sum_{i=1}^{\lceil \frac{j}{2}\rceil}\Delta_{i,k,j}(2,1)+\sum_{j=0}^{2k+1}\xi_j\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor}\Delta_{i,k,j}(1,1)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(1)\right|^2\right\}.\end{aligned}$$

Thus, the operator $T_{\phi,\xi}$ is contractive on $N^2(\mathbb{P})$ if and only if

$$\begin{aligned}&\sum_{k=0}^{\infty}\left\{\left|\sum_{j=0}^{2k}\phi_j\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor}\Delta_{i,k,j}(0,0)+\sum_{j=1}^{2k}\xi_j\sum_{i=1}^{\lceil \frac{j}{2}\rceil}\Delta_{i,k,j}(1,0)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(0)\right|^2\right.\\ &\left.+\left|\sum_{j=1}^{2k+1}\phi_j\sum_{i=1}^{\lceil \frac{j}{2}\rceil}\Delta_{i,k,j}(2,1)+\sum_{j=0}^{2k+1}\xi_j\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor}\Delta_{i,k,j}(1,1)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(1)\right|^2\right\}\leqslant\sum_{i=0}^{\infty}|c_i|^2.\end{aligned}$$

Similarly, we obtain the result for expansivity. □

Remark 3.13. On using Theorem 3.12, we can obtain the necessary and sufficient condition for contractivity and expansivity of Toeplitz operator having harmonic symbol ϕ. Indeed, Toeplitz operator, T_ϕ is said to be contractive if and only if

$$\sum_{k=0}^{\infty}\left|\sum_{j=0}^{k}\phi_j\sum_{i=0}^{j}c_{k-i}b_k(k-i,j)+\sum_{j=1}^{\infty}\phi_{-j}\sum_{i=\max\{k,j\}}^{k+j}c_ib_i(j,k)\right|^2\leqslant\sum_{i=0}^{\infty}|c_i|^2.$$

Example 3.14. (i) Let $\phi_\xi(z)=\frac{1}{2}N_1(z)+\frac{1}{3}N_2(z)+\overline{N_1(z)}+\frac{1}{2}\overline{N_2(z)}$, $\xi_\phi(z)=\frac{1}{4}N_1(z)+\frac{1}{5}N_2(z)+\overline{N_1(z)}+\frac{1}{3}\overline{N_2(z)}$ and $f(z)=1+\frac{1}{2}N_1(z)$. Then $\|f\|^2=\frac{5}{4}$ and

$$\begin{aligned}&\sum_{k=0}^{\infty}\left\{\left|\sum_{j=0}^{2k}\phi_j\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor}\Delta_{i,k,j}(0,0)+\sum_{j=1}^{2k}\xi_j\sum_{i=1}^{\lceil \frac{j}{2}\rceil}\Delta_{i,k,j}(1,0)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(0)\right|^2\right.\\ &\left.+\left|\sum_{j=1}^{2k+1}\phi_j\sum_{i=1}^{\lceil \frac{j}{2}\rceil}\Delta_{i,k,j}(2,1)+\sum_{j=0}^{2k+1}\xi_j\sum_{i=0}^{\lfloor \frac{j}{2}\rfloor}\Delta_{i,k,j}(1,1)+\sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(1)\right|^2\right\}\\ &=|\phi_{-1}c_1|^2+|\phi_1c_0+\phi_{-1}c_1b_1(1,1)+\xi_1c_1b_1(1,1)|^2+\\ &\qquad|\xi_1c_1b_2(1,1)+\xi_2c_1b_2(2,1)+\phi_2c_0|^2+|\xi_2c_1b_3(2,1)|^2=\frac{7237}{14400}<\frac{5}{4}.\end{aligned}$$

Thus, the operator $T_{\phi,\xi}$ is contractive.

(ii) Let $\phi_\xi(z) = \frac{3}{2}N_1(z) + \frac{4}{5}N_2(z) + \frac{11}{10}\overline{N_1(z)} + \frac{7}{10}\overline{N_2(z)}$, $\xi_\phi(z) = \frac{9}{10}N_1(z) + \frac{2}{5}N_2(z) + \frac{6}{5}\overline{N_1(z)} + \frac{3}{5}\overline{N_2(z)}$ and $f(z) = 1 + \frac{1}{2}N_1(z)$. Then $\|f\|^2 = \frac{5}{4}$ and

$$\sum_{k=0}^{\infty}\left\{\left|\sum_{j=0}^{2k}\phi_j\sum_{i=0}^{\lfloor\frac{j}{2}\rfloor}\Delta_{i,k,j}(0,0) + \sum_{j=1}^{2k}\xi_j\sum_{i=1}^{\lceil\frac{j}{2}\rceil}\Delta_{i,k,j}(1,0) + \sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(0)\right|^2\right.$$

$$\left. + \left|\sum_{j=1}^{2k+1}\phi_j\sum_{i=1}^{\lceil\frac{j}{2}\rceil}\Delta_{i,k,j}(2,1) + \sum_{j=0}^{2k+1}\xi_j\sum_{i=0}^{\lfloor\frac{j}{2}\rfloor}\Delta_{i,k,j}(1,1) + \sum_{j=1}^{\infty}{}^{i}\nu_{-j}\delta_{i,2k,j}(1)\right|^2\right\}$$

$$= |\phi_{-1}c_1|^2 + |\phi_1 c_0 + \phi_{-1}c_1 b_1(1,1) + \xi_1 c_1 b_1(1,1)|^2 +$$

$$|\xi_1 c_1 b_2(1,1) + \xi_2 c_1 b_2(2,1) + \phi_2 c_0|^2 + |\xi_2 c_1 b_3(2,1)|^2 = \frac{209}{80} > \frac{5}{4}.$$

Thus, the operator $T_{\phi,\xi}$ is expansive.

Remark 3.15. In general, the converse of Theorem 3.10 does not hold, which can be seen for the functions ϕ_ξ, ξ_ϕ and f discussed in Example 3.14 (i), for which the operator $T_{\phi,\xi}$ satisfies the condition in Theorem 3.10 but it is not expansive.

The following theorem delineates a necessary and sufficient condition for ensuring the adjoint of 2-Toeplitz operator to be contractive and expansive with harmonic symbols.

Theorem 3.16. *Let* $(\phi,\xi) \in D_\infty$ *such that* $\phi_\xi(z) = \sum_{i=0}^{\infty}\phi_i N_i(z) + \sum_{i=1}^{\infty}\phi_{-2i}\overline{N_{2i}}(z) + \sum_{i=1}^{\infty}\xi_{-(2i-1)}\overline{N_{2i-1}}(z)$ *and* $\xi_\phi(z) = \sum_{i=0}^{\infty}\xi_i N_i(z) + \sum_{i=1}^{\infty}\xi_{-2i}\overline{N_{2i}}(z) + \sum_{i=1}^{\infty}\phi_{-(2i-1)}\overline{N_{2i-1}}(z)$. *Then the following holds:*

(i) $T^*_{\phi,\xi}$ *is contractive if and only if*

$$\left|\sum_{j=0}^{\infty}\overline{\phi_j}c_j\right|^2 + \sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k}\overline{\alpha_{-j}}\sum_{i=0}^{j}c_{2k-i}b_{2k}(2k-i,j) + \sum_{j=0}^{\infty}\overline{\phi_j}\delta_{i,2k,j}(0)\right|^2$$

$$+\sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k-1}\overline{\beta_{-j}}\sum_{i=0}^{j}c_{2k-1-i}b_{2k-1}(2k-1-i,j) + \sum_{j=0}^{\infty}\overline{\xi_j}\delta_{i,2k,j}(-1)\right|^2 \leqslant \sum_{i=0}^{\infty}|c_i|^2.$$

(ii) $T^*_{\phi,\xi}$ *is expansive if and only if*

$$\left|\sum_{j=0}^{\infty}\overline{\phi_j}c_j\right|^2 + \sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k}\overline{\alpha_{-j}}\sum_{i=0}^{j}c_{2k-i}b_{2k}(2k-i,j) + \sum_{j=0}^{\infty}\overline{\phi_j}\delta_{i,2k,j}(0)\right|^2$$

$$+\sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k-1}\overline{\beta_{-j}}\sum_{i=0}^{j}c_{2k-1-i}b_{2k-1}(2k-1-i,j) + \sum_{j=0}^{\infty}\overline{\xi_j}\delta_{i,2k,j}(-1)\right|^2 \geqslant \sum_{i=0}^{\infty}|c_i|^2$$

where $\alpha_j = \begin{cases}\phi_j & \text{if } j \text{ is even;}\\ \xi_j & \text{if } j \text{ is odd}\end{cases}$, $\beta_j = \begin{cases}\xi_j & \text{if } j \text{ is even;}\\ \phi_j & \text{if } j \text{ is odd}\end{cases}$ *and* $c_i \in \mathbb{C}$.

Proof. Let $f(z) = \sum\limits_{i=0}^{\infty} c_i N_i(z) \in N^2(\mathbb{P})$, then we have $\|f(z)\|^2 = \sum\limits_{i=0}^{\infty} |c_i|^2$. For $m \in \mathbb{N}_0$, Theorem 2.12 gives that the m-th row of the matrix of the adjoint operator $T^*_{\phi,\xi}$ coincides with the m-th row of $T_{{}^m k_{\phi,\xi}}$. On using Lemma 2.5 and Lemma 2.7, we compute

$$\begin{aligned}
T_{{}^m k_{\phi,\xi}} f(z) =& P\left({}^m k_{\phi,\xi} f(z)\right) = P\left(\sum_{i=1}^{\infty} \overline{{}^m a_{-i}} N_i(z) \sum_{j=0}^{\infty} c_j N_j(z)\right. \\
&\left. + \sum_{i=0}^{\infty} {}^m\lambda_i \overline{N_i(z)} \sum_{j=0}^{\infty} c_j N_j(z)\right) \\
=& \sum_{j=0}^{\infty} {}^m\lambda_j c_j b_j(j,0) + \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} \overline{{}^m a_{-j}} \sum_{i=0}^{j} c_{k-i} b_k(k-i,j) \right. \\
&\left. + \sum_{j=0}^{\infty} {}^m\lambda_j \sum_{i=\max\{k,j\}}^{k+j} c_i b_i(j,k) \right\} N_k(z).
\end{aligned} \tag{3.3}$$

Now, using Parseval's identity and equation (3.3), we have

$$\begin{aligned}
\left\|T^*_{\phi,\xi} f(z)\right\|^2 =& \sum_{m=0}^{\infty} \left|\left\langle T^*_{\phi,\xi} f(z), N_m(z)\right\rangle\right|^2 = \sum_{m=0}^{\infty} \left|\left\langle T_{{}^m k_{\phi,\xi}} f(z), N_m(z)\right\rangle\right|^2 \\
=& \sum_{m=0}^{\infty} \left| \left\langle \sum_{j=0}^{\infty} {}^m\lambda_j c_j + \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} \overline{{}^m a_{-j}} \sum_{i=0}^{j} c_{k-i} b_k(k-i,j) \right.\right.\right. \\
&\left.\left.\left. + \sum_{j=0}^{\infty} {}^m\lambda_j \sum_{i=\max\{k,j\}}^{k+j} c_i b_i(j,k) \right\} N_k(z), N_m(z) \right\rangle \right|^2 \\
=& \left| \sum_{j=0}^{\infty} \overline{\phi_j} c_j \right|^2 + \sum_{k=1}^{\infty} \left| \sum_{j=0}^{k} \overline{{}^k a_{-j}} \sum_{i=0}^{j} c_{k-i} b_k(k-i,j) + \sum_{j=0}^{\infty} {}^k\lambda_j \sum_{i=\max\{k,j\}}^{k+j} c_i b_i(j,k) \right|^2.
\end{aligned}$$

This further implies that

$$\begin{aligned}
\left\|T^*_{\phi,\xi} f(z)\right\|^2 =& \left| \sum_{j=0}^{\infty} \overline{\phi_j} c_j \right|^2 + \sum_{k=1}^{\infty} \left| \sum_{j=0}^{2k} \overline{\alpha_{-j}} \sum_{i=0}^{j} c_{2k-i} b_{2k}(2k-i,j) + \sum_{j=0}^{\infty} \overline{\phi_j} \delta_{i,2k,j}(0) \right|^2 \\
&+ \sum_{k=1}^{\infty} \left| \sum_{j=0}^{2k-1} \overline{\beta_{-j}} \sum_{i=0}^{j} c_{2k-1-i} b_{2k-1}(2k-1-i,j) + \sum_{j=0}^{\infty} \overline{\xi_j} \delta_{i,2k,j}(-1) \right|^2.
\end{aligned}$$

Thus, the operator $T^*_{\phi,\xi}$ is contractive on $N^2(\mathbb{P})$ if and only if

$$\left| \sum_{j=0}^{\infty} \overline{\phi_j} c_j \right|^2 + \sum_{k=1}^{\infty} \left| \sum_{j=0}^{2k} \overline{\alpha_{-j}} \sum_{i=0}^{j} c_{2k-i} b_{2k}(2k-i,j) + \sum_{j=0}^{\infty} \overline{\phi_j} \delta_{i,2k,j}(0) \right|^2$$

$$+\sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k-1}\overline{\beta_{-j}}\sum_{i=0}^{j}c_{2k-1-i}b_{2k-1}(2k-1-i,j)+\sum_{j=0}^{\infty}\overline{\xi_j}\delta_{i,2k,j}(-1)\right|^2\leqslant\sum_{i=0}^{\infty}|c_i|^2.$$

Analogously, we obtain the result for expansivity. This completes the result. $\square$

Example 3.17. (i) Let $f(z)=1$, $\phi_\xi(z)=\frac{6}{5}N_1(z)+\frac{3}{5}\overline{N_1(z)}$ and $\xi_\phi(z)=\frac{2}{5}N_1(z)+\frac{9}{10}\overline{N_1(z)}$. Then $\|f\|^2=1$ and

$$\left|\sum_{j=0}^{\infty}\overline{\phi_j}c_j\right|^2+\sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k}\overline{\alpha_{-j}}\sum_{i=0}^{j}c_{2k-i}b_{2k}(2k-i,j)+\sum_{j=0}^{\infty}\overline{\phi_j}\delta_{i,2k,j}(0)\right|^2$$

$$+\sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k-1}\overline{\beta_{-j}}\sum_{i=0}^{j}c_{2k-1-i}b_{2k-1}(2k-1-i,j)+\sum_{j=0}^{\infty}\overline{\xi_j}\delta_{i,2k,j}(-1)\right|^2=\frac{81}{100}<1.$$

Thus, the operator $T^*_{\phi,\xi}$ is contractive.

(ii) Let $f(z)=1$, $\phi_\xi(z)=\frac{5}{2}N_1(z)+\frac{9}{5}\overline{N_1(z)}$ and $\xi_\phi(z)=\frac{7}{2}N_1(z)+\frac{6}{5}\overline{N_1(z)}$. Then $\|f\|^2=1$ and

$$\left|\sum_{j=0}^{\infty}\overline{\phi_j}c_j\right|^2+\sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k}\overline{\alpha_{-j}}\sum_{i=0}^{j}c_{2k-i}b_{2k}(2k-i,j)+\sum_{j=0}^{\infty}\overline{\phi_j}\delta_{i,2k,j}(0)\right|^2$$

$$+\sum_{k=1}^{\infty}\left|\sum_{j=0}^{2k-1}\overline{\beta_{-j}}\sum_{i=0}^{j}c_{2k-1-i}b_{2k-1}(2k-1-i,j)+\sum_{j=0}^{\infty}\overline{\xi_j}\delta_{i,2k,j}(-1)\right|^2=\frac{36}{25}>1.$$

Thus, the operator $T^*_{\phi,\xi}$ is expansive.

Remark 3.18. In general, the converse of Theorem 3.11 does not hold, that can be seen for the functions ϕ_ξ, ξ_ϕ and f discussed in Example 3.17 (i), for which the operator $T^*_{\phi,\xi}$ satisfies the condition in Theorem 3.11 but it is not expansive.

4. **Conclusion and Future remarks**

We dedicate this paper to the study of an invariant associated with the 2-Toeplitz operator $T_{\phi,\xi}$ defined on the Newton space $N^2(\mathbb{P})$. Our investigation includes an analysis of the contractivity and expansivity of $T_{\phi,\xi}$, for which we derive characterizations in the cases of analytic and co-analytic symbols. We note that the characterization of these properties for other symbol types, such as harmonic and non-harmonic, is a subject for future exploration. Additionally, we propose the examination of other operator properties, including spectral properties, as a direction for subsequent research.

Conflict of interest

The authors declare that there is no conflict of interest.

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