Аннотация

УДК 517.955, 517.958

Аннотация: Рассматривается система уравнений Власова–Пуассона с однородным внешним магнитным полем, описывающая кинетику двухкомпонентной разреженной плазмы в трёхмерном пространстве. Для любых начальных функций распределения с компактными носителями получены достаточные условия для внешнего магнитного поля, которые обеспечивают глобальное существование функций распределения плотности с произвольно малым ростом их носителей по пространственным переменным.

Ключевые слова: система уравнений Власова–Пуассона, задача Коши, удержание плазмы.

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Abstract. We consider the Vlasov-Poisson system with external homogeneous magnetic field describing kinetics of two-component rarefied plasma in the three-dimensional case. For any initial density distribution functions with compact supports, we obtain sufficient conditions for external magnetic field that provide global existence of density distribution functions with arbitrary small growth of their supports with respect to space variables.

Keywords: Vlasov-Poisson system, Cauchy problem, plasma confinement.

The Cauchy Problem for the Vlasov-Poisson System with External Magnetic Field

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1 Introduction

The Vlasov equations were derived in [1]. Global classical solutions to the Cauchy problem for the Vlasov-Poisson system were studied in [2–6]. The Vlasov-Poisson equations have many important applications: to the theory of high temperature plasma, to stellar dynamics, etc. One of the most exciting applications is connected with controlled thermonuclear fusion. Since plasma in a fusion reactor has a very high temperature, collisions of charged particles with the wall of the reactor may lead to a destruction of the wall. Therefore one of the most important problems in plasma physics is confinement of high temperature plasma at some distance from the wall. In order to provide plasma confinement physicists are using external magnetic field of a special form [7].

In this paper we consider a simplest model of plasma confinement: the Cauchy problem for the Vlasov-Poisson system with external magnetic field. Section 2 deals with the statement of problem, notation and formulation of the main result. Section 3 is devoted to the existence of global classical solution. In Section 4, we obtain a priori estimate for the norm of electric field strength. Section 5 deals with the properties of characteristics for the Vlasov equations. In Section 6, we obtain sufficient conditions for external magnetic field, which provide existence of density distribution functions with arbitrary small growth of their supports with respect to space variables as the time changes on a finite interval.

Conditions for existence of classical solutions to mixed problems for the Vlasov-Poisson system with external magnetic field providing confinement of two-component plasma in the cases of half-space, infinite cylinder ("mirror trap") and torus ("tokamak") were studied in [8–15].

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2 Statement of Problem. Main Result

We consider the Vlasov-Poisson system of equations

$$-\Delta\varphi(x,t) = 4\pi \int\limits_{\mathbb{R}^3} \sum_{\beta} \beta f^{\beta}(x,v,t)dv, \quad x \in \mathbb{R}^3,\, t \in (0,T),\, \beta = \pm 1, \tag{2.1}$$

$$\frac{\partial f^{\beta}}{\partial t} + (v, \nabla_x f^{\beta}) + \beta(-\nabla_x \varphi + [v, B(x)], \nabla_v f^{\beta}) = 0, \tag{2.2}$$

$$x, v \in \mathbb{R}^3,\ t \in (0,T),\ \beta = \pm 1,$$

with the initial conditions

$$f^{\beta}(x,v,t)\big|_{t=0} = f_0^{\beta}(x,v), \quad x, v \in \mathbb{R}^3, \quad \beta = \pm 1. \tag{2.3}$$

Here $f^{\beta} = f^{\beta}(x,v,t) \geq 0$ is the density distribution function of positively charged ions if $\beta = +1$ and negatively charged electrons if $\beta = -1$, at the point $x = (x_1, x_2, x_3)$ with velocity $v = (v_1, v_2, v_3)$ at the moment t; $\varphi = \varphi(x,t)$ is a potential of self-consistent electric field; ∇_x and ∇_v are gradients with respect to x and v, respectively; $B = B(x)$ is the induction of external magnetic field; $(.,.)$ is the scalar product in $\mathbb{R}^3$; $[.,.]$ is the vector product in $\mathbb{R}^3$.

Suppose that the potential φ satisfies the decreasing condition at infinity

$$\lim_{|x|\to\infty} \varphi(x,t) = 0, \quad 0 \leq t \leq T. \tag{2.4}$$

Denote by $C^s(\mathbb{R}^n)$, $s \geq 0$, $n \in \mathbb{N}$, the Hölder space of continuous functions on $\mathbb{R}^n$ having all continuous derivatives on $\mathbb{R}^n$ up to the k-th order, $k = [s]$, with the finite norm

$$\|u\|_s = \max_{|\alpha|\leq k} \sup_x |D^{\alpha}u(x)| \quad \text{if } s = k \in \mathbb{Z},\ 0 \leq k, \tag{2.5}$$

$$\|u\|_s = \|u\|_k + \|u\|_{\sigma} \quad \text{if } s = k + \sigma,\ 0 \leq k \in \mathbb{Z},\ 0 < \sigma < 1, \tag{2.6}$$

where

$$|u|_{\sigma} = \max_{|\alpha|=k} \sup_{x\neq y} |x-y|^{-\sigma} |D^{\alpha}u(x) - D^{\alpha}u(y)|,$$

$$D^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \quad \alpha = (\alpha_1, \ldots, \alpha_n),\ |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Let $C(\mathbb{R}^n) = C^0(\mathbb{R}^n)$, and let $\mathring{C}^k(\mathbb{R}^n)$, $k, n \in \mathbb{N}$, be a set of continuously differentiable functions in $\mathbb{R}^n$ with compact supports.

We introduce the Banach space $C([0,T], C^s(\mathbb{R}^3))$, $s \geq 0$, consisting of continuous functions $[0,T] \ni t \mapsto \varphi(.,t) \in C^s(\mathbb{R}^3)$ with the norm

$$\|\varphi\|_{s;T} = \sup_{0\leq t\leq T} \|\varphi(.,t)\|_s. \tag{2.7}$$

Let $B_\rho(x^0) = \{x \in \mathbb{R}^3 : |x - x^0| < \rho\}$, and let $B_\rho = B_\rho(0)$. Denote by $|B_\rho| = \dfrac{4\pi\rho^3}{3}$ the volume of the ball B_ρ.

Further, we denote by k_i, c_j, positive constants in inequalities, which do not depend on functions contained in these inequalities.

Denote by $\hat{C}^s(\mathbb{R}^3)$ the space of vector-functions $Y = (Y_1, Y_2, Y_3)^T$, having coordinates $Y_i \in C^s(\mathbb{R}^3)$, with the norm

$$\langle Y \rangle_s = \Big\{ \sum_{i=1}^{3} \|Y_i\|_s^2 \Big\}^{1/2}. \tag{2.8}$$

Assume that the following conditions hold.

Condition 2.1. Let $B \in \hat{C}^1(\mathbb{R}^3)$.

Condition 2.2. Let $f_0^\beta \in \mathring{C}^1(\mathbb{R}^6)$, and let

$$\operatorname{supp} f_0^\beta \subset D_0 := B_\lambda \times B_\rho,$$

where $\lambda, \rho > 0$.

Condition 2.3. Let $B(x) = (0, 0, b)$ for $x \in \{x \in \mathbb{R}^3 : |x'| < 2\lambda\}$, where $x = (x', x_3)$.

The main result of this paper can be formulated as following.

Theorem 2.1. *Let Conditions 2.1–2.3 hold. Then for any $\delta, 0 < \delta < \lambda$, there is a $b > 0$ such that there exists a solution of problem (2.1)– (2.4) $\{\varphi, f^\beta\}$, $\varphi \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$, $f^\beta \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T])$ having the following property: $\operatorname{supp} f^\beta \subset (\{x \in \mathbb{R}^3 : |x'| < \lambda + \delta\} \cap B_{\lambda_1}) \times B_{\rho_1}$, where $\lambda < \lambda_1 < \infty$, $\rho < \rho_1 < \infty$.*

Further we obtain Theorem 2.1 as a corollary from Theorem 6.1, in which we describe explicit condition for a constant b, see (5.9). A proof of Theorem 6.1 consists of three steps. First, we shall formulate lemma on the existence of global classical solution to the problem (2.1)– (2.4). Second, we shall obtain a priori estimate for the norm of electric field strength through the norm of density distribution function for charged particles. At the third step we shall estimate deviations of trajectories of particles from the Larmor trajectories. From these estimates we derive the conclusion of Theorem 2.1.

3 Existence of Global Classical Solutions

Lemma 3.1. *Let Conditions 2.1, 2.2 hold. Then there exists a solution of problem (2.1)–(2.4) $\{\varphi, f^\beta\}$ such that: $\varphi \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$ and $f^\beta \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T])$. Moreover, f^β have compact supports, $\beta = \pm 1$.*

Proof. By virtue of Theorem 1 from [2] and Theorem from [4], there exists a solution of the problem (2.1)–(2.4) $\varphi \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$, $f^\beta \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T])$, $\beta = \pm 1$. Moreover,

$$R(T) := \Big\{ 1 + \max_{\beta} \sup_{v \in \mathbb{R}^3} |v| : \text{there exist } x \in \mathbb{R}^3 \text{ and } t \in [0,T] \\ \text{such that } f^\beta(x,v,t) \neq 0 \Big\} < \infty. \quad (3.1)$$

We note that unlike papers [2] and [4] we have the additional term $\beta\left([v, B(x)], \nabla_v f^\beta\right)$ in the Vlasov equations, which provides plasma confinement. However, we still can use a standard proof for the existence of global solution, since it is based on the invariance of the Lebesgue measure with respect to transformations generated by vector–field

$$\left\{V_\varphi^\beta(\tau), -\beta \nabla_x \varphi\left(X_\varphi^\beta(\tau), \tau\right) + \beta\left[V_\varphi^\beta(\tau), B(X_\varphi^\beta(\tau))\right]\right\}$$

in characteristic equations and a priori estimates of velocities following from the equality $\left(V_\varphi^\beta(\tau), [V_\varphi^\beta(\tau), B(X_\varphi^\beta(\tau))]\right) = 0$. □

Remark 3.1. Generally speaking $R(T)$ depends also on f_0^β $(\beta = \pm 1)$.

4 A Priori Estimate for the Norm of Electric Field Strength

We define the space $C([0,T], C^s_\Omega(\mathbb{R}^3))$, where $C^s_\Omega(\mathbb{R}^3) = \{w \in C^s(\mathbb{R}^3) : \operatorname{supp} w \subset \Omega\}$, $\Omega \subset \mathbb{R}^3$ is a bounded domain.

Let $C_0(\mathbb{R}^3) = \{g \in C(\mathbb{R}^3); g(x) \to 0 \text{ as } |x| \to \infty\}$.

We consider the equation

$$-\Delta u(x) = F(x), \quad x \in \mathbb{R}^3\ , \quad (4.1)$$

with the decreasing condition at infinity

$$u(x) \to 0 \quad \text{as} \quad |x| \to \infty. \quad (4.2)$$

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Then, for any function $F \in C^\sigma_\Omega(\mathbb{R}^3)$, $0 < \sigma < 1$, there exists a unique solution of equation (4.1) with condition (4.2) $u \in C^{2+\sigma}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$.*

Moreover,

$$\||\nabla u|\|_0 \leqslant c_1 \|F\|_0, \quad (4.3)$$

where

$$0 < c_1 = \frac{1}{4\pi} \sup_{x \in \mathbb{R}^3} \left\{ \sum_{i=1}^3 \left(\int_\Omega \frac{|x_i - y_i|}{|x-y|^3} dy \right)^2 \right\}^{1/2} < \infty.$$

Proof. We consider the Newtonian potential

$$w(x) = \int\limits_{\Omega} \Gamma(x-y)F(y)\,dy, \tag{4.4}$$

where $\Gamma(x) = \dfrac{1}{4\pi|x|}$.

By virtue of Lemma 2.2 from [16, Chap. 3, Sect. 2] and Lemma 4.2 from [17, Chap. 4, Sect. 4.2], the function w belongs to $C^{2+\sigma}(\mathbb{R}^3)$ and satisfies equation (4.1). From the boundedness of Ω, it follows that

$$|w(x)| \le k_1(1+|x|)^{-1}, \quad x \in \mathbb{R}^3, \tag{4.5}$$

where $k_1 = k_1(F) > 0$ does not depend on x, i.e., $w \in C_0(\mathbb{R}^3)$. Therefore, the problem (4.1), (4.2) has the classical solution $u = w \in C^{2+\sigma}(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$.

Differentiating the right hand side of (4.4), we obtain the expression for constant c_1.

In order to prove a uniqueness of solution to the problem (4.1), (4.2), we put $F(x) = 0$. Then, by virtue of Liouville theorem, $u(x)$ is a polynomial. However, a nontrivial polynomial does not satisfy the decreasing condition (4.2). □

Let $\{\varphi, f^\beta\}$, $\varphi \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$, $f^\beta \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T])$, $\beta = \pm 1$, be a solution of problem (2.1)–(2.4). For the present function $\varphi \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$, the Vlasov equations (2.2) with the initial conditions (2.3) can be solved with the help of characteristics method. For this purpose, we consider the following system of ordinary differential equations:

$$\frac{dX_\varphi^\beta(\tau)}{d\tau} = V_\varphi^\beta(\tau), \quad 0 < \tau < T, \quad \beta = \pm 1, \tag{4.6}$$

$$\frac{dV_\varphi^\beta(\tau)}{d\tau} = -\beta \nabla_x \varphi(X_\varphi^\beta(\tau), \tau) + \beta[V_\varphi^\beta(\tau), B(X_\varphi^\beta(\tau))], \tag{4.7}$$
$$0 < \tau < T, \quad \beta = \pm 1,$$

with initial condition

$$X_\varphi^\beta(\tau)|_{\tau=t} = x, \quad \beta = \pm 1, \tag{4.8}$$

$$V_\varphi^\beta(\tau)|_{\tau=t} = v, \quad \beta = \pm 1. \tag{4.9}$$

We denote the solution of problem (4.6)–(4.9) by $(X_\varphi^\beta(\tau,x,v,t), V_\varphi^\beta(\tau,x,v,t)) := S_\varphi^\beta(\tau,x,v,t)$.

Lemma 4.2. *Let the vector function $B(x)$ and functions $f_0^\beta(x,v)$ satisfy Conditions 2.1 and 2.2. Then the following estimate holds*

$$|||\nabla\varphi|||_{0,T} \le c_1 |B_{R(T)}| 4\pi \max_\beta \|f_0^\beta\|_0, \tag{4.10}$$

where the number $R(T)$, $0 < R(T) < \infty$, is given by (3.1), and $c_1 > 0$ is a constant from inequality (4.3).

Proof. Clearly,

$$f^\beta(x,v,t) = f_0^\beta(S_\varphi^\beta(0,x,v,t)), \quad x \in \mathbb{R}^3,\ v \in \mathbb{R}^3,\ t \in (0,T). \tag{4.11}$$

From Lemma 4.1 and inequality (3.1) we obtain

$$\begin{aligned}
|||\nabla\varphi|||_{0,\tau} &\le c_1 4\pi \sup_{x,t} \int_{\mathbb{R}^3} |\sum_\beta \beta f^\beta(x,v,t)| dv \le \\
&\le c_1 4\pi \sup_{x,t} \int_{\mathbb{R}^3} \max_\beta f^\beta(x,v,t)\, dv = \\
&= c_1 4\pi \sup_{x,t} \int_{\mathbb{R}^3} \max_\beta f_0^\beta(S_\varphi^\beta(0,x,v,t)), dv \le \\
&\le c_1 |B_{R(T)}| 4\pi \max_\beta \sup_{(y,w)\in B_{R_0}} |f_0^\beta(y,w)|, \qquad x \in \mathbb{R}^3, \quad t \in [0,T],
\end{aligned}$$

where $R_0 = (\lambda^2 + \rho^2)^{1/2}$. From this it follows inequality (4.10). □

5 Some Properties of Characteristics

We again consider the system of ordinary differential equations (4.6), (4.7) with initial conditions (4.8), (4.9).

Lemma 5.1. *Let Conditions 2.1– 2.3 hold. Then for every* $(x,v) \in D_0$ *and* $0 \le t \le T$ *the following estimates take place*

$$|V_\varphi^\beta(t,x,v,0)| < \rho_1, \tag{5.1}$$

$$|X_\varphi^\beta(t,x,v,0)| < \lambda_1, \tag{5.2}$$

where $\rho_1 = \rho + Tc_1|B_{R(T)}|4\pi \max_\beta \|f_0^\beta\|_0$, $\lambda_1 = \lambda + \rho_1 T$.

Proof. Multiplying the left and the right sides of equation (4.7) by V_φ^β, we have

$$\frac{1}{2}\frac{d}{d\tau}|V_\varphi^\beta(\tau,x,v,0)|^2 = -\beta(\nabla_x\varphi(X_\varphi^\beta(\tau,x,v,0), V_\varphi^\beta(\tau,x,v,0)), \qquad 0 \le \tau \le T.$$

From this equality and from the Cauchy-Bunyakovskii inequality we obtain

$$\frac{1}{2}\frac{d}{d\tau}|V_\varphi^\beta(\tau,x,v,0)|^2 \le |\nabla_x\varphi(X_\varphi^\beta(\tau,x,v,0)| \cdot |V_\varphi^\beta(\tau,x,v,0)|, \qquad 0 \le \tau \le T.$$

Hence,

$$\frac{d}{d\tau}|V_\varphi^\beta(\tau,x,v,0)| \le |\nabla_x \varphi(X_\varphi^\beta(\tau,x,v,0))|, \quad 0 \le \tau \le T. \tag{5.3}$$

Integrating (5.3) by τ from 0 to t, by virtue of Lemma 4.2 and Condition 2.2, we have

$$\left|V_\varphi^\beta(t,x,v,0)\right| \le |v| + \int\limits_0^t |\nabla_x \varphi(X_\varphi^\beta(\tau,x,v,0))| d\tau <$$

$$\rho + Tc_1|B_{R(T)}|4\pi \max_\beta \|f_0^\beta\|_0. \tag{5.4}$$

Integrating equation (4.6) by τ from 0 to t and using inequality (5.4), we obtain (5.2). □

We introduce the matrix $R(\theta)$ by the formula:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

As it is known, multiplication by the matrix $R(\theta)$ corresponds to rotation by the angle θ on the plane. We now formulate some properties of this operator, which allow us to study the behaviour of characteristics of Vlasov equations.

Lemma 5.2 (see [11])**.**

(a) $R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2), \quad \theta_1,\theta_2 \in \mathbb{R}.$

(b) $R(\theta)^m = R(m\theta), \quad \theta \in \mathbb{R},\ m \in \mathbb{Z}.$

(c) $\frac{d}{d\theta}R(\theta) = R\left(\frac{\pi}{2}\right)R(\theta) = R\left(\theta + \frac{\pi}{2}\right), \quad \theta \in \mathbb{R}.$

(d) $|R(\theta)x| = |x|, \quad \theta \in \mathbb{R},\ x \in \mathbb{R}^2.$

(e) $\exp(tR(\theta)) = \exp(t\cos\theta)R(t\sin\theta).$

Proof. The properties (a)-(d) are obvious. We prove property (e). From property (b) and definition of the matrix $R(\theta)$, it follows that

$$\exp(tR(\theta)) = \sum_{n=0}^\infty \frac{t^n}{n!}R(\theta)^n = \sum_{n=0}^\infty \frac{t^n}{n!}R(n\theta) = \sum_{n=0}^\infty \frac{t^n}{n!}\begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}. \tag{5.5}$$

From the Euler formula we have

$$\exp(t\exp(i\theta)) = \exp(t(\cos\theta + i\sin\theta)) =$$

$$= \exp(t\cos\theta)[\cos(t\sin\theta) + i\sin(t\sin\theta)]. \tag{5.6}$$

It easy to see that

$$\exp(t\exp(i\theta)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\exp(in\theta) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\cos(n\theta) + i\sum_{n=0}^{\infty} \frac{t^n}{n!}\sin(n\theta). \tag{5.7}$$

From equations (5.6) and (5.7), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!}\cos(n\theta) &= \exp(t\cos\theta)\cos(t\sin\theta), \\ \sum_{n=0}^{\infty} \frac{t^n}{n!}\sin(n\theta) &= \exp(t\cos\theta)\sin(t\sin\theta). \end{aligned} \tag{5.8}$$

The equalities (5.5) and (5.8) imply that

$$\begin{aligned} \exp(tR(\theta)) &= \exp(t\cos\theta)\begin{pmatrix} \cos(t\sin\theta) & -\sin(t\sin\theta) \\ \sin(t\sin\theta) & \cos(t\sin\theta) \end{pmatrix} = \\ &= \exp(t\cos\theta)R(t\sin\theta). \end{aligned}$$

□

Let $x' = (x_1, x_2)$, and let $X_\lambda^{\beta'}(\tau, x, v, 0) := \left(X_{\lambda,1}^\beta(\tau, x, v, 0), X_{\lambda,2}^\beta(\tau, x, v, 0)\right)$. Assume that the following condition is fulfilled.

Condition 5.1. Let a constant b in Condition 2.3 satisfy the inequality

$$\frac{2}{\delta}(\rho + Tc_1|B_{R(T)}|4\pi \max_\beta \|f_0^\beta\|_0) < b, \tag{5.9}$$

where $c_1 > 0$ is a constant from Lemma 4.1.

The next lemma is a generalization of Lemma 3.3 from [11].

Lemma 5.3. *Let Conditions 2.1– 2.3 and 5.1 hold. Then the solution of problem (4.6)–(4.9) given by* $\left(X_\varphi^\beta(\tau, x, v, 0), V_\varphi^\beta(\tau, x, v, 0)\right) := S_\varphi^\beta(\tau, x, v, 0)$ *has the following properties: for all* $(x, v) \in D_0$ *and* $\tau \in [0, T]$,

$$\left|X_\varphi^{\beta'}(\tau, x, v, 0) - x'\right| < \delta, \qquad X_\varphi^\beta(\tau, x, v, 0) \in B_{\lambda_1}, \qquad V_\varphi^\beta(\tau, x, v, 0) \in B_{\rho_1},$$

where $0 < \delta < \lambda$, $\lambda_1 = \lambda + \rho_1 T$, $\rho_1 = \rho + Tc_1|B_{R(T)}|4\pi \max_\beta \|f_0^\beta\|_0$.

Proof. We prove that

$$\left|X_\varphi^{\beta'}(\tau, x, v, 0) - x'\right| < \delta \text{ for any } \tau \in [0, T]. \tag{5.10}$$

Assume to the contrary that there is a $\tau_0 \in [0, T]$ such that

$$\left|X_\varphi^{\beta'}(\tau, x, v, 0) - x'\right| \geq \delta.$$

Since $X_\varphi^{\beta'}(0,x,v,0) = x'$, then for some $0 < \tau_1 \leqslant \tau_0 \leqslant T$, we have

$$\left| X_\varphi^{\beta'}(\tau_1, x, v, 0) - x' \right| = \delta, \tag{5.11}$$

$$\left| X_\varphi^{\beta'}(\tau, x, v, 0) - x' \right| < \delta, \ \tau \in [0, \tau_1). \tag{5.12}$$

Since $0 < \delta < \lambda$, Condition 2.3 allows us to rewrite characteristic equation (4.7) in the form

$$\frac{dV_\varphi^\beta(\tau)}{d\tau} = -\beta \nabla_x \varphi(X_\varphi^\beta, \tau) + \beta \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_\varphi^\beta(\tau), \quad \tau \in (0, \tau_1).$$

Therefore we have

$$\frac{d}{d\tau} \begin{pmatrix} V_{\varphi,1}^\beta(\tau) \\ V_{\varphi,2}^\beta(\tau) \end{pmatrix} + \beta\, b\, R\left(\frac{\pi}{2}\right) \begin{pmatrix} V_{\varphi,1}^\beta(\tau) \\ V_{\varphi,2}^\beta(\tau) \end{pmatrix} = -\beta \nabla_{(x_1,x_2)} \varphi(X_\varphi^\beta, \tau),$$

$$\tau \in (0, \tau_1).$$

If we multiply the last equation by $\exp(\tau\beta b R(\frac{\pi}{2}))$, we get

$$\frac{d}{d\tau}\left[\exp\left(\tau\beta b R\left(\frac{\pi}{2}\right)\right) \begin{pmatrix} V_{\varphi,1}^\beta(\tau) \\ V_{\varphi,2}^\beta(\tau) \end{pmatrix} \right] = -\beta \exp(\tau\beta b R(\frac{\pi}{2})) \nabla_{(x_1,x_2)} \varphi(X_\varphi^\beta, \tau),$$

$$\tau \in (0, \tau_1). \tag{5.13}$$

Integrating equation (5.13) from 0 to t, $t \in (0, \tau_1]$, we obtain

$$\exp\left(t\beta b R\left(\frac{\pi}{2}\right)\right) \begin{pmatrix} V_{\varphi,1}^\beta(t) \\ V_{\varphi,2}^\beta(t) \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$

$$= -\beta \int_0^t \exp\left(\tau\beta b R\left(\frac{\pi}{2}\right)\right) \nabla_{(x_1,x_2)} \varphi(X_\varphi^\beta, \tau)\, d\tau. \tag{5.14}$$

Lemma 5.2(e) implies that

$$\exp\left(t\beta b R\left(\frac{\pi}{2}\right)\right) = R(t\beta b). \tag{5.15}$$

From (5.14), (5.15) it follows that

$$\begin{pmatrix} V_{\varphi,1}^\beta(t) \\ V_{\varphi,2}^\beta(t) \end{pmatrix} = R(-t\beta b) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \beta \int_0^t R\left((\tau - t)\beta b\right) \nabla_{(x_1,x_2)} \varphi(X_\varphi^\beta, \tau)\, d\tau,$$

$$t \in (0, \tau_1]. \tag{5.16}$$

From (4.6) and (5.16) it follows that

$$\begin{pmatrix} X^{\beta}_{\varphi,1}(\tau_1) \\ X^{\beta}_{\varphi,2}(\tau_1) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + I_1 + I_2, \tag{5.17}$$

where

$$I_1 = \int\limits_0^{\tau_1} R(-t\beta b) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dt,$$

$$I_2 = -\beta \int\limits_0^{\tau_1} dt \int\limits_0^{t} R((\tau - t)\beta b) \nabla_{(x_1,x_2)} \varphi(X^{\beta}_{\varphi}, \tau)\, d\tau.$$

We now estimate the integrals I_1 and I_2.
Lemma 5.2(c) implies that

$$R(-t\beta b) = -\frac{1}{\beta b}\frac{d}{dt}\left(R(-t\beta b - \frac{\pi}{2})\right).$$

Therefore, by virtue of Lemma 5.2(a), we have

$$\begin{aligned} I_1 &= \frac{1}{\beta b}\left\{-R(-\tau_1\beta b) + E\right\} R\left(-\frac{\pi}{2}\right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \\ &= \frac{1}{\beta b} \begin{pmatrix} 1-\cos(\tau_1\beta b) & -\sin(\tau_1\beta b) \\ \sin(\tau_1\beta b) & 1-\cos(\tau_1\beta b) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \\ &= \frac{1}{\beta b} \begin{pmatrix} \beta\sin(\tau_1 b)v_1 + (1-\cos(\tau_1 b))v_2 \\ -(1-\cos(\tau_2 b))v_1 + \beta\sin(\tau_1 b)v_2 \end{pmatrix}, \end{aligned}$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus we have

$$\begin{aligned} |I_1| &= \frac{1}{b}\Big\{\{\beta\sin(\tau_1 b)v_1 + (1-\cos(\tau_1 b))v_2\}^2 + \\ &\quad + (-(1-\cos(\tau_1 b))v_1 + \beta\sin(\tau_1 b)v_2)^2\Big\}^{1/2} = \\ &= \frac{1}{b}\left\{\left(v_1^2 + v_2^2\right)\left(((1-\cos(\tau_1 b))^2 + \sin^2(\tau_1 b)\right)^{\frac{1}{2}}\right\} = \\ &= \frac{1}{b}|v|\sqrt{2}\sqrt{1-\cos(\tau_1 b)} \le \frac{2}{b}|v|. \end{aligned} \tag{5.18}$$

Changing the order of integration and using again Lemma 5.2(c), we obtain

$$\begin{aligned} I_2 &= -\beta \int\limits_0^{\tau_1} \left\{ \int\limits_{\tau}^{\tau_1} R((\tau - t)\beta b)\, dt \right\} \nabla_{(x_1,x_2)}\varphi(X^{\beta}_{\varphi}, \tau)\, d\tau = \\ &= \frac{1}{b}\int\limits_0^{\tau_1} \left\{R((\tau-\tau_1)\beta b) - E\right\} R(-\tfrac{\pi}{2}) \nabla_{(x_1,x_2)}\varphi(X^{\beta}_{\varphi}, \tau)\, d\tau = \end{aligned}$$

$$= \frac{1}{b}\int\limits_0^{\tau_1} \begin{pmatrix} \cos((\tau-\tau_1)b)-1 & -\beta\sin((\tau-\tau_1)b) \\ \beta\sin((\tau-\tau_1)b) & \cos((\tau-\tau_1)b)-1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial\varphi(X_\varphi^\beta,\tau)}{\partial x_1} \\ \dfrac{\partial\varphi(X_\varphi^\beta,\tau)}{\partial x_2} \end{pmatrix} d\tau =$$

$$= \frac{1}{b}\int_0^{\tau_1} \begin{pmatrix} \beta\sin((\tau-\tau_1)b)\ \dfrac{\partial\varphi(X_\varphi^\beta,\tau)}{\partial x_1} + \cos((\tau-\tau_1)b)-1)\ \dfrac{\partial\varphi(X_\varphi^\beta,\tau)}{\partial x_2} \\ (1-\cos((\tau-\tau_1)b))\dfrac{\partial\varphi(X_\varphi^\beta,\tau)}{\partial x_1} + \beta\sin((\tau-\tau_1)b)\dfrac{\partial\varphi(X_\varphi^\beta,\tau)}{\partial x_2} \end{pmatrix} d\tau.$$

Finally, we have

$$|I_2| \le \frac{1}{b}\int\limits_0^{\tau_1} \Bigg\{ ((1-\cos((\tau-\tau_1)b))^2 + \sin^2((\tau-\tau_1)b) \times$$
$$\times \left(\left(\frac{\partial\varphi(X_\varphi^\beta,\tau)}{\partial x_1} \right)^2 + \left(\frac{\partial\varphi(X_\varphi^\beta,\tau)}{\partial x_2} \right)^2 \right) \Bigg\}^{1/2} d\tau =$$
$$= \frac{1}{b}\int\limits_0^{\tau_1} ((1-\cos((\tau-\tau_1)b))^2 + \sin^2((\tau-\tau_1)b))^{1/2} |\nabla_{(x_1,x_2)}\varphi(X_\varphi^\beta,\tau)| d\tau \le$$
$$\le \frac{2}{b}\int\limits_0^{\tau_1} |\nabla_{(x_1,x_2)}\varphi(X_\varphi^\beta,\tau)| d\tau \le \frac{2T}{b} \||\nabla\varphi|\|_{0,T}. \quad (5.19)$$

From (5.11), (5.17)–(5.19) and Lemma 4.2 it follows that

$$\delta = |X_\varphi^{\beta'}(\tau_1,x,v,0) - x'| \le |I_1| + |I_2| \le$$
$$\le \frac{2}{b}(\rho + T\||\nabla\varphi|\|_{0,T}) \le \frac{2}{b}(\rho + Tc_1|B_{R(T)}|4\pi \max_\beta \|f_0^\beta\|). \quad (5.20)$$

By virtue of (5.9), we obtain

$$\frac{2}{b}(\rho + Tc_1|B_{R(T)}|4\pi \max_\beta \|f_0^\beta\|_0) < \delta.$$

We have contradiction with (5.20). This implies ((5.10).

From Lemma 5.1 it follows that $X_\varphi^\beta(\tau,x,v;0) \in B_{\lambda_1}$and $V_\varphi^\beta(\tau,x,v;0) \in B_{\rho_1}$ for $((x,v) \in D_0$ and $\tau \in [0,T]$. □

6 Proof of Theorem 2.1

First we shall prove the following auxiliary statement.

Lemma 6.1. *Let the assumptions of Lemma 5.3 hold. Then* $\operatorname{supp} f_0^\beta(S_\varphi^\beta(0,.,.,t)) \subset D_1 := (B_{\lambda_1} \cap \{x : |x'| < \lambda + \delta\}) \times B_{\rho_1}$ *for* $t \in [0,T]$.

Proof. By virtue of Condition 2.2 it is sufficient to prove that $S_\varphi^\beta(t,x,v,0) \in D_1$ for $(x,v) \in D_0$, $t \in [0,T]$. This statement follows from Lemma 5.3. □

We define the function $f^\beta(x,v,t)$ by the formula

$$f^\beta(x,v,t) = \begin{cases} f_0^\beta(S_\varphi^\beta(0,x,v,t), & (x,v) \in D_1, \quad t \in [0,T], \\ 0, & (x,v) \in \left(\mathbb{R}^3 \times \mathbb{R}^3\right) \setminus D_1,\ t \in [0,T]. \end{cases} \tag{6.1}$$

Let assumptions of Lemma 5.3 hold. Then, by virtue of Lemma 3.1, there exists a global classical solution of problem (2.1)–(2.4) $\left\{\varphi, f^\beta\right\}$, $\varphi \in C([0,T], C^{2+\sigma}(\mathbb{R}^3))$, $f^\beta \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T])$. From Lemma 6.1 and representation (6.1) it follows that $\operatorname{supp} f^\beta = \operatorname{supp} f_0^\beta(S_\varphi^\beta(0,.,.,t)) \subset D_1$. Thus we have proved the following result.

Theorem 6.1. *Let Conditions (2.1)–(2.3) be fulfilled. Then, for any* $\delta, 0 < \delta < \lambda$ *and* b *satisfying Condition 5.1, there exists a global classical solution of problem* (2.1)–(2.4) $\left\{\varphi, f^\beta\right\}$, $\varphi \in C([0,T], C^{2+\sigma}(\mathbb{R}^3)), f^\beta \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T])$ *such that* $\operatorname{supp} f^\beta \subset D_1, \beta = \pm 1$.

Theorem 2.1 follows from Theorem 6.1. Moreover, Condition 5.1 allows to define the external magnetic field, which provides plasma confinement.

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