On Operators Dominated by the Kantorovich–Banach and the Levi Operators in Locally Solid Lattices

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Аннотация

A linear operator T acting in a locally solid vector lattice (E, τ) is said to be: a Lebesgue operator, if $Tx_\alpha \xrightarrow{\tau} 0$ for every net in E satisfying $x_\alpha \downarrow 0$; a KB-operator, if, for every τ-bounded increasing net x_α in E_+, there exists an $x \in E$ with $Tx_\alpha \xrightarrow{\tau} Tx$; a quasi KB-operator, if T takes τ-bounded increasing nets in E_+ to τ-Cauchy ones; a Levi operator, if, for every τ-bounded increasing net x_α in E_+, there exists an $x \in E$ such that $Tx_\alpha \xrightarrow{o} Tx$; a quasi Levi operator, if T takes τ-bounded increasing nets in E_+ to o-Cauchy ones. The present article is devoted to the domination problem for the quasi KB-operators and the quasi Levi operators in locally solid vector lattices. Moreover, some properties of Lebesgue operators, Levi operators, and KB-operators are investigated. In particularly, it is proved that the vector space of regularly Lebesgue operators is a subalgebra of the algebra of all regular operators.

keywords: locally solid lattice, Lebesgue operator, Levi operator, KB-operator, lattice homomorphism

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In the present paper all vector lattices are real and Archimedean, and linear topologies are Hausdorff. The

main subject of the paper is the domination problem for operators in locally solid vector lattices dominated by Kantorovich-Banach lattice homomorpisms, quasi KB-operators, and quasi Levi operators. Recently, this topic has attracted attention (see [1] and references therein). Recall that a locally solid vector lattice (E, τ) is said to be:

i) a *Lebesgue lattice* (σ-*Lebesgue lattice*) if, for every net (sequence) $x_\alpha \downarrow 0$ in E, we have $x_\alpha \xrightarrow{\tau} 0$;

ii) a *Levi lattice* (σ-*Levi lattice*) if any increasing τ-bounded net (sequence) in E_+ has a supremum in E.

In i), the condition $x_\alpha \downarrow 0$ can be replaced by $x_\alpha \xrightarrow{o} 0$.

A normed lattice $(E, \|\cdot\|)$ is called *Kantorovich-Banach* or *KB-space* if every norm bounded upward directed set in E_+ converges in norm. Each KB-space is a Levi lattice with order continuous complete norm; and every normed Levi (σ-Levi) lattice is Dedekind (σ-Dedekind) complete.

iii) A locally solide lattice (E, τ) is called a KB (σ-KB) *lattice*, if every increasing τ-bounded net (sequence) in E_+ is τ-convergent.

It is whell known that any KB (σ-KB) lattice is Levi (σ-Levi) lattice, and every Levi (σ-Levi) lattice is Dedekind (σ-Dedekind) complete (see Theorem 2.21(c) in [2]). A linear operator T acting in a locally solid lattice (E, τ) is called *$o\tau$-continuous* (σ-*$o\tau$-continuous*), if $Tx_\alpha \xrightarrow{\tau} 0$ for all net (sequence) x_α such that $x_\alpha \xrightarrow{o} 0$ [3]. Various locally solid versions of properties of Banach lattices, like the property to be a KB-space, has been studied recently by many authors (see, a.e., [3], [4], [5], [6]). The main idea of operator versions of topological vector lattices is the redistribution of topological and order properties between domain and range of the operator. Since the order convergence is not topological in general, the most interesting

versions arise when the o- and the τ-convergence are involved simultaneously.

For the more general form of the following definition we refer to [1].

Definition 1. Let T be a linear operator in a locally solid lattice (E, τ). Then

(a) T is a *Lebesgue* (*σ-Lebesgue* operator), if $Tx_\alpha \xrightarrow{\tau} 0$ for every net (sequence) $x_\alpha \downarrow 0$ in E;

(b) T is *$o\tau$-bounded* (*$o\tau$-compact*), if the set $T[0, x]$ is a τ-bounded (τ-totally bounded) in E for each $x \in E_+$;

(c) T is a (σ-) *Kantorovich-Banach* (or (σ-) KB) operator, if, for every τ-bounded increasing net (sequence) x_α in E_+, there exists $x \in E$ such that $Tx_\alpha \xrightarrow{\tau} Tx$;

(d) T is a *quasi* KB (*quasi* σ-KB) operator, if it takes τ-bounded increasing nets (sequences) in E_+ onto τ-Cauchy ones;

(e) T is a *Levi* (*σ-Levi*) operator, if, for every τ-bounded increasing net (sequence) x_α in E_+, there exists $x \in E$ such that $Tx_\alpha \xrightarrow{o} Tx$;

(f) T is a *quasi Levi* (*quasi σ-Levi*) operator, if it takes τ-bounded increasing nets (sequences) in E_+ onto o-Cauchy ones.

Proposition 1. (Cf. Proposition 2.7 in [1]). *Every continuous linear operator T in a locally solid lattice (E, τ) is $o\tau$-bounded.*

The proposition follows from the following two classical facts:

1) In a locally solid lattice (E, τ), all order bounded subsets are τ-bounded.

2) Each τ-continuous linear operator takes τ-bounded sets to τ-bounded ones.

Definition 2. (See Definition 2 in [7]). Let P be a set of linear operators between ordered vector spaces X and Y. An operator $T : X \to Y$ is called a *regularly* P-*operator* (r-P-*operator* for short), if there exist two positive P-operators $T_1, T_2 : X \to Y$ such that $T = T_1 - T_2$. The set of all r-P-operators from X to Y is denoted by r-P(X, Y).

Proposition 2. *Each regularly Lebesgue (regularly σ-Lebesgue) operator T in a locally solid lattice (E, τ) is order continuous (order σ-continuous).*

Доказательство. As the regularly σ-Lebesgue case is similar, we restrict ourselves to the case of a regularly Lebesgue operator T. Without loss of generality suppose $T \geq 0$. In order to show that T is order continuous, it suffices to prove $Tx_\alpha \downarrow 0$ for each net x_α in E such that $x_\alpha \downarrow 0$. Let $x_\alpha \downarrow 0$ in E. Since T is positive then $Tx_\alpha \downarrow$. Since T is a Lebesgue operator then $Tx_\alpha \xrightarrow{\tau} 0$. It follows $Tx_\alpha \downarrow 0$ (e.g., by Theorem 2.21(c) of [2]), and hence T is order continuous. □

Corollary 1. *The family* r-L$_{Leb}(E)$ (r-L$^\sigma_{Leb}(E)$) *of all regularly Lebesgue (regularly σ-Lebesgue) operators in a locally solid lattice (E, τ) is a subalgebra of the algebra $L_r(E)$ of all regular operators in E. Moreover, $I \in$* r-L$_{Leb}(E)$ *($I \in$* r-L$^\sigma_{Leb}(E)$*) iff (E, τ) is a Lebesgue (σ-Lebesgue) lattice, where I is the identity of E.*

Доказательство. It suffices to prove that the set r-L$_{Leb}(E)$ is closed with respect to the composition of operators. Let $T, S \in$ r-L$_{Leb}(E)$. Without loss of generality, assume $T, S \geq 0$. Take a net $x_\alpha \downarrow 0$. Then, by Proposition 2, $Sx_\alpha \downarrow 0$ and hence $TSx_\alpha \xrightarrow{\tau} 0$. In the case of regularly σ-Lebesgue operators, the proof is similar. The rest of the proof of Corollary 1 is trivial. □

Lemma 1. (Cf. Lemma 2.1 in [1]). *A linear operator T in a locally solid lattice (E,τ) is regularly Lebesgue (regularly σ-Lebesgue) iff T is regularly $o\tau$-continuous (regularly σ-$o\tau$-continuous).*

Доказательство. Without loss of generality assume $T \geq 0$. As the σ-Lebesgue case is similar, we consider only Lebesgue operators. The sufficiency is routine. For the necessity, assume T is a Lebesgue operator and $x_\alpha \xrightarrow{o} 0$ in E. Take a net $y_\beta \downarrow 0$ in E such that, for each β, there exists α_β with $|x_\alpha| \leq y_\beta$ for $\alpha \geq \alpha_\beta$. It follows from $T \geq 0$ that $|Tx_\alpha| \leq T|x_\alpha| \leq Ty_\beta$ for $\alpha \geq \alpha_\beta$. Since T is a Lebesgue, $Ty_\beta \xrightarrow{\tau} 0$. As the topology τ is locally solid, it follows $Tx_\alpha \xrightarrow{\tau} 0$, and hence T is $o\tau$-continuous. □

Proposition 3. (cf. Proposition 2.1 in [1]). *Every regular operator T in a locally solid lattice (E,τ) is $o\tau$-bounded.*

Here we include the following list of useful properties of operators under the consideration.

a) The identity operator in E is a Lebesgue/KB/Levi operator iff E is a Lebesgue/KB/Levi lattice respectively. Every $o\tau$-continuous (σ-$o\tau$-continuous) operator is a Lebesgue (σ-Lebesgue) operator and, by Lemma 1, every regularly Lebesgue (regularly σ-Lebesgue) operator in E is regularly $o\tau$-continuous (regularly σ-$o\tau$-continuous). The authors do not know whether any regular Lebesgue (σ-Lebesgue) operator is $o\tau$-continuous (σ-$o\tau$-continuous).

b) The discontinuous operator $Tx := (\sum_{k=1}^{\infty} x_k)e_1$ in the normed lattice $(c_{00}, \|\cdot\|_\infty)$ is $o\tau$-compact and $o\tau$-continuous, yet T is not compact. Every continuous linear operator T in a discrete Dedekind complete locally

solid locally convex vector lattice is $o\tau$-compact in view of Corollary 6.57 of [2].

c) Every KB-operator is quasi KB, and every continuous linear operator in a KB-space is a KB-operator. It is well known that the identity I in a Banach lattice is a KB-operator iff I is a σ-KB-operator iff I is a quasi KB-operator. The notions “quasi KB operator"and “quasi σ-KB operator"coincide by Proposition 4 below. Every order bounded operator in a KB-space is a quasi KB-operator.

d) Every compact operator T in a Banach lattice is $o\tau$-compact. By Example 1 below, a compact operator need not to be a Lebesgue operator. In particular, an $o\tau$-compact operator is not necessarily $o\tau$-continuous.

Example 1. (See Example 3.1 of [1]). Let $E = (c_\omega(\mathbb{R}), \|\cdot\|_\infty)$ be a Banach lattice of all bounded real functions on $\mathbb{R}$ such that each $f \in E$ differs from a constant a_f on at most countable subset of $\mathbb{R}$.

Define a positive operator T in E as follows. Let Tf be a constant function $a_f \cdot \mathbb{1}_\mathbb{R} \in E$ such that $\{d \in \mathbb{R} : f(d) \neq a_f\}$ is at most countable.

(1) T is a continuous rank one operator (and consequently T is KB operator, compact, and $o\tau$-compact) in E. Let $f_n \xrightarrow{o} 0$. Since, for any $\varepsilon > 0$, there exists n_ε such that $\cup_{n \geq n_\varepsilon}\{d \in \mathbb{R} : |f_n(d)| \geq \varepsilon\}$ is at most countable, then $\|Tf_n\|_\infty < \varepsilon$ for all $n \geq n_\varepsilon$. Thus, the operator T is σ-$o\tau$-continuous and hence a σ-Lebesgue operator with respect to the norm topology on E.

(2) T is not a Lebesgue operator. Indeed, for the net $f_\alpha := \mathbb{1}_{\mathbb{R}\setminus\alpha} \in E$, indexed by the directed set Δ of all finite subsets of $\mathbb{R}$ ordered by inclusion, we have $f_\alpha \downarrow 0$. However $\|Tf_\alpha\|_\infty = \|\mathbb{1}_\mathbb{R}\|_\infty = 1$ for all $\alpha \in \Delta$.

Proposition 4. (Cf. Proposition 1.2 of [1]). *An operator T in a locally solid lattice is a quasi KB-operator iff T is quasi σ-KB.*

It follows from Proposition 4 that every topologically complete σ-KB-lattice (E,τ) is a KB-lattice (see Corollary 1.1 in [1]). Note that the sets $L_{o\tau}(E)$, $L_{o\tau b}(E)$, and $L_{o\tau c}(E)$ of $o\tau$-continuous, $o\tau$-bounded, and $o\tau$-compact operators in a locally solid lattice E are vector spaces satisfying $L_{o\tau c}(E) \subseteq L_{o\tau b}(E)$.

A vector lattice E is said to be *laterally* (σ-)*complete*, if every (countable) subset of mutually disjoint vectors in E_+ has a supremum. A laterally complete vector lattice is discrete iff it is lattice isomorphic to $\mathbb{R}^S$ for some set S.

Definition 3. (Cf. Definition 2.1 of [1]). A locally solid lattice (E,τ) is τ-*laterally* (σ-) *complete*, if every τ-bounded (countable) subset of mutually disjoint vectors in E_+ possesses a supremum.

Every laterally (σ-) complete locally solid lattice (E,τ) is τ-laterally (σ-) complete, and every order complete AM-space X with the order unit is τ-laterally complete with respect to the norm.

Example 2. (Cf. Example 2.1 in [1]). Consider a vector lattice E of real functions on $\mathbb{R}$ such that every $f \in E$ differs from a constant a_f on at most countable subset of $\mathbb{R}$, and $f - a_f\mathbb{1}_{\mathbb{R}} \in \ell_1(\mathbb{R})$ for all $f \in E$. The vector lattice E is complete with respect to the norm

$$\|f\| := |a_f| + \|f - a_f\mathbb{1}_{\mathbb{R}}\|_1.$$

Clearly, E is not a K_σ-space since $f_n := \mathbb{1}_{\mathbb{R}\setminus\{1,2,\dots,n\}} \downarrow \geq 0$, yet $\inf\limits_{n\in\mathbb{N}} f_n$ does not exist in E. Note that E is not τ-laterally σ-complete with respect to the norm topology on

E. Indeed, the norm bounded countable set of mutually disjoint orts $e_n = \mathbb{1}_{\{n\}} \in E_+$ has no supremum in E.

By Proposition 2.4 of [1], each Levi (σ-Levi) lattice is a τ-laterally (σ-) complete (σ-) Dedekind complete vector lattice.

Now, we turn to the domination problem. Namely, let T and S be operators on E with $0 \leq S \leq T$. Under what conditions does the assumption that T is a Lebesgue operator, $o\tau$-bounded, $o\tau$-compact, KB-operator, or Levi operator imply that S has the same property? It is clear that the problem has the positive solution for the Lebesgue, the σ-Lebesgue, and the $o\tau$-bounded operators.

Recall that every order bounded disjointness preserving operator T in a vector lattice E has a modulus $|T|$ satisfying $|T||x| = |T|x|| = |Tx|$ for all $x \in E$ (cf. Theorem 2.40 in [8]); moreover, there exist two lattice homomorphisms $R_1, R_2 : E \to E$ such that $T = R_1 - R_2$ (cf. [8, Exercise 1, p.130]).

Proposition 5. (Cf. Theorem 2.5 in [1]). *Let T be an order bounded disjointness preserving KB (σ-KB) operator in a locally solid lattice (E, τ). If $|S| \leq |T|$, then S is also a KB (σ-KB) operator.*

Доказательство. Take a τ-bounded increasing net (sequence) x_α in E_+. Then $T(x_\alpha - x) \xrightarrow{\tau} 0$ for some $x \in E$. Hence

$$|S(x_\alpha - x)| \leq |S||x_\alpha - x| \leq |T||x_\alpha - x| = |Tx_\alpha - Tx| \xrightarrow{\tau} 0.$$

Therefore $Sx_\alpha \xrightarrow{\tau} Sx$. □

Since, for a lattice homomorphism T, every operator S with $0 \leq S \leq T$ is a lattice homomorphism (cf. e.g.,

[8, Theorem 2.14]), the next theorem follows from Proposition 5.

Theorem 1. (Cf. Corollary 2.3 of [1]). *Let T be a KB (σ-KB) lattice homomorphism in a locally solid lattice. Then every operator S satisfying $0 \leq S \leq T$ is also a KB (σ-KB) lattice homomorphism.*

The following result generalizes Proposition 2.9 of [4] for locally solid lattices.

Theorem 2. (Cf. Theorem 2.6 in [1]). *Let T be a positive quasi KB-operator in a locally solid lattice (E, τ). Then every operator S satisfying $0 \leq S \leq T$ is also quasi KB.*

Доказательство. Let x_α be an increasing τ-bounded net in E_+. Then $Tx_\alpha \uparrow$, and since T is a quasi KB-operator, the net Tx_α is τ-Cauchy. Take $U \in \tau(0)$ and a solid neighborhood $V \in \tau(0)$ such that $V - V \subseteq U$. There exists α_0 satisfying $T(x_\alpha - x_\beta) \in V$ for all $\alpha, \beta \geq \alpha_0$. In particular, $T(x_\alpha - x_{\alpha_0}) \in V$ for all $\alpha \geq \alpha_0$. Since $0 \leq S \leq T$ and V is solid, $S(x_\alpha - x_{\alpha_0}) \in V$ for all $\alpha \geq \alpha_0$. Thus

$$Sx_\alpha - Sx_\beta = S(x_\alpha - x_{\alpha_0}) - S(x_\beta - x_{\alpha_0}) \in V - V \subseteq U$$

for all $\alpha, \beta \geq \alpha_0$. Since $U \in \tau(0)$ was taken arbitrarily, Sx_α is a τ-Cauchy net. □

Theorem 3. (See Theorem 2.7 of [1]). *Let T be a positive quasi Levi operator in a locally solid lattice (E, τ). Then each operator S with $0 \leq S \leq T$ is also quasi Levi.*

Доказательство. Take an increasing τ-bounded net x_α in E_+. Then the net Tx_α is o-Cauchy. Hence there exists a net $y_\beta \downarrow 0$ in E such that given β there is an α_β with

$|Tx_{\alpha_1} - Tx_{\alpha_2}| \leq y_\beta$ for all $\alpha_1, \alpha_2 \geq \alpha_\beta$. Then, for $\alpha_1, \alpha_2 \geq \alpha_\beta$, we have

$$Sx_{\alpha_1} - Sx_{\alpha_2} \leq S(x_{\alpha_1} - x_{\alpha_\beta}) \leq T(x_{\alpha_1} - x_{\alpha_\beta}) \leq y_\beta,$$

$$Sx_{\alpha_2} - Sx_{\alpha_1} \leq S(x_{\alpha_2} - x_{\alpha_\beta}) \leq T(x_{\alpha_2} - x_{\alpha_\beta}) \leq y_\beta.$$

Thus $|Sx_{\alpha_1} - Sx_{\alpha_2}| \leq y_\beta$ for all $\alpha_1, \alpha_2 \geq \alpha_\beta$, and so S is a quasi Levi operator. □

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