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# ESTIMATES OF ALEXANDROV'S $n$-WIDTH OF THE COMPACT SET OF $C^\infty$-SMOOTH FUNCTIONS ON A FINITE SEGMENT

**V. N. Belykh**

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**Abstract**—We obtain two-sided estimates for Alexandrov's $n$-width of the compact set of infinitely smooth functions boundedly embedded into the space of continuous functions on a finite segment.

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The designs of computer algorithms for numerically solving boundary value problems always involve some approximations of continual objects by finite-dimensional ones as well as constructions of some analogs of the latter that rely on the concepts admitting discrete formalization [1]. A meaningful description of an object $X$ is extracted with the tools of approximation theory if some approximation technique suitable for $X$ was found and understood. The discretization procedure is a compulsory stage, and it necessitates some loss of information; furthermore, the best discrete description of $X$ organized in a certain way as a compact metric set leads to the concept of Alexandrov's $n$-width $\alpha_n(X)$ [2, Chapter 1, Section 2, Subsection 4].

By playing a key role in estimates for the limit approximation possibilities of a compact set of functions $X$, the asymptotics for $\alpha_n(X)$ indicates the accuracy with which compact sets of topological dimension $n$ exhaust $X$ with the growth of $n$. The greater the supply of smoothness of $X$, the higher the vanishing rate of $\alpha_n(X)$. For the compact sets $X$ of functions of finite and infinite smoothness these asymptotics differ fundamentally: While in the first case $\alpha_n(X)$ decays as some fixed power of $1/n$, in the second case $\alpha_n(X)$ decays faster than every finite power of $1/n$; see [2].

Therefore, Alexandrov's $n$-width $\alpha_n(X)$ turned out a deep mathematical concept and led to the rethinking of the very status of the significance for the actual calculations of additional, including infinite, smoothness of the compact sets $X$ of solutions to problems. Eventually the introduction of the numerical parameter $\alpha_n(X)$ into everyday computer practice conducted (see [3, Chapter 3, Section 2, Subsection 5]) to the discovery of fundamentally new *nonsaturable* computational methods whose practical efficiency is directly related to the asymptotics of the decay of $\alpha_n(X)$ as $n \to \infty$.

The distinguishing feature of nonsaturable numerical methods is the capacity to adjust automatically with the growth of $n$ to arbitrary approximation possibilities of the compact solution set $X$, i.e., as the "supply" of smoothness of $X$ grows, a nonsaturable numerical method, other conditions being equal, improves itself, deriving the increment of its practical efficiency directly from the differential nature of $X$ (the *nonsaturability phenomenon* [4]). Therefore, the extraordinary smoothness of solutions to problems, previously kept on the periphery of the urgent needs of computing, becomes their active character. The peak efficiency of a nonsaturable numerical method, *exponential convergence*, is attained on the class of infinitely smooth functions. This is a fundamental difference of nonsaturable numerical methods from saturable ones, i.e., those with a leading error term: finite difference methods, finite element methods, quadrature methods, and so on.

There are classes of problems, elliptic [5, 6] for instance, whose compact solution sets consist of nonanalytic $C^\infty$-smooth functions of various nature. The classes of such functions, intermediate between analytic functions and functions of finite smoothness, are usually defined by indicating a majorant for the growth of their order $k$-derivatives as $k \to \infty$, and stating the main assumptions on the growth of the majorant becomes one of the methods for their classification.

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In this article we obtain two-sided estimates for Alexandrov's $n$-width $\alpha_n(X)$ of the compact set of $C^\infty$-smooth functions boundedly embedded into the space $C$ of continuous functions on a finite segment. The arguments rely on a new characterization of the class of $C^\infty$-smooth functions which invokes its best Chebyshev description by polynomials [4].

## 1. Compact Set: $\varepsilon$-Cover, Dimension, Alexandrov's $n$-Width

Given a compact set $X$ in some Banach space $B$, consider a finite open cover $\Upsilon$ for $X$. The choice of $\Upsilon$ is never unique because of the Borel–Lebesgue Theorem. However, which topological property of the elements of $\Upsilon$ could be used as the basis for a procedure for selecting the required cover uniquely?

It stands to reason to extract the following from the collection of general topological properties of the elements of $\Upsilon$. Given an integer $m \geq 0$, the cover $\Upsilon$ has *multiplicity* $m$ whenever each collection of its arbitrary $m+1$ elements has empty intersection, while there exist some collection of $m$ elements with nonempty intersection.

Previously speaking about the discretization of a compact set $X$, we somewhat vaguely characterized by indicating only those elements of $X$ which are determined by a finite collection of independent numerical parameters. A priori it is far from obvious that we can mathematically state the idea of the number of measurements (or dimension) for such general objects as compact sets of functions. We can uniquely relate the concept of multiplicity to the approximation dimension of $X$, which makes the perception of the number of measurements internally consistent.

Definition (see [7, Chapter 1, Section 1.7, Subsection 1.7.3]). A compact set $X$ has *topological dimension* $m$, in symbols $\dim X = m$ whenever, given $\varepsilon > 0$, there exists an $\varepsilon$-cover for $X$ by open sets of diameter less than $\varepsilon$ and multiplicity at most $m+1$, and $m$ is the minimal integer with this property.

Therefore, to each $\varepsilon$-cover for $X$ we can always associate some positive integer $m$ for which $X$ contains a point that belongs to $m$ distinct elements of $\Upsilon$. This definition of dimension corresponds fully to the intuitive understanding of the dimension of the cube $\mathbb{I}^m$ of side $d > 0$,

$$\mathbb{I}^m \equiv \mathbb{I}^m_d = \{\boldsymbol{\xi} \in \mathbb{R}^m : |\xi_r| \leq d/2,\ r = 0, 1, \dots, m-1\},$$

as reinforced by the Brouwer–Lebesgue Tiling Theorem: $\dim \mathbb{I}^m = m$.

*Alexandrov's $m$-width* of $X$ is defined as follows [2, Chapter 1, Section 2, Subsection 4]:

$$\alpha_m(X, B) = \inf_{(X^m,\nu)} \sup_{f \in X \subset B} \|f - \nu(f)\|, \tag{1.1}$$

where inf is taken over all possible pairs $(X^m, \nu)$ consisting of a compact subset $X^m$ of topological dimension $m$ in the Banach space $B$ and a continuous mapping $\nu : X \to X^m$.

The introduction by Alexandrov [8] of the concept of $m$-width witnesses the fundamentally important relation of a compact subset $X$ of $B$ to the elementary geometric transformation of $X^m$. In other words, we can prescribe each point $f$ in $X \subset B$ with accuracy $\varepsilon$ by precisely $m$ "coordinates" without fail if we want the latter to depend continuously on a point while the point itself, on the coordinates.

Given an integer $m \geq 0$, put

$$l^m_\infty = \{\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_{m-1}) \in \mathbb{R}^m,\ |\boldsymbol{\xi}|_\infty = \max_{r=0,1,\dots,m-1} |\xi_r|\}.$$

The following lemma will be useful in the sequel; see [1].

**Lemma 1.** *If $\mathbb{I}^m_d$ is a cube of side $d$ and $\dim \mathbb{I}^m_d = m$ then $\alpha_{m-1}(\mathbb{I}^m_d, l^m_\infty) = d/2$.*

According to the corollary to Theorem 1.4 of [1], Lemma 1 follows from Theorem 2 (see [7, Chapter 4, Section 4.1, Subsection 4.1.1]) and the corollary to Theorem 4 (see [7, Chapter 4, Section 4.1, Subsection 4.1.2]).

## 2. Periodic Case: the Main Facts, Definitions, and the Result

Before estimating $\alpha_m(X)$, we give a series of the auxiliary results related directly to approximations of $C^\infty$-smooth periodic functions by trigonometric polynomials. Let us start with definitions.

Consider the space $\widetilde{C}[0,2\pi]$ of $2\pi$-periodic continuous functions on the whole axis $R$ which is equipped with the norm $\|\varphi\| = \max_{t\in[0,2\pi]} |\varphi(t)|$. We treat $\widetilde{C}[0,2\pi]$ as the space $C \equiv C[S]$ of continuous functions on the unit circle $S$ which remain continuous when extended $2\pi$-periodically to the whole axis $R$.

Consider the class $\mathscr{T}^m \subset \widetilde{C}[0,2\pi]$ of trigonometric polynomials of degree at most $m$ and put

$$e_m(\varphi) = \inf_{\iota_m \in \mathscr{T}^m} \|\varphi - \iota_m\|, \quad m \geq 0,\ \varphi \in C. \tag{2.1}$$

Furthermore, the infimum is always attained at some element in $\mathscr{T}^m$.

Denote the space of $2\pi$-periodic infinitely differentiable functions on $S$ by $C^\infty \equiv C^\infty[S]$. Note that $C^\infty$ has an absolute topological basis [9]:

$$\{\pi_p(t)\} \equiv \{\pi_0(t) = 1/2,\ \pi_{2p-1}(t) = \sin pt,\ \pi_{2p}(t) = \cos pt \text{ for all } p > 0\}.$$

The presence of a basis means that in $C^\infty$ we can consider the series $\sum_{p=0}^{\infty} c_p \pi_p(t)$ with the following convention about convergence: The series converges absolutely, i.e.,

$$\sum_{p=0}^{\infty} \|c_p \pi_p^{(k)}\| < \infty \quad \text{for all } k \geq 0,$$

and its sum is an element $\varphi$ of $C^\infty$. Here $\pi_p^{(k)}(t) = d^k \pi_p(t)/dt^k$.

Thus, we can identify an arbitrary periodic $C^\infty$-smooth function $\varphi(t)$ with its trigonometric series

$$\varphi(t) = \sum_{p=0}^{+\infty} c_p \pi_p(t) \equiv \frac{c_0}{2} + \sum_{p=1}^{+\infty} (c_{2p} \cos pt + c_{2p-1} \sin pt). \tag{2.2}$$

Series (2.2) converges uniformly in $C$; it also converges in the sense of the Hilbert space $L_2[S]$ and presents the Fourier series of $\varphi(t)$; i.e.,

$$c_0 = \frac{1}{\pi} \int_0^{2\pi} \varphi(t)\,dt, \quad \begin{pmatrix} c_{2p} \\ c_{2p-1} \end{pmatrix} \equiv \frac{1}{\pi} \int_0^{2\pi} \varphi(t) \begin{pmatrix} \cos pt \\ \sin pt \end{pmatrix} dt \quad \text{for all } p \geq 1,$$

because $\{\pi_p(t)\}$ is a complete orthogonal system.

As we already pointed out, the compact sets of $C^\infty$-smooth functions are prescribed by indicating the majorants of the growth of their order $k$ derivatives depending on the integer parameter $k \geq 0$; moreover, stating the main assumptions on the growth of the majorant $G(k)$ as $k \to \infty$ is one of the classification methods for the classes of infinitely smooth functions. Furthermore, all closed bounded subsets of $C^\infty$ are compact [10]; i.e.,

$$\widetilde{K}_G^\infty \equiv \{f \in C^\infty : \|f\| \leq G(0),\ \|f^{(k)}(t)\| \leq G(k) \text{ for all } k > 0\} \tag{2.3}$$

with a positive numerical sequence $\{G(k)\}$ is a compact set in $C^\infty$, while the classical Arzelà Theorem shows that $\widetilde{K}_G^\infty$ is also a compact subset of $C$.

Consider some new approach to describing periodic $C^\infty$-smooth functions. It bases on using the classical Jackson's inequalities (see [4])

$$e_m(f) \leq \Lambda_k \cdot \frac{\|f^{(k)}\|}{m^k} \quad (m > 0),\ k \geq 0,\ \Lambda_k = \frac{4}{\pi} \sum_{\nu=0}^{+\infty} \frac{(-1)^{(k+1)\nu}}{(2\nu+1)^{k+1}} \leq \pi/2, \tag{2.4}$$

and consists in indicating the pairs of monotone functions $\mu(r)$ and $\vartheta(r)$ of a real argument $r$ constructed efficiently from the tuple $\{G(k)\}$ and possessing some useful properties on the semiaxis $r \geq 0$.

Take some sequence $\{G(k)\}$ of positive integers with

$$\lim_{k\to\infty} \sqrt[k]{G(k)} = \infty. \tag{2.5}$$

Associate to $\{G(k)\}$ the two functions of a numerical argument $r \in [0, \infty)$:

$$\mu(r) = \begin{cases} G(0) & \text{for} \quad 0 \leq r < 1, \\ \inf\limits_{k\geq 0} \frac{G(k)}{r^k} & \text{for} \quad r \geq 1, \end{cases} \qquad \vartheta(r) = \begin{cases} 0 & \text{for} \quad 0 \leq r < 1, \\ \max\bigl\{k \mid \mu(r) = \frac{G(k)}{r^k}\bigr\} & \text{for} \quad r \geq 1. \end{cases}$$

By (2.5), we can always replace the infimum in the definition of $\mu(r)$ by the minimum, and so

$$\mu(r) = \min_{k\geq 0} \frac{G(k)}{r^k} = \frac{G[\vartheta(r)]}{r^{\vartheta(r)}} \quad \text{and} \quad e_m(f) \leq \frac{\pi}{2}\mu(m). \tag{2.6}$$

We have the following (see [4]):

**Theorem 1.** *For $r \geq 1$ the function $\vartheta(r)$ is integer-valued, nonnegative, nondecreasing, right-continuous, and tending to infinity together with $r$. The function $\mu(r)$ is strictly decreasing, continuous, and vanishing as $r \to \infty$. Furthermore,*

$$\mu(r) = G(0)e^{-\int_1^r \frac{\vartheta(t)}{t}\,dt}, \quad r \geq 1. \tag{2.7}$$

**Corollary 1.** *The function $\mu(r)$ vanishes as $r \to \infty$ faster than an arbitrary power of $r$; i.e., $\lim_{r\to\infty} r^p\mu(r) = 0$ for every $p \geq 0$.*

The classes of periodic $C^\infty$-smooth functions we mentioned are nonempty: they include, for instance, the Gevrey classes with majorant $H(k) = cA^k k^{\alpha k}$, where $\alpha > 1$, $c > 0$, and $A \geq 1$ are constants.

As a formal advantage of the expansion in (2.2), we should point out the circumstance that it is nonlocal and the allowed form of approximate description of an element $\varphi(t)$ of $C^\infty$ is realized by the partial Fourier sum $\varphi_N(t)$, which involves finitely many elements of the basis $\{\pi_p(t)\}$. The practical benefit of the description is characterized by the number of associated coefficients in the expansion of $\varphi(t)$ in this basis, and so the description is determined only by constraints on the class $C^\infty$ itself of a general mathematical nature. Proceed to stating the constraints of this kind.

Since the compact set $\widetilde{K}_G^\infty \subset C^\infty$ is defined (see (2.3)) by indicating a majorant for the order $k$ derivatives of its elements and embeds into the space of uniformly converging series (2.2), the following question arises: How much are the orders of decay of the coefficients $c_p$ in the expansion of an element $\varphi$ of $\widetilde{K}_G^\infty$ in the trigonometric basis $\{\pi_p(t)\}$ agree with the order of growth of the majorant $G(k)$ defining the compact set $\widetilde{K}_G^\infty$?

**Lemma 2** [10]. *If $\varphi$ lies in the compact set $\widetilde{K}_G^\infty$ then*

$$|c_0| \leq 2G(0), \quad |c_{2p-1}| \leq 2\mu(p), \quad |c_{2p}| \leq 2\mu(p) \quad \textit{for all } p \geq 1.$$

*Conversely, if the coefficients in the expansion of $\varphi \in C^\infty$ satisfy*

$$|c_0| \leq G(0), \quad |c_{2p-1}| \leq \frac{1}{8}\frac{\mu(p)}{p^2}, \quad |c_{2p}| \leq \frac{1}{8}\frac{\mu(p)}{p^2} \quad \textit{for all } p \geq 1$$

*then $\varphi$ lies in $\widetilde{K}_G^\infty$; see (2.3).*

Identifying the compact set $\widetilde{K}_G^\infty$ with the set of trigonometric series (2.2), we create a constructive tool for obtaining the required estimates.

Observe firstly that approximation to periodic functions by trigonometric polynomials of order at most $m-1$ determines the subspace $\mathscr{T}^{m-1} \subset C$ of polynomials of topological dimension $\dim \mathscr{T}^{m-1} = 2m-1$.

Some upper bound for $\alpha_{2m-1}(\widetilde{K}_G^\infty, C)$ follows from the definitions in (1.1), (2.1), and (2.6):

$$\alpha_{2m-1}(\widetilde{K}_G^\infty, C) \le \inf_{(\mathscr{T}^{m-1},\nu)} \sup_{f\in \widetilde{K}_G^\infty \subset C} \|f-\nu(f)\| = \sup_{f\in \widetilde{K}_G^\infty \subset C} e_{m-1}(f) \le \frac{\pi}{2}\mu(m-1). \tag{2.8}$$

Let us find a lower bound for $\alpha_{2m-1}(\widetilde{K}_G^\infty, C)$. The idea to use the properties that the width $\alpha_{2m-1}(\widetilde{K}_G^\infty, C)$ is monotone by inclusion and nonincreasing as a function of the index $m$ plays some role here. Thus, if we can linearly and isometrically send some compact set $X_0$ into the compact set $\widetilde{K}_G^\infty$ then

$$\alpha_{2m}(X_0, C) \le \alpha_{2m-1}(X_0, C) \le \alpha_{2m-1}(\widetilde{K}_G^\infty, C).$$

Choose as $X_0$ the cube $\mathbb{Q}^{2m+1} \subset \widetilde{K}_G^\infty$ for which we will estimate $\alpha_{2m}(\mathbb{Q}^{2m+1}, C)$ from below.

Indeed, take $f$ in $\widetilde{K}_G^\infty$ and expand $f$ as $f(t) = \sum_{r=0}^\infty c_r \pi_r(t)$. Then for every we have $k \ge 0$

$$\|f^{(k)}\| \le G(k), \quad \mu(r) = \min_{k\ge 0} \frac{G(k)}{r^k}, \quad \|\pi_r^{(k)}(t)\| = \left\| \begin{pmatrix} \cos rt \\ \sin rt \end{pmatrix}^{(k)} \right\| \le r^k.$$

Introducing the function

$$\phi_r(t) \equiv \frac{\mu(m)}{2(m+1)}\pi_r(t) = \frac{\mu(m)}{2(m+1)} \begin{pmatrix} \cos rt \\ \sin rt \end{pmatrix} \equiv \begin{pmatrix} \phi_r^c(t) \\ \phi_r^s(t) \end{pmatrix}, \quad 0 \le r \le m,$$

we obtain

$$\begin{aligned} \|\phi_r^{(k)}(t)\| &= \frac{\mu(m)}{2(m+1)}\|\pi_r^{(k)}(t)\| \le \frac{G(k)\mu(m)}{2(m+1)} \cdot \frac{r^k}{G(k)} \le \frac{G(k)\mu(m)}{2(m+1)} \cdot \left( \min_{k\ge 0} \frac{G(k)}{r^k} \right)^{-1} \\ &= \frac{G(k)\mu(m)}{2(m+1)} \cdot (\mu(r))^{-1} \le \frac{G(k)}{2(m+1)} \cdot \frac{\mu(m)}{\mu(r)} \le \frac{G(k)}{2(m+1)} \quad \text{for all } k \ge 0. \end{aligned}$$

The functions $\phi_r(t)$ are linearly independent and belong to $\widetilde{K}_G^\infty$, and their linear combination

$$\omega(t) = \sum_{r=0}^m \big(\xi_r \phi_r^c(t) + \eta_r \phi_r^s(t)\big)$$

for $|\xi_r|, |\eta_r| \le 1$ also lies in the compact set $\widetilde{K}_G^\infty$:

$$\|\omega^{(k)}(t)\| \le \sum_{r=0}^m \big(|\xi_r|\|\phi_r^{c(k)}(t)\| + |\eta_r|\|\phi_r^{s(k)}(t)\|\big) \le \sum_{r=0}^m \frac{G(k)}{m+1} \le G(k) \quad \text{for all } k \ge 0.$$

Define in $\widetilde{K}_G^\infty$ the family of functions

$$\mathbb{Q}^{2m+1} = \left\{ \omega(t) = \sum_{r=0}^m (\xi_r \cos rt + \eta_r \sin rt) : |\xi_r|, |\eta_r| \le \frac{\mu(m)}{2(m+1)}, r = 0, 1, \dots, m \right\}, \tag{2.9}$$

where $\xi_0, \xi_1, \dots, \xi_m; \eta_1, \dots, \eta_m$ are reals.

Put $|\boldsymbol{\xi}|_\infty = \max\{|\xi_0|/2, |\xi_1|, \dots, |\xi_m|\}$ and $|\boldsymbol{\eta}|_\infty = \max\{|\eta_1|, |\eta_2|, \dots, |\eta_m|\}$.

The cube

$$\mathbb{I}^{2m+1} = \left\{ \boldsymbol{\zeta} \in \mathbb{R}^{2m+1} : |\zeta_r| \le \frac{\mu(m)}{2(m+1)}, \ r = 0, 1, \dots, 2m \right\}$$

of topological dimension $2m+1$ and side $d = \frac{\mu(m)}{m+1}$ embeds into $\widetilde{K}^\infty_G$ linearly and homeomorphically with nondecreasing distances; and so isometrically; see [3, Chapter 3, Section 7, Proposition 3], and the image of the cube is the set of (2.9) because

$$|\boldsymbol{\zeta}|_\infty \equiv \max(|\boldsymbol{\xi}|_\infty, |\boldsymbol{\eta}|_\infty) \le \sqrt{2}\left\|\xi_0/2 + \sum_{r=1}^{m}(\xi_r \cos rt + \eta_r \sin rt)\right\|. \tag{$*$}$$

Estimate $(*)$ is easy from Bessel's inequalities; see [3, Chapter 2, Section 3, Subsection 1]:

$$|\boldsymbol{\zeta}|_\infty^2 \equiv \left(\max\left\{\frac{|\xi_0|}{2}, |\xi_1|, \dots, |\xi_m|, |\boldsymbol{\eta}|_\infty\right\}\right)^2 \le \left(\max\left\{\frac{|\xi_0|}{\sqrt{2}}, |\xi_1|, \dots, |\xi_m|, |\boldsymbol{\eta}|_\infty\right\}\right)^2$$
$$\le \frac{\xi_0^2}{2} + \sum_{r=1}^{m}(\xi_r^2 + \eta_r^2) \le \frac{1}{\pi}\int_0^{2\pi} \omega^2(t)\,dt \le 2\max_{t\in[0,2\pi]}|\omega(t)|^2 = 2\|\omega\|^2.$$

Using the argument above and Lemma 1, we arrive at the required lower bound:

$$\alpha_{2m}\big(\widetilde{K}^\infty_G, C\big) \ge \alpha_{2m}(\mathbb{Q}^{2m+1}, C) \ge \frac{1}{\sqrt{2}}\alpha_{2m}\big(\mathbb{I}^{2m+1}, l_\infty^{2m+1}\big) = \frac{1}{\sqrt{2}}\frac{\mu(m)}{2(m+1)}. \tag{2.10}$$

**Theorem 2.** *We have the following estimates for Alexandrov's width of the compact sets $\widetilde{K}^\infty_G$ of periodic $C^\infty$-smooth functions boundedly embedded into the space $C$ of continuous periodic functions on the circle $S$, see* (2.8), (2.10), *and* (2.7)*:*

$$\frac{1}{2\sqrt{2}}\frac{\mu(m)}{(m+1)} \le \alpha_{2m}\big(\widetilde{K}^\infty_G, C\big) \le \alpha_{2m-1}\big(\widetilde{K}^\infty_G, C\big) \le \frac{\pi}{2}\mu(m-1) \quad \text{for integer } m \ge 1.$$

**Corollary 2.** *The estimates of the widths of the Gevrey classes with majorant $H(k) = cA^k k^{\alpha k}$, where $\alpha > 1$, $c > 0$, and $A \ge 1$, admit the estimates* (*see* [4] *for details*)

$$\frac{c}{2\sqrt{2}}\frac{1}{(m+1)}e^{-\varrho\sqrt[\alpha]{m}} \le \alpha_{2m}\big(\widetilde{K}^\infty_H, C\big) \le \alpha_{2m-1}\big(\widetilde{K}^\infty_H, C\big) \le \pi\frac{c}{2}\beta e^{-\varrho\sqrt[\alpha]{m-1}}.$$

*Here $\varrho = \alpha e^{-1}/\sqrt[\alpha]{A}$ and $\beta = \exp(\alpha e\sqrt[\alpha]{A}/2)$ are real constants.*

## 3. Aperiodic Case: Definitions and the Result

For the class of $C^\infty$-smooth functions not periodic on the segment $I \equiv [-1,1]$ the situation is partly similar to the periodic case considered above.

In order to estimate the width $\alpha_n(X, C)$ of a compact set $X$ of aperiodic $C^\infty$-smooth functions on $I$, we use the series of the properties related to their approximation by algebraic polynomials. Let us start with definitions.

Denote by $C \equiv C[I]$ the space of aperiodic continuous real-valued functions $\psi(t)$ on the segment $I$ equipped with the norm $\|\psi\| = \max_{t\in[-1,1]}|\psi(t)|$; by $\mathscr{P}^n \subset C$ the subspace of algebraic polynomials of degree at most $n$, where $n \ge 0$ is an integer; and by $C^k \equiv C^k[I]$, the space of $k$ times continuously differentiable functions on $I$.

Denote by $E_n(\psi) \equiv \inf_{\upsilon_n\in\mathscr{P}^n}\|\psi - \upsilon_n\|$ the best Chebyshev approximation to a function $\psi$ in $C$ by a polynomial. The infimum on the subspace $\mathscr{P}^n$ is always attained.

Denote the space of aperiodic infinitely differentiable functions on $I$ by $C^\infty \equiv C^\infty[I]$. It is known [9, 11] that the tuple of functions

$$\{T_n(t)\} \equiv \{T_n(t) = \cos(n\cos^{-1}t) \text{ for all } n \ge 0\}$$

constitutes an absolute topological basis for $C^\infty$, meaning that we can expand every function $\psi(t)$ in $C^\infty$ in the basis $\{T_n(t)\}$:

$$\psi(t) = \frac{c_0}{2} T_0(t) + \sum_{n=1}^{+\infty} c_n T_n(t). \tag{3.1}$$

Series (3.1) converges uniformly in $C$ as well as in the Hilbert space $L_2[I,\phi]$ with weight $\phi(t) = (1-t^2)^{-1/2}$. Note that (3.1) is the Fourier–Chebyshev series of $\psi(t)$ because $\{T_n(t)\}$ is a complete orthogonal system; i.e.,

$$c_0 = \frac{2}{\pi}\int_{-1}^{1} \frac{\psi(t)}{\sqrt{1-t^2}}\,dt, \quad c_n = \frac{2}{\pi}\int_{-1}^{1} \frac{\psi(t)T_n(t)}{\sqrt{1-t^2}}\,dt \quad \text{for all } n > 0.$$

To describe the functions $\psi$ in the space $C^k$ of algebraic polynomials constructively, use the Jackson–Sinwel Theorem [12]:

$$E_n(\psi) \le \Lambda_k \|\psi^{(k)}\| \frac{(n-k)!}{n!}, \quad n \ge k \ge 0, \quad \Lambda_k = \frac{4}{\pi}\sum_{\nu=0}^{+\infty} \frac{(-1)^{(k+1)\nu}}{(2\nu+1)^{k+1}} \le \frac{\pi}{2}. \tag{3.2}$$

Here $\Lambda_k$ are the classical Favard constants: $2/\pi \le \Lambda_k \le \pi/2$.

It is convenient to rearrange the right-hand side of (3.2) (see [11]) as

$$E_n(\psi) \le \frac{\pi}{2}\left(\sqrt[k]{\frac{n^k(n-k)!}{n!}}\right)^k \frac{\|\psi^{(k)}\|}{n^k} \le \frac{\pi}{2}\frac{e^k\|\psi^{(k)}\|}{n^k}, \quad n \ge k \ge 0. \tag{3.3}$$

As was mentioned, one of the factors classifying $C^\infty$-smooth functions on the finite segment $I$ is the character of growth of their majorant $G(k)$:

$$\lim_{k\to\infty} \sqrt[k]{G(k)} = \infty. \tag{3.4}$$

Some new method for describing aperiodic $C^\infty$-smooth functions is suggested in [4]. It consists in specifying the pairs of monotone functions $\lambda(r)$ and $\theta(r)$ of a real argument $r \ge 0$ which are constructed effectively from the tuple $\{G(k)\}$ and related to the majorant of $G(k)$ by (3.3). Recall the definitions of these functions.

Take $f \notin \mathscr{P}^n$ with $\|f\| \le G(0)$ and $\|f^{(k)}\| \le G(k)$ and assume (3.4). Associate to the sequence $\{G(k)\}$ the two functions of a real argument $r \ge 0$:

$$\lambda(r) = \begin{cases} G(0), & 0 \le r < 1, \\ \min\limits_{0\le k\le r} \frac{G(k)}{r^k}, & r \ge 1, \end{cases} \qquad \theta(r) = \begin{cases} 0, & 0 \le r < 1, \\ \max\bigl\{k \mid 1 \le k \le r \;\&\; \lambda(r) = \frac{G(k)}{r^k}\bigr\}, & r \ge 1. \end{cases}$$

**Theorem 3** [4]. *For $r \ge 1$ the function $\theta(r)$ is integer-valued, nonnegative, nondecreasing, right-continuous, and tends to infinity together with $r$. The function $\lambda(r)$ is strictly decreasing, right-continuous, and vanishes as $r \to \infty$. The points where $\lambda(r)$ has discontinuities from the left are the discontinuity points $r = r_i$ of $\theta(r)$. Furthermore, for every $r \ge 0$ we have*

$$\lambda(r) = G(0) e^{-\int_1^r \frac{\theta(t)}{t}\,dt} e^{-\sum_{1<r_i\le r} |\sigma_i|}, \quad r \ge 1. \tag{3.5}$$

*Here $\sigma_0 = 0$ and $\sigma_i = \ln\lambda(r_i - 0) - \ln\lambda(r_i)$ for all $i > 0$.*

**Corollary 3.** *The function $\lambda(r)$ vanishes as $r \to \infty$ faster than every power of $r$: for every $p \geq 0$ we have $\lim_{r\to\infty} r^p \lambda(r) = 0$.*

Along with the above, rearrange (3.3) as

$$E_n(f) \leq \frac{\pi}{2}\lambda\left(\frac{n}{e}\right) \quad \text{and} \quad \lambda(x) = \min_{0\leq k\leq x} \frac{G(k)}{x^k} = \frac{G[\theta(x)]}{x^{\theta(x)}}. \tag{3.6}$$

Here the function $\theta(x)$ indicates, with the growth of $x$, the order of decay of the Chebyshev approximation process, while the function $\lambda(x)$ determines the accuracy of the approximation.

The above-mentioned classes of $C^\infty$-smooth functions are nonempty: They include the available Gevrey classes with majorant $H(k) = cA^k k^{\alpha k}$, where $\alpha > 1$, $c > 0$, and $A \geq 1$ are constants.

A closed bounded set in $C^\infty$ is compact [10, 11], i.e., the set

$$K_G^\infty \equiv \{f \in C^\infty : \|f\| \leq G(0),\ \|f^{(k)}\| \leq G(k) \text{ for all } k > 0\} \tag{3.7}$$

with a prescribed sequence $\{G(k)\}$ of reals is a compact subset of $C^\infty$, and so by Arzelà's Theorem $K_G^\infty$ is a compact subset of $C$ as well.

Before deriving estimates for Alexandrov's width $\alpha_n\big(K_G^\infty, C\big)$, let us present a series of results on the expansion of a function in a basis for $C^\infty$. As indicated, the practical benefit of the Fourier expansion of a function $\psi(t)$ in a basis is determined by the number of terms in the partial sum $\psi_N(t)$, and so by some constraints on the class $C^\infty$ itself of a general mathematical nature.

Proceed to stating these constraints by describing the compact set $K_G^\infty$ in terms of the coefficients in the expansions of its elements with respect to the Chebyshev polynomial system $\{T_n(t)\}$; see (3.1).

We have a nontrivial lemma.

**Lemma 4** [11]. *If $\psi$ belongs to the compact set $K_G^\infty$ then*

$$|c_0| \leq G(0), \quad |c_n| \leq \pi\lambda\left(\frac{n}{e}\right) \quad \textit{for all } n > 0.$$

*Conversely, if the coefficients in the expansion of a function $\psi$ in $C^\infty$ satisfy*

$$|c_0| \leq \frac{1}{2}G(0), \quad |c_n| \leq \frac{3}{\pi^2}\frac{\lambda(n^2)}{n^2} \quad \textit{for all } n > 0$$

*then $\psi$ belongs to the compact set $K_G^\infty$.*

An upper bound for Alexandrov's width $\alpha_n\big(K_G^\infty, C\big)$ can be extracted from its definition in (1.1). Indeed, we successively find (see (3.6))

$$\alpha_n(K_G^\infty, C) = \inf_{(X^n,\nu)} \sup_{f\in K_G^\infty\subset C} \|f - \nu(f)\| \leq \sup_{f\in K_G^\infty\subset C} E_n(f) \leq \frac{\pi}{2}\lambda\left(\frac{n}{e}\right). \tag{3.8}$$

To estimate $\alpha_n\big(K_G^\infty, C\big)$ from below, construct the $(n+1)$-dimensional cube $\mathbb{F}^{n+1}$ in $K_G^\infty$. Indeed, take $f$ in $K_G^\infty$ and impose the conditions (see [13])

$$f(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n T_n(t), \quad \|f^{(k)}\| \leq G(k), \quad \lambda(r) = \min_{0\leq k\leq r} \frac{G(k)}{r^k}, \quad \big\|T_r^{(k)}(t)\big\| \leq r^{2k}.$$

If $n \geq r \geq 0$; then, since all derivatives of orders $k > r$ of the functions $\phi_r(t) = \lambda(n^2)(n+1)^{-1}T_r(t)$ vanish identically, we may assume that $k \leq r$ and obtain

$$\begin{aligned}\big\|\phi_r^{(k)}(t)\big\| &= \frac{\lambda(n^2)}{n+1}\big\|T_r^{(k)}(t)\big\| \leq \frac{G(k)\lambda(n^2)}{n+1}\frac{r^{2k}}{G(k)} = \frac{G(k)\lambda(n^2)}{n+1}\left(\frac{G(k)}{r^{2k}}\right)^{-1} \\ &\leq \frac{G(k)\lambda(n^2)}{n+1}\left(\min_{0\leq k\leq r^2}\frac{G(k)}{(r^2)^k}\right)^{-1} \leq \frac{G(k)\lambda(n^2)}{n+1}(\lambda(r^2))^{-1} \leq \frac{G(k)}{n+1}\frac{\lambda(n^2)}{\lambda(r^2)} \leq \frac{G(k)}{n+1}.\end{aligned}$$

The functions $\phi_r(t)$ for $0 \le r \le n$ are linearly independent and lie in $K_G^\infty$. Their linear combination $\varkappa(t) = \sum_{r=0}^n \varpi_r \phi_r(t)$ with $|\varpi_r| \le 1$ also lies in $K_G^\infty$:

$$\|\varkappa^{(k)}(t)\| \le \sum_{r=0}^n |\varpi_r| \big\|\phi_r^{(k)}(t)\big\| \le \sum_{r=0}^n \frac{G(k)}{n+1} \le G(k).$$

Following the strategy of the proof of Theorem 2, define in $K_G^\infty$ the set of functions

$$\mathbb{F}^{n+1} = \left\{\varkappa(t) = \sum_{r=0}^n \varpi_r T_r(t) : |\varpi_r| \le \frac{\lambda(n^2)}{n+1},\ r = 0, 1, \dots, n\right\}, \tag{3.9}$$

where $\varpi_0, \varpi_1, \dots, \varpi_n$ are reals.

Put $|\boldsymbol{\varpi}|_\infty = \max(|\varpi_0|/2, |\varpi_1|, \dots, |\varpi_n|)$.

The cube $\mathbb{J}^{n+1} = \left\{\boldsymbol{\varpi} \in \mathbb{R}^{n+1} : |\varpi_r| \le \frac{\lambda(n^2)}{n+1},\ r = 0, 1, \dots, n\right\}$ of topological dimension $n+1$ and side $d = 2\frac{\lambda(n^2)}{n+1}$ embeds into $K_G^\infty$ linearly and homeomorphically with nondecreasing distances; thus, isometrically (see [3, Chapter 3, Section 7, Proposition 3]) and its image is the set in (3.9) because

$$|\boldsymbol{\varpi}|_\infty \le \sqrt{2}\left\|\frac{\varpi_0}{2} + \sum_{r=1}^n \varpi_r T_r(t)\right\|. \tag{$**$}$$

Estimate $(**)$ follows from Bessel's inequality for orthogonal series with the weight

$$|\boldsymbol{\varpi}|_\infty^2 \equiv (\max\{|\varpi_0|/2, ||\varpi_1|, \dots, |\varpi_n|\})^2 \le (\max\{|\varpi_0|/\sqrt{2}, |\varpi_1|, \dots, |\varpi_n|\})^2$$
$$\le \frac{\varpi_0^2}{2} + \sum_{r=1}^n \varpi_r^2 \le \frac{2}{\pi}\int_{-1}^{1} \frac{\varkappa^2(t)}{\sqrt{1-t^2}}\, dt \le 2 \max_{t\in[-1,1]} |\varkappa(t)|^2 = 2\|\varkappa\|^2.$$

Using the argument above and Lemma 1, we arrive at the lower bound

$$\alpha_n\big(K_G^\infty, C\big) \ge \alpha_n(\mathbb{F}^{n+1}, C) \ge \frac{1}{\sqrt{2}}\alpha_n\big(\mathbb{J}^{n+1}, l_\infty^{n+1}\big) = \frac{1}{\sqrt{2}}\frac{\lambda(n^2)}{n+1}, \quad n \ge 0. \tag{3.10}$$

**Theorem 4.** *Estimates for Alexandrov's width $\alpha_n(K_G^\infty, C)$ of the compact set $K_G^\infty$ of aperiodic $C^\infty$-smooth functions embedded boundedly into the space $C$ of continuous functions on the segment $I = [-1, 1]$ are of the form (see (3.8), (3.10), and (3.5)):*

$$\frac{1}{\sqrt{2}}\frac{1}{n+1}\lambda(n^2) \le \alpha_n\big(K_G^\infty, C\big) \le \frac{\pi}{2}\lambda\left(\frac{n}{e}\right) \quad \text{for integer } n \ge 0.$$

**Corollary 4.** *The Gevrey classes with majorant $H(k) = cA^k k^{\alpha k}$, where $\alpha > 1$, $c > 0$, and $A \ge 1$, admit the following estimates for their widths (see [4] for the details)*

$$\frac{c}{\sqrt{2}}\frac{1}{n+1}e^{-\varrho\sqrt[\alpha]{n^2}} \le \alpha_n\big(K_H^\infty, C\big) \le \pi\frac{c}{2}\beta e^{-\varrho\sqrt[\alpha]{n/e}} \quad \text{for integer } n \ge 0.$$

*Here $\varrho = \alpha e^{-1}/\sqrt[\alpha]{A}$ and $\beta = \exp\left(\alpha e\sqrt[\alpha]{A}/2\right)$ are real constants.*

Remark. Being rather general, Theorem 4 also applies to the case of compact sets of analytic functions, and its proof avoids analyticity.

The compact set $K_H^\infty \subset C^\infty$ consisting of Gevrey functions is interesting because under the assumption $\sum_{n=0}^\infty \frac{1}{\sigma_n} < \infty$, where $\sigma_n = \inf_{k\ge n} \sqrt[k]{H(k)}$, it consists of completely supported functions [14].

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### CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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V. N. Belykh
Sobolev Institute of Mathematics, Novosibirsk, Russia
*E-mail address*: belykh@math.nsc.ru