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THE INJECTIVITY RADIUS AND SHORTEST ARCS OF THE OBLATE ELLIPSOID OF REVOLUTION

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Abstract—We found the geodesics, shortest arcs, cut loci, and injectivity radius of any oblate ellipsoid of revolution in three-dimensional Euclidean space.

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1. Introduction

Corollary 4.14 of [1] states that the injectivity radius $i(M)$ of a compact Riemannian manifold M is equal to

$$i(M) = \min\{t_0, l_0/2\}, \tag{1}$$

where t_0 is the minimum of the first conjugate values along all arclength-parametrized geodesics on M and l_0 is the minimum of lengths of nontrivial geodesic loops on M.

The referee of a preliminary version of the book [2] posed the question of the existence of a compact Riemannian manifold M such that $i(M) = t_0 < l_0/2$.

Theorem 1.5.55 of [2] states that the ellipsoid of revolution

$$x^2 + y^2 + \frac{z^2}{a^2} = 1, \quad a > 0, \tag{2}$$

yields the affirmative answer if $a < 4/(3\pi)$.

The maximum of the Gaussian curvature of such ellipsoid is attained on its equator and equals $1/a^2$. Therefore by the available results of Riemannian geometry, t_0 is equal to πa; i.e., the first conjugate value along the equator. The inequality $l_0/2 > \pi a$ for $0 < a < 4/3\pi$ is obtained in [2] by a simple but sufficiently rough evaluation of l_0 from below through lengths of some inscribed broken lines of the orthogonal projections of geodesics onto the equatorial plane.

The ellipsoid of revolution (2) is *oblate* or *prolate* if $0 < a < 1$ or $a > 1$ respectively.

The main result of this paper is that $i(M) = \pi a < l_0/2$ if M is ellipsoid (2), with $0 < a < 1$, furnished with the induced Riemannian metric from ambient three-dimensional Euclidean space $\mathbb{R}^3$. It is proved in Theorem 2.

We find the shortest arcs of oblate ellipsoids of revolutions in Theorem 3 and the cut loci for all its points in Corollary 6.

It is clear that the above-mentioned ellipsoid is (metrically) similar with the coefficient c to an oblate ellipsoid of revolution with semiaxes $0 < b < c$, where $b/c = a$.

Let us distinguish two papers whose results are connected with Corollary 6.

The authors of [3] proved that the cut locus of every point on each ellipsoid in $\mathbb{R}^3$ is an arc on the curvature line through the antipodal point. They proved also that the conjugate locus of every point on the ellipsoid with three different semiaxes has exactly four cusps which is known as the last geometric statement of Jacobi.

Notice that the (only) curvature lines on an ellipsoid of revolution (distinct from a sphere) are meridians and parallels.

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The authors of [4] consider a surface of revolution M (of class C^∞) in $\mathbb{R}^3$ diffeomorphic to the 2-sphere with the center at zero and poles p and q which are mirror symmetric in a parallel (equator). Their main theorem states that if the Gaussian curvature of M is a nondecreasing (nonincreasing) function on the distance to p along a meridian from p to the equator, then the cut locus C_x for every point $x \in M \setminus \{p, q\}$ is $\{-x\}$ or an arc of the parallel (meridian) passing through $-x$. Furthermore, if $C_x = \{-x\}$ for such point x, then the Gaussian curvature of M is constant.

Let us note that the Gaussian curvature of M at p and q is nonnegative.

Unlike our paper, neither [3] nor [4] indicates the sizes of arcs C_x for the case of an oblate ellipsoid of revolution.

2. Main Results

The ellipsoid of revolution (2) is defined by the parametric equations

$$R(u, \varphi) = (\cos u \cos \varphi, \cos u \sin \varphi, a \sin u), \quad 0 < a, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2\pi. \tag{3}$$

The upper half of the ellipsoid may be set by the equation

$$z = a\sqrt{1 - r^2}, \quad r = \sqrt{x^2 + y^2} = \cos u, \quad 0 \leq r \leq 1. \tag{4}$$

The Gaussian curvature of a surface of revolution of a view $z = z(r)$ is computed as in Section 2.7 from [5]:

$$K(r) = \frac{z'(r) z''(r)}{r(1 + z'^2(r))^2}. \tag{5}$$

For (4) we obtain

$$z'(r) = -ar(1 - r^2)^{-1/2}, \quad z''(r) = -a(1 - r^2)^{-3/2},$$

$$1 + z'^2(r) = (1 + (a^2 - 1)r^2)(1 - r^2)^{-1}, \quad z'(r)z''(r) = a^2 r (1 - r^2)^{-2},$$

$$K(r) = \frac{a^2}{(1 + (a^2 - 1)r^2)^2}, \tag{6}$$

$$a^2 = K(0) \leq K(r) \leq K(1) = \frac{1}{a^2}, \quad 0 < a < 1, \tag{7}$$

$$K(r) \equiv 1, \quad a = 1,$$

$$\frac{1}{a^2} = K(1) \leq K(r) \leq K(0) = a^2, \quad 1 < a. \tag{8}$$

The classical *Clairaut rule* states (see subsection (2.12.101) of [5]) that for every geodesic $\gamma = \gamma(t)$ on arbitrary surface of revolution we have

$$r(t) \cos \psi(t) = \frac{r^2(t) \varphi'(t)}{|\gamma'(t)|} = \text{const} := \mathrm{I}. \tag{9}$$

Here $r(t)$ is the radius of the parallel (on the surface) passing through $\gamma(t)$, while $\psi(t)$ is the angle between the tangent vector of the geodesic and the parallel at the same point, and $\varphi(t)$ is the polar angle around the rotation axis of the surface; we will assume that $\varphi'(t) \geq 0$.

Corollary 1. *Every geodesic $\gamma = \gamma(t)$, with $t \in \mathbb{R}$, on ellipsoid (2) with $|\gamma'(t)| \equiv 1$, $\mathrm{I} := r_0 > 0$ and $\phi'(t) > 0$ is uniquely defined by r_0, the minimal radius of the parallels intersecting the geodesic up to a shift of t. This geodesic coincides with the equator for $r_0 = 1$ but intersects the equator and touches both parallels with the radius r_0 for $r_0 \in (0, 1)$.*

Proof. It is easy to see that all claims of the corollary follows from the Clairaut rule and the fact that the rotations around z-axis and the mirror reflections relative to the meridional and the equatorial planes are isometries of the ellipsoid. □

Definition 1. Let $\gamma = \gamma(t)$, with $t \in \mathbb{R}$, be a arclength-parametrized geodesic on a complete Riemannian manifold (M, g) with the origin $\gamma(0) = p$. A point $\gamma(t)$, with $t \neq 0$, is a conjugate point to p along γ if the differential $(\mathrm{dExp}_p)_{t\gamma'(0)}(\cdot)$ (with Exp_p the exponential mapping at p) is a generate linear mapping. The first conjugate value to p along γ is the infimum of $t > 0$ such that $\gamma(t)$ is a conjugate point to p along γ.

Corollary 2. *For every point on the geodesic of* (2), *the distance along this geodesic to a nearest conjugate* (*with respect to the geodesic*) *point is less than* $\frac{\pi}{a}$ *and is not less than* πa, *if* $0 < a < 1$, *and more than* $\frac{\pi}{a}$ *and is not more than* πa, *if* $a > 1$. *It is equal to* πa *only for one geodesic, the equator. The length of every closed geodesic consisting of two antipodal meridians* (*the double meridian*) *is less than* 2π *and more than* 4 *if* $0 < a < 1$, *and more than* 2π *if* $a > 1$.

Proof. It follows from (6)–(8) that the sectional curvature of the ellipsoid is an increasing positive function on r, attains the maximal value $\frac{1}{a^2}$ on the equator and the minimal value a^2 at poles if $0 < a < 1$, and is a decreasing positive function on r, attains the minimal value $\frac{1}{a^2}$ at the equator and the maximal value a^2 at poles if $a > 1$. This, Corollary 1, and Theorem 1.5.26 in [2] imply the both parts of the first claim.

The last claim is a consequence of the fact that each double meridian is a closed geodesic by the Clairaut rule, passes through the poles for $r_0 = 0$, while one of the double meridians, the ellipse $x^2+\frac{z^2}{a^2} = 1$, is a convex plane curve inscribed into the circle with the unit radius and the length 2π for $0 < a < 1$ and circumscribed around this circle for $a > 1$. Also the length of the double meridian is always more than 4, the doubled length of its orthogonal projection onto the equatorial plane. $\square$

Theorem 1. *Consider a geodesic on ellipsoid* (2) *which is distinct from the equator and double meridians. The difference of two consecutive values of the polar angle* φ *for the intersection of this geodesic with the equator is less than* π *if* $0 < a < 1$, *and is more than* π *if* $a > 1$.

Proof. Assume for simplicity that this geodesic is parametrized by arclength, i.e., $|\gamma'(t)| = 1$. Since $z = a\sqrt{1-r^2}$, it suffices to find the orthogonal projection of the geodesic onto the plane $z = 0$ with polar coordinates (r, ϕ); in other words, to find the function $r = r(\varphi)$ along the geodesic. It follows from (9) that

$$r\cos\psi = r_0 \Rightarrow \cos\psi = \frac{r_0}{r}, \quad \sin\psi = \frac{\sqrt{r^2 - r_0^2}}{r}, \quad \varphi' = \frac{\cos\psi}{r} = \frac{r_0}{r^2}. \tag{10}$$

Moreover,

$$\gamma'(t) = R'_\varphi(u(t),\varphi(t))\varphi'(t) + R'_u(u(t),\varphi(t))u'(t), \quad \langle R'_\varphi, R'_u\rangle \equiv 0,$$

$$\sin^2\psi = |R'_u|^2 u'^2 = (\sin^2 u + a^2\cos^2 u)u'^2 = (1+(a^2-1)r^2)u'^2, \quad u' = \frac{-\sin\psi}{\sqrt{1+(a^2-1)r^2}}, \tag{11}$$

if $u = u(t) > 0$ and $r(t) = \cos u(t)$ increases from r_0 to 1. Then

$$r' = -\sin u \cdot u' = \frac{\sqrt{1-r^2}\sqrt{r^2-r_0^2}}{r\sqrt{1+(a^2-1)r^2}}, \quad \frac{dr}{d\varphi} = \frac{dr}{dt}\frac{dt}{d\varphi} = \frac{r\sqrt{(1-r^2)(r^2-r_0^2)}}{r_0\sqrt{1+(a^2-1)r^2}}.$$

The last expression is more (less) than $\frac{r\sqrt{(1-r^2)(r^2-r_0^2)}}{r_0}$ (i.e. $dr/d\varphi$ for $a = 1$, for the unit sphere) if $0 < a < 1$ ($a > 1$). Since for the unit sphere the difference is equal to π, Theorem 1 is proved. $\square$

Proposition 1. *Let a geodesic* $\gamma = \gamma(t) = R(u(t), \varphi(t))$ *on ellipsoid* (2) *which is distinct from the equator and double meridians start at the equator for* $\varphi_0 = 0$. *Then under increasing of* φ *this geodesic intersects the equator at the first time for* φ_1, *where* $\pi a < \varphi_1 < \pi$ *if* $0 < a < 1$, *and* $\pi < \varphi_1 < \pi a$ *if* $a > 1$.

Proof. The inequality $\varphi_1 < \pi$ for $0 < a < 1$ and the inequality $\varphi_1 > \pi$ for $a > 1$ were proved in Theorem 1. Let $\varphi = \varphi(r)$ and $\theta = \theta(r)$ be the polar angles for the geodesic from the proposition

for $a \neq 1$ and an analogous geodesic for $a = 1$, i.e., on the unit sphere $z = \sqrt{1-r^2}$, with initial data $\varphi(r=1) = \theta(r=1) = 0$. The proof of Theorem 1 for the decreasing of r from 1 to r_0 implies

$$\frac{dr}{d\varphi} = \frac{-r\sqrt{(1-r^2)(r^2-r_0^2)}}{r_0\sqrt{1+(a^2-1)r^2}}, \quad \frac{dr}{d\theta} = \frac{-r\sqrt{(1-r^2)(r^2-r_0^2)}}{r_0}.$$

Therefore we can consider the function $\varphi(\theta) := \varphi(r(\theta))$. Note that

$$\frac{d\varphi}{d\theta} = \frac{dr}{d\theta}\Big/\frac{dr}{d\varphi} = \sqrt{1+(a^2-1)r^2}. \tag{12}$$

The last expression is more (less) than $\sqrt{1+(a^2-1)} = \sqrt{a^2} = a$ if $0 < a < 1$ $(a > 1)$. It is well known that under decreasing of r from 1 to r_0 we get $\theta(r_0) = \pi/2$ and further under increasing of r from r_0 to 1 we get $\theta_1 = \theta(1) = \pi$. Then

$$\varphi_1 = 2\int\limits_0^{\pi/2} \sqrt{1+(a^2-1)r^2(\theta)}d\theta = \int\limits_0^{\pi} \sqrt{1+(a^2-1)r^2(\theta)}d\theta, \tag{13}$$

which, by the above, is more than πa for $0 < a < 1$ and less than πa for $a > 1$. $\square$

Lemma 1. *If* $0 < r_0 < 1$ *and* $0 \leq \theta \leq \pi/2$, *then* $r(\theta) = r_0(1+(r_0^2-1)\cos^2\theta)^{-1/2}$.

Proof. By the Cosine Theorem of spherical geometry for the rectangular spherical triangle with legs θ, $u(\theta)$, and hypotenuse l we obtain

$$\cos l = \cos u(\theta) \cdot \cos\theta = r(\theta)\cos\theta.$$

By the Sinus Theorem of spherical geometry,

$$\sin l = \sqrt{1-r^2(\theta)\cos^2\theta} = \frac{\sin u(\theta)}{\sin\psi_0} = \frac{\sqrt{1-r^2(\theta)}}{\sqrt{1-r_0^2}}.$$

This implies

$$1-r^2(\theta) = (1-r_0^2)(1-r^2(\theta)\cos^2\theta)$$

and then the formula from Lemma 1. $\square$

Corollary 3. $\varphi_1 = \varphi_1(r_0)$, *with* $0 < r_0 < 1$, *may be arbitrary real in indicated intervals from Proposition* 1.

Proof. It follows from Lemma 1 and (13) that $\varphi_1(r_0)$, with $0 \leq r_0 \leq 1$, is a continuous function. Corollary 3 follows from the fact that by Lemma 1, $r(\theta) \equiv 0$ for $r_0 = 0$ and $r(\theta) \equiv 1$ for $r_0 = 1$. $\square$

Using Lemma 1, we arrive by direct computation to

Lemma 2. *If* $0 < r_0 < 1$, $0 \leq \theta \leq \pi/2$, *then*

$$\frac{\partial r(\theta)}{\partial r_0} = \sin^2(\theta)(1+(r_0^2-1)\cos^2\theta)^{-3/2},$$

which is equal to 0 *for* $\theta = 0$ *and positive for* $0 < \theta \leq \pi/2$.

We will put $v = \varphi_1$. It is clear that $v = v(r_0)$.

Lemma 3. *The function $v = v(r_0)$ is continuously differentiable, its derivative $v'(r_0)$ is negative for $0 < a < 1$ and positive for $a > 1$. There exists an inverse continuously differentiable decreasing (increasing) function $r_0 = r_0(v)$, where $a\pi < v < \pi$ for $0 < a < 1$ (and $\pi < v < a\pi$ for $a > 1$).*

Proof. By (13) together with Lemmas 1 and 2, we have

$$v'(r_0) = 2\int_0^{\pi/2} \frac{\partial}{\partial r_0}(\sqrt{1+(a^2-1)r^2(\theta)})d\theta$$

$$= 2\int_0^{\pi/2} (1+(a^2-1)r^2(\theta))^{-1/2}(a^2-1)r(\theta)\frac{\partial r(\theta)}{\partial r_0}d\theta$$

$$= 2\int_0^{\pi/2} (1+(a^2-1)r^2(\theta))^{-1/2}(a^2-1)r(\theta)\sin^2(\theta)(1+(r_0^2-1)\cos^2\theta)^{-3/2}d\theta.$$

This implies all claims of Lemma 3. □

Lemma 4. *Let a segment of an arclength-parametrized geodesic $\gamma = \gamma(t)$ on ellipsoid (2) which is distinct from double meridians and the equator starts and ends at the equator; $\gamma(0) = R(0,0)$ and $\gamma(t(r_0)) = R(0, v(r_0))$, where*

$$0 < \cos\psi_0 = r_0 < 1, \quad \psi_0 = \angle(\gamma'(0), R'_\varphi(0,0)), \quad v(r_0) > 0.$$

Then the length of this segment is equal to

$$l(r_0) = t(r_0) = 2\int_0^{\pi/2} \frac{r_0}{1-(1-r_0^2)\cos^2\theta}\sqrt{1+\frac{(a^2-1)r_0^2}{1-(1-r_0^2)\cos^2\theta}}d\theta. \tag{14}$$

Proof. The rightmost equality in (10), equality (12), and Lemma 1 imply that

$$\frac{dt}{d\theta} = \frac{dt}{d\varphi}\cdot\frac{d\varphi}{d\theta} = \frac{r^2(\theta)}{r_0}\sqrt{1+(a^2-1)r^2(\theta)}$$

$$= \frac{r_0}{1-(1-r_0^2)\cos^2\theta}\sqrt{1+\frac{(a^2-1)r_0^2}{1-(1-r_0^2)\cos^2\theta}}.$$

This yields (14). □

Remark 1. Using (14) and the indefinite integral **1** of Section **5.14** of [6], we get

$$l(r_0) = t(r_0) = 2\int_0^{\pi/2} \frac{r_0 d\theta}{1-(1-r_0^2)\cos^2\theta} = \frac{-2r_0}{\sqrt{r_0^2}}\arctan\left(\sqrt{r_0^2}\cot\theta\right)\Big|_0^{\pi/2}$$

$$= -2\arctan(r_0\cot\theta)|_0^{\pi/2} = 2(\arctan(+\infty) - \arctan(0)) = 2\cdot\frac{\pi}{2} = \pi$$

if $0 < r_0 < 1$ and $a = 1$, as it must be.

Remark 2. Formula (14) could be rewritten shorter:

$$l(r_0) = 2\int_0^{\pi/2} r_0(\sin^2\theta + r_0^2\cos^2\theta)^{\frac{-3}{2}}(\sin^2\theta + (a^2-\sin^2\theta)r_0^2)^{\frac{1}{2}}d\theta. \tag{15}$$

Proposition 2. *The geodesic segments of ellipsoid* (2)

$$\gamma(t, r_0) = \mathrm{Exp}_p\left(t\left(r_0 R'_\varphi(0,0) + \left(\sqrt{1-r_0^2}/a\right) R'_u(0,0)\right)\right), \quad 0 \le t \le l(r_0), \tag{16}$$

where $p = R(0,0)$, $0 < r_0 < 1$, are parametrized by arclength, constitute a family of class C^∞, start and end on the equator.

Proof. First of all, $R'_\varphi(0,0) = (0,1,0)$ and $R'_u(0,0) = (0,0,a)$. This implies that geodesic segments (16) are arclength-parametrized. It is clear that ellipsoid (2) with the metric induced from $\mathbb{R}^3$ is a compact real-analytic Riemannian manifold. Therefore, by Proposition 10.5 from [7] the exponential mapping Exp_p is real analytic. It is not difficult to see from (15) that $l(r_0)$, with $0 < r_0 < 1$, is a function of class C^∞. Consequently, (16) is a family of class C^∞. By the definition of $l(r_0)$, all segments of the family (16) start and end on the equator (while their interiors are located inside the upper half) of ellipsoid (2). □

Corollary 4. *The family $\gamma(t, v) = \gamma(t, r_0(v))$, $0 \le t \le l(r_0(v))$, where $\pi a < v < \pi$ for $0 < a < 1$ and $\pi < v < \pi a$ for $a > 1$, of the geodesic segments defined by means of* (16) *and the function $r_0(v)$ from Lemma 3, is the mapping of class C^∞. Moreover, $l(v) = l(r_0(v))$ is the length of the geodesic segment $\gamma(\cdot, v)$.*

Proof. In consequence of Proposition 2, $v(r_0) = \varphi(\gamma(l(r_0), r_0))$ is a C^∞-function. Then the inverse function $r_0(v)$, defined by Lemma 3 on the corresponding interval, is infinitely differentiable. The second claim is a corollary of the fact that $\gamma(\cdot, v)$ is arclength-parametrized. □

Remark 3. In fact, the function $l(r_0)$, with $0 < r_0 < 1$, and geodesic segments from Proposition 2 and Corollary 4 are real-analytic.

Proposition 3. *We have the derivatives $l'(v) = r_0(v) > 0$ for $0 < a < 1$, $\pi a < v < \pi$ and $l'(v) = r_0(v) > 0$ for $a > 1$ and $\pi < v < \pi a$.*

Proof. Choose an arbitrary $v_0 \in (\pi a, \pi)$ for $0 < a < 1$ and $v_0 \in (\pi, \pi a)$ for $a > 1$. Define some family of arclength-parametrized geodesics $\gamma(t, v)$, with $0 \le t \le l(v_0)$ such that the tangent vector field $X = X(t, v) = \frac{\partial \gamma(t,v)}{\partial t} = \gamma_t(t, v)$ to the geodesic $\gamma(t, v)$ (under fixed v) has the length $|X(t, v)| = |\gamma_t(t, v)| = c(v) = l(v)/l(v_0) > 0$. By Corollary 4, the family $\gamma(t, v)$, the vector field $X(t, v)$, and the function $l(v)$ are infinitely differentiable. It is clear that also the vector field $Y = Y(t, v) = \frac{\partial \gamma(t,v)}{\partial v} = \gamma_v(t, v)$ along the family $\gamma(t, v)$ is infinitely differentiable.

By the formula of the first variation of the curve length ((1.58) from [2]) we obtain

$$l'(v) = \frac{1}{c(v)} \langle Y, X\rangle\Big|_{t=0}^{t=l(v_0)} - \frac{1}{c(v)} \int_0^{l(v_0)} \langle Y(t, v), \nabla_X X(t, v)\rangle dt.$$

Moreover, for all v under consideration, $Y(0, v) = 0$, since $\gamma(0, v) = R(0,0)$ and $\nabla_X X(t, v) = 0$, because $\gamma(\cdot, v)$ are geodesics. Thus,

$$l'(v) = \frac{1}{c(v)} \langle Y(l(v_0), v), X(l(v_0), v)\rangle = \cos\psi_0(v) = r_0(v) > 0$$

for $0 < a < 1$ and $a > 1$. □

Now it is time to remind some notions of Riemannian geometry.

Definition 2. Let (M, g) be a complete Riemannian manifold with metric tensor g. The injectivity radius $i_p(M)$ of (M, g) at $p \in M$ is defined as the supremum of $r > 0$ such that the exponential mapping Exp_p of (M, g) at p is a diffeomorphism on the open ball $U(0, r) \subset (M_p, g_p)$, where (M_p, g_p)

is the tangent Euclidean space to (M,g) at p. By definition, the injectivity radius $i(M)$ of (M,g) is $i(M) = \inf_{p\in M} i_p(M)$.

Definition 3. Let (M,g) be a complete Riemannian manifold, $p \in M$. By definition, $t_0(p)$ is the supremum of $r > 0$ such that the differential $(\mathrm{dExp}_p)_v(\cdot)$ is a nondegenerate linear mapping for every $v \in U(0,r) \subset (M_p, g_p)$, while $t_0 = \inf_{p\in M} t_0(p)$.

Remark 4. $t_0(p) > 0$, and $t_0 > 0$ if M is compact.

Theorem 2. *Let $0 < a < 1$. Then the length of every geodesic loop on ellipsoid* (2) *is more than $2\pi a$, and its injectivity radius is equal to πa.*

Proof. By (7), the Gaussian curvature of ellipsoid (2) attains the maximum $1/a^2$ only at $r = 1$. Therefore by Theorem 1.5.26 from [2] the first conjugate value attains the minimum πa on the geodesic ray $R(0, \varphi(t) = t)$, with $t \geq 0$, and every equatorial segment $R(0,t)$, with $0 \leq t \leq v$, where $v > \pi a$, is not the shortest arc by Proposition 1.5.29 from [2].

In the proof of Corollary 3, it was arranged that the decreasing by Lemma 3 the function $v = v(r_0)$, with $0 < r_0 < 1$, is continuous on $[0,1]$ and $v(r_0 = 0) = \pi$, $v(r_0 = 1) = \pi a$. Consequently, so we have the inverse decreasing continuous function $r_0(v)$, with $v \in [\pi a, \pi]$; moreover, $r_0(\pi a) = 1$, $r_0(\pi) = 0$.

It follows from the above, Proposition 2, and Corollary 4 that $l(v) = d(R(0,0), R(0,v))$ for $v \in (\pi a, \pi)$, where d is the intrinsic metric on ellipsoid (2). Then

$$d(R(0,0), R(0,\pi a)) = l(\pi a) = \pi a, \quad d(R(0,0), R(0,\pi)) = l(\pi) < \pi,$$

because d is continuous. Notice that here $l(\pi)$ is the length of a meridian (the half of a double meridian). Then by Proposition 3

$$\pi a < l(v) < l(\pi) < \pi, \quad \pi a < v < \pi,$$

the minimal length of geodesic loop on ellipsoid (2), $l_0 > 2l(v) > 2\pi a$ and by (1), its injectivity radius is equal to $\pi a < l_0/2$. $\square$

We will compute the meridian length l of ellipsoid (2). If $\varphi = 0$ in (3), then

$$l = 2\int_0^{\pi/2} \sqrt{(x'_u)^2 + (z'_u)^2}du = 2\int_0^{\pi/2} \sqrt{\sin^2 u + a^2\cos^2 u}du$$

$$= 2\int_0^{\pi/2} \sqrt{1 + (a^2-1)\cos^2 u}du = 2\int_0^{\pi/2} \sqrt{1 + (a^2-1)r^2(u)}du.$$

If $a = 1$ then $l = \pi$. Let us calculate l for $a > 0$ and $a \neq 1$.

Let $a > 1$. Then, taking into account the last two equalities,

$$l = 2\int_0^{\pi/2} \sqrt{1 + (a^2-1)(1-\sin^2 u)}du = 2\int_0^{\pi/2} \sqrt{a^2 - (a^2-1)\sin^2 u}du$$

$$= 2a\int_0^{\pi/2} \sqrt{1 - (1-1/a^2)\sin^2 u}du = 2a\int_0^{\pi/2} \sqrt{1 - k^2\sin^2 u}du = 2aE(\pi/2, k),$$

where $k^2 = (1 - 1/a^2) = (a^2-1)/a^2$, $E(u,k)$ is the *normal elliptic Legendre integral of the second kind* (see (21.6–30) in [8]), and $2aE(\pi/2,k) := 2aE(k)$, while

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)\frac{k^{2n}}{1-2n} \tag{17}$$

is the *full elliptic Legendre integral of the second kind* ((21.6–33) in [8]), with $0! = 0^0 := 1$.

Let $0 < a < 1$. Then

$$l = 2\int_0^{\pi/2} \sqrt{1+(a^2-1)\cos^2 u}du = 2\int_0^{\pi/2} \sqrt{1+(a^2-1)\sin^2(\pi/2-u)}du$$

$$= -2\int_{\pi/2}^{0} \sqrt{1+(a^2-1)\sin^2(\pi/2-u)}d(\pi/2-u) = 2\int_0^{\pi/2} \sqrt{1-(1-a^2)\sin^2\theta}d\theta$$

$$= 2\int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}d\theta = 2E(\pi/2,k) = 2E(k),$$

where $k^2 = 1-a^2$.

3. Shortest Arcs of the Oblate Ellipsoid of Revolution

Theorem 3. *For ellipsoid* (2), *with* $0 < a < 1$, *the following hold:*

(i) *Every segment of a geodesic distinct from equator and located in the upper or the lower part of the ellipsoid, i.e., for* $[0 \le u \le \pi/2]$ *or for* $[-\pi/2 \le u \le 0]$ *in equation* (3), *is a shortest arc. Moreover, the segment is the unique shortest arc with given ends if and only if at least one of the segment ends does not lie on the equator.*

(ii) *The equatorial segment is a* (*unique*) *shortest arc* (*with given ends*) *if and only if its length is not more than* πa.

(iii) *The segment of a double meridian is a shortest arc if and only if its length is not more than* $2E(k)$, $k^2 = 1-a^2$, (17); *it is unique shortest arc with given ends if and only if its length is less than* $2E(k)$.

(iv) *A geodesic segment with the length* l *and ends* $p_1 = R(u_1,\varphi_1)$ *and* $p_2 = R(u_2,\varphi_2)$:

$$-\pi/2 < u_1 < 0 < u_2 < \pi/2, \quad 0 < |\varphi_1 - \varphi_2| = \omega < \pi; \tag{18}$$

is a shortest arc if and only if the geodesic corresponds to some parameter r_0, *where*

$$0 < r_0 \le \min(\cos u_1, \cos u_2), \quad \omega \le v(r_0), \quad l \le l(r_0). \tag{19}$$

Moreover, the following are equivalent:

$$v(r_0) = \omega, \quad l = l(r_0). \tag{20}$$

This segment is the unique shortest arc with given ends if and only if one of the following is satisfied:

$$\cos u_1 \ne \cos u_2, \quad \omega < v(r_0), \quad l < l(r_0), \quad r_0 = \cos u_1 = \cos u_2. \tag{21}$$

Proof. (i): Using Proposition 1, Corollary 3, Lemma 3, Corollary 4, and the invariance of the ellipsoid under rotations around z-axis and the reflection relative to equatorial plane, every two points on the equator which are the ends of a segment of the equator with the length v, where $\pi a < v \le \pi$, can be joined only by two nonequatorial geodesic segments of equal length $l(v)$ which are mirror symmetric to each other relative to the equatorial plane. This implies (i).

(ii): This is a corollary of the second phrase from the proof of Theorem 2.

(iii): This follows from $l(v=\pi) = 2E(k)$ and (17).

(iv): Applying, if need be, the mirror reflection in the plane containing the meridian with $\varphi = \frac{\varphi_1+\varphi_2}{2}$, we can assume that $\varphi_1 < \varphi_2$.

There exists at least one shortest arc joining p_1 and p_2; let $\gamma = \gamma(t)$, $0 \leq t \leq l$, be one of these shortest arcs which is parametrized by arclength. Notice that by conditions (18), p_1 and p_2 do not lie on the equator and cannot lie simultaneously on the same double meridian. Therefore the geodesic segment γ corresponds to some parameter r_0, where $0 < r_0 \leq \min(\cos u_1, \cos u_2)$.

Moreover, $\omega \leq v(r_0)$. Otherwise $\omega > v(r_0)$ and $l(r_0) < l$. Next,

a) $r_0 < \min(\cos u_1, \cos u_2)$ or b) $r_0 = \min(\cos u_1, \cos u_2)$.

In case a), let us consider the shortest arc

$$\gamma_1 = \gamma(t) = R(u(t), \varphi(t)), \quad 0 \leq t \leq l(r_0). \tag{22}$$

Then $\cos(u(l(r_0))) = \cos(u(0))$ and for some $t_1 \in (0, l(r_0))$,

α) $r_0 = \cos(u(t_1))$, $u(t_1) < 0$ or β) $r_0 = \cos(u(t_1))$, $u(t_1) > 0$.

In case α), $u(2t_1) = u_1$, and the formula $\gamma_2(t) = R(\tilde{u}(t), \tilde{\varphi}(t))$, where

$$(\tilde{u}(t), \tilde{\varphi}(t))) = (u(t + 2t_1), \varphi(t + 2t_1) + \varphi_1 - \varphi(2t_1)), \quad 0 \leq t \leq l(r_0) - 2t_1;$$

$$(\tilde{u}(t), \tilde{\varphi}(t)) = (-u(t - l(r_0) + 2t_1), \varphi(l(r_0)) + \varphi(t - l(r_0) + 2t_1) - \varphi(2t_1)),$$

if $l(r_0) - 2t_1 \leq t \leq l(r_0))$; define another continuous curve with the same length $l(r_0)$ and the same ends $\gamma(0)$ and $\gamma(l(r_0))$, as for γ_1, i.e., a shortest arc. Then the geodesic segment γ cannot be a shortest arc.

Considerations in case β) are analogous.

Let condition b) is fulfilled. Applying, if need be, the composition of the mirror reflections in the equatorial plane and some vertical plane, including z-axis, we can suppose that $r_0 = \cos u_1$. Let $0 < \delta < \min(l(r_0)/2, l - l(r_0))$. Then the geodesic segment

$$\gamma_3(t) = \gamma(t + \delta) = R(u_3(t), \varphi_3(t)), \quad 0 \leq t \leq l(r_0),$$

is a shortest arc and

$$-\pi/2 < u_3(0) < 0 < u_3(l(r_0)) < \pi/2, \quad r_0 < \cos u_3(0) = \cos u_3(l(r_0)).$$

This means that this shortest arc satisfies condition a). We proved earlier that this is impossible. Thus, $\omega \leq v(r_0)$ and $l \leq l(v(r_0)) = l(r_0)$.

It is clear that under conditions (19) the two equalities of (20) are equivalent.

Let us prove that under conditions (20) a geodesic segment with the parameter r_0, $0 < r_0 < 1$, of the view (22), which joins p_1 and p_2, is a shortest arc.

Notice at first that then $\cos u_1 = \cos u_2$ and it is easy to deduce from the above that there exist at most two such geodesic segments with the parameter r_0, and only one if and only if $r_0 = \cos u_1 = \cos u_2$.

Assume that p_1 and p_2 can be joined by a shortest arc $\gamma_4 = \gamma_4(t) = R(u_4(t), \varphi_4(t))$, with $0 \leq t \leq l_4$, parametrized by arclength, where $l_4 \leq l(r_0)$, with the parameter $r_4 \neq r_0$ and $0 < r_4 < 1$. By the above, it must be $v(r_0) = \omega \leq v(r_4)$ and by Lemma 3, $v(r_0) = \omega < v(r_4)$, $0 < r_4 < r_0$. By Proposition 3, $l(r_0) = l(v(r_0)) < l(v(r_4)) = l(r_4)$ and $l_4 < l(r_4)$.

Arguing as above, we can suppose that $u_1'(0) \geq 0$ and $u_4'(0) \geq 0$. In consequence of this, inequalities $0 < r_4 < r_0 < 1$, and the Clairaut rule, $u_4'(0) > u_1'(0) \geq 0$. Then by (i) the geodesic segments γ_1 and γ_4 intersect only at p_1 and p_2, and elsewhere γ_4 is located above γ_1. Hence, since $u_1'(l(r_0)) \leq 0$, by the Clairaut rule it must be $u_4'(l_4) < u_1'(l(r_0)) \leq 0$. Thus, there exists $\bar{t}$ such that $0 < \bar{t} < l_4$, $u_4(\bar{t}) = u_2$, and $r(\gamma((\bar{t} + l_4)/2)) = r_4$. This implies that $u_4(\bar{t}/2) = 0$ and the geodesic segments $\gamma_4(t)$, with $0 \leq t \leq \bar{t}/2$, and $\gamma_4(t)$, $\bar{t}/2 \leq t \leq \bar{t}$, have equal lengths.

Therefore $l_4 = l(r_4)$; a contradiction.

Thus, the above claim is true.

Consequently, a geodesic segment with conditions (19) is a shortest arc as well as the last claim of the theorem. $\square$

Corollary 5. *A geodesic segment with the parameter* r_0 *on the ellipsoid of revolution* (2), *with* $0 < a < 1$, *is a shortest arc if and only if its length is not more than* $l(r_0)$, *where* $l(1) = \pi a$, $l(0) = 2E(k)$, *and* $k^2 = 1 - a^2$.

Definition 4. Let (M, g) be a compact Riemannian manifold, $p \in M$, and $0 \neq w \in M_p$, while $\mu(w)$ is the maximum of positive numbers μ such that $\mathrm{Exp}_p(tw)$, $0 \leq t \leq \mu$, is a shortest arc; $\tilde{C}_p = \{\mu(w)w,\ 0 \neq w \in M_p\}$ is called the *cut locus* in M_p, and $C_p = \mathrm{Exp}_p(\tilde{C}_p)$ is said to be the *cut locus* for p.

Corollary 6. *If* $p = R(\varphi_0, u_0)$ *is a point of ellipsoid* (2), *with* $0 < a < 1$; *then*

$$C_p = \{R(\varphi, -u_0), \quad \varphi \in [\varphi_0 - \pi, \varphi_0 - v(\cos u_0)] \cup [\varphi_0 + v(\cos u_0), \varphi_0 + \pi]\}$$

provided that $u_0 \neq \pm\pi/2$, $v(\cos 0) = \pi a$, *and* $C_p = \{(0, 0, \mp a)\}$, *if* $p = (0, 0, \pm a)$.

Remark 5. Under small changes, Theorem 3, Corollaries 5, 6, and the proofs given here, including proofs for the claims on which they are based, are valid also for the surfaces from [4] if $v'(r_0) < 0$ for nonequatorial parallels and the Gaussian curvature is a nondecreasing function on r_0.

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CONFLICT OF INTEREST

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References

1. Sakai T., *Riemannian Geometry*, Amer. Math. Soc., Providence (1996) (Transl. Math. Monogr.; vol. 149).
2. Berestovskii V. and Nikonorov Yu., *Riemannian Manifolds and Homogeneous Geodesics*, Springer Nature Switzerland AG, Cham (2020) (Springer Monogr. Math.).
3. Itoh J. and Kiyohara K., "The cut loci and the conjugate loci on ellipsoids," Manuscripta Math., vol. 114, 247–264 (2004).
4. Sinclair R. and Tanaka M., "The cut locus of a two sphere of revolution and Toponogov's comparison theorem," Tôhoku Math. J., vol. 59, 379–399 (2007).
5. Toponogov V.A., *Differential Geometry of Curves and Surfaces*, Birkhäuser, Boston, Basel, and Berlin (2006).
6. Brychkov Yu.A., Marichev O.I., and Prudnikov A.P., *Tables of Indefinite Integrals*, Gordon and Breach, Ann Arbor (1989).
7. Helgason S., *Differential Geometry and Symmetric Spaces*, Academic, New York and London (1962).
8. Korn G.A. and Korn T.M., *Mathematical Handbook*, McGraw-Hill, New York, San Francisco, Toronto, London, and Sydney (1968).

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