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KOLMOGOROV EQUATIONS FOR DEGENERATE ORNSTEIN–UHLENBECK OPERATORS

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Abstract—We consider Kolmogorov operators with constant diffusion matrices and linear drifts, i.e., Ornstein–Uhlenbeck operators, and show that all solutions to the corresponding stationary Fokker–Planck–Kolmogorov equations (including signed solutions) are invariant measures for the generated semigroups. This also gives a relatively explicit description of all solutions.

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1. Introduction

In the theory of Fokker–Planck–Kolmogorov equations the simplest case is that of a constant diffusion matrix and a linear drift, i.e., the case of an Ornstein–Uhlenbeck operator. If the diffusion matrix is nondegenerate, then in this case the principal objects of study (solutions to stationary equations and Cauchy problems, semigroups, etc.) admit explicit representations, stationary solutions coincide with invariant probability measures of the semigroups and are unique. However, the situation is more complicated for a degenerate diffusion matrix. Here some of the facts known in the nondegenerate case have remained unclarified, and the goal of this work is their justification. Although the validity of the results established below is rather predictable for experts, the proofs turn out far from obvious.

Let $A = (a^{ij})_{i,j\le d}$ be a nonnegative definite symmetric constant matrix and let b be a smooth vector field on $\mathbb{R}^d$. We recall that for the differential operator (referred as a *Kolmogorov operator*)

$$L_{A,b}f(x) = \operatorname{trace}(AD^2f(x)) + \langle b(x), \nabla f(x)\rangle = a^{ij}\partial_{x_i}\partial_{x_j}f(x) + b^i(x)\partial_{x_i}f(x),$$

where the standard summation rule over repeated indices is meant, the stationary Fokker–Planck–Kolmogorov equation

$$L^*_{A,b}\mu = 0 \qquad (1.1)$$

arises with respect to bounded Borel measures μ on $\mathbb{R}^d$. This equation is understood in the sense of the integral identity

$$\int_{\mathbb{R}^d} L_{A,b}f\,d\mu = 0 \qquad (1.2)$$

for functions f from the space $C_0^\infty(\mathbb{R}^d)$ of smooth functions with bounded support, i.e., the equality

$$\partial_{x_i}\partial_{x_j}(a^{ij}\mu) - \partial_{x_i}(b^i\mu) = 0$$

in the sense of distributions. If A is nondegenerate, then μ is given by a smooth density ϱ with respect to the Lebesgue measure and this density satisfies the equation $\partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho) = 0$ in the usual sense.

If the mapping b is linear, i.e., $b(x) = Bx$, where B is a linear operator; then we write $L_{A,B}$ in place of $L_{A,b}$. Such operators are called *Ornstein–Uhlenbeck operators*. Many papers devote to the operators and address highly nontrivial relevant questions (see, for example, [1–4], where additional references can be found).

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The case of a nondegenerate matrix A reduces to the case $A = I$ by a linear change of variable. Of particular interest is the situation where there is a probability solution μ, i.e., $\mu \geq 0$ and $\mu(\mathbb{R}^d) = 1$. Even for $A = I$ and a smooth mapping b it remains unknown whether the existence of a nonzero signed solution always implies the existence of a probability solution. It is shown below that the answer is positive for a linear coefficient b and an arbitrary matrix A.

If a probability solution exists, then it can be nonunique even for $A = I$ and a smooth nonlinear mapping b (see [5, 6] and [7, Chapter 4]). If b is linear and A is nondegenerate, then a probability solution is unique (if existent); see [7, Theorem 4.1.6]. Moreover, there are no signed solutions (see [7, § 4.3]). This is false in case of a degenerate matrix A. For example, for $A = 0$ and $b = 0$ all measures are solutions. If $d = 2$, $A = 0$, and b is the rotation by $\pi/2$, i.e., $b(x) = (-x_2, x_1)$, then every rotationally invariant measure satisfies the equation. Indeed, $\mathrm{div}(b\mu) = b^1\partial_{x_1}\mu + b^2\partial_{x_2}\mu = 0$ if μ has a smooth density depending on $|x|^2$. Hence this equality is valid for the convolutions $\mu * p_\varepsilon$, where $p_\varepsilon = (2\pi\varepsilon)^{-1}\exp(-|x|^2/(2\varepsilon^2))$. Letting $\varepsilon \to 0$ we obtain the equality for μ. It follows from the sequel that in this case all solutions consist of rotationally invariant measures.

In the case of a general smooth mapping b and a nondegenerate matrix A the existence of a probability solution μ implies the existence of a strongly continuous semigroup of operators T_t on $L^1(\mu)$ with the properties: (a) the generator of the semigroup is an extension of $L_{A,b}$ on $C_0^\infty(\mathbb{R}^d)$; (b) $\|T_t\| \leq 1$; (c) the operators T_t are sub-Markov, i.e., $0 \leq T_t f \leq 1$ whenever $0 \leq f \leq 1$. It is unknown whether the semigroup with the listed properties is unique; however, if we omit (b) and (c), then there is no uniqueness in case of nonuniqueness of a probability solution to (1.1) (see [7, Chapter 5]). The so-called *canonical semigroup* $\{T_t^\mu\}_{t\geq 0}$, associated with $L_{A,b}$, is determined by the property that for its generator $L_{A,b}^\mu$ the operator $(I - L_{A,b}^\mu)^{-1}$ is defined as follows: $(I - L_{A,b}^\mu)^{-1}f$ for $f \in C_0^\infty(\mathbb{R}^d)$ is the limit of the solutions to the Dirichlet problems $u - L_{A,b}u = f$ with zero boundary conditions on the balls of radius n centered at the origin. In the case of a linear drift there is the explicitly defined transition semigroup $\{T_t\}_{t\geq 0}$ (see (1.5) below).

Appearance of the canonical semigroup leads to the question of the invariance of μ with respect to the semigroup, i.e., the validity of the identity

$$\int_{\mathbb{R}^d} T_t^\mu f \, d\mu = \int_{\mathbb{R}^d} f \, d\mu$$

for all functions f from the space $C_b(\mathbb{R}^d)$ of bounded continuous functions (in our case this is equivalent to the equality for all $f \in C_0^\infty(\mathbb{R}^d)$). However, such invariance holds only in case of uniqueness of a probability solution to (1.1). We will show that for a linear drift coefficient all solutions to (1.1) (including signed solutions) are always invariant under the transition semigroup $\{T_t\}_{t\geq 0}$.

The Kolmogorov operator generates the Cauchy problem for the evolution Fokker–Planck–Kolmogorov equation

$$\partial_t \mu_t = L_{A,b}^* \mu_t, \quad \mu_0 = \nu$$

on the interval $[0, T]$ with a given $T > 0$ and a given initial condition ν which is a bounded measure. A solution is a family of bounded measures μ_t (possibly, signed), Borel measurably depending on t, i.e., the functions $t \mapsto \mu_t(E)$ must be Borel for all Borel sets E, for which the total variation $\|\mu_t\|$ of μ_t is integrable in t on $[0, T]$ and

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t - \int_{\mathbb{R}^d} \varphi \, d\nu = \int_0^t \int_{\mathbb{R}^d} L_{A,b}\varphi \, d\mu_s \, ds \tag{1.3}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. A probability solution is a solution for which μ_t are probabilistic.

With the Kolmogorov operator $L_{A,b}$ we also associate the stochastic differential equation

$$d\xi_t = \sqrt{2A}\, dw_t + b(\xi_t)dt$$

with a Wiener process w_t, which in the case of a linear drift b has the linear form

$$d\xi_t = \sqrt{2A}\, dw_t + B\xi_t dt. \tag{1.4}$$

We recall (see, for example, [8, § 8.1] and [9, § 5.1]) that for arbitrary linear operators S and B and a Wiener process w_t in $\mathbb{R}^d$ the stochastic differential equation

$$d\xi_t = S\, dw_t + B\xi_t dt, \quad \xi_0 = x$$

with a nonrandom initial value x has the unique solution

$$\xi_{t,x} = e^{tB}x + \int_0^t e^{(t-s)B} S\, dw_s,$$

where the vector stochastic integral is a Gaussian vector with zero mean and covariance operator

$$K_t = \int_0^t e^{sB} SS^* e^{sB^*}\, ds.$$

Denote by G_t the centered Gaussian measure with this covariance operator which is the distribution of the indicated stochastic integral at time t. The transition semigroup of the process is defined on $C_b(\mathbb{R}^d)$ by the formula

$$T_t f(x) = \mathrm{E}(f(\xi_{t,x})) = \int_{\mathbb{R}^d} f(e^{tB}x + y)\, G_t(dy). \tag{1.5}$$

This semigroup is contracting on $C_b(\mathbb{R}^d)$ but not strongly continuous (see [7, Example 5.1.2]). The symmetric nonnegative operator $A = SS^*/2$ defines the differential operator $L_{A,B}$, for which for all $f \in C_0^\infty(\mathbb{R}^d)$ we have

$$\partial_t T_t f = L_{A,B} T_t f = T_t L_{A,B} f.$$

The semigroup $\{T_t\}_{t\geq 0}$ generates the semigroup $\{T_t^*\}_{t\geq 0}$ on the space of bounded measures by the formula

$$\int_{\mathbb{R}^d} f d(T_t^*\nu) = \int_{\mathbb{R}^d} T_t f\, d\nu.$$

In other words, in terms of convolution

$$T_t^*\nu = (\nu \circ S_t^{-1}) * G_t, \quad S_t x = e^{tB},$$

where $\nu \circ S_t^{-1}$ is the image of the measure ν under the operator S_t. The family of measures $T_t^*\nu$ gives a solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation

$$\partial_t \mu_t = L_{A,B}^* \mu_t, \quad \mu_0 = \nu.$$

It is known (see [10] and [7, § 9.8(iii)]) that for every probability measure ν this Cauchy problem has the unique solution consisting of probability measures, and in Remark 3.2 below we will explain uniqueness in the class of signed solutions.

For a nondegenerate operator A, in case of existence of a probability solution μ to the equation $L_{A,B}^*\mu = 0$, the canonical semigroup $\{T_t^\mu\}_{t\geq 0}$ coincides on $C_b(\mathbb{R}^d)$ with the explicitly defined semigroup $\{T_t\}_{t\geq 0}$. Indeed, it is known for a long time (see [11]) that the Cauchy problem for the direct equation

$$\partial_t u(x,t) = L_{A,B} u(x,t), \quad u(x,0) = u_0(x)$$

with a nondegenerate operator A has a unique solution in $C_b(\mathbb{R}^d)$ for every initial function from this space (uniqueness holds even in the much larger class of functions with the estimate $|u(x,t)| \le C\exp(k|x|^2)$). The coincidence of the semigroups also follows from the uniqueness of solutions to the Cauchy problem for the Fokker–Planck–Kolmogorov equation, because both semigroups generate adjoint semigroups on the space of bounded measures giving solutions to this Cauchy problem. The adjoint semigroup to the canonical semigroup acts on measures by the formula

$$(T_t^{\mu})^*\nu(E) = \int\limits_{\mathbb{R}^d} T_t^{\mu} I_E \, d\nu,$$

where we use a continuous version of the function $T_t^{\mu} I_E$ (which exists by the properties of canonical semigroups; see [7, Theorem 5.4.5]). The action of T_t^* was defined above. Hence the adjoint semigroups are equal, which is equivalent to the equality of these semigroups.

In the general case (for a nonlinear mapping b) it is not true that a solution to the stationary equation $L_{A,b}^*\mu = 0$ is invariant under $\{T_t\}_{t\ge 0}$, but in the case under consideration both properties are equivalent, which is one of the main results of this paper.

Some full description of all stationary probability measures of the process $\xi_{x,t}$ was obtained in [12]. The general form of stationary distributions of the process is as follows: $\sigma * G$, where G is the centered Gaussian measure with covariance operator

$$K = \int\limits_0^{\infty} e^{sB} SS^* e^{sB^*} \, ds,$$

which equals the limit of K_t as $t \to \infty$ in the case of existence of stationary distributions, σ is a probability measure invariant for the process with the zero diffusion, i.e., for the deterministic dynamical system $x'(t) = Bx(t)$, in other words, a probability measure invariant under all operators e^{tB}. This is equivalent to the identity $\widetilde{\sigma}(y) = \widetilde{\sigma}(e^{tB^*}y)$ for its Fourier transform. According to [12], the existence of stationary distributions is equivalent to the existence of a nonnegative symmetric operator Q satisfying

$$BQ + QB^* = -2A = -SS^*. \tag{1.6}$$

Equivalent algebraic conditions are given in [12, 13]. The operator S in the above-stated results need not be symmetric, but the operator $A = SS^*/2$ arising in (1.1) is always symmetric and nonnegative definite, hence for our purposes we can assume that $S = \sqrt{2A}$.

2. Main Results

Below we consider the equation $L_{A,B}^*\mu = 0$ in the case where $A = (a^{ij})_{i,j\le d}$ and $B = (b^{ij})_{i,j\le d}$ are constant linear operators on $\mathbb{R}^d$ and A is symmetric and nonnegative definite.

The main results of this work are as follows:

Theorem 2.1. *A bounded Borel measure μ on $\mathbb{R}^d$ (possibly, signed) satisfies the equation $L_{A,B}^*\mu = 0$ precisely when μ is invariant under the semigroup $\{T_t\}_{t\ge 0}$ on $C_b(\mathbb{R}^d)$.*

Corollary 2.2. *Every probability solution to the equation $L_{A,B}^*\mu = 0$ has the form*

$$\mu = \sigma * G, \tag{2.1}$$

where G is the centered Gaussian measure with covariance operator

$$K = 2\int\limits_0^{\infty} e^{sB} A e^{sB^*} \, ds$$

and σ is a probability measure invariant under all operators e^{tB}; moreover, all measures of this form satisfy the given equation.

If A is nondegenerate; then, as already noted, a probability solution to the equation $L^*_{A,B}\mu = 0$ is unique (up to a constant factor this is the unique solution in the class of all bounded measures); hence, this is the measure G. A necessary and sufficient condition for the existence and uniqueness of a probability solution is the negativity of the real parts of all eigenvalues of B; see [12] and a more general result for a nonconstant matrix A in [14].

Corollary 2.3. *A signed Borel measure μ satisfies the equation $L^*_{A,B}\mu = 0$ precisely when μ has the form $\mu = c_1\mu_1 - c_2\mu_2$, where c_1 and c_2 are nonnegative constants and μ_1 and μ_2 are probability measures satisfying the same equation and hence having the form* (2.1).

Corollary 2.4. *The existence of a nonzero solution* (*then also of a probability solution*) *to the equation $L^*_{A,B}\mu = 0$ is equivalent to the existence of a nonnegative symmetric operator Q satisfying* (1.6).

It is clear from the above that the total variation $|\mu|$ of a solution to the equation $L^*_{A,B}\mu = 0$ is a solution too. Note that if the equation $L^*_{I,b}\mu = 0$ has two different probability solutions μ_1 and μ_2, then $|\mu_1 - \mu_2|$ cannot be a solution, since μ_1 and μ_2 have smooth densities ϱ_1 and ϱ_2; hence, $|\mu_1 - \mu_2|$ is given by the continuous density $|\varrho_1 - \varrho_2|$ and this density has no zeros (see [7, § 3.4]), which is impossible, because the function $\varrho_1 - \varrho_2$ has the zero integral over the whole space.

The proofs are given in the next section.

3. Auxiliary Results and Proofs

We start with a simple direct justification of a particular case of Corollary 2.4; namely, we show that the existence of a probability solution with finite second moment is equivalent to (1.6). Suppose that μ is a probability solution to (1.1) with finite second moment; i.e., there is a nonnegative symmetric operator $Q = (Q^{jk})_{j,k\le d}$ for which

$$\langle Qu, v\rangle = \int_{\mathbb{R}^d} \langle u, x\rangle\langle v, x\rangle\, \mu(dx), \quad u, v \in \mathbb{R}^d.$$

Then, as is readily seen (see the proof of the theorem), in the defining identity (1.2) we can insert $\varphi(x) = \langle Cx, x\rangle$, where C is a symmetric operator, which by the equality

$$L_{A,B}\varphi(x) = 2\,\mathrm{trace}(AC) + 2\langle Bx, Cx\rangle$$

yields the relationships

$$\mathrm{trace}(AC) = -\int_{\mathbb{R}^d} \langle Bx, Cx\rangle\mu(dx) = -\frac{1}{2}\,\mathrm{trace}[Q(CB + B^*C)] - \frac{1}{2}\,\mathrm{trace}[(BQ + QB^*)C],$$

implying (1.6). Thus, the existence of a nonnegative definite operator Q satisfying this equality is a necessary condition for the existence of a probability solution with finite second moment. Conversely, if such operator exists, then the symmetric Gaussian measure σ_Q with covariance operator Q satisfies (1.1). Indeed, for a Gaussian measure, it suffices to verify (1.2) for the functions of the form $\varphi(x) = \exp(i\langle x, y\rangle)$, since this implies the equality for all functions φ that are periodic extensions of smooth functions with support in cubes, whence the equality is readily derived for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. For functions of the indicated form we have

$$L_{A,B}\varphi(x) = -\langle Ay, y\rangle \exp(i\langle x, y\rangle) + i\exp(i\langle x, y\rangle)\langle Bx, y\rangle,$$

where the integral of the first term equals $-\langle Ay, y\rangle\widetilde{\sigma_T}(y)$. Hence it remains to calculate

$$i\int_{\mathbb{R}^d} \exp(i\langle x, y\rangle)\langle Bx, y\rangle\, \sigma_Q(dx),$$

which equals

$$i\int_{\mathbb{R}^d} \exp(i\langle x,y\rangle)\langle x,B^*y\rangle\,\sigma_Q(dx) = \frac{d}{dt}\Big|_{t=0}\int_{\mathbb{R}^d} \exp(i\langle x,y+tB^*y\rangle)\,\sigma_Q(dx)$$

$$= \frac{d}{dt}\Big|_{t=0}\exp\Big(-\frac{1}{2}\langle Q(y+tB^*y),y+tB^*y\rangle\Big) = -\frac{1}{2}\widetilde{\sigma_T}(y)\big[\langle Qy,B^*y\rangle + \langle QB^*y,y\rangle\big].$$

By (1.6) the sum of the integrals of the terms in this expression for $L_{A,B}\varphi(x)$ equals zero.

If B is negative definite, i.e., $\langle Bx,x\rangle \le -\alpha|x|^2$, where $\alpha>0$, then there exists a unique probability solution, and, moreover, it has all moments. In the general case this is not true, for example, as it was mentioned earlier, if $d=2$, $A=0$, and B is the operator of rotation by $\pi/2$, then all rotationally invariant measures are solutions, so that there are solutions without moments. Corollary 2.4 asserts the equivalence of (1.6) and the existence of a nonzero and possibly signed solution, with respect to which $|x|^2$ need not be integrable. In that case the simple reasoning above does not work and another justification of (1.6) is needed. It is as follows: We will show that although not all solutions have second moment, the existence of some nonzero solution yields a Gaussian solution.

We will need the obvious assertion: Let $C_\infty(\mathbb{R}^d)$ denote the space of continuous functions on $\mathbb{R}^d$ with zero limit at infinity which is equipped with the sup-norm and let $\mathscr{S}(\mathbb{R}^d)$ denote the standard space of smooth rapidly decreasing functions.

Lemma 3.1. (i) *Let γ be a Gaussian measure on $\mathbb{R}^d$, let Λ and T be linear operators on $\mathbb{R}^d$ such that Λ is invertible, and let f be a Borel function on $\mathbb{R}^d$ such that $|f(x)|\le C(1+|x|)^{-k}$ for some $C,k>0$. Then*

$$g(x) = \int_{\mathbb{R}^d} f(\Lambda x + Ty)\,\gamma(dy)$$

satisfies the estimate

$$|g(x)| \le C_1(1+|x|)^{-k},\quad C_1 = C2^{k+1}(1+\|\Lambda^{-1}\|)^k(1+I_k),\quad I_k = \int_{\mathbb{R}^d} |Ty|^k\,\gamma(dy).$$

If $f\in C_\infty(\mathbb{R}^d)$ then $g\in C_\infty(\mathbb{R}^d)$. If $f\in\mathscr{S}(\mathbb{R}^d)$ then $g\in\mathscr{S}(\mathbb{R}^d)$.

(ii) *If we are given a family of operators Λ_t on $\mathbb{R}^d$ with the uniformly bounded inverse operators Λ_t^{-1}, then for the corresponding functions g_t the number C_1 can be chosen common. In particular, if $\Lambda_t = e^{tB}$, where B is a linear operator and $t\in[0,T]$, then for every $f\in\mathscr{S}(\mathbb{R}^d)$ the function $g_t(x)$ on $\mathbb{R}^d\times[0,T]$ is twice continuously differentiable and its first and second derivatives in x are estimated by functions of the form $C_m(1+|x|)^{-m}$ for all $m>0$.*

Proof. (i) Let $|Ty|\le|\Lambda x|/2$. Then $|\Lambda x+Ty|\ge|\Lambda x|/2$ and $|f(\Lambda x+Ty)|\le C(1+|\Lambda x|/2)^{-k}$. Therefore,

$$|g(x)| \le C(1+|\Lambda x|/2)^{-k} + C\gamma(y\colon |Ty|\ge|\Lambda x|/2).$$

The first term is not greater than $C2^k(1+|\Lambda x|)^{-k}$. The second term for $|\Lambda x|\ge 1$ is estimated by $C2^k|\Lambda x|^{-k}I_k \le C2^{k+1}I_k(1+|\Lambda x|)^{-k}$ by the Chebyshev inequality. For $|\Lambda x|\le 1$ the second term does not exceed $C2^k(1+|\Lambda x|)^{-k}$. Thus,

$$|g(x)| \le C2^{k+1}(1+I_k)(1+|\Lambda x|)^{-k}.$$

Owing to the inequality $|x|\le\|\Lambda^{-1}\||\Lambda x|$, we obtain $\frac{1}{1+|\Lambda x|}\le(1+\|\Lambda^{-1}\|)\frac{1}{1+|x|}$, which gives the announced estimate.

Similarly we obtain $g\in C_\infty(\mathbb{R}^d)$ if $f\in C_\infty(\mathbb{R}^d)$. If $f\in\mathscr{S}(\mathbb{R}^d)$, then

$$\partial_{x_i} g(x) = \int_{\mathbb{R}^d} \langle\nabla f(\Lambda x+Ty), Se_i\rangle\,\gamma(dy),$$

where $\{e_i\}$ is the standard basis in $\mathbb{R}^d$; whence by induction we conclude that $g \in \mathscr{S}(\mathbb{R}^d)$. Assertion (ii) is obvious from the proof. $\square$

PROOF OF THEOREM 2.1. Let μ be a bounded measure on $\mathbb{R}^d$ with $L^*_{A,B}\mu = 0$ and let $\varphi \in C_0^\infty(\mathbb{R}^d)$. By (1.5) and Lemma 3.1, applied to the invertible operator $\Lambda = e^{tB}$, we have $\psi = T_t\varphi \in \mathscr{S}(\mathbb{R}^d)$. Let us verify that

$$\int_{\mathbb{R}^d} L_{A,B}\psi \, d\mu = 0.$$

Take $\eta \in C_0^\infty(\mathbb{R}^d)$ equal to 1 on the unit ball. Set $\eta_n(x) = \eta(x/n)$. The integrals of $L_{A,B}(\eta_n\psi)$ against the measure μ vanish. Moreover,

$$L_{A,B}(\eta_n\psi) = \eta_n L_{A,B}\psi + \psi L_{A,B}\eta_n + 2\langle A\nabla\psi, \nabla\eta_n\rangle.$$

The function $L_{A,B}\psi$ is bounded and the integral of $\eta_n L_{A,B}\psi$ against μ tends to the integral of $L_{A,B}\psi$. The functions $\psi L_{A,B}\eta_n$ converge pointwise to zero and are uniformly bounded, since $\psi \in \mathscr{S}(\mathbb{R}^d)$. Hence the integral of $\psi L_{A,B}\eta_n$ against μ converges to zero. The same is true for the uniformly bounded functions $2\langle A\nabla\psi, \nabla\eta_n\rangle$. Thus, the integral of $L_{A,B}T_t\varphi$ against μ is zero. It follows that the integral of $T_t\varphi$ is constant, because $\partial_t(T_t\varphi) = L_{A,B}T_t\varphi$ and $L_{A,B}T_t\varphi = T_tL_{A,B}\varphi$ is uniformly bounded due to the compactness of the support of φ. Since φ in $C_0^\infty(\mathbb{R}^d)$ was arbitrary, we can conclude that the measures μ and $T_t^*\mu$ are equal. It is seen from this reasoning that $T_t^*\mu = \mu$ implies $L^*_{A,B}\mu = 0$. $\square$

PROOF OF COROLLARY 2.2. Let μ be a probability measure satisfying the equation $L^*_{A,B}\mu = 0$ and let G_t be the centered Gaussian measure with covariance operator

$$K_t = 2\int_0^t e^{sB}Ae^{sB^*}\, ds.$$

By the theorem the integrals of the functions $\exp(i\langle x, y\rangle)$ against $T_t^*\mu$ do not depend on t; i.e., for the Fourier transform of the measure μ we have the identity

$$\begin{aligned}\widetilde{\mu}(y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp(i\langle e^{tB}x + u, y\rangle)\, G_t(du)\,\mu(dx) &= \int_{\mathbb{R}^d} \exp(i\langle e^{tB}x, y\rangle)\widetilde{G_t}(y)\,\mu(dx)\\ &= \widetilde{G_t}(y)\widetilde{\mu}(e^{tB^*}y).\end{aligned}$$

Let us show that K_t converge to some nonnegative symmetric operator K as $t \to +\infty$. It suffices to verify that the increasing nonnegative quadratic forms $\langle K_ty, y\rangle$ have a finite limit for every $y \in \mathbb{R}^d$. Suppose that for some y the limit is infinite. Then $\widetilde{G_t}(y) = \exp\bigl(-\langle K_ty, y\rangle/2\bigr) \to -\infty$. Since $|\widetilde{\mu}(e^{tB^*}y)| \le 1$, we conclude from the above identity that $\widetilde{\mu}(y) = 0$. This also remains true for all vectors αy with $\alpha > 0$. Thus, $\widetilde{\mu}(\alpha y) = 0$; whence, letting $\alpha \to 0$, we obtain $\widetilde{\mu}(0) = 0$, which is a contradiction, since $\widetilde{\mu}(0) = 1$.

Denote by G the centered Gaussian measure with covariance operator K. It follows from the above that the functions $\widetilde{\mu}(e^{tB^*}y)$, i.e., the Fourier transforms of $\mu \circ S_t^{-1}$, converge pointwise to the continuous function $\widetilde{\mu}(y)\exp\bigl(\langle Ky, y\rangle/2\bigr)$. Hence, by Bochner's Theorem, this function is the Fourier transform of some probability measure ν (see [15, Theorem 7.13.1]). Thus, $\widetilde{\mu}(y) = \widetilde{\nu}(y)\widetilde{G}(y)$, whence $\mu = \nu * G$. The measure ν is a solution to the equation with the zero diffusion matrix, and so $\widetilde{\nu}(e^{tB^*}y) = \widetilde{\nu}(y)$ for all $y \in \mathbb{R}^d$ and $t \ge 0$. Indeed,

$$\widetilde{\nu}(e^{tB^*}y) = \frac{\widetilde{\mu}(e^{tB^*}y)}{\widetilde{G}(e^{tB^*}y)} = \frac{\widetilde{\mu}(e^{tB^*}y)\widetilde{G_t}(y)}{\widetilde{G}(e^{tB^*}y)\widetilde{G_t}(y)} = \frac{\widetilde{\mu}(y)}{\widetilde{G}(y)} = \widetilde{\nu}(y),$$

because

$$\widetilde{G}(e^{tB^*}y)\widetilde{G_t}(y) = \exp\Bigl(-\int\limits_0^\infty \langle e^{sB}Ae^{sB^*}e^{tB^*}y, e^{tB^*}y\rangle\,ds - \int\limits_0^t \langle e^{sB}Ae^{sB^*}y, y\rangle\,ds\Bigr)$$
$$= \exp\Bigl(-\int\limits_0^\infty \langle e^{sB}Ae^{sB^*}y, y\rangle\,ds\Bigr)$$

by the equality

$$\int\limits_0^\infty \langle e^{sB}Ae^{sB^*}e^{tB^*}y, e^{tB^*}y\rangle\,ds = \int\limits_0^\infty \langle e^{(s+t)B}Ae^{(s+t)B^*}y, y\rangle\,ds$$
$$= \int\limits_t^\infty \langle e^{sB}Ae^{sB^*}y, y\rangle\,ds,$$

obtained by the change of variable. □

Proof of Corollary 2.3. According to the theorem the measure μ is invariant under the semigroup $\{T_t\}_{t\geq 0}$, hence its positive and negative parts are also invariant (see [7, Lemma 5.1.4(iii)]). Therefore, they are proportional to some probability measures of the above-indicated form. □

Proof of Corollary 2.4. According to the above, the existence of a nonzero solution to (1.1) implies the existence of a Gaussian solution, which by virtue of the argument at the beginning of this section gives (1.6). Of course, here we could simply refer to [12], since it is shown that this Gaussian solution is invariant for the semigroup, but the given justification proves (1.6) directly. □

Remark 3.2. For every measure ν on $\mathbb{R}^d$ the Cauchy problem (1.3) on the interval $[0,1]$ has the unique solution in the class of measures of bounded variation on $\mathbb{R}^d \times [0,1]$ having the form $\mu(dx\,dt) = \mu_t(dx)\,dt$. This follows from a more general result on uniqueness of signed solutions which is proved in our separate paper, but in the particular case under consideration is proved simpler. To this end we observe that for $\nu = 0$ by [7, Proposition 6.1.2] for every continuous function ψ on $\mathbb{R}^d \times [0,1]$ with two continuous derivatives that vanishes for all x outside of some ball, we have

$$\int\limits_{\mathbb{R}^d} \psi(x,t)\,\mu_t(dx) = \int\limits_0^t \int\limits_{\mathbb{R}^d} [\partial_s\psi(x,s) + L_{A,B}\psi(x,s)]\,\mu_s(dx)\,ds.$$

This equality remains valid also for twice continuously differentiable functions ψ for which the condition of boundedness of the support in x is replaced by the estimates by $C_2(1+|x|)^{-2}$ for the function itself and the first and second derivatives in x. Indeed, as above, with the aid of these estimates in the equality for the functions $\eta_n(x)\psi(x,t)$ we can pass to the limit as $n\to\infty$ and use that the integrals with the terms $\psi(x,t)L_{A,B}\eta_n(x)$ and $\langle A\nabla_x\psi, \nabla\eta_n\rangle$ converge to zero. In particular, by the lemma we can take for ψ the function $\psi(x,t) = T_{\tau-t}\varphi(x)$, where $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $\tau\in[0,1]$. Then the right-hand side equals zero; hence, the integral of $\psi(x,\tau) = \varphi(x)$ against μ_τ converges to zero, which gives the equality $\mu_\tau = 0$, because τ is arbitrary.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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