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ON THE APPROXIMATIVE PROPERTIES OF FOURIER SERIES IN LAGUERRE–SOBOLEV POLYNOMIALS

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Abstract—Considering the approximation of a function f from a Sobolev space by the partial sums of Fourier series in a system of Sobolev orthogonal polynomials generated by classical Laguerre polynomials, we obtain an estimate for the convergence rate of the partial sums to f.

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1. Introduction

Assume that $\alpha > -1$, $r \in \mathbb{N}$, and $1 \le p < \infty$. Let $\rho(x) = e^{-x}x^\alpha$ be a weight function, and let L^p_ρ be the space of measurable functions f on $[0, \infty)$ such that

$$\|f\|_{L^p_\rho} = \left(\int_0^\infty |f(x)|^p \rho(x)\,dx \right)^{\frac{1}{p}} < \infty.$$

Assume further that $W^r_{L^p_\rho}$ is the space of all $(r-1)$ times continuously differentiable functions f with $f^{(r-1)}$ absolutely continuous at an arbitrary segment $[a, b] \subset [0, \infty)$ and $f^{(r)} \in L^p_\rho$. Let W^r stand for f in $W^r_{L^2_\rho}$ for which $|f^{(r)}(x)|e^{-\frac{x}{2}} \le 1$. Define the Sobolev-type inner product on $W^r_{L^2_\rho}$ as follows:

$$\langle f, g\rangle_S = \sum_{\nu=0}^{r-1} f^{(\nu)}(0)g^{(\nu)}(0) + \int_0^\infty f^{(r)}(x)g^{(r)}(x)\rho(x)\,dx. \tag{1}$$

In [1], there was introduced the system of polynomials

$$l^\alpha_{r,r+n}(x) = \frac{1}{(r-1)!\sqrt{h^\alpha_n}} \int_0^x (x-t)^{r-1} L^\alpha_n(t)\,dt, \quad n = 0, 1, \dots,$$

$$l^\alpha_{r,n}(x) = \frac{x^n}{n!}, \quad n = 0, 1, \dots, r-1,$$

orthonormal for $\alpha > -1$ with respect to (1) and generated by the Laguerre polynomials $\{L^\alpha_n(x)\}_{n=0}^\infty$. It was shown in [2] that $\{l^\alpha_{r,k}(x)\}_{k=0}^\infty$ is complete in $W^r_{L^2_\rho}$. The Fourier series of $f \in W^r_{L^2_\rho}$ in this system has the form

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(0)\frac{x^k}{k!} + \sum_{k=r}^\infty \widehat{f}^\alpha_{r,k} l^\alpha_{r,k}(x), \tag{2}$$

where

$$\widehat{f}^\alpha_{r,k} = \frac{1}{\sqrt{h^\alpha_{k-r}}} \int_0^\infty f^{(r)}(t) L^\alpha_{k-r}(t)\rho(t)\,dt, \quad k = r, r+1, \dots. \tag{3}$$

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Also in [2], the following was proved:

Theorem A. *Assume that $-1 < \alpha < 1$, $f \in W^r_{L^2_\rho}$, and $0 \le A < \infty$. Then, for every $x \in [0,\infty)$, we have*

$$f(x) = \sum_{k=0}^{r-1} f^{(k)}(0)\frac{x^k}{k!} + \sum_{k=r}^{\infty} \widehat{f}^{\alpha}_{r,k} l^{\alpha}_{r,k}(x),$$

where the Fourier series of f in $l^{\alpha}_{r,k}(x)$ converges uniformly in $x \in [0, A]$.

Theorem A was generalized with respect to p in [3].

Theorem B. *Assume that $-1 < \alpha < 1$. If $f \in W^r_{L^p_\rho}$ then for $p \ge 2$ series* (2) *converges uniformly to f on every segment $[0, A]$. If $1 \le p < 2$ then there exists $f \in W^r_{L^p_\rho}$ whose Fourier series diverges at $x = \pi^2$.*

In case $\alpha = 0$. The polynomials $l^0_{r,r+n}(x)$ satisfy the equality [2, Corollary 3.1]

$$l^0_{r,r+n}(x) = \frac{x^r L^r_n(x)}{(n+r)^{[r]}},$$

where $(n+r)^{[r]} = (n+r)(n+r-1)\cdots(n+1)$. Then (2) takes the form

$$f(x) \sim \sum_{\nu=0}^{r-1} f^{(\nu)}(0)\frac{x^k}{k!} + x^r \sum_{k=0}^{\infty} \frac{\widehat{f}^{0}_{r,k+r}}{(k+r)^{[r]}} L^r_k(x).$$

Denote the partial sum of this series by $S_{r,n+r}(f,x)$:

$$S_{r,n+r}(f,x) = \sum_{\nu=0}^{r-1} f^{(\nu)}(0)\frac{x^k}{k!} + x^r \sum_{k=0}^{n} \frac{\widehat{f}^{0}_{r,k+r}}{(k+r)^{[r]}} L^r_k(x). \tag{4}$$

It follows from (4) that $S_{n+r}(f,x)$ satisfies

$$(S_{r,n+r}(f,0))^{(\nu)} = f^{(\nu)}(0), \quad 0 \le \nu \le r-1.$$

Moreover, if $f(x) = p_{n+r}(x)$ is an algebraic polynomial of degree $n+r$ then

$$S_{r,n+r}(p_{n+r},x) \equiv p_{n+r}(x). \tag{5}$$

Using (5), the authors of [2] studied the approximation properties of the partial sums $S_{r,n+r}(f,x)$. In particular, it was proved (see [2, Theorem 5.2]) for $f \in W^r_{L^2_\omega}$ and $\omega(x) = e^{-x}$ that

$$e^{-x/2}x^{-r/2+1/4}|f(x) - S_{r,n+r}(f,x)| \le (1 + \lambda_{r,n}(x))E^r_{n+r}(f),$$

where

$$\lambda_{r,n}(x) \le c(r)\begin{cases} \ln(n+1), & x \in [0,\kappa/2];\\ \ln(n+1) + (x/(\kappa^{1/3} + |x-\kappa|))^{1/4}, & x \in [\kappa/2, 3\kappa/2];\\ n^{-r/2+5/4}e^{-x/4}, & x \in [3\kappa/2,\infty). \end{cases}$$

Here $c(r)$ is a positive constant depending only on r, $\kappa = 4n + 2r + 2$. Note that $E^r_{n+r}(f)$ is defined as

$$E^r_{n+r}(f) = \inf_{q_{n+r}} \sup_{x>0} |q_{n+r}(x) - f(x)|e^{-x/2}x^{-r/2+1/4},$$

where the infimum is taken over all algebraic polynomials q_{n+r} of degree $n+r$ for which $f^{(\nu)}(0) = q^{(\nu)}_{n+r}(0)$, with $\nu = \overline{0, r-1}$.

The above estimates for $|f(x) - S_{r,n+r}(f,x)|$ contain the best approximation $E^r_{n+r}(f)$, whose behavior has not been studied yet. In the present article, for $r = 1$, we obtain the convergence rate of $S_{1,n+1}(f,x)$ to $f(x)$ not containing the best approximation $E^r_{n+r}(f)$. More exactly, we have

Theorem 1. *Assume that $f \in W^1$ and $x \in [0,\infty)$. Then*

$$\frac{x^{-\frac{1}{4}}e^{-\frac{x}{2}}}{\sqrt{x+1}}|f(x) - S_{1,n+1}(f,x)| \le c\frac{\ln(n+1)}{n^{\frac{1}{4}}},$$

where c is a positive constant.

The proof of Theorem 1 is given in Section 4.

Analogous problems about approximating the functions of a Sobolev space by algebraic polynomials were addressed by various authors (see [4–7] and the bibliography therein). We will recall some results of [4–7]. The space

$$W^r_{L^p_w} = W^r_{L^p_w}[-1,1] = \{f \in C^{r-1}[-1,1] : f^{(r)} \in L^p_w\},$$

with $w(x) = w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ and $1 \le p < \infty$, was considered in [4] with the norm

$$\|f\|_{W^r_{L^p_w}} = \left(\sum_{k=0}^{r} \|f^{(k)}\|^p_{L^p_w}\right)^{1/p}.$$

The following theorems were proved for $W^r_{L^p_w}$:

Theorem C. *Assume that $\alpha,\beta > -1$. Assume further that $f \in W^r_{L^p_w}$ for $1 \le p < \infty$ or $f \in C^r[-1,1]$ for $p = \infty$. Then there exists a polynomial p_n such that*

$$\|f - p_n\|_{W^r_{L^p_w}} \le cE_n(f^{(r)})_{L^p_w},$$

where $E_n(f^{(r)})_{L^p_w} = \inf_{p\in\Pi_n} \|f^{(r)} - p\|_{L^p_w}$ is the best approximation in the metric of L^p_w.

Theorem D. *Assume that $\alpha,\beta > -1$. Assume further that $f \in W^r_{L^p_w}$ for $1 \le p < \infty$ or $f \in C^r[-1,1]$ for $p = \infty$. Then there exists a polynomial p_n such that*

$$\left\|f^{(k)} - p_n^{(k)}\right\|_{L^p_w} \le cn^{-r+k}E_n(f^{(r)})_{L^p_w}$$

provided that either $\alpha = 0$ or $\beta = 0$.

Furthermore, in [6] there was considered the space

$$H^r_\omega = H^r_\omega(a,b) = \{f \in L^2_\omega(a,b) : f^{(m)} \in L^2_\omega(a,b),\ 1 \le m \le r\}$$

with the norm

$$\|f\|_{H^r_\omega} = \left(\sum_{m=0}^{r} \|f^{(m)}\|^2_{L^2_w}\right)^{1/2}.$$

Here $(a,b) = \mathbb{R}$ for $\omega(x) = e^{-x^2}$, while $(a,b) = (0,\infty)$ for $\omega(x) = e^{-x}x^\alpha$ and $\alpha > -1$, and $(a,b) = (-1,1)$ for $\omega(x) = (1-x)^\alpha(1+x)^\beta$, with $\alpha,\beta > -1$. The article [6] studied the approximation of functions in H^r_ω in the metric of L^2_ω by means of the partial sums $\mathscr{S}^N_n f(x)$ of the Fourier series in a system of polynomials $\{q_j(x)\}$ orthogonal with respect to the Sobolev inner product. Namely, the following was proved:

Theorem E. *Let $r \ge N+1$ and let $f \in H^r_\omega$ be such that $f \in L^2_{v_{r-N-1}}$. Then*

$$\left\|f^{(m)} - (\mathscr{S}^N_n f)^{(m)}\right\|_{L^2_\omega} \le c\begin{cases} \frac{(-\lambda_{n-N,0})^{(m-N)/2}}{(-\lambda_{n-r,r-N-1})^{(r-N-1)/2}}E_{n-r}(f^{(r)})_{L^2_{v_{r-N-1}}}, & N \le m \le r, \\ \frac{(-\lambda_{n-N,0})^{N/2}}{(-\lambda_{n-r,r-N-1})^{(r-N-1)/2}}E_{n-r}(f^{(r)})_{L^2_{v_{r-N-1}}}, & 0 \le m \le N-1, \end{cases}$$

for a Hermite weight function; if $\alpha > -1$ then the estimates hold for a Laguerre weight function; and if $\alpha,\beta \ge 0$ then the estimates hold for a Jacobi weight function.

2. Some Information about Laguerre Polynomials

Let α be an arbitrary real. Then the Laguerre polynomials $L_n^\alpha(x)$ satisfy the relations [8]:

• Rodrigues' formula

$$L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \left(x^{n+\alpha} e^{-x}\right)^{(n)};$$

• the orthogonality relation

$$\int_0^\infty L_n^\alpha(x) L_m^\alpha(x) \rho(x)\, dx = h_n^\alpha \delta_{n,m}, \quad \alpha > -1,$$

where $\delta_{n,m}$ is the Kronecker symbol, and $h_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}$;

• the Christoffel–Darboux formula

$$K_n^\alpha(x,t) = \sum_{k=0}^{n} \frac{L_k^\alpha(x) L_k^\alpha(t)}{h_k^\alpha} = \frac{n+1}{h_n^\alpha} \frac{L_n^\alpha(x) L_{n+1}^\alpha(t) - L_{n+1}^\alpha(x) L_n^\alpha(t)}{x-t}; \tag{6}$$

• the recurrent formula

$$L_0^\alpha(x) = 1, \quad L_1^\alpha(x) = -x + \alpha + 1,$$

$$nL_n^\alpha(x) = (-x + 2n + \alpha - 1) L_{n-1}^\alpha(x) - (n + \alpha - 1) L_{n-2}^\alpha(x), \quad n \geq 2; \tag{7}$$

• the equalities

$$nL_n^\alpha(x) = (n+\alpha) L_{n-1}^\alpha(x) - x L_{n-1}^{\alpha+1}(x), \tag{8}$$

$$L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x); \tag{9}$$

• the weighted estimate [9, 10]

$$e^{-\frac{x}{2}} |L_n^\alpha(x)| \leq c(\alpha) A_n^\alpha(x), \quad \alpha > -1. \tag{10}$$

Here and below, c and $c(\alpha)$ are constants depending on the above parameters,

$$A_n^\alpha(x) = \begin{cases} \theta^\alpha, & 0 \leq x \leq \frac{1}{\theta}; \\ \theta^{\frac{\alpha}{2} - \frac{1}{4}} x^{-\frac{\alpha}{2} - \frac{1}{4}}, & \frac{1}{\theta} < x \leq \frac{\theta}{2}; \\ [\theta(\theta^{\frac{1}{3}} + |x - \theta|)]^{-\frac{1}{4}}, & \frac{\theta}{2} < x \leq \frac{3\theta}{2}; \\ e^{-\frac{x}{4}}, & \frac{3\theta}{2} < x, \end{cases}$$

where

$$\theta = \theta_n(\alpha) = 4n + 2\alpha + 2.$$

The orthonormal Laguerre polynomials $l_n^\alpha(x) = \frac{1}{\sqrt{h_n^\alpha}} L_n^\alpha(x)$ satisfy the estimates

$$e^{-\frac{x}{2}} |l_{n+1}^\alpha(x) - l_{n-1}^\alpha(x)| \leq c(\alpha) \begin{cases} \theta^{\frac{\alpha}{2} - 1}, & 0 \leq x \leq \frac{1}{\theta}; \\ \theta^{-\frac{3}{4}} x^{-\frac{\alpha}{2} + \frac{1}{4}}, & \frac{1}{\theta} < x \leq \frac{\theta}{2}; \\ x^{-\frac{\alpha}{2}} \theta^{-\frac{3}{4}} (\theta^{\frac{1}{3}} + |x - \theta|)^{\frac{1}{4}}, & \frac{\theta}{2} < x \leq \frac{3\theta}{2}; \\ e^{-\frac{x}{4}}, & \frac{3\theta}{2} < x. \end{cases} \tag{11}$$

3. Auxiliary Assertions

Put

$$\mathscr{K}_n(x,t) = \sum_{k=0}^{n} L_k^1(x) L_k(t).$$

Lemma 1. *We have*

$$(x-t)\mathscr{K}_n(x,t) = (n+1)\left(L_n^1(x)L_{n+1}(t) - L_{n+1}^1(x)L_n(t)\right) + \sum_{k=0}^{n} L_k(x)L_k(t). \tag{12}$$

Proof. From (7), for $\alpha = 0$, we infer that

$$kL_k(t) + (k-1)L_{k-2}(t) = (2k-1)L_{k-1}(t) - tL_{k-1}(t);$$

and, for $\alpha = 1$,

$$kL_k^1(x) + kL_{k-2}^1(x) = 2kL_{k-1}^1(x) - xL_{k-1}^1(x).$$

Hence,

$$kL_k(t) + (k-1)L_{k-2}(t) = 2kL_{k-1}(t) - tL_{k-1}(t) - L_{k-1}(t), \tag{13}$$

$$kL_k^1(x) + (k-1)L_{k-2}^1(x) = 2kL_{k-1}^1(x) - xL_{k-1}^1(x) - L_{k-2}^1(x). \tag{14}$$

Next, multiply (13) by $L_{k-1}^1(x)$ and (14) by $L_{k-1}(t)$. Then subtract from the first so-obtained equality the second and use (9):

$$\begin{gathered} k\left(L_{k-1}^1(x)L_k(t) - L_k^1(x)L_{k-1}(t)\right) - (k-1)\left(L_{k-2}^1(x)L_{k-1}(t) - L_{k-1}^1(x)L_{k-2}(t)\right) \\ = (x-t)L_{k-1}^1(x)L_{k-1}(t) - L_{k-1}(x)L_{k-1}(t). \end{gathered}$$

Hence,

$$\begin{gathered} (x-t)L_{k-1}^1(x)L_{k-1}(t) = L_{k-1}(x)L_{k-1}(t) \\ +k\left(L_{k-1}^1(x)L_k(t) - L_k^1(x)L_{k-1}(t)\right) - (k-1)\left(L_{k-2}^1(x)L_{k-1}(t) - L_{k-1}^1(x)L_{k-2}(t)\right). \end{gathered}$$

Summing this equality over k from 1 to $n+1$ and putting $L_{-1}^1(x) = L_{-1}(t) = 0$, we obtain (12). $\square$

Lemma 2. *We have*

$$(x-t)\mathscr{K}_n(x,t) = xL_n^2(x)L_n(t) - tL_n^1(x)L_n^1(t) - L_n^1(x)L_n(t) + \sum_{k=0}^{n} L_k(x)L_k(t).$$

Proof. From (8), for $\alpha = 0$,

$$L_{n+1}(t) = L_n(t) - \frac{t}{n+1}L_n^1(t);$$

and, for $\alpha = 1$,

$$L_{n+1}^1(x) = \frac{n+2}{n+1}L_n^1(x) - \frac{x}{n+1}L_n^2(x).$$

Then

$$\begin{gathered} L_n^1(x)L_{n+1}(t) - L_{n+1}^1(x)L_n(t) \\ = L_n^1(x)\left(L_n(t) - \frac{t}{n+1}L_n^1(t)\right) - L_n(t)\left(\frac{n+2}{n+1}L_n^1(x) - \frac{x}{n+1}L_n^2(x)\right) \\ = \frac{x}{n+1}L_n^2(x)L_n(t) - \frac{t}{n+1}L_n^1(x)L_n^1(t) - \frac{1}{n+1}L_n^1(x)L_n(t). \end{gathered}$$

This and Lemma 1 give Lemma 2. $\square$

Lemma 3. *$K_n^0(x,t) = \sum_{k=0}^n L_k(x)L_k(t)$ admits the representation*

$$K_n^0(x,t) = \frac{n+1}{2n+1}L_n(x)L_n(t)$$
$$+\frac{n(n+1)}{2n+1}\frac{1}{t-x}[L_n(t)(L_{n+1}(x)-L_{n-1}(x)) - L_n(x)(L_{n+1}(t)-L_{n-1}(t))]. \tag{15}$$

Proof. Indeed, from (6), for $\alpha = 0$, we have

$$\frac{1}{n+1}K_n^0(x,t) = \frac{1}{t-x}\left(L_{n+1}(x)L_n(t) - L_n(x)L_{n+1}(t)\right),$$
$$\frac{1}{n}K_n^0(x,t) = \frac{1}{t-x}\left(L_n(x)L_{n-1}(t) - L_{n-1}(x)L_n(t)\right) + \frac{1}{n}L_n(x)L_n(t).$$

Hence,

$$\left(\frac{1}{n+1}+\frac{1}{n}\right)K_n^0(x,t) = \frac{1}{n}L_n(x)L_n(t)$$
$$+\frac{1}{t-x}[L_n(t)(L_{n+1}(x)-L_{n-1}(x)) - L_n(x)(L_{n+1}(t)-L_{n-1}(t))].$$

Multiplying both sides of the last equality by $\frac{n(n+1)}{2n+1}$, we get (15). $\square$

Lemma 4. *We have*

$$\sum_{k=0}^{\infty}\left(\frac{x^{\frac{3}{4}}L_k^1(x)}{k+1}\right)^2 = \frac{e^x-1}{\sqrt{x}}, \quad x\in[0,\infty).$$

Proof. Indeed, using the equality [11, p. 623, formula 6]

$$\sum_{k=0}^{\infty}\frac{k!L_k^{\alpha}(x)L_k^{\alpha}(y)}{(\alpha+1)_k(k+1)} = \frac{\alpha}{(xy)^{\alpha}}e^{x+y}\gamma(\alpha,x)\Gamma(\alpha,y), \quad 0<x\le y,$$

where $\gamma(\alpha,x)$ and $\Gamma(\alpha,y)$ are the incomplete gamma-functions defined as

$$\gamma(\alpha,x) = \int_0^x t^{\alpha-1}e^{-t}\,dt, \quad \Gamma(\alpha,x) = \int_x^{\infty} t^{\alpha-1}e^{-t}\,dt,$$

we infer

$$\sum_{k=0}^{\infty}\left(\frac{x^{\frac{3}{4}}L_k^1(x)}{k+1}\right)^2 = x^{\frac{3}{2}}\frac{1}{x^2}e^{2x}\gamma(1,x)\Gamma(1,x)$$
$$= \frac{e^{2x}}{\sqrt{x}}\int_0^x e^{-t}\,dt\cdot\int_x^{\infty}e^{-t}\,dt = \frac{e^{2x}}{\sqrt{x}}(1-e^{-x})e^{-x} = \frac{e^x-1}{\sqrt{x}}. \quad \square$$

We will also need the following lemma that is proved in [12, Lemma 1]:

Lemma 5. *Suppose that $\alpha > -1$, $n \in \mathbb{N}$, and $x\in[0,\infty)$. Then*

$$e^{-x}K_n^{\alpha}(x,x) \le c(\alpha)\begin{cases} n^{-\alpha}, & x\in[\theta_n/2, 3\theta_n/2],\\ n^{1-\alpha}(A_n^{\alpha}(x))^2, & x\in[0,\theta_n/2]\cup[3\theta_n/2,\infty).\end{cases}$$

4. Proof of Theorem 1

Consider

$$I = \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \int\limits_0^\infty e^{-\frac{x+t}{2}} |\mathscr{K}_n(x,t)|\, dt$$

and estimate the behavior of I for $x \in [0,\infty)$.

Lemma 6. *Suppose that*

$$\nu = \nu_n = 4n+2, \quad X_1 = \left[0, \quad \frac{3}{\nu}\right], \quad X_2 = \left[\frac{3}{\nu}, \quad \frac{\nu}{2}\right], \quad X_3 = \left[\frac{\nu}{2}, \quad \frac{3\nu}{2}\right], \quad X_4 = \left[\frac{3\nu}{2}, \infty\right).$$

Then

$$I \le c \begin{cases} \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \nu^{\frac{3}{2}}, & x \in X_1; \\ \frac{\nu^{\frac{3}{4}}}{\sqrt{x+1}} \ln(n+1), & x \in X_2; \\ \nu^{\frac{7}{12}} \ln(n+1), & x \in X_3; \\ \frac{n^{\frac{3}{2}} x^{\frac{3}{4}} e^{-\frac{x}{4}}}{\sqrt{x+1}}, & x \in X_4. \end{cases} \tag{16}$$

Proof. Take $x \in X_1$. Then

$$I = \left(\int\limits_0^{4/\nu} + \int\limits_{4/\nu}^\infty \right) \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} e^{-\frac{x+t}{2}} \left|\mathscr{K}_n(x,t)\right|\, dt = I_1 + I_2.$$

From (10) we obtain

$$I_1 \le c \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \int\limits_0^{4/\nu} \sum_{k=0}^n \nu_k \, dt \le c \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \nu^2 \frac{4}{\nu} = c \frac{x^{\frac{3}{4}} \nu}{\sqrt{x+1}}. \tag{17}$$

For estimating I_2, rewrite it as

$$I_2 = \left(\int\limits_{4/\nu}^{\nu/2} + \int\limits_{\nu/2}^{3\nu/2} + \int\limits_{3\nu/2}^\infty \right) \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} e^{-\frac{x+t}{2}} \left|\mathscr{K}_n(x,t)\right|\, dt = I_2^1 + I_2^2 + I_2^3. \tag{18}$$

Using Lemma 2 for I_2^1, write down the inequality

$$I_2^1 \le I_2^{11} + I_2^{12} + I_2^{13} + I_2^{14}.$$

Estimate I_2^{11}, I_2^{12}, and I_2^{13} with the use of (10):

$$I_2^{11} \le c \frac{x^{\frac{7}{4}}}{\sqrt{x+1}} \nu^2 \int\limits_{4/\nu}^{\nu/2} \frac{\nu^{-\frac{1}{4}} t^{-\frac{1}{4}}}{t-x}\, dt \le c \frac{x^{\frac{7}{4}}}{\sqrt{x+1}} \nu^{\frac{7}{4}} \int\limits_{4/\nu}^{\nu/2} t^{-\frac{5}{4}}\, dt \le c \frac{\nu^2 x^{\frac{7}{4}}}{\sqrt{x+1}},$$

$$I_2^{12} \le c \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \nu \int\limits_{4/\nu}^{\nu/2} \frac{t \nu^{\frac{1}{4}} t^{-\frac{3}{4}}}{t-x}\, dt \le c \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \nu^{\frac{5}{4}} \int\limits_{4/\nu}^{\nu/2} t^{-\frac{3}{4}}\, dt \le c \frac{\nu^{\frac{3}{2}} x^{\frac{3}{4}}}{\sqrt{x+1}},$$

$$I_2^{13} \le c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\nu \int\limits_{4/\nu}^{\nu/2} \frac{\nu^{-\frac{1}{4}} t^{-\frac{1}{4}}}{t-x}\, dt \le c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\nu^{\frac{3}{4}} \int\limits_{4/\nu}^{\nu/2} t^{-\frac{5}{4}}\, dt \le c\frac{\nu x^{\frac{3}{4}}}{\sqrt{x+1}}.$$

For estimating I_2^{14}, apply (15), (10), and (11):

$$I_2^{14} \le c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \left(\int\limits_{4/\nu}^{\nu/2} \frac{\nu^{-\frac{1}{4}} t^{-\frac{1}{4}}}{t-x}\, dt + n\frac{1}{\nu} \int\limits_{4/\nu}^{\nu/2} \frac{\nu^{-\frac{1}{4}} t^{-\frac{1}{4}}}{(t-x)^2}\, dt + n \int\limits_{4/\nu}^{\nu/2} \frac{\nu^{-\frac{3}{4}} t^{\frac{1}{4}}}{(t-x)^2}\, dt \right)$$

$$\le c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \left(\nu^{-\frac{1}{4}} \int\limits_{4/\nu}^{\nu/2} t^{-\frac{5}{4}}\, dt + \nu^{-\frac{1}{4}} \int\limits_{4/\nu}^{\nu/2} t^{-\frac{9}{4}}\, dt + n^{\frac{1}{4}} \int\limits_{4/\nu}^{\nu/2} t^{-\frac{7}{4}}\, dt \right) \le c\frac{\nu x^{\frac{3}{4}}}{\sqrt{x+1}}.$$

Then

$$I_2^1 \le c\frac{\nu^{\frac{3}{2}} x^{\frac{3}{4}}}{\sqrt{x+1}}. \tag{19}$$

The quantities I_2^2 and I_2^3 are estimated likewise; they satisfy the estimates

$$I_2^2 \le c\frac{\nu^{\frac{3}{2}} x^{\frac{3}{4}}}{\sqrt{x+1}}, \quad I_2^3 \le c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\nu^2 e^{-\frac{3\nu}{8}}. \tag{20}$$

From (18)–(20) we infer

$$I_2 \le c\frac{\nu^{\frac{3}{2}} x^{\frac{3}{4}}}{\sqrt{x+1}}.$$

This and (17) yield

$$I \le c\frac{\nu^{\frac{3}{2}} x^{\frac{3}{4}}}{\sqrt{x+1}}, \quad x \in X_1.$$

Now, let $x \in X_2$. Put

$$D_1 = [0, x - \sqrt{x/\nu}], \quad D_2 = (x - \sqrt{x/\nu}, x + \sqrt{x/\nu}), \quad D_3 = [x + \sqrt{x/\nu}, \infty). \tag{21}$$

Then I can be represented as

$$I = \left(\int\limits_{D_1} + \int\limits_{D_2} + \int\limits_{D_3} \right) \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} e^{-\frac{x+t}{2}} \left| \mathscr{K}_n(x,t) \right| dt = J_1 + J_2 + J_3.$$

Estimate J_2. To this end, we use Lemma 5 and obtain

$$J_2 = \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \int\limits_{D_2} e^{-\frac{x+t}{2}} \left| \mathscr{K}_n(x,t) \right| dt$$

$$\le \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} ((n+1)e^{-x} K_n^1(x,x))^{1/2} \int\limits_{D_2} \left(e^{-t} K_n^0(t,t) \right)^{1/2} dt$$

$$\le c\frac{x^{\frac{3}{4}}\sqrt{n+1}}{\sqrt{x+1}} (\nu^{\frac{1}{2}} x^{-\frac{3}{2}})^{1/2} \int\limits_{D_2} (n\nu^{-\frac{1}{2}} t^{-\frac{1}{2}})^{1/2}\, dt$$

$$\leq c\frac{n}{\sqrt{x+1}}x^{-\frac{1}{4}}\sqrt{\frac{x}{\nu}} \leq c\frac{\sqrt{n}x^{\frac{1}{4}}}{\sqrt{x+1}}. \tag{22}$$

Turn to estimating J_1. Assume that

$$D_1^1 = \left[0, \frac{1}{\nu}\right] \quad \text{and} \quad D_1^2 = \left(\frac{1}{\nu}, x - \sqrt{\frac{x}{\nu}}\right].$$

Then

$$J_1 = \left(\int\limits_{D_1^1} + \int\limits_{D_1^2}\right)\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}e^{-\frac{x+t}{2}}|\mathscr{K}_n(x,t)|\,dt = J_1^1 + J_1^2.$$

For estimating J_1^2, use Lemma 2 and write

$$J_1^2 \leq J_1^{21} + J_1^{22} + J_1^{23} + J_1^{24}.$$

Estimate J_1^{21}. From (10) we infer

$$J_1^{21} \leq c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}x\nu^{\frac{3}{4}}x^{-\frac{5}{4}}\int\limits_{D_1^2}\frac{\nu^{-\frac{1}{4}}t^{-\frac{1}{4}}}{x-t}\,dt$$

$$\leq c\frac{x^{\frac{1}{2}}}{\sqrt{x+1}}\nu^{\frac{1}{2}}\int\limits_{D_1^2}\frac{t^{-\frac{1}{4}}}{x-t}\,dt = c\frac{x^{\frac{1}{4}}}{\sqrt{x+1}}\nu^{\frac{1}{2}}\int\limits_{1/\nu x}^{1-\sqrt{1/\nu x}}\frac{y^{-\frac{1}{4}}}{1-y}\,dy$$

$$= c\frac{\sqrt{\nu}x^{\frac{1}{4}}}{\sqrt{x+1}}\left(\int\limits_{1/\nu x}^{1/3}y^{-\frac{1}{4}}\,dy + \int\limits_{1/3}^{1-\sqrt{1/\nu x}}\frac{1}{1-y}\,dy\right) \leq c\frac{\sqrt{\nu}x^{\frac{1}{4}}}{\sqrt{x+1}}\ln\sqrt{\nu x}.$$

The quantities J_1^{22} and J_1^{23} can be estimated likewise:

$$J_1^{22} \leq c\frac{\sqrt{\nu}x^{\frac{1}{4}}}{\sqrt{x+1}}\ln\sqrt{\nu x}, \quad J_1^{23} \leq c\frac{x^{-\frac{1}{4}}}{\sqrt{x+1}}\ln\sqrt{\nu x}.$$

Estimate J_1^{24}. Lemma 3 and estimates (10) and (11) give

$$J_1^{24} \leq c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\left[\nu^{-\frac{1}{2}}x^{-\frac{1}{4}}\int\limits_{D_1^2}\frac{t^{-\frac{1}{4}}}{x-t}\,dt + x^{\frac{1}{4}}\int\limits_{D_1^2}\frac{t^{-\frac{1}{4}}}{(x-t)^2}\,dt + x^{-\frac{1}{4}}\int\limits_{D_1^2}\frac{t^{\frac{1}{4}}}{(x-t)^2}\,dt\right]$$

$$= c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\left[(\nu x)^{-\frac{1}{2}}\int\limits_{1/\nu x}^{1-\sqrt{1/\nu x}}\frac{y^{-\frac{1}{4}}}{1-y}\,dy + \frac{1}{x}\int\limits_{1/\nu x}^{1-\sqrt{1/\nu x}}\frac{y^{-\frac{1}{4}}}{(1-y)^2}\,dy + \frac{1}{x}\int\limits_{1/\nu x}^{1-\sqrt{1/\nu x}}\frac{y^{\frac{1}{4}}}{(1-y)^2}\,dy\right]$$

$$\leq c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\left[(\nu x)^{-\frac{1}{2}}\ln\sqrt{\nu x} + \frac{1}{x}\sqrt{\nu x} + \frac{1}{x}\sqrt{\nu x}\right] \leq c\frac{\sqrt{\nu}x^{\frac{1}{4}}}{\sqrt{x+1}}.$$

The estimates for J_1^{2i} yield

$$J_1^2 \leq c\frac{\sqrt{\nu}x^{\frac{1}{4}}}{\sqrt{x+1}}\ln\sqrt{\nu x}.$$

Arguing by analogy, we can show that

$$J_1^1 \le c\frac{\nu^{-\frac{1}{4}}x^{-\frac{1}{2}}}{\sqrt{x+1}}.$$

Thus,

$$J_1 \le c\frac{\sqrt{\nu}x^{\frac{1}{4}}}{\sqrt{x+1}}\ln\sqrt{\nu x}. \tag{23}$$

Turn to estimating J_3. We put

$$D_3^1 = [x+\sqrt{x/\nu}, \nu/2+\sqrt{x/\nu}], \quad D_3^2 = (\nu/2+\sqrt{x/\nu}, 3\nu/2], \quad D_3^3 = (3\nu/2, \infty).$$

Using these relations, we can write

$$J_3 = \left(\int\limits_{D_3^1} + \int\limits_{D_3^2} + \int\limits_{D_3^3}\right)\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}e^{-\frac{x+t}{2}}|\mathscr{K}_n(x,t)|\,dt = J_3^1 + J_3^2 + J_3^3.$$

Repeating the arguments that led us to the estimate for J_1^2, we can show that

$$J_3^1 \le c\frac{\nu^{\frac{1}{2}}}{\sqrt{x+1}}(x^{\frac{1}{4}}\ln\sqrt{\nu x} + \nu^{\frac{1}{4}}),$$

$$J_3^2 \le c\frac{\nu^{\frac{3}{4}}}{\sqrt{x+1}}\ln(n+1), \quad J_3^3 \le c\frac{x^{\frac{1}{2}}\nu^{\frac{3}{4}}}{\sqrt{x+1}}e^{-\frac{3\nu}{16}}.$$

Hence,

$$J_3 \le c\frac{\nu^{\frac{3}{4}}}{\sqrt{x+1}}\ln(n+1). \tag{24}$$

From (22)–(24) we infer

$$I \le c\frac{\nu^{\frac{3}{4}}}{\sqrt{x+1}}\ln(n+1), \quad x \in X_2.$$

Turn to the case of $x \in X_3$. Using (21), write

$$I = \left(\int\limits_{D_1} + \int\limits_{D_2} + \int\limits_{D_3}\right)\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}e^{-\frac{x+t}{2}}|\mathscr{K}_n(x,t)|\,dt = H_1 + H_2 + H_3.$$

The quantity H_2 satisfies the same estimate as J_2:

$$H_2 \le c\frac{\sqrt{n}x^{\frac{1}{4}}}{\sqrt{x+1}}. \tag{25}$$

For estimating H_3, we partition D_3 into the intervals

$$\mathscr{D}_3^1 = [x+\sqrt{x/\nu}, 3\nu/2+\sqrt{x/\nu}], \quad \mathscr{D}_3^2 = (3\nu/2+\sqrt{x/\nu}, \infty).$$

Then $H_3 = H_3^1 + H_3^2$. Lemma 2 implies that

$$H_3^1 \le H_3^{11} + H_3^{12} + H_3^{13} + H_3^{14}.$$

For estimating H_3^{1i}, with $i = 1, 2, 3, 4$, turn to (10) and (11), which give

$$H_3^{11} \le c\frac{x^{\frac{7}{4}}}{\sqrt{x+1}}\frac{1}{\nu^{\frac{1}{2}}(\nu^{\frac{1}{3}}+|x-\nu|)^{\frac{1}{4}}}\int\limits_{\mathscr{D}_3^1}\frac{dt}{(t-x)(\nu^{\frac{1}{3}}+|t-\nu|)^{\frac{1}{4}}}$$

$$\le c\frac{x^{\frac{7}{4}}}{\sqrt{x+1}\nu^{\frac{1}{2}}\nu^{\frac{1}{6}}}\int\limits_{\mathscr{D}_3^1}\frac{dt}{t-x} \le cn^{\frac{7}{12}}\ln(n+1),$$

$$H_3^{12} \le c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\frac{1}{\nu^{\frac{1}{2}}(\nu^{\frac{1}{3}}+|x-\nu|)^{\frac{1}{4}}}\int\limits_{\mathscr{D}_3^1}\frac{t\,dt}{(t-x)(\nu^{\frac{1}{3}}+|t-\nu|)^{\frac{1}{4}}}$$

$$\le c\frac{x^{\frac{3}{4}}\nu}{\sqrt{x+1}\nu^{\frac{2}{3}}}\int\limits_{\mathscr{D}_3^1}\frac{dt}{t-x} \le cn^{\frac{7}{12}}\ln(n+1),$$

$$H_3^{13} \le c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\frac{1}{\nu^{\frac{2}{3}}}\int\limits_{\mathscr{D}_3^1}\frac{dt}{t-x} \le c\frac{\ln(n+1)}{n^{\frac{5}{12}}},$$

$$H_3^{14} \le \frac{cx^{\frac{3}{4}}}{\sqrt{x+1}}\Bigg[\frac{1}{\nu^{\frac{2}{3}}}\int\limits_{\mathscr{D}_3^1}\frac{dt}{t-x}$$

$$+n\frac{(\nu^{\frac{1}{3}}+|x-\nu|)^{\frac{1}{4}}}{\nu^{\frac{13}{12}}}\int\limits_{\mathscr{D}_3^1}\frac{dt}{(t-x)^2}+\frac{n}{\nu^{\frac{13}{12}}}\int\limits_{\mathscr{D}_3^1}\frac{(\nu^{\frac{1}{3}}+|t-\nu|)^{\frac{1}{4}}}{(t-x)^2}\,dt\Bigg]$$

$$\le c\frac{x^{\frac{3}{4}}}{\sqrt{x+1}}\left[\frac{\ln(n+1)}{\nu^{\frac{2}{3}}}+\nu^{\frac{1}{6}}\sqrt{\frac{\nu}{x}}+\nu^{\frac{1}{6}}\sqrt{\frac{\nu}{x}}\right] \le c\nu^{\frac{5}{12}}.$$

Consequently, $H_3^1 \le cn^{\frac{7}{12}}\ln(n+1)$. The quantity H_3^2 satisfies the same estimate as J_3^3; therefore,

$$H_3 \le cn^{\frac{7}{12}}\ln(n+1). \tag{26}$$

Estimate H_1. Put

$$\mathscr{D}_1^1=[0,1/\nu], \quad \mathscr{D}_1^2=(1/\nu,\nu/2-\sqrt{x/\nu}], \quad \mathscr{D}_1^3=(\nu/2-\sqrt{x/\nu},x-\sqrt{x/\nu}]$$

and write

$$H_1 = H_1^1 + H_1^2 + H_1^3.$$

Repeating the arguments of estimating J_1^1 and J_1^2 almost verbatim, we can show that

$$H_1^1 \le c\frac{1}{\nu^{\frac{13}{12}}}, \quad H_1^2 \le c\nu^{\frac{5}{12}}\ln(n+1), \quad H_1^3 \le c\nu^{\frac{7}{12}}\ln(n+1).$$

Then

$$H_1 \le c\nu^{\frac{7}{12}}\ln(n+1).$$

This and estimates (25) and (26) give

$$I \le c\nu^{\frac{7}{12}}\ln(n+1), \quad x \in X_3.$$

Finally, consider the case when $x \in X_4$. Lemma 5 and (10) yield

$$
\begin{aligned}
I &\le \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \left(e^{-x} \sum_{k=0}^{n} \big(L_k^1(x)\big)^2 \right)^{\frac{1}{2}} \int_0^\infty \left(e^{-t} \sum_{k=0}^{n} (L_k(t))^2 \right)^{\frac{1}{2}} dt \\
&\le \frac{x^{\frac{3}{4}}\sqrt{n}}{\sqrt{x+1}} \big(e^{-x} K_n^1(x,x)\big)^{\frac{1}{2}} \int_0^\infty \big(e^{-t} K_n^0(t,t)\big)^{\frac{1}{2}} dt \\
&\le c\frac{x^{\frac{3}{4}} n}{\sqrt{x+1}} e^{-\frac{x}{4}} \left(\int_0^{1/\nu} + \int_{1/\nu}^{\nu/2} + \int_{\nu/2}^{3\nu/2} + \int_{3\nu/2}^{\infty} \right) A_n(t)\, dt \\
&\le c\frac{x^{\frac{3}{4}} n}{\sqrt{x+1}} e^{-\frac{x}{4}} \left(\frac{1}{\nu} + \nu^{-\frac{1}{4}} \nu^{\frac{3}{4}} + \nu^{-\frac{1}{4}} \nu^{\frac{3}{4}} + c \right) \le c\frac{n^{\frac{3}{2}} x^{\frac{3}{4}} e^{-\frac{x}{4}}}{\sqrt{x+1}}. \quad \square
\end{aligned}
$$

Return to the proof of Theorem 1. From (4), for $r = 1$, we have

$$
S_{1,n+1}(f,x) = f(0) + x \sum_{k=0}^{n} \frac{\widehat{f}^{\,0}_{1,k+1}}{k+1} L_k^1(x). \tag{27}
$$

Reckoning with (3), we obtain

$$
\begin{aligned}
R_n(f,x) &= \frac{x^{-\frac{1}{4}} e^{-\frac{x}{2}}}{\sqrt{x+1}} (f(x) - S_{1,n+1}(f,x)) \\
&= \frac{x^{\frac{3}{4}} e^{-\frac{x}{2}}}{\sqrt{x+1}} \sum_{k=n+1}^{\infty} \frac{L_k^1(x)}{k+1} \int_0^\infty f'(t) e^{-t} L_k(t)\, dt \\
&= \frac{x^{\frac{3}{4}} e^{-\frac{x}{2}}}{\sqrt{x+1}} \int_0^\infty f'(t) e^{-t} \sum_{k=n+1}^{\infty} \frac{L_k^1(x) L_k(t)}{k+1}\, dt.
\end{aligned} \tag{28}
$$

Placing the sum under the integral in (28) can be explained as follows: Let

$$
A_{r,n,m} = x^{\frac{3}{4}} e^{-\frac{x}{2}} \sum_{k=n+1}^{m} \frac{L_k^1(x) L_k(t)}{k+1}, \quad A_{r,n} = \lim_{m\to\infty} A_{r,n,m}.
$$

Consider the difference

$$
\int_0^\infty f'(t) A_{r,n} e^{-t}\, dt - \int_0^\infty f'(t) A_{r,n,m} e^{-t}\, dt
$$

and demonstrate that it vanishes as $m \to \infty$. We have

$$
\int_0^\infty f'(t) A_{r,n} e^{-t}\, dt - \int_0^\infty f'(t) A_{r,n,m} e^{-t}\, dt \le \|f'\|_{L^2_w} \|A_{r,n} - A_{r,n,m}\|_{L^2_w}.
$$

We prove that

$$
\|A_{r,n} - A_{r,n,m}\|_{L^2_w} \to 0 \quad \text{as } m \to \infty.
$$

To this end, show that $A_{r,n,m}$ is a Cauchy sequence:

$$\|A_{r,n,m+p} - A_{r,n,m}\|^2_{L^2_w} = \int\limits_0^\infty \left(x^{\frac{3}{4}} e^{-\frac{x}{2}} \sum_{k=m+1}^{m+p} \frac{L_k^1(x) L_k(t)}{k+1} \right)^2 e^{-t}\, dt$$

$$= e^{-x} \sum_{k=m+1}^{m+p} \left(\frac{x^{\frac{3}{4}} L_k^1(x)}{k+1} \right)^2 < \varepsilon$$

since

$$\sum_{k=0}^{\infty} \left(\frac{x^{\frac{3}{4}} L_k^1(x)}{k+1} \right)^2$$

converges (see Lemma 4).

Next, from (28) we infer

$$|R_n(f,x)| \le \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \int\limits_0^\infty e^{-\frac{x+t}{2}} \left| \sum_{k=n+1}^{\infty} \frac{L_k^1(x) L_k(t)}{k+1} \right| dt$$

$$= \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \int\limits_0^\infty e^{-\frac{x+t}{2}} \lim_{m\to\infty} \left| \sum_{k=n+1}^{m} \frac{L_k^1(x) L_k(t)}{k+1} \right| dt.$$

Use the Abel transform:

$$|R_n(f,x)| \le \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \int\limits_0^\infty e^{-\frac{x+t}{2}} \lim_{m\to\infty} \left| \frac{\mathscr{K}_m(x,t)}{m+1} - \frac{\mathscr{K}_n(x,t)}{n+2} + \sum_{k=n+1}^{m-1} \frac{\mathscr{K}_k(x,t)}{(k+1)(k+2)} \right| dt$$

$$\le \frac{x^{\frac{3}{4}}}{\sqrt{x+1}} \int\limits_0^\infty e^{-\frac{x+t}{2}} \left(\lim_{m\to\infty} \frac{|\mathscr{K}_m(x,t)|}{m+1} + \frac{|\mathscr{K}_n(x,t)|}{n+2} + \sum_{k=n+1}^{\infty} \frac{|\mathscr{K}_k(x,t)|}{(k+1)(k+2)} \right) dt$$

$$= \mathscr{I}_1 + \frac{1}{n+2}\mathscr{I}_2 + \mathscr{I}_3.$$

Fatou's Lemma (see [13, p. 170] and estimates (16) imply that

$$\lim_{m\to\infty} \mathscr{I}_1 = 0.$$

For $\mathscr{I}_2$, we have estimates (16). Now, estimate $\mathscr{I}_3$. From Lemma 6 we have

$$\mathscr{I}_3 \le c \sum_{k=n+1}^{\infty} \frac{k^{\frac{3}{4}} \ln(k+1)}{k^2} = c \sum_{k=n+1}^{\infty} \frac{\ln(k+1)}{k^{\frac{5}{4}}} \le c\frac{\ln(n+1)}{n^{\frac{1}{4}}}.$$

Theorem 1 is proved.

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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