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ON THE LEVI CLASS OF THE QUASIVARIETY OF RIGHT-ORDERABLE GROUPS

A. V. Zenkov

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Abstract—We show that the Levi class of the quasivariety of right-orderable groups strictly includes this quasivariety.

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1. Introduction

Recall that a group G is *right-orderable* if G can be equipped with a linear order $\geq$ stable under right multiplication; i.e., $x \geq y$ implies $xz \geq yz$ for all $x, y, z \in G$.

The definition of right-orderable group G implies that G is torsion-free. It is also well known (see, for example, [1]) that the class $\mathscr{D}_r$ of all right-orderable groups is a quasivariety.

Remark. If in the definition of right-ordered group, we replace the requirement of stability under right multiplication by stability under two-sided multiplication then we obtain the definition of *orderable group*. Of course, every orderable group is right-orderable.

Let G be an arbitrary group and take $x \in G$. Denote the normal closure of x in G by $G_x = (x^G)$. Given an arbitrary class of groups $\mathscr{K}$, define the *Levi class* $L(\mathscr{K})$ of $\mathscr{K}$ as follows: a group G belongs to $L(\mathscr{K})$ if and only if $G_x \in \mathscr{K}$ for every $x \in G$.

The notion of Levi class was introduced in [2] under the influence of Levi's work [3] devoted to the groups with *abelian* normal closures of elements. Morse proved in [4] that the Levi class of each variety is a variety. The same holds for quasivarieties (see [5]). The Levi classes of particular (mainly nilpotent) quasivarieties were investigated by Lodeishchikova in [6–8] and Shakhova in [9, 10].

The definition of Levi class implies directly that (1) the operator L preserves the set-theoretical inclusion; and (2) if a class $\mathscr{K}$ is closed under subgroups then $\mathscr{K} \subseteq L(\mathscr{K})$.

Hence, $\mathscr{D}_r \subseteq L(\mathscr{D}_r)$. The main result states that the last inclusion is in fact *strict*. Namely, there exists a non-right-orderable torsion-free group in which the normal closure of every element is right-orderable.

All needed information from the theory of right-ordered groups can be found in [1]; and from group theory, in [11].

2. The Main Result

Consider the two mappings $a : \mathbb{R}^3 \to \mathbb{R}^3$ and $b : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the rule

$$a(x, y, z) = (x + 1, 1 - y, -z), \quad b(x, y, z) = (-x, y + 1, 1 - z). \tag{1}$$

Denote the group of transformations of $\mathbb{R}^3$ which are generated by these elements by $\mathbb{G}$. This group was distinguished in [12, p. 13]. Let us calculate the action of ab in $\mathbb{R}^3$; i.e.,

$$ab(x, y, z) = a(-x, y + 1, 1 - z) = (1 - x, -y, z - 1).$$

Hence,

$$(ab)^2(x, y, z) = (x, y, z - 2). \tag{2}$$

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The following are checked likewise:

$$(ba)^2(x,y,z) = (x,y,z+2), \tag{3}$$

$$a^2(x,y,z) = (x+2,y,z), \tag{4}$$

$$b^2(x,y,z) = (x,y+2,z), \tag{5}$$

$$(b^2)^a(x,y,z) = (x,y-2,z), \tag{6}$$

$$(a^2)^b(x,y,z) = (x-2,y,z), \tag{7}$$

$$((ab)^2)^a(x,y,z) = (x,y,z+2), \tag{8}$$

$$((ab)^2)^b(x,y,z) = (x,y,z+2). \tag{9}$$

Let $c = (ab)^2$. Then by (3) $(ba)^2 = c^{-1}$. Formulas (2), (4), and (5) imply that a^2, b^2, and c commute pairwise; i.e.,

$$[a^2, c] = [b^2, c] = [a^2, b^2] = e. \tag{10}$$

Relations (6)–(9) show that

$$(a^2)^b = a^{-2}, \quad (b^2)^a = b^{-2}, \tag{11}$$

$$c^a = c^b = c^{-1}. \tag{12}$$

Consider $g \in \mathbb{G}$. By (11), we can assert that $g = g' a^{2\alpha} b^{2\beta}$, where $\alpha, \beta \in \mathbb{Z}$ while

$$g' = a^{\varepsilon_1} b \cdot \ldots \cdot ab^{\delta_s}, \tag{13}$$

ε_1 and δ_s are equal to 0 or 1.

Consider the case $\varepsilon_1 = 1$ (the unassembled part g' of g begins with a). Furthermore, if $\delta_s = 1$ then $g' = (ab)^s$. Assume that $s = 2\gamma + r$, where r is 0 or 1. Then $g' = c^\gamma (ab)^r$. Consequently,

$$g = \begin{cases} c^\gamma a^{2\alpha} b^{2\beta} & \text{if } s = 2\gamma, \\ c^\gamma a^{-2\alpha} b^{-2\beta}(ab) & \text{if } s = 2\gamma + 1. \end{cases}$$

Now, let $\delta_s = 0$. Then $g' = (ab)^s b^{-1}$, and so $g = (ab)^s a^{-2\alpha} b^{2\beta-2} b$. Therefore,

$$g = \begin{cases} c^\gamma a^{-2\alpha} b^{2\beta-2} b & \text{if } s = 2\gamma, \\ c^\gamma a^{2\alpha} b^{-2\beta} a & \text{if } s = 2\gamma + 1. \end{cases}$$

Analogous formulas can also be obtained in the case when the unassembled part g' of g begins with b ($\varepsilon_1 = 0$). Thus, each element $g \in \mathbb{G}$ is representable as

$$g = c^\gamma a^{2\alpha} b^{2\beta} a^\varepsilon b^\delta, \tag{14}$$

where $\alpha, \beta, \gamma \in \mathbb{Z}$, while ε and δ are equal to 0 or 1.

Put $\mathbb{H} = gr(c, a^2, b^2)$. The above implies that $\mathbb{H} = (c) \times (a^2) \times (b^2)$ and $\mathbb{H} \triangleleft \mathbb{G}$.

Show that $[a,b] \in \mathbb{H}$. Indeed, $[a,b] = a^{-1}b^{-1}ab = a^{-2}b^2abab = cb^2a^{-2} \in \mathbb{H}$ which implies immediately that the commutator subgroup $\mathbb{G}'$ of $\mathbb{G}$ lies in $\mathbb{H}$. Therefore, $\mathbb{H}ab = \mathbb{H}ba$. Obviously, $\mathbb{H}b = \mathbb{H}b^{-1}$ and $\mathbb{H}a = \mathbb{H}a^{-1}$. Thus, we have the following decomposition of $\mathbb{G}$ into the union of right cosets:

$$\mathbb{G} = \mathbb{H} \cup \mathbb{H}a \cup \mathbb{H}b \cup \mathbb{H}ab. \tag{15}$$

Clearly, $g^2 \in \mathbb{H}$ for every $g \in \mathbb{G}$. Therefore, for proving the absence of torsion, it suffices to observe that the square of each nonidentity element is nonidentity too. Assume for example that $g = h(ab)$, where $h = c^\gamma a^{2\alpha} b^{2\beta} \in \mathbb{H}$. Then

$$g^2 = h(ab)h(ab) = c^\gamma a^{2\alpha} b^{2\beta} c^\gamma a^{-2\alpha} b^{-2\beta} (ab)^2 = c^{2\gamma+1} \neq e.$$

The remaining cases are checked similarly.

Show that $\mathbb{G}$ is not right-orderable. Suppose not, while $\mathbb{G}$ is endowed by a right linear order $\geq$. Without loss of generality, we may assume that $c \geq e \geq c^{-1}$. Owing to the isolatedness of the order, $ab \geq e \geq ba$. Therefore, a and b have different signs. Suppose that a is positive. Then $a \geq b^{-1} \geq e$. Hence, $a^2b^{-1} \geq b^{-1}ab^{-1}$. Therefore, $b^{-1}a^{-1} \geq b^{-1}ab^{-1}a \geq e$, which leads to the contradiction $(ab)^{-1} \geq e$. The case of $a^{-1} \geq b \geq e$ is examined likewise.

As usual, denote by $sgr(y_1, y_2, \ldots, y_n)$ the subsemigroup in G generated by $y_1, y_2, \ldots, y_n$. We will need the following (semigroup) right orderability criterion (see [1, p. 48]): *A group G is right-orderable if and only if for any nonidentity $y_1, y_2, \ldots, y_n \in G$ there are $\varepsilon_i = \pm 1$, with $i = 1, \ldots, n$, such that $e \notin sgr\big(y_1^{\varepsilon_1}, \ldots, y_n^{\varepsilon_n}\big)$.*

The proof of the fact that the group $\mathbb{G}_x$ is right-orderable for every $x \in \mathbb{G}$ splits into the four cases (by the number of right cosets).

Case 1: $x \in \mathbb{H}$. In this case $\mathbb{G}_x \leq \mathbb{H}$, and so $\mathbb{G}_x$ is right-orderable.

Case 2: $x \in \mathbb{H}a$, i.e., $x = ha$, where $h \in \mathbb{H}$. Take $g \in \mathbb{G}$. Then $x^g = (h^g[g, a^{-1}])a = h_1a$, where $h_1 = h^g[g, a^{-1}] \in \mathbb{H}$. Since $x^{-1} = \widetilde{h}a$, where $\widetilde{h} = (h^{-1})^a a^{-2}$; it follows on repeating the arguments above for x^{-1} that $(x^{\pm 1})^g = h^*a$ for a suitable $h^* \in \mathbb{H}$.

Consequently, in this case every $y \in \mathbb{G}_x$ looks as

$$y = ha^{\delta}, \tag{16}$$

where $h = c^{\gamma}a^{2\alpha}b^{2\beta} \in \mathbb{H}$, with δ equal to 0 or 1. More exactly, let us write down y^{-1} when $\delta = 1$. Now, $y = c^{\gamma}b^{2\beta}a^{2\alpha+1}$ and so $y^{-1} = c^{\gamma}b^{2\beta}a^{-(2\alpha+1)}$. If $\delta = 0$; then, obviously, $y^{-1} = c^{-\gamma}b^{-2\beta}a^{-2\alpha}$. Henceforth, we denote the degree of a in the representation of y by $\deg_y(a)$. Using this notation, the above can be written down as follows:

$$\deg_y(a) \geq 0 \Longleftrightarrow \deg_{y^{-1}}(a) \leq 0.$$

Suppose that we have nonzero $y_1, y_2, \ldots, y_n \in \mathbb{G}_x$. Given y_i, choose the degree ε_i in the order:

(1) if $\deg_{y_i}(a) \neq 0$ then we choose ε_i so that $\deg_{y_i^{\varepsilon_i}}(a) > 0$;

(2) if $\deg_{y_i}(a) = 0$ and $\deg_{y_i}(b) \neq 0$ then we choose ε_i so that $\deg_{y_i^{\varepsilon_i}}(b) > 0$;

(3) if $\deg_{y_i}(a) = \deg_{y_i}(b) = 0$ then $\deg_{y_i}(c) \neq 0$, and thus ε_i is such that $\deg_{y_i^{\varepsilon_i}}(c) > 0$.

Note that $z \in sgr(y_1^{\varepsilon_1}, \ldots, y_n^{\varepsilon_n})$ has the form

$$z = y_{i_1}^{\varepsilon_{i_1}} \cdot \ldots \cdot y_{i_t}^{\varepsilon_{i_t}}, \tag{17}$$

where $t > 0$. If there exists $y_{i_j}^{\varepsilon_{i_j}}$ such that $\deg_{y_{i_j}^{\varepsilon_{i_j}}}(a) > 0$; then $\deg_z(a) > 0$ and so $z \neq e$. Suppose now that the representation of each $y_{i_j}^{\varepsilon_{i_j}}$ does not contain at least one a but has at least one element whose representation includes b. Then, of course, $\deg_z(a) = 0$ but $\deg_z(b) > 0$. Hence, $z \neq e$. Finally, if the representation of each $y_{i_j}^{\varepsilon_{i_j}}$ contains neither a nor b then $\deg_z(c) > 0$. The examination of Case 2 is over.

Case 3: $x \in \mathbb{H}b$. This is analogous to the previous case.

Case 4: $x \in \mathbb{H}ab$. Arguing by analogy with Case 2, we conclude that for each $g \in \mathbb{G}$ we have $(x^{\pm 1})^g = h^*ab$ for a suitable $h^* \in \mathbb{H}$. Therefore, $y \in \mathbb{G}_x$ looks as

$$y = h(ab)^{\delta}, \tag{18}$$

where $h = c^{\gamma}a^{2\alpha}b^{2\beta} \in \mathbb{H}$, while δ is equal to 0 or 1. Since $c^{\gamma} = (ab)^{2\gamma}$, we have $y = a^{2\alpha}b^{2\beta}(ab)^{2\gamma+\delta}$. Given nonidentity $y_1, y_2, \ldots, y_n \in \mathbb{G}_x$, the choice of the signs is carried out by analogy with Case 2 but in the following order:

(1) if $\deg_{y_i}(ab) \neq 0$ then choose ε_i so that $\deg_{y_i^{\varepsilon_i}}(ab) > 0$;

(2) if $\deg_{y_i}(ab) = 0$ and $\deg_{y_i}(a) \neq 0$ then we choose ε_i so that $\deg_{y_i^{\varepsilon_i}}(a) > 0$;

(3) if $\deg_{y_i}(ab) = \deg_{y_i}(a) = 0$ then $\deg_{y_i}(b) \neq 0$, and thus ε_i is such that $\deg_{y_i^{\varepsilon_i}}(b) > 0$.

Arguing as above, we see that

$$e \notin sgr(y_1^{\varepsilon_1}, \ldots, y_n^{\varepsilon_n}),$$

i.e., $\mathbb{G}_x$ is right-orderable.

Thus, $\mathbb{G}_x$ is right-orderable for every $x \in \mathbb{G}$. Hence, $\mathscr{D}_r \subsetneqq L(\mathscr{D}_r)$.

Recall that a group is locally indicable if its every finitely generated nonidentity subgroup admits a homomorphism into an additive group of integers. It is well known that every locally indicable group is right-orderable and the class $\mathscr{J}$ of all locally indicable groups is a quasivariety (see [1]). The class $\mathscr{D}_o$ of all orderable groups is also a quasivariety. Moreover, $\mathscr{D}_o \subsetneqq \mathscr{J}$.

In this connection, of interest is the study of the properties of the quasivarieties $\mathscr{D}_o$, $\mathscr{J}$, and $\mathscr{D}_r$, and their Levi classes as elements of the lattice of quasivarieties of torsion-free groups.

For example, the inclusion $\mathscr{J} \subseteqq L(\mathscr{D}_o)$ is impossible because the group $\mathbb{K} = \langle a, b | a^b = a^{-1} \rangle$ is locally indicable (admits four right orders) but the normal closure $\mathbb{K}_{ab}$ of its element ab is nonorderable. However, the above implies that $\mathscr{D}_o \subseteqq L(\mathscr{D}_o) \cap \mathscr{J}$. Is coincidence possible here, or does there exist nonorderable locally indicable group in which the normal closure of any element is orderable? Analogous questions can also considered in other cases.

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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A. V. Zenkov
Altai State University of Agriculture, Barnaul, Russia
E-mail address: alexey_zenkov@yahoo.com