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TOPOLOGICAL PROPERTIES OF MAPPINGS WITH FINITE DISTORTION ON CARNOT GROUPS

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Abstract—We prove that every mapping with finite distortion on a Carnot group is open and discrete provided that it is quasilight and the distortion coefficient is integrable. Also, we estimate the Hausdorff dimension of the preimages of points for mappings on a Carnot group with a bounded multiplicity function and summable distortion coefficient. Furthermore, we give some example showing that the obtained estimates cannot be improved.

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1. Introduction

Given a domain Ω in $\mathbb{R}^n$ with $n \geq 2$, consider a mapping $f : \Omega \to \mathbb{R}^n$ in the Sobolev space $W^1_{1,\mathrm{loc}}(\Omega, \mathbb{R}^n)$ which has the locally integrable Jacobian $J(x, f) = \det Df(x)$. Refer to f as a *mapping with finite distortion* whenever $J(x, f) \geq 0$ almost everywhere and $Df(x) = 0$ almost everywhere on the zero set of the Jacobian. The function

$$K_O(x) = \begin{cases} \frac{\|Df(x)\|^n}{J(x,f)} & \text{if } J(x, f) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is the *outer distortion coefficient*. Here $\|Df(x)\|$ is the operator norm of the linear mapping $Df(x)$.

If K_O is a bounded function and $f \in W^1_{n,\mathrm{loc}}(\Omega, \mathbb{R}^n)$ then f is a *mapping with bounded distortion*. Reshetnyak established the fundamental topological properties of mappings with bounded distortion on the Euclidean space; i.e., continuity, openness, and discreteness (see [1]). Vodopyanov and Goldshtein proved the continuity of mappings with finite distortion in [2]. Many articles seek the conditions for mappings with finite distortion to be open and discrete. In the planar case Iwaniec and Šverák proved the openness and discreteness of mappings with finite distortion for $K_O \in L_1$ and $f \in W^1_{2,\mathrm{loc}}(\Omega, \mathbb{R}^2)$ in [3]. Villamor and Manfredi showed the openness and discreteness of mappings with finite distortion on $\mathbb{R}^n$ for $n \geq 3$ under the condition $K_O \in L_{p,\mathrm{loc}}$ with $p > n - 1$ and $f \in W^1_{n,\mathrm{loc}}(\Omega, \mathbb{R}^n)$ in [4]. In the boundary case $K_O \in L_{n-1}$ the openness and discreteness are justified only under additional assumptions; i.e., the topological condition of quasilightness [5] and the integrability of the inner distortion coefficient (see [6]). In the case $K_O \in L_p$ with $p < n - 1$ Ball constructed an example of a continuous quasilight mapping of Sobolev class W^1_n with finite distortion that is neither open nor discrete in [7].

The goal of this article is to study the topological properties of mappings with finite distortion on Carnot groups with sub-Riemannian metric. Recently Vodopyanov established the continuity of mappings with finite distortion on Carnot groups in [8]. By now openness and discreteness in the sub-Riemannian case are established only on the two-step Carnot groups of $\mathbb{H}$-type for mappings with finite distortion [9] and on the two-step Carnot groups for mappings with bounded distortion (see [10, 11]).

A *stratified homogeneous group*, or alternatively a *Carnot group*, is a connected simply-connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra V is the direct sum $V_1 \oplus \cdots \oplus V_m$ of vector spaces with

$$\dim V_1 \geq 2, \quad [V_1, V_k] = V_{k+1} \text{ for } 1 \leq k \leq m - 1, \quad [V_1, V_m] = \{0\}.$$

The Euclidean space is an example of an abelian Carnot group. Fix an inner product on V.

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The distance d (intrinsic *Carnot–Carathéodory metric*) between two points $x, y \in \mathbb{G}$ is defined as the greatest lower bound of the lengths of *horizontal* curves connecting x and y. The Hausdorff dimension of $\mathbb{G}$ with respect to the Carnot–Carathéodory metric is denoted by ν and equals $\sum_{i=1}^{m} i \dim V_i$.

DEFINITION. Given a domain Ω of a Carnot group $\mathbb{G}$, consider a mapping $f : \Omega \to \mathbb{G}$ of the Sobolev class $W^1_{1,\mathrm{loc}}(\Omega, \mathbb{G})$ with locally integrable Jacobian $J(x, f)$. Say that f has *finite distortion* whenever $J(x, f) \geq 0$ almost everywhere and $D_h f(x) = 0$ almost everywhere on the zero set of the Jacobian.

Define the *distortion coefficient* K_p as

$$K_p(x) = \inf\{k > 0 : \|D_h f(x)\| \leq kJ(x, f)^{1/p}\}.$$

It is obvious that $(K_\nu)^\nu = K_O$, where

$$K_O(x) = \begin{cases} \frac{\|D_h f(x)\|^\nu}{J(x,f)} & \text{if } J(x, f) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is the *outer distortion coefficient*. The Sobolev class, Jacobian, and horizontal differential $D_h f$ are defined in Section 2.

Recall that a mapping $f : \Omega \to \mathbb{G}$ with $\Omega \subset \mathbb{G}$ is *open* whenever the image of each open set is open, f is *discrete* whenever the preimage $f^{-1}(y)$ of each $y \in f(\Omega)$ consists of isolated points, and f is *light* whenever the preimage of every point of $\mathbb{G}$ is totally disconnected. Call $f : \Omega \to \mathbb{G}$ *quasilight* whenever all components of $f^{-1}(y)$ are compact for each $y \in \mathbb{G}$.

The main result is as follows:

Theorem 1. *Given an open subset Ω of a Carnot group $\mathbb{G}$, consider a quasilight mapping $f \in W^1_{\nu,\mathrm{loc}}(\Omega, \mathbb{G})$ with finite distortion not constant on each connected component and $K_\nu \in L_{\nu(\nu-1),\mathrm{loc}}(\Omega)$ (or equivalently $K_O \in L_{\nu-1,\mathrm{loc}}(\Omega)$). Then we can redefine f on a measure zero set so that*

(1) f becomes continuous, open, and discrete;
(2) f becomes $\mathscr{P}$-differentiable almost everywhere;
(3) f has Luzin's $\mathscr{N}$ and $\mathscr{N}^{-1}$-properties;
(4) $J(x, f) > 0$ almost everywhere.

Recently Vodopyanov showed in [8, Proposition 19] that we can redefine each mapping of Sobolev class $W^1_{\nu,\mathrm{loc}}(\Omega, \mathbb{G})$ with finite distortion on a measure zero set so that it becomes continuous, $\mathscr{P}$-differentiable almost everywhere, and has Luzin's $\mathscr{N}$-property. From the classical proof of Reshetnyak's Theorem [12] we know that if f is differentiable almost everywhere and has Luzin's $\mathscr{N}$-property then the lightness of f implies the discreteness and openness of f. To prove Theorem 1, we also have to show that f is light; i.e., $\mathscr{H}^1(f^{-1}(y)) = 0$ for all y.

The measure of the preimages of points is estimated in the following theorem which is of interest in its own right.

Theorem 2. *Suppose that $1 \leq q < p \leq \nu$. Given a connected open set $\Omega \subset \mathbb{G}$, consider a continuous mapping $f \in W^1_{p,\mathrm{loc}}(\Omega, \mathbb{G})$ with finite distortion, locally summable Jacobian, and $K_p \in L_{\varkappa,\mathrm{loc}}(\Omega)$, where $\varkappa = \frac{pq}{p-q}$, and f satisfies Luzin's $\mathscr{N}$-property. If the multiplicity of f is essentially bounded in a neighborhood of 0 then either $f = 0$ almost everywhere or $f^{-1}(0)$ is of $\mathscr{H}^{\nu-q}$-measure zero.*

Theorem 2 addresses the larger class $W^1_{p,\mathrm{loc}}$ of Sobolev mappings with $1 \leq p \leq \nu$. Therefore, to its statement we add the conditions of continuity, summability of the Jacobian, and Luzin's $\mathscr{N}$-property.

The proof of Theorem 2 rests on the ideas of [5] which established a similar result in the Euclidean case. The main technical part is Theorem 3 of [5] on estimating the oscillations of Sobolev functions. Although this theorem is stated and proved in $\mathbb{R}^n$, it remains valid on Carnot groups. Moreover, Theorem 3 of [5] is valid on an arbitrary homogeneous metric space where the analogs of Sobolev functions are defined and Poincaré's inequality holds. We will present the version without proof; see Theorem 5. However,

the Euclidean theory is not directly applicable: We have to separately establish Luzin's $\mathscr{N}^{-1}$-property and the nonvanishing of the Jacobian almost everywhere (see Theorem 4 in Section 4) and to study the properties of Sobolev functions on Carnot groups (see Proposition 2 in Section 3).

Note that the mapping of Theorem 2 induces the bounded embedding operator for Sobolev classes by composition; see Theorem 4 in Section 4. Vodopyanov with coauthors (see [2, 13]) introduced and studied this class of mappings on the Euclidean space; this is the so-called functional approach to quasiconformal analysis. Homeomorphisms on Carnot groups which induce bounded embedding operators of Sobolev spaces are studied in [14]. To prove Theorem 4, we rely on [13] where the similar result is established in the Euclidean case.

Moreover, in this article we exhibit some example of a mapping with finite distortion on a Carnot group which demonstrates that the exponent in Theorem 2 is optimal. It is inspired by Ball's example [7].

For mappings with finite distortion, the "bad set" W is studied without the additional assumption of quasilightness. Given a mapping $f : \Omega \to \mathbb{G}$ with finite distortion, denote by W the set of points in Ω belonging to a nontrivial component $f^{-1}(y)$ for some $y \in \mathbb{G}$; i.e., $W = f^{-1}(W')$, where W' consists of the points of $f(\Omega) \subset \mathbb{G}$ at which discreteness is violated: $y \in W'$ whenever $f^{-1}(y)$ is not comprised of isolated points.

Theorem 3. *Given an open subset Ω of a Carnot group $\mathbb{G}$, consider a nonconstant continuous mapping $f \in W^1_{\nu,\mathrm{loc}}(\Omega, \mathbb{G})$ with finite distortion and $K_O \in L_{\nu-1,\mathrm{loc}}(\Omega)$. Then W, defined above, is a closed set in Ω, while f is open and discrete outside W.*

In order to prove Theorem 3, it suffices to show that f is quasilight outside W. In the Euclidean case Rajala obtained a similar result; see [6, Lemma 2.4].

This article has the following structure. Section 2 presents the needed definitions, notation, and auxiliary results. Section 3 addresses the fractional maximal function. In Section 4 we prove Luzin's $\mathscr{N}^{-1}$-property. The proofs of Theorems 1–3 occupy Section 5. The example is given in Section 6.

2. Definitions and Auxiliary Results

Carnot groups. Consider a Carnot group $\mathbb{G}$ with Lie algebra V. Fix on V some inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ with $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$ for $\xi \in V$.

Take some vector fields $X_1, \dots, X_n$ constituting a basis for the horizontal space V_1. Since they generate the whole V, we can include them in an orthonormal basis for V together with the fields $X_{n+1}, \dots, X_N$ formed by commutators of the fields in V_1. Moreover, assume that

$$1 = \sigma_1 = \dots = \sigma_n < \sigma_{n+1} \le \dots \le \sigma_N = m,$$

where $\sigma_i = \{k : X_i \in V_k\}$ is the *degree* of the field X_i.

Identify $g \in \mathbb{G}$ with $x \in \mathbb{R}^N$ via the exponential mapping $\exp\bigl(\sum x_i X_i\bigr) = g$. Furthermore, it is obvious that $0 = (0, \dots, 0)$ is the identity element of $\mathbb{G}$, while $x^{-1} = (-x_1, \dots, -x_N)$ for $x = (x_1, \dots, x_N) \in \mathbb{G}$.

A *dilation* is defined as $\delta_t x = (t^{\sigma_1} x_1, t^{\sigma_2} x_2, \dots, t^{\sigma_N} x_N)$ for $t > 0$. The Carnot–Carathéodory distance is homogeneous: $d(\delta_t x, \delta_t y) = t d(x, y)$.

Measures on a Carnot group. This article considers the s-dimensional Hausdorff measure on $\mathbb{G}$ with respect to the Carnot–Carathéodory distance:

$$\mathscr{H}^s(A) = \lim_{\delta \to 0} \inf \Bigl\{ \sum_i (\operatorname{diam} E_i)^s \mid A \subseteq \bigcup_i E_i,\ \operatorname{diam} E_i < \delta \Bigr\},$$

where $\operatorname{diam} E_i = \sup\{d(x, y) : x, y \in E_i\}$ and $A \subset \mathbb{G}$.

We identified the points of $\mathbb{G}$ with the points of $\mathbb{R}^N$ via the exponential mapping. The Lebesgue measure on $\mathbb{R}^N$ is a bi-invariant Haar measure on $\mathbb{G}$ [15, Proposition 1.2]. Therefore, the classical Lebesgue integral is defined on $\mathbb{G}$, which we denote by $\int_\Omega f(x)\, dx$, where Ω is a measurable subset of $\mathbb{G}$

and f is a measurable function on Ω. Moreover, denote by $L_q(\Omega)$ the space of q-integrable functions on Ω; and by $\|\cdot\|_{q,\Omega}$, the L_q-norm of a function on Ω.

Denote the Lebesgue measure of a measurable set $A \subset \mathbb{G}$ by $|A|$. Up to a factor, $|A|$ coincides with the ν-dimensional Hausdorff measure $\mathscr{H}^\nu(A)$.

Sobolev-class functions.

Definition. For $1 \leq q \leq \infty$ the *Sobolev space* $W_q^1(\Omega)$, or $L_q^1(\Omega)$, consists of the locally summable functions $f : \Omega \to \mathbb{R}$, with generalized derivatives $X_i f$ along the vector fields X_i for $i = 1, \dots, n$ and the finite norm (seminorm)

$$\|f \mid W_q^1(\Omega)\| = \|f\|_{q,\Omega} + \|\nabla_h f\|_{q,\Omega} \quad \left(\|f \mid L_q^1(\Omega)\| = \|\nabla_h f\|_{q,\Omega}\right),$$

where $\nabla_h f = (X_1 f, \dots, X_n f)$ is the *subgradient* of f.

Recall that a locally summable function $g_i : \Omega \to \mathbb{R}$ is the *generalized derivative* of a function f along the vector field X_i, for $i = 1, \dots, n$, whenever $\int_\Omega g_i \psi \, dx = -\int_\Omega f X_i \psi \, dx$ for every test function $\psi \in C_0^\infty(\Omega)$. If $f \in W_q^1(U)$ for each bounded open set U with $\overline{U} \subset \Omega$ then we say that f *belongs to the class* $W^1_{q,\mathrm{loc}}(\Omega)$.

Definition. A domain (a connected open set) $\Omega \subset \mathbb{G}$ is a *John domain* with inner radius α and outer radius β, where $0 < \alpha \leq \beta < \infty$ [16], whenever there exists a distinguished point $x_0 \in U$ such that every other point $x \in \Omega$ can be connected in Ω to x_0 by a rectifiable curve $\gamma(s)$, for $0 \leq s \leq l \leq \beta$, where s is the arc length, such that $\gamma(0) = x_0$ and $\gamma(l) = x$, while

$$\operatorname{dist}(\gamma(s), \partial\Omega) \geq \frac{\alpha}{l} s \quad \text{for all } s \in [0, l].$$

The ball $B(a, r)$ in the Carnot–Carathéodory metric is a John domain with inner radius r and outer radius r. Each domain with smooth boundary is a John domain.

Lemma 1 (Poincaré's inequality). *Suppose that $1 \leq p < \infty$. If $\Omega \subset \mathbb{G}$ is a John domain then every $u \in W_p^1(\Omega)$ satisfies*

$$\|u - u_\Omega\|_{q,\Omega} \leq C (\operatorname{diam} \Omega)^{1 - \frac{\nu}{p} + \frac{\nu}{q}} \|\nabla_h u\|_{p,\Omega}, \tag{1}$$

where $1 \leq q \leq \frac{\nu p}{\nu - p}$ for $p < \nu$, while $1 \leq q \leq \infty$ for $p = \nu$ and $1 \leq q \leq \infty$ for $p > \nu$. Here $u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(x) \, dx$ is the average value of u on Ω.

If $u = 0$ on a measurable set $T \subset \Omega$ of positive measure then

$$\|u\|_{q,\Omega} \leq C \frac{|\Omega|^{\frac{1}{q}}}{|T|^{\frac{1}{q}}} (\operatorname{diam} \Omega)^{1 - \frac{\nu}{p} + \frac{\nu}{q}} \|\nabla_h u\|_{p,\Omega}. \tag{2}$$

Proof. Poincaré's inequality (1) on balls in Carnot groups is established by Lu in [17]; while on John domains, by Vodopyanov and Isangulova in [18, Theorem 4].

The proof of (2) follows the strategy of the proof of Lemma 3 of [19], in the case of balls and $q = p$. Put $M = \frac{1}{|\Omega|^{1/q}} \|u\|_{q,\Omega} > 0$. The case $M = 0$ is trivial. Assume that $u_\Omega > 0$. If $u_\Omega < 0$ then take $-u$, and if $u_\Omega = 0$ then (2) holds. We have

$$\|M - u_\Omega\|_{q,\Omega} = |\Omega|^{1/q} |M - u_\Omega|$$
$$= |\Omega|^{1/q} \left| \frac{1}{|\Omega|^{1/q}} \|u\|_{q,\Omega} - \frac{1}{|\Omega|^{1/q}} \|u_\Omega\|_{q,\Omega} \right| \leq \|u - u_\Omega\|$$

and

$$|T|^{\frac{1}{q}} \leq \|1 - M^{-1} u\|_{q,\Omega} = \frac{1}{M} \|u - M\|_{q,\Omega}$$
$$\leq \frac{1}{M} (\|u - u_\Omega\|_{q,\Omega} + \|M - u_\Omega\|_{q,\Omega}) \leq \frac{2}{M} \|u - u_\Omega\|_{q,\Omega}.$$

Consequently,

$$\|u\|_{q,\Omega} = |\Omega|^{1/q} M \le \frac{2|\Omega|^{\frac{1}{q}}}{|T|^{\frac{1}{q}}} \|u - u_\Omega\|_{q,\Omega}$$

and (1) yields the required inequality. $\square$

Sobolev-class mappings.

Definition. Given a domain $\Omega \subset \mathbb{G}$, a mapping $f : \Omega \to \mathbb{G}$ *belongs to the Sobolev class* $W^1_{q,\mathrm{loc}}(\Omega, \mathbb{G})$ whenever the following are met:

(A) $[f]_z : x \in \Omega \mapsto d(f(x), z)$ belongs to $W^1_{q,\mathrm{loc}}(\Omega)$ for all $z \in \mathbb{G}$.

(B) The family of $\{\nabla_h [f]_z\}_{z\in\mathbb{G}}$ has a majorant in $L_{q,\mathrm{loc}}(\Omega)$; i.e., there is a function $g \in L_{q,\mathrm{loc}}(\Omega)$ independent of z such that $|\nabla_h[f]_z(x)| \le g(x)$ for almost all $x \in \Omega$.

Reshetnyak introduced this definition for mappings from a Euclidean space to a metric space in [20]. Vodopyanov used this approach for mappings of Carnot groups in [21]. Represent a mapping in the coordinates of the first kind $f = (f_1, \dots, f_N) : \Omega \to \mathbb{G}$. It is known (see [19] for instance) that $X_j f(x) \in V_1(f(x))$ for every Sobolev-class mapping and $j = 1, \dots, n$; hence, the matrix $D_h f(x)$ with entries $(X_i f_j(x))$ for $i, j = 1, \dots, n$ determines a linear mapping from the horizontal space $V_1(x)$ into $V_1(f(x))$ for almost all $x \in \Omega$. The mapping is called the *formal horizontal differential*. Denote by $\|D_h f(x)\|$ its norm

$$\|D_h f(x)\| = \sup_{\xi \in V_1,\, \|\xi\|=1} \|(D_h f(x))(\xi)\|.$$

We can take $\|D_h f\|$ as $g \in L_{q,\mathrm{loc}}(\Omega)$ in the definition of Sobolev mappings; see [19, 22].

In turn, the horizontal differential $D_h f : V_1 \to V_1$ induces the graded homomorphism of Lie algebras $Df : V \to V$ called the *formal differential* [19]. The determinant of $Df(x)$ is the (*formal*) *Jacobian* of f which is denoted by $J(x, f)$.

The change-of-variables formula. Take $f : \Omega \to \mathbb{G}$ and $E \subset \Omega$. The function $\mathscr{N}(y, f, E) : \mathbb{G} \to \mathbb{N} \cup \{0, \infty\}$ defined as

$$\mathscr{N}(y, f, E) = \begin{cases} 0 & \text{if the preimage } f^{-1}(y) \cap E \text{ is empty,} \\ \infty & \text{if the preimage } f^{-1}(y) \cap E \text{ is infinite,} \\ \#(f^{-1}(y) \cap E) & \text{otherwise} \end{cases}$$

is the *Banach indicatrix* or *multiplicity function* of f. Here $\#(f^{-1}(y) \cap E)$ stands for the size of the preimage $f^{-1}(y) \cap E$ of y.

Recall that a mapping has *Luzin's $\mathscr{N}$-property* whenever the image of each measure zero set has measure zero.

Definition. Given a domain Ω in $\mathbb{G}$, a mapping $f : \Omega \to \mathbb{G}$ is *absolutely continuous on lines*, in symbols $f \in ACL(\Omega)$, whenever for each domain U with $\overline{U} \subset \Omega$ and the foliation Γ_j determined by the left-invariant vector field X_j the mapping f is absolutely continuous on $\gamma \cap U$ with respect to the $\mathscr{H}^1$-Hausdorff measure for $d\gamma_j$-almost all curves $\gamma \in \Gamma_j$ for $j = 1, \dots, n$. For such f, the derivatives $X_j f$ along the horizontal vector fields X_j for $j = 1, \dots, n$ such that $X_j f(x) \in V_1(x)$ exist $\mathscr{H}^\nu$-almost everywhere in Ω [23, Proposition 4.1].

Proposition 1 [14, Proposition 3]. *Given an open set $D \subset \mathbb{G}$, if $f : D \to \mathbb{G}$ is a mapping in $W^1_{1,\mathrm{loc}}(D, \mathbb{G})$ or $ACL(D)$ then*

(1) *there exists a Borel set $\Sigma \subset D$ of measure zero such that $f : D \setminus \Sigma \to \mathbb{G}$ has Luzin's $\mathscr{N}$-property;*

(2) *for every measurable set $A \subset D \setminus \Sigma$ we have the area formula*

$$\int_A |J(x, f)|\, dx = \int_{\mathbb{G}} \mathscr{N}(y, f, A)\, dy; \tag{3}$$

(3) *for every nonnegative measurable function u we have the following change-of-variables formula in Lebesgue integrals:*

$$\int\limits_{D\setminus\Sigma} u(x)\,|J(x,f)|\,dx = \int\limits_{\mathbb{G}} \sum_{x\in f^{-1}(y)\setminus\Sigma} u(x)\,dy. \tag{4}$$

3. The Fractional Maximal Function

Take $\alpha \in [0,\nu)$ and $g \in L_1(\mathbb{G})$ with $g \geq 0$. Introduce the fractional maximal function

$$M_\alpha g(z) := \sup_{r>0} r^\alpha \fint\limits_{B(z,r)} g\,dx.$$

Lemma 2. *Suppose that $1 \leq q < p \leq \nu$. If $g \in L_p(\mathbb{G})$ is a nonnegative function on a Carnot group $\mathbb{G}$ then $M_{\frac{q}{p}} g(z) < \infty$ for $\mathscr{H}^{\nu-q}$-almost all $z \in \mathbb{G}$.*

Proof. Demonstration uses Lemma 3.2 of [24] on the maximal functions of finite measures. Put

$$S = \{z \in \mathbb{G} : M_{\frac{q}{p}} g(z) > \sigma\}, \quad \sigma > 0.$$

For every $z \in S$ there is a real $r_z > 0$ such that

$$r_z^{q/p} \fint\limits_{B(z,r_z)} g\,dx > \sigma.$$

Hence, Hölder's inequality yields

$$\sigma < \frac{r_z^{q/p}}{|B(z,r_z)|}\Bigg(\int\limits_{B(z,r_z)} g^p\,dx\Bigg)^{1/p} |B(z,r_z)|^{1-1/p} = \frac{r_z^{(q-\nu)/p}}{|B(0,1)|^{1/p}}\Bigg(\int\limits_{B(z,r_z)} g^p\,dx\Bigg)^{1/p},$$

and consequently

$$\sigma^p < \frac{r_z^{q-\nu}}{|B(0,1)|}\int\limits_{B(z,r_z)} g^p\,dx.$$

We obtain the uniform boundedness of r_z; i.e.,

$$r_z^{\nu-q} < \frac{1}{\sigma^p |B(0,1)|}\int\limits_{B(z,r_z)} g^p\,dx \leq \frac{1}{\sigma^p |B(0,1)|}\int\limits_{\mathbb{G}} g^p\,dx.$$

By the Vitali Covering Lemma, we can choose a countable subfamily of $\{B(z,r_z), z \in S\}$ which consists of disjoint balls $B_i = B(z_i, r_i)$ satisfying $S \subset \bigcup_i 5B_i$.

Therefore,

$$\mathscr{H}^{\nu-q}(S) \leq \sum_i (10r_i)^{\nu-q} < \frac{10^{\nu-q}}{\sigma^p|B(0,1)|}\sum_i \int\limits_{B_i} g^p\,dx \leq \frac{10^{\nu-q}}{\sigma^p|B(0,1)|}\int\limits_{\mathbb{G}} g^p\,dx.$$

This implies immediately that $\mathscr{H}^{\nu-q}\{z \in \mathbb{G} : M_{\frac{q}{p}} g(z) = \infty\} = 0$. $\square$

Lemma 3. *Take $\beta \in (0,1]$. If g is a nonnegative summable function on $\mathbb{G}$ then*

$$\int\limits_{B(y,r)} \frac{g(z)}{d(y,z)^{\nu-1}}\,dz \le Cr^{\beta} M_{1-\beta} g(y)$$

for all $y \in \mathbb{G}$ and $r > 0$. The constant C is independent of y, r, and g.

PROOF. Put $r_k = 2^{-k} r$ and $B_k = B(y, r_k)$. Then

$$\int\limits_{B(y,r)} \frac{g(z)}{d(y,z)^{\nu-1}}\,dz \le \sum_{k=0}^{\infty} (r_{k+1})^{1-\nu} \int\limits_{B_k \setminus B_{k+1}} g(z)\,dz$$

$$\le \sum_{k=0}^{\infty} (r_k/2)^{1-\nu} \int\limits_{B_k} g(z)\,dz = 2^{\nu-1}|B(0,1)| \sum_{k=0}^{\infty} r_k \fint\limits_{B_k} g(z)\,dz$$

$$\le 2^{\nu-1}|B(0,1)| M_{1-\beta} g(y) \sum_{k=0}^{\infty} r_k^{\beta} = Cr^{\beta} M_{1-\beta} g(y). \quad \square$$

Proposition 2. *Suppose that $1 \le q < p \le \nu$. Given an open set $\Omega \subset \mathbb{G}$, if $u \in W_p^1(\Omega)$ is a continuous function then for $\mathscr{H}^{\nu-q}$-almost all $z \in \Omega$ we have*

$$\limsup_{r \to 0} r^{-\beta} \fint\limits_{B(z,r)} |u(x) - u(z)|\,dx < \infty, \quad \text{where} \quad \beta = 1 - \frac{q}{p}.$$

PROOF. Without loss of generality we may assume that $\Omega = \mathbb{G}$. Almost all points are Lebesgue points for u and $M_{1-\beta}\|\nabla_h u\|(z) < \infty$ for $\mathscr{H}^{\nu-q}$-almost all $z \in \mathbb{G}$ by Lemma 2. Choose $z \in \mathbb{G}$ such that $M_{1-\beta}\|\nabla_h u\|(z) < \infty$ and z is a Lebesgue point for u.

Put $B = B(z,r)$. Then

$$|u(x) - u_B| \le C \int\limits_{2B} \frac{\|\nabla_h u(y)\|}{d(x,y)^{\nu-1}}\,dy$$

for almost all $x \in B$; see [25, Lemma 3.1] for instance.

This yields

$$\fint\limits_{B} |u(x) - u(z)|\,dx \le \fint\limits_{B} \big(|u(x) - u_B| + |u(z) - u_B|\big)\,dx$$

$$\le C_1 \fint\limits_{B} dx \int\limits_{2B} \frac{\|\nabla_h u(y)\|}{d(x,y)^{\nu-1}}\,dy + C_2 \fint\limits_{B} dx \int\limits_{2B} \frac{\|\nabla_h u(y)\|}{d(z,y)^{\nu-1}}\,dy = I_1 + I_2.$$

Since $x \in B = B(z,r)$ and $y \in 2B = B(z,2r)$, we have $d(x,y) \le d(x,z) + d(z,y) \le 3r$ and

$$\int\limits_{B} \frac{dx}{d(x,y)^{\nu-1}} \le \int\limits_{B(y,3r)} \frac{dx}{d(x,y)^{\nu-1}} \le C \int\limits_{0}^{3r} \frac{t^{\nu-1}dt}{t^{\nu-1}} = C'r.$$

By Fubini's Theorem,

$$I_1 = \frac{C_1}{|B|} \int\limits_{2B} \|\nabla_h u(y)\|\,dy \int\limits_{B} \frac{dx}{d(x,y)^{\nu-1}}$$

$$\le \frac{C_1' r}{|B|} \int\limits_{2B} \|\nabla_h u(y)\|\,dy \le C_1'' r^{\beta} M_{1-\beta}\|\nabla_h u\|(z).$$

To estimate I_2, use Lemma 2:

$$I_2 = C_2 \fint\limits_{B} dx \int\limits_{2B} \frac{\|\nabla_h u(y)\|}{d(y,z)^{\nu-1}} dy = C_2' \int\limits_{2B} \frac{\|\nabla_h u(y)\|}{d(y,z)^{\nu-1}} dy \le C_2'' r^{\beta} M_{1-\beta}\|\nabla_h u\|(z). \quad \square$$

4. Finiteness of Distortion and Boundedness of the Multiplicity Function

In this section we consider mappings with finite distortion and essentially bounded multiplicity function (i.e., bounded on a set of full measure). We verify that every mapping of this kind has Luzin's $\mathscr{N}^{-1}$-property and their Jacobians are positive almost everywhere. Recall that $f : \Omega \to \mathbb{G}$ *has Luzin's* $\mathscr{N}^{-1}$*-property* whenever the preimage of every measure zero set by f is a measure zero set.

We follow the idea of the proof of Theorem 5 of [13], where Luzin's $\mathscr{N}^{-1}$-property and the nonvanishing of the Jacobian almost everywhere are established for the mappings inducing the bounded embedding operators of Sobolev functions on the Euclidean spaces. The $\mathscr{N}^{-1}$-property for mappings with finite distortion and bounded multiplicity function in the Euclidean case is also obtained in Theorem 1.2 of [26].

Definition. Given two open sets D and D' in a group $\mathbb{G}$, say that $f : D \to D'$ induces the *bounded embedding operator* $f^* : L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D') \to L^1_q(D)$, where $1 \le q \le p \le \infty$, as the composition $f^*u = u \circ f$ whenever there exists a constant $K < \infty$ such that

$$\|f^*u \mid L^1_q(D)\| \le K\|u \mid L^1_p(D')\| \text{ for every function } u \in L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D').$$

Here $\mathrm{Lip}_{\mathrm{loc}}(D')$ is the set of all locally Lipschitz functions on D'.

Put $Z = \{x \in D \mid J(x, f) = 0\}$ and take a Borel set $\Sigma \subset D$ of measure zero such that $f : D \setminus \Sigma \to \mathbb{G}$ has Luzin's $\mathscr{N}$-property. Put

$$H_q(y) = \begin{cases} \Big(\sum\limits_{x\in f^{-1}(y)\setminus(\Sigma\cup Z)} \frac{\|D_h f(x)\|^q}{|J(x,f)|}\Big)^{1/q}, \\ 0, \quad \{f^{-1}(y) \setminus (\Sigma \cup Z)\} = \varnothing. \end{cases}$$

Theorem 4. *Suppose that* $1 \le q \le p$. *Consider two open sets* D *and* D' *in* $\mathbb{G}$ *and a mapping* $f : D \to D'$ *not constant on any connected component and satisfying the following:*

(a) f *belongs to the class* $ACL(D)$*;*

(b) f *has finite distortion;*

(c) $\mathscr{N}(y, f, D) \in L_\infty(D')$*;*

(d) $K_p \in L_\varkappa(D)$*, where* $\varkappa = \frac{pq}{p-q}$ *and* $\varkappa = \infty$ *if* $p = q$.

Then

(1) H_q *belongs to* $L_\varkappa(D')$ *and* $\|H_q\|_{\varkappa,D'} \le \|\mathscr{N}(\cdot, f, D)\|^{1/p}_{\infty,D'}\|K_p\|_{\varkappa,D}$*;*

(2) f *induces the bounded embedding operator*

$$f^* : L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D') \to L^1_q(D)$$

and

$$\|f^*u \mid L^1_q(D)\| \le \|H_q \mid L_\varkappa(D')\| \cdot \|u \mid L^1_p(D')\| \quad \textit{for all } u \in L^1_p(D') \cap \mathrm{Lip}_{\mathrm{loc}}(D');$$

(3) f *has Luzin's* $\mathscr{N}^{-1}$*-property and the Jacobian* $J(x, f)$ *is nonzero almost everywhere.*

Proof. (1): For $p = q$ the estimates are obvious because $\varkappa = \infty$:

$$\begin{aligned} H^p_p(y) &= \sum_{x\in f^{-1}(y)\setminus(Z\cup\Sigma)} \frac{\|D_h f(x)\|^p}{|J(x,f)|} \\ &\le \mathscr{N}(y, f, D') \left\| \frac{\|D_h f(\cdot)\|^p}{|J(\cdot, f)|} \right\|_{\infty,D} = \mathscr{N}(y, f, D')\|K_p\|^p_{\infty,D} \end{aligned}$$

for y with $f^{-1}(y) \setminus (Z \cup \Sigma) \ne \varnothing$.

Consider the case $q < p$. Formula (4) yields

$$\begin{aligned} \|K_p\|^\varkappa_{\varkappa,D} &= \int\limits_D \left(\frac{\|D_h f(x)\|}{|J(x,f)|^{1/p}}\right)^{\frac{pq}{p-q}} dx = \int\limits_{D\setminus(Z\cup\Sigma)} \left(\frac{\|D_h f(x)\|^q}{|J(x,f)|}\right)^{\frac{p}{p-q}} |J(x,f)|\, dx \\ &= \int\limits_{D'} \sum_{x\in f^{-1}(y)\setminus(Z\cup\Sigma)} \left(\frac{\|D_h f(x)\|^q}{|J(x,f)|}\right)^{\frac{p}{p-q}} dy. \end{aligned}$$

On the other hand, since the multiplicity function is bounded, we have

$$\Bigg(\sum_{x\in f^{-1}(y)\setminus(Z\cup\Sigma)}\frac{\|D_hf(x)\|^q}{|J(x,f)|}\Bigg)^{\frac{p}{p-q}}\leq \mathscr{N}(y,f,D)^{\frac{\varkappa}{p}}\sum_{x\in f^{-1}(y)\setminus(Z\cup\Sigma)}\bigg(\frac{\|D_hf(x)\|^q}{|J(x,f)|}\bigg)^{\frac{p}{p-q}}$$

for almost all $y\in D'$. Hence, we immediately obtain

$$\|H_q\|^{\varkappa}_{\varkappa,D'}\leq\|\mathscr{N}(\cdot,f,D)\|^{\frac{\varkappa}{p}}_{\infty,D'}\|K_p\|^{\varkappa}_{\varkappa,D}.$$

(2): To verify that f induces a bounded embedding operator of Sobolev spaces, take $u\in L^1_p(D')\cap\mathrm{Lip}_{\mathrm{loc}}(D')$. Then $u\circ f\in ACL(D)$. The rule for differentiating composition, the change-of-variables formula, and Hölder's inequality imply that

$$\begin{aligned}\big\|f^*u\mid L^1_q(D)\big\|^q&=\int\limits_{D\setminus(\Sigma\cup Z)}(\|\nabla_h u(f(x))\|\cdot\|D_hf(x)\|)^q\,dx\\&\overset{(4)}{=}\int\limits_{D'}\|\nabla_h u(y)\|^q\Bigg(\sum_{x\in f^{-1}(y)\setminus(Z\cup\Sigma)}\frac{\|D_hf(x)\|^q}{|J(x,f)|}\Bigg)\,dy\\&\leq\Bigg(\int\limits_{D'}\|\nabla_h u(y)\|^p\,dy\Bigg)^{q/p}\Bigg(\int\limits_{D'}H_q^{\varkappa}(y)\,dy\Bigg)^{q/\varkappa}=\big\|u\mid L^1_p(D')\big\|^q\,\|H_q\|^q_{\varkappa,D'}.\end{aligned}$$

In the first equality here we use the finiteness of distortion and the property that $|\Sigma|=0$.

(3): Let us establish Luzin's $\mathscr{N}^{-1}$-property. Without loss of generality we may assume that D is a bounded set and $q<\nu$. Construct an auxiliary function $\eta\in C^\infty_O(\mathbb{G})$ such that $\eta=1$ on $B(0,1)$ and $\eta=0$ outside $B(0,2)$.

Given $y_0\in D'$, put $u(y)=\eta\big(\delta_{\frac{1}{r}}\big(y_0^{-1}\cdot y\big)\big)$. Then $u\in L^1_p(D')$ if $\overline{B(y_0,2r)}\subset D'$ and

$$\big\|u\mid L^1_p(D')\big\|^p=\int\limits_{D'}\big\|\nabla_h\eta\big(\delta_{\frac{1}{r}}\big(y_0^{-1}\cdot y\big)\big)\big\|^p r^{-p}\,dy=\int\limits_{B(0,2)}\|\nabla_h\eta(z)\|^p r^{\nu-p}\,dz=Cr^{\nu-p}.$$

Consequently,

$$\big\|f^*u\mid L^1_q(D)\big\|\leq\|H_q\|_{\varkappa,B(y_0,2r)}\big\|u\mid L^1_p(D')\big\|=Cr^{\frac{\nu}{p}-1}\|H_q\|_{\varkappa,B(y_0,2r)}.$$

Fix a measure zero set E in D'. We have to show that $|f^{-1}(E)|=0$. The finiteness of distortion yields $|D\setminus f^{-1}(E)|\neq0$. Indeed, otherwise $J(x,f)=0$ for almost all $x\in D$, and by the finiteness of distortion $D_hf=0$ almost everywhere. The latter means that f is a constant function on D, which contradicts the hypotheses. Take a subdomain A of D with smooth boundary. Observe that the measure of $D\setminus A$ can be arbitrarily small. Then $|A\setminus f^{-1}(E)|>0$ and by Luzin's Theorem there is a compact set $T\subset A\setminus f^{-1}(E)$ of positive measure on which f is continuous. The image $f(T)\subset D'$ is compact and $f(T)\cap E=\varnothing$.

Consider an open set $U\subset D'$ such that $E\subset U$ and $f(T)\cap U=\varnothing$. Construct a countable cover for U by some balls $B_i=B(y_i,r_i)$ such that $\{2B_i=B(y_i,2r_i)\}_{i\in\mathbb{N}}$ is a cover for U of finite multiplicity. Given $i\in\mathbb{N}$, put $u_i(y)=\eta\big(\delta_{\frac{1}{r_i}}\big(y_i^{-1}\cdot y\big)\big)$. Then $f^*u_i=1$ on $f^{-1}(B_i)$ and $f^*u_i=0$ outside $f^{-1}(2B_i)$. In particular, $f^*u_i=0$ on T. By the version of Poincaré's inequality (2), Lemma 1 shows that

$$\int\limits_V|f^*u_i(x)|^{q'}\,dx\leq C\Bigg(\int\limits_V\|\nabla_h(f^*u_i)(x)\|^q\,dx\Bigg)^{q'/q},\quad q'=\frac{\nu q}{\nu-q}.$$

Together with (4), this implies that

$$|f^{-1}(B_i)\cap V| \le C\|H_q\|_{\varkappa,2B_i}^{q'}|B_i|^{(1/p-1/\nu)q'} = C\|H_q\|_{\varkappa,2B_i}^{q'}|B_i|^{\frac{q(\nu-p)}{p(\nu-q)}}.$$

If $q<p<\nu$; then, summing over all i and applying Hölder's inequality, we obtain

$$\sum_{i=1}^{\infty}|f^{-1}(B_i)\cap V| \le C\left(\sum_{i=1}^{\infty}\|H_q\|_{\varkappa,2B_i}^{\varkappa}\right)^{\frac{\nu(p-q)}{p(\nu-q)}}\left(\sum_{i=1}^{\infty}|B_i|\right)^{\frac{q(\nu-p)}{p(\nu-q)}}.$$

Since $\{B_i\}_{i\in\mathbb{N}}$ and $\{2B_i\}_{i\in\mathbb{N}}$ are covers of finite multiplicity and $2B_i\subset U$ for all $x\in\mathbb{N}$, it follows that

$$|f^{-1}(E)\cap A| \le C\|H_q\|_{\varkappa,U}^{q'}|U|^{\frac{q(\nu-p)}{p(\nu-q)}}.$$

If $q=p<\nu$ then $\varkappa=\infty$ and

$$|f^{-1}(U)\cap A| \le \sum_{i=1}^{\infty}|f^{-1}(B_i)\cap A| \le C\sum_{i=1}^{\infty}\|H_q\|_{\infty,2B_i}^{q'}|B_i| \le C\|H_q\|_{\infty,U}^{q'}|U|.$$

If $q<p=\nu$ then $\varkappa=q'$ and

$$|f^{-1}(U)\cap A| \le \sum_{i=1}^{\infty}|f^{-1}(B_i)\cap A| \le C\sum_{i=1}^{\infty}\|H_q\|_{\varkappa,2B_i}^{\varkappa} \le C\|H_q\|_{\varkappa,U}^{\varkappa}.$$

In all cases we have $|f^{-1}(E)\cap A|=0$ because U is chosen arbitrarily. Recall that by construction the measure of $D\setminus A$ can be arbitrarily small; consequently, $|f^{-1}(E)\cap D|=0$.

It remains to show that $J(x,f)\neq 0$ almost everywhere. Assume on the contrary that $|Z|>0$, where $Z=\{x\in D \mid J(x,f)=0\}$ as above. Since $|\Sigma|=0$, it follows that $|Z\setminus\Sigma|>0$. Then Luzin's $\mathscr{N}^{-1}$-property yields $|f(Z\setminus\Sigma)|>0$, which contradicts the area formula (3):

$$0=\int\limits_{Z\setminus\Sigma}|J(x,f)|\,dx \ge |f(Z\setminus\Sigma)|.$$

The proof of Theorem 4 is complete. $\square$

5. Proofs of Theorems 1–3

Theorem 5 [5, Theorem 3]. *Suppose that $1\le q<\nu$, $\mu>0$, $\beta\in(0,1)$, and $\gamma>0$. Consider an open set $\Omega\subset\mathbb{G}$ and $\varphi\in W_q^1(\Omega)$ such that $\varphi>0$ almost everywhere and put*

$$Z=\Bigg\{z\in\Omega : \limsup_{r\to 0} r^{-\beta} \fint\limits_{B(z,r)}\varphi\,dx<\gamma\Bigg\}.$$

If $\mathscr{H}^{\nu-q}(Z)>\mu$ then there exists a compact set $F\subset\Omega\setminus Z$ such that $|F|>0$ and

$$\sup_{x\in F}\varphi(x)^q \le C\int\limits_F \|\nabla_h\varphi(x)\|^q\,dx,$$

where $C=C(\nu,q,\mu,\beta)$.

We omit the proof because it repeats verbatim the proof of Theorem 3 of [5] in the Euclidean case.

Proof of Theorem 2. Without loss of generality, assume that Ω is bounded. Consider $f\in W_p^1(\Omega,\mathbb{G})$ with finite distortion and $K_p\in L_\varkappa(\Omega)$, where $\varkappa=\frac{pq}{p-q}$, as well as $J(x,f)\in L_1(\Omega)$ and

$\mathscr{N}(y, f, D) \in L_\infty(D')$. Suppose that f is nonzero on a set of positive measure. Theorem 4 shows that f is nonzero almost everywhere and that $J(x, f) > 0$ almost everywhere.

Assume on the contrary that $\mathscr{H}^{\nu-q}(f^{-1}(0)) > 0$ and find a contradiction.

Put $\varphi(x) = d(f(x), 0)$. Then $\varphi \in W^1_{p,\mathrm{loc}}(\Omega)$ and $\|\nabla_h \varphi(x)\| \le \|D_h f(x)\|$ by the definition of Sobolev mappings [22, Remark A.3]; furthermore, $\varphi(z) = 0$ if and only if $f(z) = 0$. By Proposition 2, we have

$$\limsup_{r\to 0} r^{-\beta} \fint_{B(z,r)} |\varphi(x) - \varphi(z)|\, dx < \infty, \quad \beta = 1 - \frac{q}{p}$$

for $\mathscr{H}^{\nu-q}$-almost all $z \in \Omega$. By assumption, $\mathscr{H}^{\nu-q}(f^{-1}(0)) > 0$, and so there exists a real $\gamma \in (0, \infty)$ such that

$$\mathscr{H}^{\nu-q}(Z_\gamma) > 0, \quad \text{where} \quad Z_\gamma = \Big\{ z \in f^{-1}(0) : \limsup_{r\to 0} r^{-\beta} \fint_{B(z,r)} \varphi(x)\, dx < \gamma \Big\}.$$

Consequently, we can apply Theorem 5. Construct a recurrent sequence of open sets Ω_k which include Z_γ, as well as a sequence F_k of compact sets. Put $\Omega_1 = \Omega$.

By Theorem 5, for φ there is a compact set $F_1 \subset \Omega_1 \setminus Z_\gamma$ such that $|F_1| > 0$ and

$$\sup_{F_1} \varphi^q \le C \int_{F_1} \|\nabla_h \varphi(x)\|^q\, dx \le C \int_{F_1} \|D_h f(x)\|^q\, dx.$$

On step k with $k > 1$ choose $\Omega_k = \Omega_{k-1} \setminus F_{k-1}$. By Theorem 5 again there is a compact set $F_k \subset \Omega_k \setminus Z_\gamma$ such that $|F_k| > 0$ and

$$\sup_{F_k} \varphi^q \le C \int_{F_k} \|D_h f(x)\|^q\, dx. \tag{5}$$

The constant C is independent of k.

Put

$$\rho_k = \sup_{F_k} \varphi = \sup_{x \in F_k} d(f(x), 0).$$

It is obvious that $f(F_k) \subseteq \overline{B(0, \rho_k)}$ and $|f(F_k)| \le C\rho_k^\nu$. Since F_k are disjoint, (5) implies that $\rho_k \to 0$ as $k \to \infty$. Then there are $k_0 \in \mathbb{N}$ and $\rho_0 > 0$ such that $\rho_k < \rho_0$ for all $k > k_0$. The multiplicity of f is essentially bounded by some constant M on $B(0, \rho_0)$. The area formula (3) yields

$$\int_{F_k} J(x, f)\, dx \le M|f(F_k)| \le C \cdot M \cdot \rho_k^\nu. \tag{6}$$

By (5) and (6), the definition of distortion coefficient and Hölder's inequality imply that

$$\rho_k^q \le C \int_{F_k} \|D_h f(x)\|^q\, dx \le C \int_{F_k} (K_p)^q J(x, f)^{q/p}\, dx$$
$$\le C \Big(\int_{F_k} (K_p)^\varkappa\, dx \Big)^{(p-q)/p} \Big(\int_{F_k} J(x, f)\, dx \Big)^{q/p} \le C M^{q/p} \rho_k^{\frac{\nu q}{p}} \Big(\int_{F_k} (K_p)^\varkappa\, dx \Big)^{(p-q)/p}.$$

Since $\varphi > 0$ almost everywhere in F_k and $|F_k| > 0$, we infer that $\rho_k > 0$, and therefore

$$\rho_k^{1-\frac{\nu}{p}} \le C M^{1/p} \|K_p\|_{\varkappa, F_k}.$$

Since $\frac{\nu}{p} \geq 1$ and $\rho_k \to 0$, there exists a constant $\kappa > 0$ independent of k such that

$$\int\limits_{F_k} (K_p)^{\varkappa} \, dx > \kappa.$$

This is the required contradiction because F_k are disjoint by construction. The proof of Theorem 2 is complete. $\square$

Proof of Theorem 1. Consider a quasilight mapping f of class $W^1_{\nu,\mathrm{loc}}(\Omega, \mathbb{G})$ with finite distortion. By Proposition 19 of [8], redefining f on some measure zero set, we can make f continuous and enjoying Luzin's $\mathscr{N}$-property. The lightness of f and Luzin's $\mathscr{N}^{-1}$-property follow from Theorems 2 and 4 with $q = \nu - 1$ and $p = \nu$ under the condition that quasilightness implies the boundedness of the multiplicity function.

Fix $x_0 \in \Omega$ and put $y_0 = f(x_0)$. We have to show that the multiplicity function is bounded in some neighborhood of y_0. Using quasilightness, find an open set $\Omega' \subset \Omega$ which includes the connected component of $f^{-1}(y_0)$ containing x_0 so that $y_0 \notin f(\partial\Omega')$. Take $\rho > 0$ for which $\overline{B(y_0, \rho)} \cap f(\partial\Omega') = \varnothing$ and define Ω'' as the connected component of $\Omega' \cap f^{-1}(\overline{B(y_0, \rho)})$ containing x_0. Then

$$\mu(y, f, \Omega'') = \begin{cases} \mu(y_0, f, \Omega''), & y \in B(y_0, \rho), \\ 0, & y \notin \overline{B(y_0, \rho)}. \end{cases}$$

Here $\mu(y, f, \Omega'')$ is the degree of f at y.

Proposition 20 of [8] yields

$$\mu(y, f, \Omega'') = \mathscr{N}(y, f, \Omega'')$$

for almost all $y \in B(y_0, \rho) \subset f(\Omega'') \setminus f(\partial\Omega'')$, which means that the multiplicity function is essentially bounded in a neighborhood of y_0.

Thus, $\mathscr{H}^1(f^{-1}(y)) = 0$ for all $y \in f(\Omega)$ by Theorem 2, which means that f is a light mapping. It remains to justify the openness and discreteness of f. Since f is orientation-preserving and has Luzin's $\mathscr{N}$-property, and so the change-of-variables formula holds, we can carry over the classical proof of openness and discreteness due to Reshetnyak (see [1]) to the case of Carnot groups without any changes. $\square$

Proof of Theorem 3. Consider a continuous mapping $f : \Omega \to \mathbb{G}$ of class $W^1_{\nu,\mathrm{loc}}$ with finite distortion and $K_O \in L_{\nu-1,\mathrm{loc}}(\Omega)$. Denote by W' the set of points in $f(\Omega) \subset \mathbb{G}$ where discreteness is violated: $y \in W'$ whenever $f^{-1}(y)$ does not consist of isolated points. Then $W = f^{-1}(W')$. In order to prove Theorem 3, we have to show that for each point outside W there is a neighborhood on which f is quasilight. Consequently, we can apply Theorem 1.

Consider some domain D compactly embedded into Ω. We may assume that $f \in W^1_\nu(D, \mathbb{G})$ and $K_O \in L_{\nu-1}(D)$.

Fix $x \in D \setminus W$ and put $y = f(x)$. The set $f^{-1}(y)$ consists of isolated points, and the connected component containing x amounts to x itself. Take the connected component U_j of $f^{-1}(B(y, 1/j))$ containing x. Since $\overline{U_j} \subset \mathbb{G}$ are compact and connected, their intersection E is also connected. On the other hand, $x \in E \setminus \partial D \subset f^{-1}(y)$; hence, $E = \{x\}$. This implies that $\mathrm{diam}(U_j) \to 0$ as $j \to \infty$. Fix j such that U_j is compactly embedded into D.

Verify that f is quasilight on U_j for j sufficiently large; i.e., the connected components of $f^{-1}(y) \cap U_j$ are compact for all $y \in \mathbb{G}$. Assume on the contrary that for some $z \in U_j$ the connected component of $f^{-1}(f(z))$ containing z meets ∂U_j at some point b. Since $f(b) = f(z) \in B(y, 1/j)$, there exists $t > 0$ such that $f(B(b, t)) \subset B(y, 1/j)$. This contradicts the definition of U_j. Consequently, f is quasilight on U_j. Theorem 1 shows that f is open and discrete on U_j. $\square$

6. Example

Consider the Carnot group $\mathbb{G}$ which is the product $\mathbb{R}^n \times \mathbb{G}_0$, where $\mathbb{G}_0$ is a Carnot group of homogeneous dimension ν_0 with Carnot–Carathéodory metric d_0. The group $\mathbb{G}$ is of homogeneous dimension $\nu = n + \nu_0$. We denote the points of $\mathbb{G}$ by the pairs (x, y), where $x \in \mathbb{R}^n$ and $y \in \mathbb{G}_0$. Denote the Euclidean norm of $x \in \mathbb{R}^n$ by $\|x\|$, while the homogeneous norm on $\mathbb{G}_0$, by $\rho(y) = d_0(y, 0)$.

Consider the mapping F on $A = \{(x, y) \mid \|x\| < 2, \rho(y) < 1\} \subset \mathbb{G}$ as

$$F(x,y) = \begin{cases} (\rho(y)x, y) & \text{if } \|x\| \le 1, \\ \big((2(\|x\|-1) + (2-\|x\|)\rho(y))\frac{x}{\|x\|}, y\big) & \text{if } 1 < \|x\| < 2. \end{cases}$$

Note that F is continuous and fixes A and its boundary:

$$F(\overline{A}) = \overline{A} \quad \text{and} \quad F|_{\partial A} = \mathrm{id}\,.$$

Furthermore, F is differentiable for $y \neq 0$ and $\|x\| \neq 1$, satisfies the contact condition, and, moreover, $\|D_h F(x, y)\| \le C$ on A. Consequently, $F \in W_p^1(A, \mathbb{G})$ for all $p \in [1, \infty)$.

It is not difficult to verify that at the points of differentiability

$$J((x,y), F) = \rho(y)^n \quad \text{for } \|x\| < 1$$

and

$$J((x,y), F) = a^n + 2a^{n-1}(1-\rho(y))\|x\|^{-1} \quad \text{for } 1 < \|x\| < 2,$$

where

$$a = (2(\|x\|-1) + (2-\|x\|)\rho(y))\|x\|^{-1}.$$

The distortion coefficient satisfies $K_\nu \sim \rho(y)^{-\frac{n}{\nu}}$ for $\|x\| < 1$ and $K_\nu \sim |a|^{-\frac{n-1}{\nu}}$ for $1 < \|x\| < 2$. Since $a \sim \rho(y)$ as $\rho(y) \to 0$ and $\|x\| \to 1$, it follows that $K_\nu \in L_s$ for all s provided that

$$\frac{-ns}{n+\nu_0} > -\nu_0,$$

or equivalently $s < \frac{\nu_0(n+\nu_0)}{n}$. The latter condition coincides with $\varkappa = \frac{\nu q}{\nu - q}$ for $q = \nu_0$.

The multiplicity function of F equals 1 identically except at $(0, 0)$; in particular, the mapping is quasilight. At the points of differentiability the Jacobian does not vanish, and so F has finite distortion. Put

$$M := F^{-1}\big((0,0)\big) = \{(x, 0) \mid \|x\| \le 1\}.$$

Theorem 2 yields $\mathscr{H}^{\nu-q}(M) = 0$ for all $q < \nu_0$. Since M is a set of dimension $n = \nu - \nu_0$ and

$$\mathscr{H}^n(M) = \mathscr{L}^n(B_{\mathbb{R}^n}(0,1)) > 0,$$

the example we constructed shows that the conditions of Theorem 2 cannot be improved.

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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