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FINITE GROUPS WITH S-CONDITIONALLY PERMUTABLE SCHMIDT SUBGROUPS

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Abstract—A subgroup H in a finite group G is S-conditionally permutable if for every $p \in \pi(G)$ there exists a Sylow p-subgroup P in G such that $HP = PH$. We study the structure of a finite group G whose all Schmidt subgroups are S-conditionally permutable.

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1. Introduction

We will consider only finite groups.

A *Schmidt group* is a nonnilpotent group whose all proper subgroups are nilpotent. An easy check shows that each nonnilpotent group contains at least one Schmidt subgroup (i.e., a subgroup that is a Schmidt group). In this connection, the natural problem appears of studying the structure of a group with a system of Schmidt subgroups having given properties.

Groups with constraints on Schmidt subgroups were studied in many works. In particular, the groups with subnormal Schmidt subgroups were described in [1]. The groups were studied whose all Schmidt subgroups are σ-subnormal for each partition σ of the set of all primes in [2, 3]. The solubility of a group G with hereditary G-permutable Schmidt subgroups was proved in [4].

In the present article, we analyze the normal structure of a finite group whose all Schmidt subgroups are S-conditionally permutable.

Definition 1. A subgroup H in a group G is *S-conditionally permutable* if for every $p \in \pi(G)$ there exists a Sylow p-subgroup P in G such that $HP = PH$.

The notion of S-conditionally permutable subgroup was introduced in [5] in 2007. This notion develops the concept of S-permutable subgroup which was proposed by Kegel in [6] (a subgroup H in a group G is *S-permutable* if H is permutable with any Sylow subgroup in G).

The influence of S-conditionally permutable subgroups on the structure of the group was inspected in [7, 8]. In particular, some sufficiency tests were proposed in [7] for the p-nilpotency and supersolubility of a group G whose all maximal subgroups in the Sylow subgroups of G are S-conditionally permutable. The structure was studied of a soluble group whose all primary subgroups are S-conditionally permutable in [8].

Our main goal is to prove Theorems 1 and 2 which develop the results of [4].

Theorem 1. *Let G be a group whose all Schmidt subgroups are S-conditionally permutable. Then the socle of G is abelian. In particular, G is not a nonabelian simple group.*

Definition 2. Let A and B be subgroups in G. The subgroup A is *G-permutable with B* if $AB^x = B^xA$ for some $x \in G$. A subgroup A in G is G-permutable in G if A is G-permutable with all subgroups in G.

[†)] To the eightieth anniversary of Viktor Danilovich Mazurov.

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Corollary 1 [4, Theorem A]. *Let G be a group whose all Schmidt subgroups are G-permutable. Then the socle of G is abelian. In particular, G is not a nonabelian simple group.*

The following question arises naturally in connection with Theorem 1:

Question. *Is a group G soluble if all its subgroups are S-conditionally permutable?*

Some partial answer to this question is given by

Theorem 2. *Let G be a group whose every Schmidt subgroup is S-conditionally permutable in every subgroup in G which includes the Schmidt subgroup. Then G is soluble.*

Definition 3. Let A and B be subgroups in G. The subgroup A is *hereditarily G-permutable with B* if $AB^x = B^xA$ for some $x \in \langle A, B\rangle$. A subgroup A in G is *hereditarily G-permutable* in G if A is hereditarily G-permutable with all subgroups in G.

Corollary 2 [4, Theorem B]. *Let G be a group whose all Schmidt subgroups are hereditarily G-permutable. Then G is soluble.*

2. Terminology and Preliminaries

We use the standard definitions and notations from group theory which can be found in [9].

The essential structure of Schmidt groups (see Lemma 1) was established in [10, 11].

Lemma 1. *Let S be a Schmidt group. Then*
(1) $\pi(S) = \{p, q\}$;
(2) $S = [P]\langle a\rangle$, where P is a normal Sylow p-subgroup in S, while $\langle a\rangle$ is a Sylow q-subgroup of S and $\langle a^q\rangle \subseteq Z(S)$;
(3) P is the $\mathfrak{N}$-residual of S; i.e., the least normal subgroup in S the quotient group by which belongs to $\mathfrak{N}$ ($\mathfrak{N}$ is the class of all nilpotent groups);
(4) $P/\Phi(P)$ is the minimal normal subgroup in $S/\Phi(P)$, while $\Phi(P) = P' \subseteq Z(S)$;
(5) $\Phi(S) = Z(S) = P' \times \langle a^q\rangle$;
(6) $C_P(a) = \Phi(P)$;
(7) if $Z(S) = 1$ then $|S| = p^m q$, where m is the exponent of p modulo q.

Henceforth, we refer to a Schmidt group with a normal Sylow p-subgroup and a nonnormal cyclic Sylow q-subgroup as an *$S_{\langle p,q\rangle}$-group.*

Lemma 2 [12, Lemma 2]. *If K and D are subgroups in G, while D is normal in K, and K/D is an $S_{\langle p,q\rangle}$-subgroup; then the minimal supplement L to D in K possesses the properties*
(1) L is a p-closed $\{p, q\}$-subgroup;
(2) all proper normal subgroups in L are nilpotent;
(3) L has an $S_{\langle p,q\rangle}$-subgroup $P : Q$ such that Q does not lie in D and $L = (P : Q)^L = Q^L$.

Lemma 3. *If H is an S-conditionally permutable subgroup in a group G and H lies in a subnormal subgroup N of G then H is S-conditionally permutable in N.*

Proof. Let $p \in \pi(G)$. By hypothesis, there exists a Sylow p-subgroup P in G such that $HP = PH$. Since N is subnormal in G, it follows that $P \cap N = P_0$ is a Sylow p-subgroup in N. Hence,

$$HP_0 = H(P \cap N) = HP \cap N = PH \cap N = (P \cap N)H = P_0H.$$

Thus, by the arbitrary choice of $p \in \pi(G)$, we infer that H is S-conditionally permutable subgroup in N. The lemma is proved.

Lemma 4. *If a group G has no p-closed Schmidt subgroups then G is p-nilpotent.*

Proof. Demonstration follows from Ito's Theorem (see [13]) which states that each minimal non-p-nilpotent group is a Schmidt subgroup.

Lemma 5 [14, Lemma 3]. *Let G be a simple Chevalley group not in the list*

$$\{A_5(2), C_3(2), D_4(2), {}^2A_3(2)\}.$$

Then there exists a prime divisor of the order of G that does not divide the order of any proper parabolic subgroup in G.

Lemma 6 [15, Lemma 1.6]. *Let G be a simple Chevalley group and let U be a unipotent subgroup of G. If L is a maximal subgroup in G and $U \subseteq L$ then L is a parabolic subgroup in G.*

Recall that the *soluble graph* corresponding to a group G (which is denoted by $\Gamma_{sol}(G)$) is the simple graph whose vertices are prime divisors of the order of G and two different primes p and q are joined by an edge if and only if G has a soluble subgroup whose order divides by pq. A vertex in a graph is *central* if it is adjacent to all other vertices. The center of a graph is the set of all its central vertices.

Lemma 7. *Suppose that G is a group whose all Schmidt subgroups are S-conditionally permutable and $r = \max\{p \mid p \in \pi(G)\}$. If $r \geq 19$ then G is not nonabelian simple.*

Proof. Let G be a nonabelian simple group. By Lemma 4, G has an r-closed Schmidt subgroup S. Let $p \in \pi(G)$ and $P \in Syl_p(G)$. Without loss of generality, we may assume that $SP = PS$ and so SP is a subgroup in G. Obviously, $|\pi(SP)| \leq 3$. Suppose that the subgroup SP is not soluble. Then SP contains a simple section isomorphic to some triprimary group. Nonabelian simple triprimary groups are described (see, for example, [16, p. 20]): They belong to the list

$$G \in \{L_2(5), L_2(7), L_2(8), L_2(9), L_2(17), L_3(3), U_3(3), U_4(2)\}.$$

Checking the groups in this list shows that $r \leq 17$. We get a contradiction with $r \geq 19$.

Consequently, SP is soluble. Therefore, the vertices r and p are adjacent in $\Gamma_{sol}(G)$. Since p is an arbitrary prime divisor of $|G|$, we have

$$r \in Z(\Gamma_{sol}(G)).$$

By [17, Theorem 1.3], G is not a nonabelian simple group.

The lemma is proved.

3. Proof of Theorem 1

Let G be a group of the least order for which the theorem fails. Suppose that G has a proper nonabelian simple subnormal subgroup N. By Lemma 3, all Schmidt subgroups in N are S-conditionally permutable in N. This and $|N| < |G|$ imply that the socle of N is abelian. We get a contradiction to the choice of G.

Consequently, G is a nonabelian simple group. Basing on the classification of simple groups, we exclude all possible cases.

1. G is the alternating group A_n, where $n \geq 5$.

By Lemma 7, it suffices to consider the cases when $n \leq 18$.

(1) $G \in \{A_5, A_6\}$. Then G has a Schmidt subgroup $3:2$ (see, for example, [18, p. 2, 4]) that is not permutable with any of the Sylow 5-subgroups in G, which contradicts the hypothesis.

(2) $G \cong A_7$. Then G has a Schmidt subgroup $3:2$ that is not permutable with any of the Sylow 7-subgroups in G [18, p. 10]. We again get a contradiction to the hypothesis.

(3) $G \cong A_8$. Then G has a dihedron $D \cong 5:2$. Therefore, for some Sylow 2-subgroup S in G, the group G has the subgroup DS of order $2^6 \cdot 5$, which is impossible since A_8 has no maximal subgroups whose order divides by $2^6 \cdot 5$ [18, p. 22].

(4) $G \cong A_9$. The group A_9 has a unique class of maximal subgroups of odd index isomorphic to A_8 [18, p. 37]. Now, item (3) leads to a contradiction.

(5) $G \cong A_{10}$. The group A_{10} has two classes of maximal subgroups of odd index whose representatives are isomorphic to S_8 and $2^4 : S_5$ [18, p. 48]. The group G has some dihedron $D \cong 5 : 2$. Therefore, for some Sylow 2-subgroup S of G, the group G has the subgroup DS of order $2^7 \cdot 5$. If DS lies in a subgroup isomorphic to S_8 then this subgroup has a subgroup of order $2^7 \cdot 5$, which is impossible (see item (4) and [18, p. 22]). If DS lies in a subgroup isomorphic to $2^4 : S_5$ then S_5 has a subgroup of order $5 \cdot 2^3$, which is impossible either (see [18, p. 2]).

(6) $G \cong A_{11}$. The group G has some dihedron $D \cong 11 : 5$. Therefore, for some Sylow 2-subgroup S in G, the group G has the subgroup DS of order $11 \cdot 5 \cdot 2^7$. But G has no maximal subgroups whose order divides by $11 \cdot 5 \cdot 2^7$ (see [18, p. 75]); a contradiction.

(7) $G \cong A_{12}$. Then G has some dihedron $D \cong 11 : 5$. Therefore, for some Sylow 2-subgroup S in G, the group G has the subgroup DS of order $11 \cdot 5 \cdot 2^9$. which is impossible since A_{12} has no maximal subgroups of odd index whose order divides by $11 \cdot 5 \cdot 2^9$ (see [18, p. 91]).

(8) $G \cong A_{13}$. The group A_{13} has a maximal subgroup $13 : 6$ (see [18, p. 104]), and so G has a Schmidt subgroup $D \cong 13 : 3$. Therefore, for some Sylow 2-subgroup S in G, the group G has the subgroup DS of order $13 \cdot 3 \cdot 2^9$. This is impossible since A_{13} has no maximal subgroups whose order divides by $13 \cdot 3 \cdot 2^9$ (see [18, p. 104]).

(9) $G \cong A_{14}$. Let H be a maximal subgroup of odd index in G. Then by [19] H is isomorphic to one in the groups in the list

$$\{(S_2 \times S_{12}) \cap A_{14}, (S_4 \times S_{10}) \cap A_{14}, (S_6 \times S_8) \cap A_{14}, (S_2 \wr S_7) \cap A_{14}\}.$$

Consequently, $|G : H|$ divides by 13. The group A_{14} has some Schmidt subgroup $D \cong 13 : 3$. Therefore, for some Sylow 2-subgroup S in G, the group G has the subgroup DS of order $13 \cdot 3 \cdot 2^9$ having odd index not dividing by 13 in G; a contradiction.

(10) $G \cong A_{15}$. The group A_{13} has a maximal subgroup $13 : 6$ (see [18, p. 104]), and so G has some Schmidt subgroup $D \cong 13 : 2$. Therefore, for some Sylow 2-subgroup S in G, the group G has the subgroup DS that is a Hall $\{2, 13\}$-subgroup in G, which is impossible in view of [20].

(11) $G \cong A_{16}$. If H is a maximal subgroup of odd index in G then by [19] H is isomorphic either to $(S_2 \wr S_8) \cap A_{16}$ or $(S_4 \wr S_4) \cap A_{16}$. Consequently, $|G : H|$ divides by 13. Furthermore, arguing by analogy with item (9), we arrive at a contradiction.

(12) $G \cong A_{17}$. If H is a maximal subgroup of odd index in G then by [19] H is isomorphic to A_{16}. Consequently, $|G : H| = 17$. Moreover, G has some dihedron $D \cong 17 : 2$ [19, Table 1]. Therefore, for some Sylow 2-subgroup S in G, the group G has the subgroup DS of odd index not dividing by 17 in G, which is impossible.

(13) $G \cong A_{18}$. If H is a maximal subgroup of odd index in G then by [19] H is isomorphic either to $(S_2 \times S_{16}) \cap A_{18}$ or $(S_2 \wr S_9) \cap A_{18}$. Consequently, $|G : H|$ divides by 17. Further, arguing by analogy with item (12), we arrive at a contradiction.

2. G is a simple sporadic group.

By Lemma 7, G belongs to the list

$$\{M_{11},\ M_{12},\ M_{22},\ J_2,\ HS,\ M^cL,\ Suz,\ He,\ Fi_{22}\}.$$

Consider each of the possible cases:

(1) $G \cong M_{11}$. The group M_{11} has some Schmidt subgroup $D \cong 11 : 5$ (see [18, p. 18]). Then G has the subgroup DS, where S is a Sylow 2-subgroup in G. The order of this subgroup is equal to $11 \cdot 5 \cdot 2^4$. We get a contradiction to the fact that M_{11} has no maximal subgroups whose order divides by $11 \cdot 5 \cdot 2^4$ (see [18, p. 18]).

(2) $G \cong M_{12}$. The group M_{12} has some Schmidt subgroup $D \cong 11 : 5$ [18, p. 33]. Then by hypothesis G has the subgroup DS, where S is a Sylow 2-subgroup in G. The order of this subgroup is

equal to $11{\cdot}5{\cdot}2^6$. However, M_{12} has no maximal subgroups whose order divides by $11{\cdot}5{\cdot}2^6$ (see [18, p. 33]). We again get a contradiction.

(3) $G \cong M_{22}$. The group M_{22} has some Schmidt subgroup $D \cong 11 : 5$ (see [18, p. 39]). Then by hypothesis G has the subgroup DS, where S is a Sylow 2-subgroup of G. The order of this subgroup is equal to $11{\cdot}5{\cdot}2^7$. Since M_{22} has no maximal subgroups whose order divides by $11{\cdot}5{\cdot}2^7$ (see [18, p. 39]), we get a contradiction.

(4) $G \cong J_2$. The group J_2 has some Schmidt subgroup $D \cong 7 : 3$ [18, p. 42]. Then for some Sylow 2-subgroup S of G, the group G has the subgroup DS. Its order is equal to $7 \cdot 3 \cdot 2^7$. We obtain a contradiction to the fact that J_2 has no maximal subgroups whose order divides by $7 \cdot 3 \cdot 2^7$ (see [18, p. 42]).

(5) $G \cong HS$. The group HS has some Schmidt subgroup $D \cong 11 : 5$ (see [18, p. 80]). Then by hypothesis G has the subgroup DS, where S is a Sylow 2-subgroup of G. The order of DS is equal to $11 \cdot 5 \cdot 2^9$. However, HS has no maximal subgroups whose order divides by $11 \cdot 5 \cdot 2^9$ (see [18, p. 80]); a contradiction.

(6) $G \cong M^cL$. The group M^cL has some Schmidt subgroup $D \cong 11 : 5$ (see [18, p. 100]). Then, by hypothesis, for some Sylow 2-subgroup S of G, the group G has the subgroup DS and $|DS| = 11{\cdot}5{\cdot}2^7$. The only class of maximal subgroups in M^cL whose order divides by $11 \cdot 5 \cdot 2^7$ is the class of subgroups isomorphic to M_{22}. Item (3) implies that M_{22} has no subgroups of order $11 \cdot 5 \cdot 2^7$. We again get a contradiction.

(7) $G \cong Suz$. The group Suz has some Schmidt subgroup $D \cong 11 : 5$ (see [18, p. 131]). Then, as in item (6), we can show that G has a subgroup of order $11 \cdot 5 \cdot 2^{13}$. We get a contradiction since Suz has no maximal subgroups whose order divides by $11 \cdot 5 \cdot 2^{13}$ (see [18, p. 131]).

(8) $G \cong He$. The group He has some Schmidt subgroup $D \cong 17 : 2$ (see [18, p. 104]). Then, by hypothesis, for some Sylow 2-subgroup S in G, the group has the subgroup DS whose order is equal to $17 \cdot 2^{10}$. However, He has no maximal subgroups whose order divides by $17 \cdot 2^{10}$ (see [18, p. 104]); a contradiction.

(9) $G \cong Fi_{22}$. The group Fi_{22} has some Schmidt subgroup $D \cong 11 : 5$ (see [18, p. 163]). Then, by hypothesis, for some Sylow 2-subgroup S, the group G has the subgroup DS, and its order is equal to $11 \cdot 5 \cdot 2^{17}$. The only class of maximal subgroups in Fi_{22} whose order divides by $11 \cdot 5 \cdot 2^{17}$ is the class of subgroups isomorphic to $2^{10} : M_{22}$. But item (3) implies that M_{22} has no subgroups of order $11 \cdot 5 \cdot 2^7$. We again get a contradiction.

3. G is a simple group of Lie type over a field of characteristic p.

Let us first consider the case when G is not in the list

$$\{A_5(2),\ C_3(2),\ D_4(2),\ {}^2A_3(2)\}.$$

By Lemma 5, there is a prime $r \in \pi(G)$ that does not divide the order of any parabolic subgroup in G. Moreover, by Lemma 4, G has an r-closed Schmidt subgroup $R : T$. By hypothesis, G has the subgroup $U(R : T)$, where U is a unipotent subgroup of G. By Lemma 6, the subgroup $U(R : T)$ lies in some proper maximal parabolic subgroup P of G. If $P \neq G$ then r divides $|P|$, which is impossible. Consequently, $U(R : T) = G$. In this case,

$$G \in \{L_2(5),\ L_3(2),\ L_2(8),\ L_2(9),\ L_2(17),\ L_3(3),\ U_3(3),\ U_4(2)\}$$

(see, for instance, [16, p. 20]). Since U is a unipotent subgroup in the corresponding group of Lie type, the last factorization holds only in the two cases:

$$L_3(2) = D_8 \cdot (7 : 3) \ \ (p = 2) \quad \text{and} \quad L_2(5) = 5 \cdot A_4 \ \ (p = 5)$$

(see [18]). An easy check shows that these groups have Schmidt subgroups that are not S-conditionally permutable in G.

Thus,

$$G \in \{A_5(2),\ C_3(2),\ D_4(2),\ {}^2A_3(2)\}.$$

Consider all possible cases:

(a) $G \cong SL_6(2)$. Then

$$|G| = 2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31.$$

From [21, Tables 8.24 and 8.18] it follows that G has some Schmidt subgroup $L \cong 31 : 5$. From [21, Table 8.24] we conclude that the order of any maximal subgroup of G does not divide by $31 \cdot 5 \cdot 7^2$. Therefore, L is not permutable with any Sylow 7-subgroup in G, which contradicts the hypothesis.

(b) $G \cong Sp_6(2)$. Then $|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7$. The group $Sp_6(2)$ has some dihedron $D \cong 5 : 2$. Thus, G has the subgroup DS, where S is a Sylow 2-subgroup in G. The order of this subgroup is equal to $2^9 \cdot 5$. The only maximal subgroup in $Sp_6(2)$ whose order divides by $2^9 \cdot 5$ is $2^5 : S_6$ (see [18, p. 46]). Hence, S_6 has a subgroup of order $2^4 \cdot 5$, which is impossible (see [18, p. 4]).

(c) $G \cong \Omega_8^+(2)$. Then $|G| = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$. The group $\Omega_8^+(2)$ has some dihedron $D \cong 5 : 2$. Therefore, by hypothesis, G has the subgroup DS, where S is a Sylow 2-subgroup of G. The order of this subgroup is equal to $2^{12} \cdot 5$. The only maximal subgroup in $\Omega_8^+(2)$ whose order divides by $2^{12} \cdot 5$ is $2^6 : A_8$ (see [18, p. 85]). Consequently, A_8 has a subgroup of order $2^6 \cdot 5$. However, A_8 has no subgroup of this order (see [18, p. 22]). We again get a contradiction.

(d) $G \cong SU_4(2)$. Then $|G| = 2^6 \cdot 3^4 \cdot 5$. The group $SU_4(2)$ has some dihedron $D \cong 5 : 2$. Hence, G has a subgroup DS, where S is a Sylow 2-subgroup in G. The order of this group is equal to $2^6 \cdot 5$. The only maximal subgroup in $SU_4(2)$ whose order divides by $2^6 \cdot 5$ is $2^4 : A_5$ (see [18, p. 26]). It follows that A_5 has a subgroup of order $2^2 \cdot 5$, which is impossible. The so-obtained contradiction completes the proof of the theorem.

3. Proof of Theorem 2

Let G be a group of the least order for which the theorem fails. Then every Schmidt subgroup in G is S-conditionally permutable in each subgroup that contains it and G is not soluble. If M is a maximal subgroup in G and H is a Schmidt subgroup; then, by hypothesis, H is S-conditionally permutable in each subgroup of M that contains it. Hence, by the choice of G, we see that M is soluble.

Thus, G is a minimal nonsoluble group. An easy check shows that $G/\Phi(G)$ is a minimal simple group; i.e., a nonabelian simple group whose all proper subgroups are soluble. The complete list of minimal simple groups is given by Thompson in [22]. This list has the groups

$PSL_2(2^p)$, where p is a prime;

$PSL_2(3^p)$, where p is a prime greater than 3;

$PSL_2(p)$, where p is a prime greater than 5 and $p^2 + 1 \equiv 0 \pmod 5$;

$PSL_3(3)$;

$Sz(2^p)$, p is an odd prime.

Observe that $\Phi(G) \neq 1$ by Theorem 1.

Consider each of the possible cases:

1. Let $G/\Phi(G) \cong PSL_3(3)$. Then $G/\Phi(G)$ has an $S_{\langle 3,2\rangle}$-subgroup $K/\Phi(G)$ isomorphic to $3 : 2$ (see, for example, [18, p. 13]). By Lemma 2, the supplement L to $\Phi(G)$ in K has an $S_{\langle 3,2\rangle}$-subgroup $P : Q$ such that Q is not included in $\Phi(G)$, where $|P| = 3^\alpha$ and $|Q| = 2^\beta$. By hypothesis, G has a Sylow 13-subgroup R such that

$$R(P : Q) = (P : Q)R;$$

i.e., G has a subgroup $R(P:Q)$. Obviously, $R(P:Q)\Phi(G)/\Phi(G)$ has the Sylow 13-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have

$$Q\Phi(G)/\Phi(G) \neq 1.$$

The properties of the Frattini subgroup imply that $R(P:Q)\Phi(G)/\Phi(G)$ is a proper subgroup in $G/\Phi(G)$. Thus, $G/\Phi(G)$ has a proper subgroup whose order divides by 26, which is impossible (see [18, p. 13]).

2. Let $G/\Phi(G) \cong Sz(2^p)$, where p is an odd prime. As follows from [23], $|Sz(2^p)|$ is the product of the four pairwise coprime numbers 2^{2p}, 2^p-1, $2^p+\sqrt{2^{p+1}}+1$, and $2^p-\sqrt{2^{p+1}}+1$. Moreover, $Sz(2^p)$ has a Frobenius subgroup $2^{2p}:(2^p-1)$. Therefore, $G/\Phi(G)$ has an $S_{\langle 2,r\rangle}$-subgroup $K/\Phi(G)$, where r is a prime divisor of 2^p-1. By Lemma 2, the supplement L to the subgroup $\Phi(G)$ in K has an $S_{\langle 2,r\rangle}$-subgroup $P:Q$ such that Q does not lie in $\Phi(G)$, where $|P|=2^\alpha$ and $|Q|=r^\beta$. The group $Sz(2^p)$ has a maximal torus of order $2^p+\sqrt{2^{p+1}}+1$. Let t be a prime divisor of the order of the torus. By hypothesis, G has a Sylow t-subgroup T such that

$$T(P:Q)=(P:Q)T;$$

i.e., G has a subgroup $T(P:Q)$. Obviously, $T(P:Q)\Phi(G)/\Phi(G)$ has the Sylow t-subgroup $T\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have $Q\Phi(G)/\Phi(G)\neq 1$. The properties of the Frattini subgroup imply that $R(P:Q)\Phi(G)/\Phi(G)$ is a proper subgroup in $G/\Phi(G)$.

Thus, $G/\Phi(G)$ has a proper subgroup whose order divides by the primes in $\pi(2^p-1)$ and $\pi(2^p+\sqrt{2^{p+1}}+1)$. But this is impossible in view of [23].

Some description of the subgroups of $PSL_2(q)$ is given by the well-known Dixon Theorem (see, for example, [24, Theorem II.8.27]). In what follows, we will rely upon it without special reference.

3. Let $G/\Phi(G)\cong PSL_2(3^p)$, where p is a prime greater than 3. Observe that

$$|PSL_2(3^p)| = 3^p\cdot(3^p-1)\cdot(3^p+1)/2.$$

Moreover, $3^p-1=2m$, where $(2,m)=1$, and $3^p+1=4f$, where $(2,f)=1$. The group $PSL_2(3^p)$ has a Frobenius subgroup of order $3^p\cdot(3^p-1)/2$ with a supplementary multiplier that is a cyclic group of order $(3^p-1)/2$. Moreover, $(3^p-1)/2$ is odd. Let r be a prime divisor of the latter number. Then $G/\Phi(G)$ has an $S_{\langle 3,r\rangle}$-subgroup $K/\Phi(G)$. By Lemma 2, the supplement L to $\Phi(G)$ in K has an $S_{\langle 3,r\rangle}$-subgroup $P:Q$ such that Q does not lie in $\Phi(G)$, where $|P|=3^\alpha$ and $|Q|=r^\beta$. The group $PSL_2(3^p)$ has a cyclic subgroup of order $(3^p+1)/2$. As was observed above, the number $(3^p+1)/2$ divides by some odd prime t. The hypothesis implies that G has the subgroup $R(P:Q)$. Obviously, $R(P:Q)\Phi(G)/\Phi(G)$ has the Sylow t-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have

$$Q\Phi(G)/\Phi(G)\neq 1.$$

The group $R(P:Q)\Phi(G)/\Phi(G)$ has odd order. Consequently, it is a proper subgroup in $G/\Phi(G)$. Thus, $G/\Phi(G)$ has a proper subgroup whose order divides by an odd prime dividing 3^p-1 and by an odd prime dividing 3^p+1, which is impossible. We again arrive at a contradiction.

4. Let $G/\Phi(G)\cong PSL_2(p)$, where p is a prime greater than 5 and $p^2+1\equiv 0 \pmod 5$. The group $PSL_2(p)$ has a Schmidt subgroup isomorphic to A_4. Therefore, $G/\Phi(G)$ has an $S_{\langle 2,3\rangle}$-subgroup $K/\Phi(G)$. By Lemma 2, the supplement L to $\Phi(G)$ in K has an $S_{\langle 2,3\rangle}$-subgroup $P:Q$ such that Q does not lie in $\Phi(G)$, where $|P|=2^\alpha$ and $|Q|=3^\beta$.

Consider the two possible cases:

(a) Let $3\in\pi(p-1)$. If $p+1$ is a not a power of 2 then we choose an odd $r\in\pi(p+1)$. By hypothesis, G has a Sylow r-subgroup R such that $R(P:Q)=(P:Q)R$; i.e., G has the subgroup $R(P:Q)$. Obviously, $R(P:Q)\Phi(G)/\Phi(G)$ has the Sylow r-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have $Q\Phi(G)/\Phi(G)\neq 1$. Since the order of $R(P:Q)$ does not divide by p, it follows that

$R(P:Q)\Phi(G)/\Phi(G)$ is a proper subgroup in $G/\Phi(G)$. Thus, the group $G/\Phi(G)$ has a proper subgroup whose order divides by the number 3, which divides $p-1$ and an odd prime dividing $p+1$, which is impossible.

Now, let $p+1=2^\alpha$. By hypothesis, G has a Sylow 2-subgroup R such that $R(P:Q)$ is a subgroup in G. Obviously, $R(P:Q)\Phi(G)/\Phi(G)$ has the Sylow 2-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have

$$Q\Phi(G)/\Phi(G)\neq 1.$$

If $p>7$ then $G/\Phi(G)$ has a proper subgroup whose order divides by $3\in\pi(p-1)$ and by the order of a Sylow 2-subgroup in $G/\Phi(G)$, which is impossible.

Thus,

$$G/\Phi(G)\cong PSL_2(7).$$

Then $G/\Phi(G)$ has an $S_{\langle 3,2\rangle}$-subgroup $K/\Phi(G)$ isomorphic to $3:2$. By Lemma 2, the supplement L to $\Phi(G)$ in K has an $S_{\langle 3,2\rangle}$-subgroup $P:Q$ such that Q does not lie in $\Phi(G)$, where $|P|=3^\alpha$ and $|Q|=2^\beta$. By hypothesis, G has a Sylow 7-subgroup R such that

$$R(P:Q)=(P:Q)R;$$

i.e., G has the subgroup $R(P:Q)$. Clearly, $R(P:Q)\Phi(G)/\Phi(G)$ has the Sylow 7-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have $Q\Phi(G)/\Phi(G)\neq 1$. Thus, $G/\Phi(G)$ has a subgroup whose order is equal either to 14 or to 42, which is impossible.

(b) Let $3\in\pi(p+1)$. If $p-1$ is not a power of 2 then choose an odd $r\in\pi(p-1)$. By hypothesis, G has a Sylow r-subgroup R such that $R(P:Q)=(P:Q)R$, i.e., G has the subgroup $R(P:Q)$. Obviously, $R(P:Q)\Phi(G)/\Phi(G)$ has the Sylow r-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have $Q\Phi(G)/\Phi(G)\neq 1$. Since the order of $R(P:Q)$ does not divide by p, we have that $R(P:Q)\Phi(G)/\Phi(G)$ is a proper subgroup of $G/\Phi(G)$. Thus, $G/\Phi(G)$ has a proper subgroup whose order divides by 3, which divides by $p+1$, and an odd prime dividing $p-1$. This is impossible.

Let $p-1=2^n$. By hypothesis, G has a Sylow 2-subgroup R such that $R(P:Q)=(P:Q)R$; i.e., G has the subgroup $R(P:Q)$. Obviously, $R(P:Q)\Phi(G)/\Phi(G)$ has the Sylow 2-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have $Q\Phi(G)/\Phi(G)\neq 1$. In this case, $G/\Phi(G)$ has a proper subgroup whose order divides by $3\in\pi(p+1)$ and by the order of a Sylow 2-subgroup of $G/\Phi(G)$.

Since $p>7$, a Sylow 2-subgroup of $G/\Phi(G)$ has order greater than 8. Since $p+1\not\equiv 0 \pmod 4$, it follows that the above-mentioned proper subgroup cannot lie in any maximal subgroup of $G/\Phi(G)$. This is impossible.

5. Let $G/\Phi(G)\cong PSL_2(2^p)$, where p is a prime.

Consider the cases $p=2$, $p=3$, and $p>3$.

(a) Let $G/\Phi(G)\cong PSL_2(4)$. Assume that $5\notin\pi(\Phi(G))$. Then $G/\Phi(G)$ has an $S_{\langle 5,2\rangle}$-subgroup $K/\Phi(G)$ isomorphic to $5:2$, and by Lemma 2 the supplement L to $\Phi(G)$ in K has an $S_{\langle 5,2\rangle}$-subgroup $P:Q$ such that Q does not lie in $\Phi(G)$ and P does not lie $\Phi(G)$, where $|P|=5^\alpha$ and $|Q|=2^\beta$. By hypothesis, G has a Sylow 3-subgroup R such that $R(P:Q)=(P:Q)R$; i.e., G has the subgroup $R(P:Q)$. Obviously, $R(P:Q)\Phi(G)/\Phi(G)$ has the Sylow 3-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Clearly, $R(P:Q)\Phi(G)/\Phi(G)$ is a proper subgroup in $G/\Phi(G)$. Thus, $G/\Phi(G)$ has a proper subgroup of order 30, which is impossible. Consequently, $5\in\pi(\Phi(G))$.

Let $S/\Phi(G)$ be a subgroup of order 3 in $G/\Phi(G)$. Obviously, $S=\langle x\rangle\Phi(G)$ for some 3-element x of G. Suppose that S is not a nilpotent group. Then it is 3-nilpotent and has a 3-nilpotent Schmidt subgroup D not lying in $\Phi(G)$. Therefore, $S=D\Phi(G)$. By hypothesis, G has a Sylow 5-subgroup A such that $AD=DA$; i.e., G contains DA. Obviously, $AD\Phi(G)/\Phi(G)=AS\Phi(G)$ has the Sylow 5-subgroup $A\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Thus, $G/\Phi(G)$ has a subgroup of order 15, which is impossible.

Thus, S is a nilpotent group. Let H be a Sylow 5-subgroup in $\Phi(G)$. Obviously, $C_G(H) \trianglelefteq G$ and $C_G(H)$ does not lie in $\Phi(G)$. Therefore, $C_G(H) = G$. Thus, $H \subseteq Z(G)$. We get a contradiction to the fact that the Schur multiplier of $PSL_2(4)$ has order 2 (see, for example, [16]).

(b) Let $G/\Phi(G) \cong PSL_2(8)$. Then

$$|PSL_2(8)| = 2^3 \cdot 3^2 \cdot 7.$$

Let $S/\Phi(G)$ be a Sylow 7-subgroup of $G/\Phi(G)$. Obviously, $S = \langle x \rangle \Phi(G)$ for some 7-element x of G. Suppose that S is not nilpotent. Then it is 7-nilpotent and has a 7-nilpotent Schmidt subgroup D not lying in $\Phi(G)$. Therefore, $S = D\Phi(G)$. By hypothesis, G has a Sylow 3-subgroup A such that $AD = DA$, i.e., G contains DA. Obviously, $AD\Phi(G)/\Phi(G) = AS\Phi(G)$ has a Sylow 3-subgroup $A\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Then $G/\Phi(G)$ has a subgroup of order 63, which is impossible.

Thus, S is a nilpotent group. Let H be a Hall $\{2,3\}$-subgroup in $\Phi(G)$. Obviously, $C_G(H) \trianglelefteq G$ and $C_G(H)$ does not lie in $\Phi(G)$. Therefore, $C_G(H) = G$. Thus, $H \subseteq Z(G)$. Since the Schur multiplier of $PSL_2(8)$ is trivial (see, for example, [16]), $H = 1$.

Let $F/\Phi(G)$ be a Sylow 3-subgroup in $G/\Phi(G)$. Obviously, $F = \langle x \rangle \Phi(G)$ for some 3-element x of G. Suppose that F is not nilpotent. Then it is 7-nilpotent and has a 7-nilpotent Schmidt subgroup B not lying in $\Phi(G)$. Therefore, $F = B\Phi(G)$. By hypothesis, G has a Sylow 7-subgroup L such that $LB = BL$; i.e., G has the subgroup BL. Obviously, $BL\Phi(G)/\Phi(G) = FL\Phi(G)$ has the Sylow 7-subgroup $L\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Then $G/\Phi(G)$ has a proper subgroup whose order divides 21, which is impossible.

Thus, F is a nilpotent group. Obviously, $C_G(\Phi(G)) \trianglelefteq G$ and $C_G(\Phi(G))$ does not lie in $\Phi(G)$. Therefore,

$$C_G(\Phi(G)) = G.$$

Consequently, $H \subseteq Z(G)$. Since the Schur multiplier of $PSL_2(8)$ is trivial (see, for example, [16]), we have $\Phi(G) = 1$.

Now, by hypothesis, all Schmidt subgroups in $PSL_2(8)$ are S-conditionally permutable. We get a contradiction to Theorem 1.

(c) Let $G/\Phi(G) \cong PSL_2(2^p)$, where p is a prime greater than 3. Note that $|PSL_2(2^p)|$ is the product of three pairwise coprime numbers 2^{2p}, 2^p-1, and 2^p+1. Moreover, for $p > 3$, the number of prime divisors of the order of $PSL_2(2^p)$ is at least 4 (see, for example, [16, p. 20]). The group $PSL_2(2^p)$ has a Frobenius subgroup of order $2^p \cdot (2^p - 1)$ with supplementary multiplier that is a cyclic group of order $2^p - 1$. Let r be a prime divisor of this number. Then $G/\Phi(G)$ has an $S_{\langle 2,r \rangle}$-subgroup $K/\Phi(G)$. By Lemma 2, the supplement L to $\Phi(G)$ in K has an $S_{\langle 2,r \rangle}$-subgroup $P : Q$ such that Q does not lie in $\Phi(G)$, where $|P| = 2^\alpha$ and $|Q| = r^\beta$. The group $PSL_2(2^p)$ has a cyclic subgroup of order $2^p + 1$. Let $t \in \pi(2^p + 1)$ and let R be a Sylow t-subgroup in G. The hypothesis implies that G has the subgroup $R(P : Q)$. Obviously, $R(P : Q)\Phi(G)/\Phi(G)$ has the Sylow t-subgroup $R\Phi(G)/\Phi(G)$ of $G/\Phi(G)$. Since Q does not lie in $\Phi(G)$, we have $Q\Phi(G)/\Phi(G) \neq 1$. Since

$$|\pi(R(P : Q)\Phi(G)/\Phi(G))| \leq 3,$$

it follows that $R(P : Q)\Phi(G)/\Phi(G)$ is a proper subgroup in $G/\Phi(G)$. Thus, $G/\Phi(G)$ has a proper subgroup whose order divides by an odd prime dividing $2^p - 1$ and by an odd prime dividing $2^p + 1$, which is impossible. The contradiction finishes the proof of the theorem.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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