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A SPECTRAL CRITERION FOR POWER-LAW CONVERGENCE RATE IN THE ERGODIC THEOREM FOR $\mathbb{Z}^d$ AND $\mathbb{R}^d$ ACTIONS

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Abstract—We prove the equivalence of the power-law convergence rate in the L_2-norm of ergodic averages for $\mathbb{Z}^d$ and $\mathbb{R}^d$ actions and the same power-law estimate for the spectral measure of symmetric d-dimensional parallelepipeds: for the degrees that are roots of some special symmetric polynomial in d variables. Particularly, all possible range of power-law rates is covered for $d = 1$.

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1. Introduction

1.1. Ergodic averages. Let $\mathscr{G}$ be the group $\mathbb{Z}^d$ or $\mathbb{R}^d$, with $d \geq 1$, and let (Ω, λ) be some space with probability measure λ on which $\mathscr{G}$ acts by the invertible measure preserving transformations $\tau_g : \Omega \to \Omega$, with $g \in \mathscr{G}$; i.e., the sets $\tau_g^{-1}(E)$ and $\tau_g(E)$ are measurable for all $g \in \mathscr{G}$, measurable $E \subseteq \Omega$, and $\lambda\big(\tau_g^{-1}(E)\big) = \lambda(E)$. The group property (in the additive notation) means that $\tau_{g_1}(\tau_{g_2}\omega) = \tau_{g_1+g_2}\omega$ for all $g_1, g_2 \in \mathscr{G}$ and $\omega \in \Omega$. It is not hard to check that the group $\{\tau_g : g \in \mathbb{Z}^d\}$ is finitely generated by the commuting automorphisms $T_k := \tau_{e_{d,k}}$, where $\{e_{d,k}\}_{k=1}^d$ are the vectors of the standard basis for $\mathbb{R}^d$, and the group $\{\tau_g : g \in \mathbb{R}^d\}$ is the product of one-dimensional commuting flows, i.e., $\tau_g = T_1^{t_1} T_2^{t_2} \cdots T_d^{t_d}$, where $g = \sum_{k=1}^d t_k e_{d,k}$ and $T_k^{t_k} := \tau_{t_k e_{d,k}}$, $t_k \in \mathbb{R}$.

The subgroups generated by the transformations $T_{j_1}, T_{j_2}, \dots, T_{j_k}$ or by the one-dimensional flows $T_{j_1}^{t_{j_1}}, T_{j_2}^{t_{j_2}}, \dots, T_{j_k}^{t_{j_k}}$ respectively will be denoted by $\mathscr{G}_{\mathbf{j}}$, where $\mathbf{j} = \sum_{n=1}^k e_{d,j_n}$. Denote the set of such multi-indices, i.e., vectors in $\mathbb{R}^d$ with entries 0 or 1 (which are the nonzero vertices of the standard unit cube in $\mathbb{R}^d$) by $\mathscr{V}_d$. Clearly, $\mathscr{G}_{\mathbf{j}} \simeq \mathbb{Z}^j$ or $\mathscr{G}_{\mathbf{j}} \simeq \mathbb{R}^j$ respectively, where $j = \|\mathbf{j}\|_1$.

Let ν_d be the Lebesgue measure on $\mathbb{R}^d$ or the counting measure on $\mathbb{Z}^d$. Let $\mathscr{I}$ be a directed set of indices numbering some family of subsets $G_\alpha \subset \mathscr{G}$, with $\alpha \in \mathscr{I}$ such that

$$\bigcup_{\alpha\in\mathscr{I}} G_\alpha = \mathscr{G}, \quad 0 < \nu(G_\alpha) \leq \nu(G_\beta) < +\infty \quad \text{for } \alpha \leq \beta.$$

Clearly,

$$\lim_{\alpha\in\mathscr{I}} \nu_d(G_\alpha) = +\infty.$$

Given $f \in L_1(\Omega, \lambda)$, define the ergodic means

$$A_\alpha f(\omega) = \frac{1}{\nu_d(G_\alpha)} \int\limits_{G_\alpha} f(\tau_g\omega)\, d\nu_d(g), \quad \omega \in \Omega,\ \alpha \in \mathscr{I}.$$

In particular, for each of the groups $\mathbb{Z}^d$ or $\mathbb{R}^d$, the averages along parallelepipeds are written down as

$$A_{\mathbf{n}} f(\omega) = \frac{1}{n_1 n_2 \cdots n_d} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} f(T_1^{k_1} \cdots T_d^{k_d}\omega), \quad \mathbf{n} \in \mathbb{N}^d, \tag{1}$$

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and, respectively,

$$A_{\mathbf{t}} f(\omega) = \frac{1}{t_1 t_2 \cdots t_d} \int_0^{t_1} \cdots \int_0^{t_d} f(T_1^{\tau_1} \cdots T_d^{\tau_d} \omega)\, d\tau_1 \dots d\tau_d, \quad \mathbf{t} \in \mathbb{R}_+^d. \tag{2}$$

1.2. Ergodic theorems. In 1951, Dunford [1] and Zygmund [2] proved the convergence a.e. of averages (1) and (2) without the commutability condition of the generating automorphisms and respectively flows for $f \in L \log^{d-1} L(\Omega, \lambda)$. Moreover, the limit function f^* is defined as

$$f^*(\omega) = \mathbb{E}_1 \mathbb{E}_2 \cdots \mathbb{E}_d f(\omega),$$

where $\mathbb{E}_j$ is the operator of conditional expectation with respect to the σ-algebra of measurable T_j-invariant sets (respectively, T_j^t-invariant sets); see, for example, [3, § 6.1] and [4, Chapter VIII, § 7]. Dunford and later Dunford and Schwartz proved convergence in the mean in L_p, with $p > 1$, for $f \in L_p(\Omega, \lambda)$ and such averages (in the more general situation than that of measure-preserving transformations, of the so-called L_1–L_∞-contractions). We mention also the recent paper [5] in which convergence a.e. in the Dunford–Zygmund Theorem is proved for $f \in L \log^{n-1} L(\Omega, \lambda)$, where $n \le d$ is the rank of the dynamical system.

Additional information about ergodic theorems for other group actions can be found in Tempelman's monograph [6] and Nevo's paper [7]. In the present article, we study the convergence rate of averages (1) and (2) in $L_2(\Omega, \lambda)$ by applying the well-developed spectral theory of the unitary representations of $\mathbb{Z}^d$ and $\mathbb{R}^d$. The unitary operators under consideration are the Koopman operators

$$U_g f(\omega) = f(\tau_g \omega), \quad g \in \mathscr{G}.$$

1.3. Spectral measures. Given a unitary representation U_g of a locally compact abelian group $\mathscr{G}$ and $f \in L_2(\Omega, \lambda)$, there exists (see [8,9]) a nonnegative Borel spectral measure $\sigma_f^{\mathscr{G}}$ on the group $\mathscr{G}^\wedge$ of all characters of $\mathscr{G}$ which is defined by the equalities

$$(U_g f, f)_{L_2(\Omega,\lambda)} = \int_{\mathscr{G}^\wedge} \chi(g)\, d\sigma_f^{\mathscr{G}}(\chi), \quad g \in \mathscr{G}.$$

For $\mathbb{R}^k$, the character group is $\mathbb{R}^k$; and for $\mathbb{Z}^k$, the character group is the torus $\mathbb{T}^k = \mathbb{R}^k / 2\pi\mathbb{Z}^k = (-\pi, \pi]^k$. Consequently, for the groups $\mathscr{G}_{\mathbf{k}}$ of Koopman operators, we have

$$\big(f\big(T_{j_1}^{n_1} T_{j_2}^{n_2} \cdots T_{j_k}^{n_k} \omega\big), f(\omega)\big) = \int_{(-\pi,\pi]^k} e^{i(\mathbf{n},\mathbf{s})}\, d\sigma_f^{\mathscr{G}_{\mathbf{k}}}(\mathbf{s}), \quad \mathbf{n} \in \mathbb{Z}^k,$$

$$\big(f\big(T_{j_1}^{t_1} T_{j_2}^{t_2} \cdots T_{j_k}^{t_k} \omega\big), f(\omega)\big) = \int_{\mathbb{R}^k} e^{i(\mathbf{t},\mathbf{s})}\, d\sigma_f^{\mathscr{G}_{\mathbf{k}}}(\mathbf{s}), \quad \mathbf{t} \in \mathbb{R}^k,$$

where $(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^k x_n y_n$ is the inner product in $\mathbb{R}^k$. Considering the measure $\sigma_f^{\mathscr{G}_{\mathbf{1}}}$ and $\mathbf{1} = (1, \dots, 1)$, we will also use the notation σ_f. To differ numbers from vectors, we will denote the vectors by boldface symbols. There will be an exclusion for $x \in \mathbb{R}^d$.

1.4. Description of the results. Our goal in the present article is to find the conditions for the ergodic means (1) and (2) to decrease in norm in the power-law form. Namely, for $f \in L_2(\Omega, \lambda)$ we can find a constant $B > 0$ and $(\alpha_1, \dots, \alpha_d) \ge \mathbf{0}$ such that

$$\|A_{\mathbf{t}} f\|_2^2 \le \frac{B}{t_1^{\alpha_1} \cdots t_d^{\alpha_d}}, \quad \mathbf{t} > \mathbf{0}; \quad \|A_{\mathbf{n}} f\|_2^2 \le \frac{B}{n_1^{\alpha_1} \cdots n_d^{\alpha_d}}, \quad \mathbf{n} \in \mathbb{N}^d. \tag{3}$$

Also, for $\mathbb{R}^d$, $\mathbb{Z}^d$, and $f \in L_2(\Omega, \lambda)$, we will assume a power-law estimate for the spectral measure of symmetric intervals, i.e., for some constants $A > 0$ and $(\alpha_1, \dots, \alpha_d) \ge \mathbf{0}$ for all $\boldsymbol{\delta} = (\delta_1, \dots, \delta_d) > \mathbf{0}$

$$\sigma_f(\Pi(\delta_1, \dots, \delta_d)) \le A \delta_1^{\alpha_1} \cdots \delta_d^{\alpha_d}, \tag{4}$$

where $\Pi(\delta_1, \dots, \delta_d) := (-\delta_1, \delta_1] \times \cdots \times (-\delta_d, \delta_d]$.

The main result is the following criterion for the power-law convergence of ergodic averages, which generalizes the well-known one-dimensional situation [10, 11].

Theorem 1. *Suppose that $\alpha_k \in [0,2]$, $1 \le k \le d$, and condition* (3) *is fulfilled. Then inequality* (4) *holds with the constant*

$A = B\rho(\alpha_1)\cdots\rho(\alpha_d)$ (*discrete time*);

$A = B\frac{\rho(\alpha_1)}{2^{\alpha_1}}\cdots\frac{\rho(\alpha_d)}{2^{\alpha_d}}$ (*continuous time*),

where $\rho(\beta) = \inf_{x>0}\frac{x^{2-\beta}}{\sin^2 x}$.

Conversely, some family of special symmetric polynomials P_d^κ of d variables with parameter $\kappa \in [0,2)$ can be explicitly distinguished such that if $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d) \ge \mathbf{0}$ is a root of one of them then condition (4) *on the spectral measure implies* (3) *with the constant*

$B = 2d!\frac{\pi^{\alpha_1+\dots+\alpha_d}A}{2-\kappa}$ (*discrete time*);

$B = 2d!\frac{2^{\alpha_1+\dots+\alpha_d}A}{2-\kappa}$ (*continuous time*).

The proof of this criterion consists of the two steps: The forward implication is the contents of Theorem 2; while the backward implication is presented in Theorem 4(1). The symmetric polynomials in the statement of the criterion are discussed in Section 4 where we also prove Theorem 4. In Theorem 3 of Section 3 we carry out a detailed discussion of the case $d = 2$ by using a new approach to estimating the norms of ergodic averages. In Section 2, we give the necessary constructions and prove Theorem 2.

2. Auxiliary Assertions

2.1. The projective property of spectral measures. The spectral measures $\{\sigma_f^{\mathscr{G}_\mathbf{j}}\}_{\mathbf{j}\in\mathscr{V}_d}$ constitute a projective system of measures. This means the following (see, for example, [12, 9.12(i)]). Denote the consecutive numbering of nonzero coordinates of a multi-index $\mathbf{j} \in \mathscr{V}_d$ by $\ell(\mathbf{j})_n$. Given multi-indices $\mathbf{k},\mathbf{j} \in \mathscr{V}_d$ such that $\mathbf{j} \le \mathbf{k}$ (i.e., $\mathbf{j}_n \le \mathbf{k}_n$ for all $1 \le n \le d$), consider the projection operators

$$\pi_{\mathbf{j},\mathbf{k}} : \mathbb{R}^k \to \mathbb{R}^j \text{ and } \pi_\mathbf{k} := \pi_{\mathbf{k},\mathbf{1}} : \mathbb{R}^d \to \mathbb{R}^k,$$

where $k = \|\mathbf{k}\|_1$, $j = \|\mathbf{j}\|_1$, and $\mathbf{1} = (1,1,\dots,1)$ defined by the chain of transformations:

$$\pi_{\mathbf{j},\mathbf{k}} : \sum_{n=1}^{k} x_n e_{k,n} \mapsto \sum_{n=1}^{k} x_n e_{d,\ell(\mathbf{k})_n}$$

$$\mapsto \sum_{n=1}^{k} \mathbf{j}_{\ell(\mathbf{k})_n} x_n e_{d,\ell(\mathbf{k})_n} = \sum_{n=1}^{j} y_n e_{d,\ell(\mathbf{j})_n} \mapsto \sum_{n=1}^{j} y_n e_{j,n}.$$

The projection $\pi_{\mathbf{j},\mathbf{k}}$ associates with each k-vector $\mathbf{x}$ the j-dimensional vector $\mathbf{y}$ that is obtained as follows: First, the k-dimensional vector $\mathbf{x}$ is taken to a d-dimensional vector in which the coordinates x_n fill up the unit coordinates of the multi-index $\mathbf{k}$. Then, only those coordinates in this d-dimensional vector remain nonzero that correspond to unities in $\mathbf{j}$. The so-obtained vector $\mathbf{y}$ is taken to $\mathbb{R}^j$. It is not hard to see that if $\mathbf{h} \le \mathbf{j} \le \mathbf{k}$ then

$$\pi_{\mathbf{h},\mathbf{j}}\pi_{\mathbf{j},\mathbf{k}} = \pi_{\mathbf{h},\mathbf{k}}, \quad \pi_{\mathbf{j},\mathbf{k}}\pi_\mathbf{k} = \pi_\mathbf{j}.$$

Proposition 1. *Given multi-indices $\mathbf{j},\mathbf{k} \in \mathscr{V}_d$ such that $\mathbf{j} \le \mathbf{k}$, we have*

$$\sigma_f^{\mathscr{G}_\mathbf{j}} = (\pi_{\mathbf{j},\mathbf{k}})_*\sigma_f^{\mathscr{G}_\mathbf{k}} := \sigma_f^{\mathscr{G}_\mathbf{k}} \circ \pi_{\mathbf{j},\mathbf{k}}^{-1}.$$

PROOF. Consider the case of $\mathbb{Z}^d$ since the arguments are analogous for $\mathbb{R}^d$. Given nonnegative integers $n_1, n_2, \dots, n_j$ we obtain

$$\int\limits_{(-\pi,\pi]^j} e^{i(\mathbf{n},\mathbf{s})}\,d\sigma_f^{\mathscr{G}_\mathbf{j}}(\mathbf{s}) = \big(f\big(T_{\ell(\mathbf{j})_1}^{n_1}T_{\ell(\mathbf{j})_2}^{n_2}\cdots T_{\ell(\mathbf{j})_j}^{n_j}\omega\big), f(\omega)\big)$$

$$= \big(f\big(T_{\ell(\mathbf{k})_1}^{\widetilde{n}_1}T_{\ell(\mathbf{k})_2}^{\widetilde{n}_2}\cdots T_{\ell(\mathbf{k})_k}^{\widetilde{n}_k}\omega\big), f(\omega)\big) = \int\limits_{(-\pi,\pi]^k} e^{i(\widetilde{\mathbf{n}},\mathbf{t})}\,d\sigma_f^{\mathscr{G}_\mathbf{k}}(\mathbf{t}),$$

where $\widetilde{\mathbf{n}} \in \mathbb{Z}_+^k$ is defined as follows: Since $\{\ell(\mathbf{j})_p\}_{p=1}^j \subset \{\ell(\mathbf{k})_q\}_{q=1}^k$, if $\ell(\mathbf{k})_q = \ell(\mathbf{j})_p$ then we put $\widetilde{n}_q = n_p$. All remaining values of $\widetilde{n}_q$ are 0. It follows that

$$(\widetilde{\mathbf{n}}, \mathbf{t})_{\mathbb{R}^k} = (\mathbf{n}, \pi_{\mathbf{j},\mathbf{k}}(\mathbf{t}))_{\mathbb{R}^j}.$$

Therefore,

$$\int\limits_{(-\pi,\pi]^j} e^{i(\mathbf{n},\mathbf{s})}\, d\sigma_f^{\mathscr{G}_\mathbf{j}}(\mathbf{s}) = \int\limits_{(-\pi,\pi]^k} e^{i(\widetilde{\mathbf{n}},\mathbf{t})}\, d\sigma_f^{\mathscr{G}_\mathbf{k}}(\mathbf{t})$$
$$= \int\limits_{(-\pi,\pi]^k} e^{i(\mathbf{n},\pi_{\mathbf{j},\mathbf{k}}(\mathbf{t}))}\, d\sigma_f^{\mathscr{G}_\mathbf{k}}(\mathbf{t}) = \int\limits_{(-\pi,\pi]^j} e^{i(\mathbf{n},\mathbf{s})}\, d(\pi_{\mathbf{j},\mathbf{k}})_*\sigma_f^{\mathscr{G}_\mathbf{k}}(\mathbf{s}).$$

Thus, all Fourier coefficients of the measures $\sigma_f^{\mathscr{G}_\mathbf{j}}$ and $(\pi_{\mathbf{j},\mathbf{k}})_*\sigma_f^{\mathscr{G}_\mathbf{k}}$ coincide, and so the measures themselves coincide [13, 3.8.6]. $\square$

Corollary 1. *If $1 \le j \le d$ then*

$$\sigma_f^{\mathscr{G}_\mathbf{1}}(\{x_j = 0\}) = \|\mathbb{E}_j f\|_2^2.$$

PROOF. We have

$$\sigma_f^{\mathscr{G}_\mathbf{1}}(\{x_j = 0\}) = \sigma_f^{\mathscr{G}_\mathbf{1}}\left(\pi_{e_j}^{-1}\{\mathbf{0}\}\right) = \sigma_f^{\mathscr{G}_{e_j}}(\{\mathbf{0}\}) = \|\mathbb{E}_j f\|_2^2. \quad \square$$

Observe that condition (4) on the spectral measure implies $\|\mathbb{E}_j f\|_2 = 0$ for $\alpha_j > 0$. Indeed,

$$\begin{aligned}\|\mathbb{E}_j f\|_2^2 = \sigma_f(\{x_j = 0\}) &= \lim_{\delta_k \to \delta_\infty} (\lim_{\delta_j \to 0} \sigma_f(\Pi(\delta_1, \dots, \delta_d))) \\ &\le A \lim_{\delta_k \to \delta_\infty} (\lim_{\delta_j \to 0} \delta_1^{\alpha_1} \cdots \delta_d^{\alpha_d}) = 0.\end{aligned}$$

Here $\delta_\infty = \pi$ for $\mathbb{Z}^d$ and $\delta_\infty = \infty$ for $\mathbb{R}^d$.

2.2. The spectral representation of norms. Consider the two families of functions $\mathscr{F}_n : (-\pi, \pi] \to \mathbb{R}_+$ and $F_t : \mathbb{R} \to \mathbb{R}_+$ defined as

$$\mathscr{F}_n(x) = \frac{1}{n^2} \frac{\sin^2 \frac{nx}{2}}{\sin^2 \frac{x}{2}},\ 0 < |x| \le \pi, \quad \mathscr{F}_n(0) = 1,\ n \in \mathbb{N};$$

$$F_t(x) = \left(\frac{\sin \frac{tx}{2}}{\frac{tx}{2}} \right)^2,\ x \ne 0, \quad F_t(0) = 1,\ t \in \mathbb{R}.$$

The functions $\mathscr{F}_n$ differ from the classical Fejer kernels only by a constant.

Proposition 2. *The norms of* (1) *and* (2) *admit the integral representations*

$$\|A_\mathbf{n} f\|_2^2 = \int\limits_{(-\pi,\pi]^d} \mathscr{F}_{n_1}(x_1) \dots \mathscr{F}_{n_d}(x_d)\, d\sigma_f(x), \quad \mathbf{n} \in \mathbb{N}^d; \tag{5}$$

$$\|A_\mathbf{t} f\|_2^2 = \int\limits_{\mathbb{R}^d} F_{t_1}(x_1) \cdots F_{t_d}(x_d)\, d\sigma_f(x), \quad \mathbf{t} \in \mathbb{R}_+^d. \tag{6}$$

PROOF. The discrete variant (formula (5)) for $\mathbb{Z}^d$ was in essence examined in [14], where averages over cubes were considered (see also the discussion in [15, Lemma 4.1]). For the completeness of exposition,

we prove (6) for $\mathbb{R}^d$. Consider the general case and then pass to the particular case, averages over parallelepipeds. We infer

$$\|A_\alpha f\|_2^2 = (A_\alpha f, A_\alpha f)$$

$$= \left(\frac{1}{\nu_d(G_\alpha)} \int\limits_{G_\alpha} f(\tau_g \omega) d\nu_d(g), \frac{1}{\nu_d(G_\alpha)} \int\limits_{G_\alpha} f(\tau_g \omega)\, d\nu_d(g) \right)$$

$$= \frac{1}{\nu_d^2(G_\alpha)} \int\limits_{G_\alpha} \int\limits_{G_\alpha} (f(\tau_g \omega), f(\tau_{g'} \omega))\, d\nu_d(g) d\nu_d(g')$$

$$= \frac{1}{\nu_d^2(G_\alpha)} \int\limits_{G_\alpha} \int\limits_{G_\alpha} (f(\tau_{g-g'} \omega), f(\omega))\, d\nu_d(g) d\nu_d(g')$$

$$= \frac{1}{\nu_d^2(G_\alpha)} \int\limits_{G_\alpha} \int\limits_{G_\alpha} \int\limits_{\mathscr{G}^\wedge} e^{i(g-g',s)}\, d\sigma_f(s)\, d\nu_d(g) d\nu_d(g')$$

$$= \frac{1}{\nu_d^2(G_\alpha)} \int\limits_{\mathscr{G}^\wedge} \int\limits_{G_\alpha} e^{i(g,s)}\, d\nu_d(g) \int\limits_{G_\alpha} e^{-i(g',s)}\, d\nu_d(g')\, d\sigma_f(s)$$

$$= \frac{1}{\nu_d^2(G_\alpha)} \int\limits_{\mathscr{G}^\wedge} \left| \int\limits_{G_\alpha} e^{i(g,s)}\, d\nu_d(g) \right|^2 d\sigma_f(s).$$

Thus,

$$\|A_\alpha f\|_2^2 = \int\limits_{\mathscr{G}^\wedge} |\mathscr{F}[\nu_d | G_\alpha](s)|^2\, d\sigma_f(s), \tag{7}$$

where

$$\mathscr{F}[\nu_d | G_\alpha](s) = \int\limits_{\mathscr{G}} e^{i(g,s)}\, d\nu_d | G_\alpha(g)$$

is the Fourier transform of the probability measure $\nu_d | G_\alpha$, where

$$\nu_d | G_\alpha(E) = \frac{\nu_d(E \cap G_\alpha)}{\nu_d(G_\alpha)}.$$

For $\mathbb{R}^d$, this is the unnormalized Fourier transform in $\mathbb{R}^d$ of the indicator $I_{\mathscr{G}_\alpha}$ of $G_\alpha \subset \mathbb{R}^d$. For the averages over parallelepipeds, we have the product of the cardinal sines

$$\int\limits_0^{t_1} \cdots \int\limits_0^{t_d} e^{-i(g,s)}\, dg = \prod_{k=1}^d \int\limits_0^{t_k} e^{-ig_k s_k}\, dg_k = \frac{2\sin\left(\frac{t_1 s_1}{2}\right)}{s_1} \cdots \frac{2\sin\left(\frac{t_d s_d}{2}\right)}{s_d} e^{-i(t,s)/2}.$$

Hence, inserting the result in the general formula, we get (6). $\square$

2.3. The spectral measures of neighborhoods of zero. Let us show how the norms of the averages are estimated from below through the spectral measures of neighborhoods of zero in the form of parallelepipeds. We will need the following relations for the kernels $\mathscr{F}_n, F_t$:

$$\frac{1}{[t]^2} \frac{\sin^2 \frac{[t]x}{2}}{\sin^2 \frac{x}{2}} \geq \left(\frac{\sin \frac{tx}{2}}{\frac{tx}{2}} \right)^2, \quad x \in \mathbb{R},\ t \geq 1,\ |tx| < 2\pi. \tag{8}$$

The following assertion is an analog of familiar estimates [16, Lemma 2; 17, Lemma 2] of the norms of (1) and (2). Given $\mathbf{s} = (s_1, \ldots, s_d) \geq \mathbf{1}$, we put $[\mathbf{s}] := ([s_1], \ldots, [s_d])$.

Proposition 3. *The norms of* (1) *and* (2) *satisfy the lower estimates: For all* $\mathbf{a} \in (0,\pi]^d$, $\mathbf{s} \in \mathbb{R}^d$, *with* $\mathbf{s} \geq \mathbf{1}$, *and* $\mathbf{t} \in \mathbb{R}^d_+$,

$$F_1(a_1)\cdots F_1(a_d)\,\sigma_f\left(\Pi\left(\frac{a_1}{s_1},\ldots,\frac{a_d}{s_d}\right)\right) \leq \|A_{[\mathbf{s}]}f\|_2^2; \tag{9}$$

$$F_1(a_1)\cdots F_1(a_d)\,\sigma_f\left(\Pi\left(\frac{a_1}{t_1},\ldots,\frac{a_d}{t_d}\right)\right) \leq \|A_{\mathbf{t}}f\|_2^2. \tag{10}$$

Proof. For proving (9), we use (5) and (8):

$$\|A_{[\mathbf{s}]}f\|_2^2 = \int\limits_{(-\pi,\pi]^d} \mathscr{F}_{[s_1]}(x_1)\ldots\mathscr{F}_{[s_d]}(x_d)\,d\sigma_f(x) \geq \int\limits_{\Pi\left(\frac{a_1}{s_1},\ldots,\frac{a_d}{s_d}\right)} \mathscr{F}_{[s_1]}(x_1)\ldots\mathscr{F}_{[s_d]}(x_d)\,d\sigma_f(x)$$

$$\geq \int\limits_{\Pi\left(\frac{a_1}{s_1},\ldots,\frac{a_d}{s_d}\right)} F_{s_1}(x_1)\ldots F_{s_d}(x_d)\,d\sigma_f(x) \geq \prod_{k=1}^{d}\min_{|x_k|\leq\frac{a_k}{s_k}}\left(\frac{\sin\frac{s_kx_k}{2}}{\frac{s_kx_k}{2}}\right)^2\sigma_f\left(\Pi\left(\frac{a_1}{s_1},\ldots,\frac{a_d}{s_d}\right)\right)$$

$$= \prod_{k=1}^{d}\min_{|y|\leq\frac{a_k}{2}}\left(\frac{\sin y}{y}\right)^2\sigma_f\left(\Pi\left(\frac{a_1}{n_1},\ldots,\frac{a_d}{n_d}\right)\right) = F_1(a_1)\cdots F_1(a_d)\,\sigma_f\left(\Pi\left(\frac{a_1}{s_1},\ldots,\frac{a_d}{s_d}\right)\right).$$

Inequality (10) is proved likewise. □

Given $\beta \in [0,2]$, put

$$\rho(\beta) = \inf_{x>0}\frac{x^{2-\beta}}{\sin^2 x}.$$

It is not hard to verify that $\rho(0) = \rho(2) = 1$. For $\beta \in (0,2)$, the infimum is attained at the first positive root of the equation $\tan x = \frac{2x}{2-\beta}$. It is also clear that $1 \leq \rho(\beta) \leq \sin^{-2}(1)$.

Proposition 3 implies

Theorem 2. *Suppose that the norms of* (1) *and* (2) *decrease by power law; i..e.,* (3) *holds with constants* $B > 0$ *and* $\alpha_j \in [0,2]$, $1 \leq j \leq d$. *Then the spectral measure* σ_f *satisfies*

$$\sigma_f(\Pi(\delta_1,\ldots,\delta_d)) \leq B\rho(\alpha_1)\cdots\rho(\alpha_d)\delta_1^{\alpha_1}\cdots\delta_d^{\alpha_d}, \quad \boldsymbol{\delta} \in (0,\pi]^d \quad (\textit{discrete time});$$

$$\sigma_f(\Pi(\delta_1,\ldots,\delta_d)) \leq B\frac{\rho(\alpha_1)}{2^{\alpha_1}}\cdots\frac{\rho(\alpha_d)}{2^{\alpha_d}}\delta_1^{\alpha_1}\cdots\delta_d^{\alpha_d}, \quad \boldsymbol{\delta} > \mathbf{0} \quad (\textit{continuous time}).$$

Proof. Consider the discrete case. Fix $\boldsymbol{\delta} \in (0,\pi]^d$. Each coordinate δ_k is representable as $\delta_k = \frac{a_k}{s_k}$, where $a_k \in (0,\pi]$ and $s_k \geq 1$ are variable real parameters. Then (9) and the relation $\frac{1}{[x]} \leq \frac{2}{x}$ for $x \geq 1$ yield

$$\begin{aligned}\sigma_f(\Pi(\delta_1,\ldots,\delta_d)) &= \sigma_f\left(\Pi\left(\frac{a_1}{s_1},\ldots,\frac{a_d}{s_d}\right)\right) \leq \frac{1}{F_1(a_1)\cdots F_1(a_d)}\|A_{[\mathbf{s}]}f\|_2^2\\
&\leq \left(\frac{a_1/2}{\sin(a_1/2)}\right)^2\cdots\left(\frac{a_d/2}{\sin(a_d/2)}\right)^2\frac{B}{[s_1]^{\alpha_1}\cdots[s_d]^{\alpha_d}}\\
&\leq B\left(\frac{a_1/2}{\sin(a_1/2)}\right)^2\cdots\left(\frac{a_d/2}{\sin(a_d/2)}\right)^2\left(\frac{2}{s_1}\right)^{\alpha_1}\cdots\left(\frac{2}{s_d}\right)^{\alpha_d}\\
&= B\frac{(a_1/2)^{2-\alpha_1}}{\sin^2(a_1/2)}\cdots\frac{(a_d/2)^{2-\alpha_d}}{\sin^2(a_d/2)}\delta_1^{\alpha_1}\cdots\delta_d^{\alpha_d}.\end{aligned}$$

Choosing the parameters a_k so that the functions containing them attain their minima, we obtain the desired estimate for discrete time. For continuous time, the estimate for the spectral measure is proved likewise. In this case, we do not need to use the inequality between a number and its integer part. □

3. Upper Estimates for the Norms of the Averages: $d = 1$ and $d = 2$

Rewrite (7) through the integral of the distribution function [13, 2.9.3]:

$$\|A_\alpha f\|_2^2 = \int\limits_{\mathscr{G}^\wedge} |\mathscr{F}[\nu_d|G_\alpha](s)|^2 \, d\sigma_f(s) = 2\int\limits_0^\infty u\sigma_f(\{s \in \mathscr{G}^\wedge : |\mathscr{F}[\nu_d|G_\alpha](s)| > u\})\, du$$

$$= 2\int\limits_0^1 u\sigma_f(\{s \in \mathscr{G}^\wedge : |\mathscr{F}[\nu_d|G_\alpha](s)| > u\})\, du.$$

The last equality holds since $|\mathscr{F}[\nu_d|G_\alpha](s)| \le 1$. The analysis of the above relation makes it possible to speak of the equivalence of the power-law convergence of ergodic averages along parallelepipeds and the power-law singularity of special neighborhoods of zero. In the well-studied one-dimensional situation, symmetric intervals turned out to be suitable.

3.1. Neighborhoods of zero as intervals: $d = 1$. Consider the one-dimensional case. For $\mathbb{Z}$ and $\mathbb{R}$ and $f \in L_2(\Omega, \lambda)$, we assume the power-law estimate (4) for the spectral measure of symmetric intervals, i.e., $\sigma_f((-\delta, \delta]) \le A\delta^\alpha$ for some constants $A > 0$ and $\alpha > 0$ for all possible $\delta > 0$. Then, for continuous and discrete time, we have (see, for instance, [16, 17]):

	$\alpha \in [0, 2)$	$\alpha = 2$	$\alpha > 2$
$\|A_t f\|_2^2$	$\mathscr{O}(t^{-\alpha})$	$\mathscr{O}(t^{-2}\ln t)$	$\mathscr{O}(t^{-2})$
$\|A_n f\|_2^2$	$\mathscr{O}(n^{-\alpha})$	$\mathscr{O}(n^{-2}\ln n)$	$\mathscr{O}(n^{-2})$

We prove these relations in a new way by using the formula above. Let us first examine the case of $\mathbb{R}$. Let $\alpha \in [0, 2)$. Then, for all $t > 0$,

$$\|A_t f\|_2^2 = 2\int\limits_0^1 u\sigma_f\left(\left\{x \in \mathbb{R} : \left|\frac{\sin(tx/2)}{tx/2}\right| > u\right\}\right) du$$

$$\le 2\int\limits_0^1 u\sigma_f\left(\left\{x \in \mathbb{R} : |x| < \frac{2}{tu}\right\}\right) du$$

$$\le 2\int\limits_0^1 u\sigma_f\left(\left(-\frac{2}{tu}, \frac{2}{tu}\right]\right) du \le 2A\int\limits_0^1 u\frac{2^\alpha}{(tu)^\alpha}\, du = \frac{2^{\alpha+1}A}{(2-\alpha)t^\alpha}.$$

If $\alpha = 2$ then for an arbitrary $\varepsilon \in (0, 1)$ we obtain

$$\|A_t f\|_2^2 = 2\int\limits_0^1 u\sigma_f\left(\left\{x \in \mathbb{R} : \left|\frac{\sin(tx/2)}{tx/2}\right| > u\right\}\right) du$$

$$\le 2\int\limits_0^\varepsilon u\|f\|_2^2\, du + 2\int\limits_\varepsilon^1 u\sigma_f\left(\left(-\frac{2}{tu}, \frac{2}{tu}\right]\right) du$$

$$\le \varepsilon^2\|f\|_2^2 + 2A\int\limits_\varepsilon^1 u\frac{2^2}{(tu)^2}\, du = \varepsilon^2\|f\|_2^2 - \frac{8A}{t^2}\ln\varepsilon.$$

The minimum of the last expression is obtained for $\varepsilon^2 = \frac{4A}{\|f\|_2^2 t^2}$. Using it, for all sufficiently large $t > 0$, we obtain

$$\|A_t f\|_2^2 \le \frac{4A}{t^2} \ln\left(\frac{e\|f\|_2^2 t^2}{4A}\right).$$

If $\alpha > 2$ then, for every $\varepsilon \in (0, 1)$, we have

$$\|A_t f\|_2^2 = 2\int_0^1 u\sigma_f\left(\left\{x \in \mathbb{R} : \left|\frac{\sin(tx/2)}{tx/2}\right| > u\right\}\right) du$$

$$\le 2\int_0^\varepsilon u\|f\|_2^2\, du + 2\int_\varepsilon^1 u\sigma_f\left(\left(-\frac{2}{tu}, \frac{2}{tu}\right]\right) du$$

$$\le \varepsilon^2\|f\|_2^2 + 2A\int_\varepsilon^1 u\frac{2^\alpha}{(tu)^\alpha}\, du \le \varepsilon^2\|f\|_2^2 + \frac{2^{1+\alpha}A}{(\alpha-2)t^\alpha}\varepsilon^{2-\alpha}.$$

Putting $\varepsilon = 1/t$, for all sufficiently large $t > 0$, we get

$$\|A_t f\|_2^2 \le \left(\|f\|_2^2 + \frac{2^{1+\alpha}A}{\alpha - 2}\right) t^{-2}.$$

Similar calculations are also applicable for discrete time. We will consider only the case of $\alpha \in [0, 2)$ to emphasize a slight difference and then examine in detail only the continuous time. Reckoning with the inequality $|\sin(x/2)| \ge \frac{|x|}{\pi}$ for all $|x| \le \pi$, we obtain

$$\|A_n f\|_2^2 = \int_{(-\pi,\pi]} \left(\frac{\sin(nx/2)}{n\sin(x/2)}\right)^2 d\sigma_f(x) \le \int_{(-\pi,\pi]} \left(\frac{\pi\sin(nx/2)}{nx}\right)^2 d\sigma_f(x)$$

$$= 2\int_0^1 u\sigma_f\left(\left\{x \in (-\pi,\pi] : \left|\frac{\pi\sin(nx/2)}{nx}\right| > u\right\}\right) du \le 2\int_0^1 u\sigma_f\left(\left\{|x| < \frac{\pi}{nu}\right\}\right) du$$

$$\le 2\int_0^1 u\sigma_f\left(\left(-\frac{\pi}{nu}, \frac{\pi}{nu}\right]\right) du \le 2A\int_0^1 u\frac{\pi^\alpha}{(nu)^\alpha}\, du = \frac{2\pi^\alpha A}{(2-\alpha)n^\alpha}.$$

The calculations for discrete time differ (after the first inequality) from continuous time only by the constant π instead of 2.

Note that all estimates were obtained because the sets $\{s \in \mathscr{G}^\wedge : |\mathscr{F}[\nu_d|\nu G_\alpha](s)| > u\}$ lie in symmetric intervals of appropriate length. The multidimensional situation has its own partition of the domain of parameters. We carry out a detailed examination of the construction for continuous time first in the dimension $d = 2$.

3.2. Neighborhoods of zero in the form of rectangles: $\boldsymbol{d = 2}$. For the group $\mathbb{R}^2$ and $f \in L_2(\Omega, \lambda)$, we again assume the power-law estimate (4) of the spectral measure of symmetric rectangles, i.e., for some constants $A > 0$ and $\alpha, \beta \ge 0$, for all $\boldsymbol{\delta} = (\delta_1, \delta_2) > \mathbf{0}$,

$$\sigma_f(\Pi(\delta_1, \delta_2)) \le A\delta_1^\alpha\delta_2^\beta. \tag{11}$$

For the averages of $\mathbb{R}^2$ over rectangles $G_{t_1,t_2} = [0, t_1] \times [0, t_2]$, we have

$$|\mathscr{F}[\nu_d|G_{t_1,t_2}](x)| = \left|\frac{2\sin\left(\frac{t_1x_1}{2}\right)}{t_1x_1}\frac{2\sin\left(\frac{t_2x_2}{2}\right)}{t_2x_2}\right|.$$

It is not hard to check that the following inclusion holds for every $u \in (0,1)$:

$$\left\{x \in \mathbb{R}^2 : \left|\frac{2\sin\left(\frac{t_1x_1}{2}\right)}{t_1x_1}\frac{2\sin\left(\frac{t_2x_2}{2}\right)}{t_2x_2}\right| > u\right\}$$

$$\subset \bigcap_{k=1}^{2}\left\{x \in \mathbb{R}^2 : |x_k| < \frac{2}{ut_k}\right\} \subset \Pi\left(\frac{2}{ut_1}, \frac{2}{ut_2}\right).$$

Some more exact inclusion is given by

$$\left\{x \in \mathbb{R}^2 : \left|\frac{2\sin\left(\frac{t_1x_1}{2}\right)}{t_1x_1}\frac{2\sin\left(\frac{t_2x_2}{2}\right)}{t_2x_2}\right| > u\right\}$$

$$\subset \Pi\left(\frac{2}{ut_1}, \frac{2}{ut_2}\right) \cap \left\{x \in \mathbb{R}^2 : |x_1x_2| < \frac{4}{t_1t_2u}\right\}.$$

Denote the set on the right-hand side by $\phi_{t_1,t_2}^{-1}E_u$, where $\phi_{t_1,t_2}(x) = \left(\frac{t_1x_1}{2}, \frac{t_2x_2}{2}\right)$ and $E_u = \Pi\left(\frac{1}{u}, \frac{1}{u}\right) \bigcap \{x \in \mathbb{R}^2 : |x_1x_2| < \frac{1}{u}\}$, $u \in (0,1)$.

Consider the one-parameter family of symmetric polynomials of two variables

$$P_2^\kappa(x,y) = x^2 + xy + y^2 - \kappa(x+y), \quad \kappa \ge 0. \tag{12}$$

It is not hard to verify (see the proof of Proposition 4 for a detailed discussion) that if some nonnegative α and β are nonzero roots of P_2^κ then there exist reals $0 \le p, q \le 1$, with $p+q=1$, such that

$$\alpha + \beta p = \kappa, \quad \beta + \alpha q = \kappa. \tag{13}$$

Theorem 3. *If estimate* (11) *holds for the spectral measure of rectangular neighborhoods of zero then*

(1) *if* $P_2^\kappa(\alpha,\beta) = 0,\ \kappa \in [0,2)$, *then* $\|A_{\mathbf{t}}f\|_2^2 = \mathscr{O}\left(t_1^{-\alpha}t_2^{-\beta}\right)$ *as* $t_1, t_2 \to \infty$;

(2) *if* $P_2^2(\alpha,\beta) = 0$ *then* $\|A_{\mathbf{t}}f\|_2^2 = \mathscr{O}\left(t_1^{-\alpha}t_2^{-\beta}\ln\left(t_1^\alpha t_2^\beta\right)\right)$ *as* $t_1, t_2 \to \infty$;

(3) *if* $P_2^\kappa(\alpha,\beta) = 0$, $\kappa > 2$, *then* $\|A_{\mathbf{t}}f\|_2^2 = \mathscr{O}\left(t_1^{-\frac{2\alpha}{\kappa}}t_2^{-\frac{2\beta}{\kappa}}\right)$ *as* $t_1, t_2 \to \infty$.

Proof. Given arbitrary numbers $0 \le p, q \le 1$, with $p+q=1$, embed E_u in a union of the two rectangles:

$$E_u \subset \Pi\left(\frac{1}{u}, \frac{1}{u^p}\right) \cup \Pi\left(\frac{1}{u^q}, \frac{1}{u}\right).$$

Granted all inclusions and the power-law estimate for the spectral measures of rectangles, for every $\varepsilon \in [0,1)$, we infer

$$\|A_{\mathbf{t}}f\|_2^2 = 2\int_0^1 u\sigma_f\left(\left\{x \in \mathbb{R}^2 : \left|\frac{2\sin\left(\frac{t_1x_1}{2}\right)}{t_1x_1}\frac{2\sin\left(\frac{t_2x_2}{2}\right)}{t_2x_2}\right| > u\right\}\right)du$$

$$\le 2\int_0^\varepsilon u\|f\|_2^2\,du + 2\int_\varepsilon^1 u\sigma_f\left(\phi_{t_1,t_2}^{-1}\left(\Pi\left(\frac{1}{u}, \frac{1}{u^p}\right) \cup \Pi\left(\frac{1}{u^q}, \frac{1}{u}\right)\right)\right)du$$

$$= \varepsilon^2\|f\|_2^2 + 2\int_\varepsilon^1 u\sigma_f\left(\Pi\left(\frac{2}{t_1u}, \frac{2}{t_2u^p}\right) \cup \Pi\left(\frac{2}{t_1u^q}, \frac{2}{t_2u}\right)\right)du$$

$$\leq \varepsilon^2\|f\|_2^2 + 2\int\limits_\varepsilon^1 uA\left(\left(\frac{2}{t_1u}\right)^\alpha\left(\frac{2}{t_2u^p}\right)^\beta + \left(\frac{2}{t_1u^q}\right)^\alpha\left(\frac{2}{t_2u}\right)^\beta\right)du$$

$$= \varepsilon^2\|f\|_2^2 + \frac{2^{1+\alpha+\beta}A}{t_1^\alpha t_2^\beta}\int\limits_\varepsilon^1 (u^{1-\alpha-p\beta} + u^{1-\beta-q\alpha})\,du.$$

Inserting (13) yields

$$\|A_{\mathbf{t}}f\|_2^2 \leq \varepsilon^2\|f\|_2^2 + \frac{2^{2+\alpha+\beta}A}{t_1^\alpha t_2^\beta}\int\limits_\varepsilon^1 u^{1-\kappa}\,du. \tag{14}$$

Depending on the sign of $2-\kappa$, we will choose suitable ε as $\varepsilon = \frac{c}{t_1^a t_2^b}$ for some $a, b, c \geq 0$.

Let $\kappa \in [0,2)$. Then we put $\varepsilon = 0$ and obtain from (14):

$$\|A_{\mathbf{t}}f\|_2^2 \leq \frac{2^{2+\alpha+\beta}A}{t_1^\alpha t_2^\beta(2-\kappa)}, \quad t_1, t_2 > 0.$$

Thus, $\|A_{\mathbf{t}}f\|_2^2 = \mathscr{O}\big(t_1^{-\alpha}t_2^{-\beta}\big)$ as $t_1, t_2 \to \infty$.

Let $\kappa = 2$. Then from (14), for $\varepsilon = \frac{1}{t_1^{\alpha/2}t_2^{\beta/2}}$, we have

$$\|A_{\mathbf{t}}f\|_2^2 \leq \varepsilon^2\|f\|_2^2 - \frac{2^{2+\alpha+\beta}A}{t_1^\alpha t_2^\beta}\ln\varepsilon = \|f\|_2^2\frac{1}{t_1^\alpha t_2^\beta} + \frac{2^{1+\alpha+\beta}A}{t_1^\alpha t_2^\beta}\ln\big(t_1^\alpha t_2^\beta\big), \quad t_1, t_2 > 0.$$

Thus,

$$\|A_{\mathbf{t}}f\|_2^2 = \mathscr{O}\big(t_1^{-\alpha}t_2^{-\beta}\ln\big(t_1^\alpha t_2^\beta\big)\big) \quad \text{as } t_1, t_2 \to \infty.$$

Now, let $\kappa > 2$. Then from (14) for $\varepsilon = \frac{1}{t_1^a t_2^b}$ we infer

$$\|A_{\mathbf{t}}f\|_2^2 \leq \varepsilon^2\|f\|_2^2 + \frac{2^{2+\alpha+\beta}A}{t_1^\alpha t_2^\beta(\kappa-2)}\varepsilon^{2-\kappa} = \|f\|_2^2\frac{1}{t_1^{2a}t_2^{2b}} + \frac{2^{2+\alpha+\beta}A}{\kappa-2}\frac{1}{t_1^{\alpha+a(2-\kappa)}t_2^{\beta+b(2-\kappa)}}.$$

Taking $a = \frac{\alpha}{\kappa}$ and $b = \frac{\beta}{\kappa}$, we get the final estimate

$$\|A_{\mathbf{t}}f\|_2^2 \leq \left(\|f\|_2^2 + \frac{2^{2+\alpha+\beta}A}{\kappa-2}\right)t_1^{-\frac{2\alpha}{\kappa}}t_2^{\frac{-2\beta}{\kappa}}, \quad t_1, t_2 > 0. \quad \square$$

4. The General Case $d > 2$

4.1. Symmetric polynomials. Recall the definitions of some standard symmetric polynomials in $\mathbb{R}^d$:

$$\sigma_{d,1}(x) = \sum_{j=1}^d x_j, \quad \sigma_{d,d-1}(x) = \sum_{j=1}^d x_1\cdots\widehat{x}_j\cdots x_d, \quad \sigma_{d,d}(x) = x_1x_2\cdots x_d.$$

Here the expression $\widehat{x}_j$ means that there is no variable x_j whereas the remaining ones are present. Consider two families of symmetric polynomials defined recurrently. The first sequence of polynomials $Q_d(x)$, with $x \in \mathbb{R}^d$, is defined as follows:

$$Q_1(x_1) = 1,\; Q_2(x_1, x_2) = x_1 + x_2, \;\; Q_d(x) = \sigma_{d,d-1}(x)\prod_{j=1}^d Q_{d-1}(\widehat{x}_j).$$

Given $\kappa \geq 0$, consider one more sequence of polynomials. We put

$$P_1^\kappa(x_1) = x_1 - \kappa, \quad P_2^\kappa(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2 - \kappa(x_1 + x_2)$$

and

$$P_d^\kappa(x) = \sum_{j=1}^{d} A_j^d(x) P_{d-1}^\kappa(\widehat{x}_j) + B_d(x),$$

where

$$A_j^d(x) = x_1 \cdots \widehat{x}_j \cdots x_d \prod_{k \neq j} Q_{d-1}(\widehat{x}_k), \quad B_d(x) = \sigma_{d,d}(x) \prod_{j=1}^{d} Q_{d-1}(\widehat{x}_j).$$

In the following lemmas, we consider some properties of these polynomials. Our main attention will be paid to the zeros of P_d^κ.

Lemma 1. *The summand in P_d^κ containing κ has the form $-\kappa Q_d(x)$, i.e., $P_d^\kappa(x) = R_d(x) - \kappa Q_d(x)$. Moreover, $R_d(x), Q_d(x) \geq 0$ for $x \geq \mathbf{0}$.*

Proof. For $d = 1$ and $d = 2$, the lemma is obvious from the definition of polynomials. Suppose that it holds for all numbers up to $d - 1$ and prove it for d. Starting from the assumption and the definition of P_d^κ, we have

$$\begin{aligned} P_d^\kappa(x) &= \sum_{j=1}^{d} A_j^d(x) R_{d-1}(\widehat{x}_j) - \kappa \sum_{j=1}^{d} A_j^d(x) Q_{d-1}(\widehat{x}_j) + B_d(x) \\ &= R_d(x) - \kappa \sum_{j=1}^{d} x_1 \cdots \widehat{x}_j \cdots x_d \prod_{k \neq j} Q_{d-1}(\widehat{x}_k,) Q_{d-1}(\widehat{x}_j) \\ &= R_d(x) - \kappa \prod_{k=1}^{d} Q_{d-1}(\widehat{x}_k) \sum_{j=1}^{d} x_1 \cdots \widehat{x}_j \cdots x_d \\ &= R_d(x) - \kappa \sigma_{d,d-1}(x) \prod_{k=1}^{d} Q_{d-1}(\widehat{x}_k) = R_d(x) - \kappa Q_d(x). \end{aligned}$$

The definition of the polynomials implies that $Q_d(x)$ and $A_j^d(x)$ are nonnegative for $x \geq \mathbf{0}$. Therefore,

$$R_d(x) = \sum_{j=1}^{d} A_j^d(x) R_{d-1}(\widehat{x}_j) + B_d(x) \geq 0 \tag{15}$$

for $x \geq \mathbf{0}$ due to the inductive assumption about the nonnegativity of $R_{d-1}(x)$. $\square$

We will be interested in the nonnegative zeros of P_d^κ. Search for such zeros first on the coordinate hyperplanes. Since the polynomials are symmetric, take, for example, $x_1 = 0$. Then the equation $P_d^\kappa(0, x_2, \ldots, x_d) = 0$ takes the form $A_1^d(x) P_{d-1}^\kappa(x_2, \ldots, x_d) = 0$ since it is not hard to see that the remaining summands vanish. Since the only nonnegative zero for $Q_d(x)$ is $\mathbf{0}$, it follows that $A_1^d(x) = 0$ if and only if $x_2 \cdots x_d = 0$.

Hence, starting from the dimension $d = 3$, the polynomial P_d^κ is zero on $(d-2)$-dimensional coordinate subspaces. Call such zeros trivial, excluding the nontrivial zeros of the polynomials P_k^κ of a less dimension $k < d$. For P_1^κ and P_2^κ, we regard as nontrivial all zeros but $\mathbf{0}$. Denote the set of nonnegative nontrivial zeros of P_d^κ by $\ker P_d^\kappa$. We also put $\ker P_d^0 = \{\mathbf{0}\}$. Thus, the nonnegative nontrivial zeros of P_d^κ, with $\kappa > 0$, either have all coordinates positive or have $0 < m \leq d - 1$ zero coordinates, and their remaining coordinates are roots of P_{d-m}^κ.

Lemma 2. *The sets* $\ker P_d^\kappa$, $\kappa \geq 0$, *split* $[0,+\infty)^d$; *i.e.,*

$$[0,+\infty)^d = \bigsqcup_{\kappa \geq 0} \ker P_d^\kappa.$$

PROOF. We show first that the sets of nontrivial zeros are disjoint. To the contrary, suppose that there exist κ_1, κ_2, and a point $x_0 \geq \mathbf{0}$ contained in both sets. Let $0 \leq m \leq d-1$ be the number of zero coordinates in x_0. Then $P_{d-m}^{\kappa_1}$ and $P_{d-m}^{\kappa_2}$ vanish at the remaining nonzero coordinates. Therefore, $R_{d-m} - \kappa_1 Q_{d-m} = R_{d-m} - \kappa_2 Q_{d-m}$. Since Q_{d-m} is nonzero, $\kappa_1 = \kappa_2$.

For an arbitrary nonzero point $x_0 \geq \mathbf{0}$, the desired value of κ is found from the equality $R_{d-m} = \kappa Q_{d-m}$, where m is the number of zero coordinates in x_0. $\square$

Lemma 3. *Let* $x \in \ker P_d^\kappa$. *Then*

$$x_1 + x_2 + \dots + x_d \geq \kappa, \quad x_j \leq \kappa, \quad j = 1, \dots, d.$$

PROOF. For P_2^κ, the inequalities follow from the representations

$$P_2^\kappa(x_1, x_2) = (x_1 + x_2)(x_1 + x_2 - \kappa) - x_1 x_2 = 0,$$

$$P_2^\kappa(x_1, x_2) = x_1(x_1 + x_2 - \kappa) - x_2(\kappa - x_2) = x_2(x_1 + x_2 - \kappa) - x_1(\kappa - x_1) = 0.$$

In the general d-dimensional case, we will show inductively that there is an analog of the first inequality; namely,

$$P_d^\kappa(x) = Q_d(x)(\sigma_{d,1}(x) - \kappa) - S_d(x),$$

where $S_d(x) \geq 0$ for $x \geq \mathbf{0}$. With account taken of the equalities

$$Q_d(x) = \sum_{j=1}^d A_j^d(x) Q_{d-1}(\widehat{x}_j), \quad A_j^d(x) Q_{d-1}(\widehat{x}_j) x_j = B_d(x),$$

which were checked in the proof of Lemma 1, we infer

$$\begin{aligned}
P_d^\kappa(x) &= \sum_{j=1}^d A_j^d(x) P_{d-1}^\kappa(\widehat{x}_j) + B_d(x) \\
&= \sum_{j=1}^d A_j^d(x)(Q_{d-1}(\widehat{x}_j)(\sigma_{d-1,1}(\widehat{x}_j) - \kappa) - S_{d-1}(\widehat{x}_j)) + B_d(x) \\
&= \sum_{j=1}^d A_j^d(x)(Q_{d-1}(\widehat{x}_j)(\sigma_{d,1}(x) - \kappa - x_j) - S_{d-1}(\widehat{x}_j)) + B_d(x) \\
&= (\sigma_{d,1}(x) - \kappa)\sum_{j=1}^d A_j^d(x) Q_{d-1}(\widehat{x}_j) - \sum_{j=1}^d A_j^d(x) Q_{d-1}(\widehat{x}_j) x_j - \sum_{j=1}^d A_j^d(x) S_{d-1}(\widehat{x}_j) + B_d(x) \\
&= Q_d(x)(\sigma_{d,1}(x) - \kappa) - \sum_{j=1}^d A_j^d(x) S_{d-1}(\widehat{x}_j) - (d-1) B_d(x) \\
&= Q_d(x)(\sigma_{d,1}(x) - \kappa) - S_d(x).
\end{aligned}$$

The inductive assumption implies that if $x \geq \mathbf{0}$ then

$$S_d(x) = \sum_{j=1}^d A_j^d(x) S_{d-1}(\widehat{x}_j) + (d-1) B_d(x) \geq 0.$$

We now show that an analog of the second relation also holds. Namely, for all $1 \le k \le d$ and $x \in \mathbb{R}^d$,

$$P_d^\kappa(x) = R_d(x) - \kappa Q_d(x) = R_d(x) - x_k Q_d(x) - (\kappa - x_k) Q_d,$$

where $R_d(x) \ge x_k Q_d(x)$ for $x \ge \mathbf{0}$. Indeed, involving (15), we have by induction

$$\begin{aligned} R_d(x) &= \sum_{j=1, j\ne k}^{d} A_j^d(x) R_{d-1}(\widehat{x}_j) + A_k^d(x) R_{d-1}(\widehat{x}_k) + B_d(x) \\ &\ge x_k \sum_{j=1, j\ne k}^{d} A_j^d(x) Q_{d-1}(\widehat{x}_j) + B_d(x) = x_k Q_d(x). \end{aligned}$$

Lemma 4. *For each $1 \le k \le d$,*

$$P_d^\kappa(x) = M_d^k(x) P_{d-1}^\kappa(\widehat{x}_k) + N_d^k(x), \quad x \in \mathbb{R}^d,$$

where $M_d^k(x), N_d^k(x) \ge 0$ for $x \ge \mathbf{0}$.

PROOF. Prove by induction that

$$M_d^k(x) = \sigma_{d,d-1}(x) \prod_{j=1, j\ne k}^{d} Q_{d-1}(\widehat{x}_j)$$

and $N_d^k(x)$ is defined by a recurrent relation. It is not hard to see that the claim holds for $d = 2$:

$$P_2^\kappa(x) = (x_1 + x_2)(x_1 - \kappa) + x_2^2 = (x_1 + x_2)(x_2 - \kappa) + x_1^2;$$

i.e., $M_2^1(x) = M_2^2(x) = x_1 + x_2 = \sigma_{2,1}(x)$, $N_2^1(x) = x_1^2$, and $N_2^2(x) = x_2^2$.

Suppose that the claim holds for all numbers up to $d - 1$ and prove it for d. Starting from the assumption, we obtain

$$\begin{aligned} P_d^\kappa(x) &= \sum_{j=1, j\ne k}^{d} A_j^d(x) P_{d-1}^\kappa(\widehat{x}_j) + A_k^d(x) P_{d-1}^\kappa(\widehat{x}_k) + B_d(x) \\ &= \sum_{j=1, j\ne k}^{d} A_j^d(x) \big(M_{d-1}^k(\widehat{x}_j) P_{d-2}^\kappa(\widehat{x}_j, \widehat{x}_k) + N_{d-1}^\kappa(\widehat{x}_j)\big) + A_k^d(x) P_{d-1}^\kappa(\widehat{x}_k) + B_d(x) \\ &= A_k^d(x) P_{d-1}^\kappa(\widehat{x}_k) + \left(\sum_{j=1, j\ne k}^{d} A_j^d(x) M_{d-1}^k(\widehat{x}_j) P_{d-2}^\kappa(\widehat{x}_j, \widehat{x}_k) + B_d(x) \right) + \sum_{j=1, j\ne k}^{d} A_j^d(x) N_{d-1}^\kappa(\widehat{x}_j). \end{aligned}$$

Denote the last summand by $N_d^k(x)$ and show that the sum of the first two summands is $M_d^k(x) P_{d-1}^\kappa(\widehat{x}_k)$. Transform $A_j^d(x) M_{d-1}^k(\widehat{x}_j)$ using the equality

$$M_d^k(x) Q_{d-1}(\widehat{x}_k) = Q_d(x).$$

We infer

$$\begin{aligned} A_j^d(x) M_{d-1}^k(\widehat{x}_j) &= x_1 \cdots \widehat{x}_j \cdots x_d \Big(\prod_{l\ne j, l\ne k} Q_{d-1}(\widehat{x}_l) \Big) Q_{d-1}(\widehat{x}_k) M_{d-1}^k(\widehat{x}_j) \\ &= x_1 \cdots \widehat{x}_j \cdots x_d \Big(\prod_{l\ne j, l\ne k} Q_{d-1}(\widehat{x}_l) \Big) \Big(\sigma_{d-1,d-2}(\widehat{x}_k) \prod_{q\ne k, q\ne j} Q_{d-2}(\widehat{x}_k, \widehat{x}_q) \Big) Q_{d-2}(\widehat{x}_k, \widehat{x}_j) M_{d-1}^k(\widehat{x}_j) \end{aligned}$$

$$
\begin{aligned}
&= x_1 \cdots \widehat{x}_j \cdots x_d \Big(\prod_{l\neq j, l\neq k} Q_{d-1}(\widehat{x}_l) \Big) \Big(\sigma_{d-1,d-2}(\widehat{x}_k) \prod_{q\neq k, q\neq j} Q_{d-2}(\widehat{x}_k, \widehat{x}_q) \Big) Q_{d-1}(\widehat{x}_j) \\
&= x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) \Big(x_1 \cdots \widehat{x}_j \cdots \widehat{x}_k \cdots x_d \prod_{q\neq k, q\neq j} Q_{d-2}(\widehat{x}_k, \widehat{x}_q) \Big) \\
&= x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) A_j^{d-1}(\widehat{x}_k).
\end{aligned}
$$

Insert the so-obtained expression in the initial sum and obtain

$$
\begin{aligned}
&\sum_{j=1, j\neq k}^{d} A_j^d(x) M_{d-1}^k(\widehat{x}_j) P_{d-2}^{\kappa}(\widehat{x}_j, \widehat{x}_k) + B_d(x) \\
&= \sum_{j=1, j\neq k}^{d} \Big(x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) A_j^{d-1}(\widehat{x}_k) \Big) P_{d-2}^{\kappa}(\widehat{x}_j, \widehat{x}_k) + B_d(x) \\
&= \Big(x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) \Big) \sum_{j=1, j\neq k}^{d} A_j^{d-1}(\widehat{x}_k) P_{d-2}^{\kappa}(\widehat{x}_j, \widehat{x}_k) + B_d(x) \\
&= \Big(x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) \Big) (P_{d-1}^{\kappa}(\widehat{x}_k) - B_{d-1}(\widehat{x}_k)) + B_d(x) \\
&= \Big(x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) \Big) P_{d-1}^{\kappa}(\widehat{x}_k).
\end{aligned}
$$

In the last equality, we used the easily verifiable relation

$$
B_d(x) = x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) B_{d-1}(\widehat{x}_k).
$$

Indeed,

$$
\begin{aligned}
&x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) B_{d-1}(\widehat{x}_k) \\
&= x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) \sigma_{d-1,d-1}(\widehat{x}_k) \prod_{j\neq k} Q_{d-2}(\widehat{x}_j, \widehat{x}_k) \\
&= \sigma_{d,d}(x) \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) \prod_{j\neq k} Q_{d-2}(\widehat{x}_j, \widehat{x}_k) \\
&= \sigma_{d,d}(x) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) Q_{d-1}(\widehat{x}_k) = B_d(x).
\end{aligned}
$$

It remains to verify that the expression before $P_{d-1}^{\kappa}(\widehat{x}_k)$ is the polynomial $M_d^k(x)$. Indeed,

$$
\begin{aligned}
&A_d^k(x) + x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) \\
&= x_1 \cdots \widehat{x}_k \cdots x_d \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) + x_k \sigma_{d-1,d-2}(\widehat{x}_k) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) \\
&= \sigma_{d,d-1}(x) \prod_{l\neq k} Q_{d-1}(\widehat{x}_l) = M_d^k(x). \quad \square
\end{aligned}
$$

4.2. An iterative system of linear equations. Our goal in this subsection is to demonstrate that every point in the set $\ker P_d^\kappa$ of nontrivial zeros of P_d^κ is a solution to a system of linear equations. Moreover, the system is obtained from the equation $x_1 - \kappa = 0$ by an iterative procedure. Let us first describe the linear system and the iterative procedure.

Suppose that we already have a system $\mathscr{L}^\kappa_{\mathscr{P}_d}(x_1, \dots, x_d)$ of $d!$ linear equations for the unknowns $x_1, \dots, x_d$ with parameters $\mathscr{P}_d$. We put $\mathscr{P}_1 = \varnothing$. The set of parameters $\mathscr{P}_d$, with $d \geq 2$, consists of the collections of probability vectors

$$p^k(m|d) = \big(p_1^k(m|d), p_2^k(m|d), \dots, p_m^k(m|d)\big),$$

where $m = 2, \dots, d$ is the dimension of the probability vector and $k = 1, \dots, \frac{d!}{m!}$ is responsible for numbering the m-dimensional vectors. The probability of a vector $p^k(m|d)$ means that all its coordinates are nonnegative and their sum is 1.

Passage from the system $\mathscr{L}^\kappa_{\mathscr{P}_d}(x_1, \dots, x_d)$ to the system $\mathscr{L}^\kappa_{\mathscr{P}_{d+1}}(x_1, \dots, x_{d+1})$ consists of the two steps: At the first step, to each of the $d!$ equations, we add the expression $p^1_{d+1}(d+1|d+1)x_{d+1}$ and replace the probability parameters of the system $\mathscr{L}^\kappa_{\mathscr{P}_d}(x_1, \dots, x_d)$ by the analogous parameters from $\mathscr{P}_{d+1}$. The second step consists in symmetrizing the system obtained at the first step. Namely, d such systems of equations are added but with cyclically permuted variables and new probability parameters. Illustrate the procedure at systems of small dimension:

$$\mathscr{L}^\kappa_{\varnothing}(x_1)\colon x_1 - \kappa = 0 \overset{\text{1st step}}{\Rightarrow} p_2^1(2|2)x_2 + x_1 - \kappa = 0 \overset{\text{2nd step}}{\Rightarrow} \begin{cases} p_2^1(2|2)x_2 + x_1 - \kappa = 0, \\ p_1^1(2|2)x_1 + x_2 - \kappa = 0, \end{cases}$$

$$\mathscr{L}^\kappa_{\mathscr{P}_2}(x_1, x_2)\colon \begin{cases} p_2^1(2|2)x_2 + x_1 - \kappa = 0, \\ p_2^1(2|2)x_1 + x_2 - \kappa = 0 \end{cases} \overset{\text{1st step}}{\Rightarrow} \begin{cases} p_3^1(3|3)x_3 + p_2^1(2|3)x_2 + x_1 - \kappa = 0, \\ p_3^1(3|3)x_3 + p_1^1(2|3)x_1 + x_2 - \kappa = 0 \end{cases}$$

$$\overset{\text{2nd step}}{\Rightarrow} \begin{cases} \begin{cases} p_3^1(3|3)x_3 + p_2^1(2|3)x_2 + x_1 - \kappa = 0, \\ p_3^1(3|3)x_3 + p_1^1(2|3)x_1 + x_2 - \kappa = 0, \end{cases} \\ \begin{cases} p_2^1(3|3)x_2 + p_2^2(2|3)x_3 + x_1 - \kappa = 0, \\ p_2^1(3|3)x_2 + p_1^2(2|3)x_1 + x_3 - \kappa = 0, \end{cases} \\ \begin{cases} p_1^1(3|3)x_1 + p_2^3(2|3)x_2 + x_3 - \kappa = 0, \\ p_1^1(3|3)x_1 + p_1^3(2|3)x_3 + x_2 - \kappa = 0. \end{cases} \end{cases}$$

Note that the set of parameters $\mathscr{P}_{d+1}$ is representable as the union of the $(d+1)$th set of parameters $\mathscr{P}_d^j$, $j = 1, \dots, d+1$, and one $(d+1)$-dimensional probability vector $p^1(d+1|d+1)$. The system $\mathscr{L}^\kappa_{\mathscr{P}_{d+1}}(x_1, \dots, x_{d+1})$ is represented as the collection of $(d+1)$ systems of linear equations

$$\mathscr{L}^{\kappa - p_j^1(d+1|d+1)x_j}_{\mathscr{P}_d^j}(x_1, \dots \widehat{x}_j, \dots, x_d).$$

Proposition 4. *Suppose that $\kappa \geq 0$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \geq \mathbf{0}$. Then $\boldsymbol{\alpha}$ belongs to $\ker P_d^\kappa$ if and only if there exists a collection of parameters $\mathscr{P}_d$ such that $\boldsymbol{\alpha}$ is a solution to the system of linear equations $\mathscr{L}^\kappa_{\mathscr{P}_d}(x_1, \dots, x_d)$.*

Proof. The proposition is obvious for $\kappa = 0$. For $\kappa > 0$, we apply induction. For $d = 1$, everything is obvious. For $d = 2$, solving the system of two equations

$$p_2^1(2|2)\alpha_2 + \alpha_1 - \kappa = 0, \quad p_1^1(2|2)\alpha_1 + \alpha_2 - \kappa = 0,$$

after multiplying the first equation by α_1; the second, by α_2; and summing them up, we obtain the equality $P_2^\kappa(\alpha_1, \alpha_2) = 0$. It is clear here that the coordinates α_1 and α_2 do not vanish simultaneously. Conversely, having the equality $P_2^\kappa(\alpha_1, \alpha_2) = 0$ in which α_1 and α_2 do not vanish simultaneously, from Lemma 3 we conclude that the reals

$$p_2^1(2|2) = \frac{\kappa - \alpha_1}{\alpha_2}, \quad p_1^1(2|2) = \frac{\kappa - \alpha_2}{\alpha_1}$$

constitute a probability vector. Moreover, $\boldsymbol{\alpha}$ is a solution to the system $\mathscr{L}^\kappa_{\mathscr{P}_2}(x_1, x_2)$.

Suppose that the claim holds for all up to $d-1$ and prove it for d. Assume first that $\boldsymbol{\alpha} \geq \mathbf{0}$ is a solution to the system $\mathscr{L}^{\kappa}_{\mathscr{P}_d}(x_1, \dots, x_d)$. If $\alpha_j = \kappa$ for some j then all remaining coordinates of $\boldsymbol{\alpha}$ are zero, and so $\boldsymbol{\alpha}$ is a nontrivial zero of P^{κ}_d. Let now $0 \leq \alpha_j < \kappa$ for all j, and then $0 \leq p^1_j(d|d)\alpha_j < \kappa$. We obtain the d systems

$$\mathscr{L}^{\kappa - p^1_j(d|d)\alpha_j}_{\mathscr{P}^j_{d-1}}(x_1, \dots, \widehat{x}_j, \dots, x_d)$$

whose solution is $\boldsymbol{\alpha}$. Since these systems already have dimension $d-1$, by the inductive assumption we conclude that

$$(\alpha_1, \dots, \widehat{\alpha}_j, \dots, \alpha_d) \in \ker P^{\kappa - p^1_j(d|d)\alpha_j}_{d-1};$$

i.e., we have a nontrivial solution to the equations

$$P^{\kappa - p^1_j(d|d)\alpha_j}_{d-1}(\widehat{\alpha}_j) = 0, \quad j = 1, \dots, d.$$

By Lemma 1, these equations are rewritten as

$$R_{d-1}(\widehat{\alpha}_j) - (\kappa - p^1_j(d|d)\alpha_j) Q_{d-1}(\widehat{\alpha}_j) = P^{\kappa}_{d-1}(\widehat{\alpha}_j) + p^1_j(d|d)\alpha_j Q_{d-1}(\widehat{\alpha}_j) = 0.$$

Multiplying the above equalities by $A^d_j(\boldsymbol{\alpha})$ and then summing them up, we infer

$$\begin{aligned} &\sum_{j=1}^{d} A^d_j(\boldsymbol{\alpha}) P^{\kappa}_{d-1}(\widehat{\alpha}_j) + \sum_{j=1}^{d} A^d_j(\boldsymbol{\alpha}) p^1_j(d|d)\alpha_j Q_{d-1}(\widehat{\alpha}_j) \\ &= \sum_{j=1}^{d} A^d_j(\boldsymbol{\alpha}) P^{\kappa}_{d-1}(\widehat{\alpha}_j) + B_d(\boldsymbol{\alpha}) \sum_{j=1}^{d} p^1_j(d|d) = P^{\kappa}_d(\boldsymbol{\alpha}) = 0. \end{aligned}$$

Moreover, $\boldsymbol{\alpha}$ is a nontrivial solution since otherwise there would be two zero coordinates, and the remaining coordinates would take arbitrary values, contradicting the nontriviality of $(\alpha_1, \dots, \widehat{\alpha}_j, \dots, \alpha_d)$ for the polynomials of less dimension.

Suppose now that $\boldsymbol{\alpha} \in \ker P^{\kappa}_d$. Consider the projections $\boldsymbol{\alpha}$ to the coordinate hyperplanes $[\boldsymbol{\alpha}]_j = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j, \dots, \alpha_d)$. By Lemma 2, there exist reals $\kappa_j \geq 0$ such that

$$[\boldsymbol{\alpha}]_j \in \ker P^{\kappa_j}_{d-1}, \quad j = 1, \dots, d.$$

Lemma 4 yields

$$\begin{aligned} 0 = P^{\kappa}_d(\boldsymbol{\alpha}) = M^j_d(\boldsymbol{\alpha}) P^{\kappa}_{d-1}(\widehat{\alpha}_j) + N^j_d(\boldsymbol{\alpha}) &= M^j_d(\boldsymbol{\alpha}) \big(P^{\kappa}_{d-1}(\widehat{\alpha}_j) - P^{\kappa_j}_{d-1}([\boldsymbol{\alpha}]_j) \big) + N^j_d(\boldsymbol{\alpha}) \\ &= M^j_d(\boldsymbol{\alpha})(\kappa_j - \kappa) Q_{d-1}(\widehat{\alpha}_j) + N^j_d(\boldsymbol{\alpha}). \end{aligned}$$

From this we conclude that $\kappa_j \leq \kappa$. Therefore, there are reals $0 \leq p^1_j(d|d) \leq 1$ with $\kappa_j = \kappa - p^1_j(d|d)\alpha_j$ (we know that $0 \leq \alpha_j \leq \kappa$). Thus,

$$(\alpha_1, \dots, \widehat{\alpha}_j, \dots, \alpha_d) \in \ker P^{\kappa - p^1_j(d|d)\alpha_j}_{d-1}.$$

By the inductive assumption, there are d sets of parameters $\mathscr{P}^j_{d-1}$ such that $\boldsymbol{\alpha}$ is a solution to the collection of systems

$$\mathscr{L}^{\kappa - p^1_j(d|d)\alpha_j}_{\mathscr{P}^j_{d-1}}(x_1, \dots, \widehat{x}_j, \dots, x_d).$$

This is equivalent to $\boldsymbol{\alpha}$ being a solution to a single system $\mathscr{L}^{\kappa}_{\mathscr{P}_d}(x_1, \dots, x_d)$ with parameters

$$\mathscr{P}_d = \bigcup_{j=1}^{d} \mathscr{P}^j_{d-1} \cup \{p^1_j(d|d),\ j = 1, \dots, d\}. \tag{16}$$

Here $p^1(d|d)$ is necessary a probability vector. $\square$

4.3. The main result. For the group $\mathbb{R}^d$ and $f \in L_2(\Omega, \lambda)$, we assume the fulfillment of (4) for the spectral measure of symmetric intervals. Consider the "cross-like" sets E_u^d, $u \in (0,1)$, defined as

$$E_u^d = \{x \in \mathbb{R}^d : |x_{i_1} x_{i_2} \cdots x_{i_k}| < 1/u,\ 1 \le k \le d,\ 1 \le i_1 < i_2 < \cdots < i_k \le d\}.$$

Given an iterative system $\mathscr{L}^{\kappa}_{\mathscr{P}_d}(x_1, \dots, x_d)$ of linear equations, denote the linear functions occurring therein without the summand $-\kappa$ by $\ell^j_{\mathscr{P}_d}(x)$, $j = 1, \dots, d!$, numbered arbitrarily. We put

$$\ell^j_{\mathscr{P}_d}(x) = \sum_{k=1}^{d} q^j_{d,k} x_k.$$

Clearly, the coefficients $q^j_{d,k}$ belong to the set of parameters $\mathscr{P}_d$.

Lemma 5. *Given a set of parameters $\mathscr{P}_d$, we have*

$$E_u^d \subset \bigcup_{j=1}^{d!} \Pi\left(\frac{1}{u^{q^j_{d,1}}}, \dots, \frac{1}{u^{q^j_{d,d}}}\right). \tag{17}$$

Proof. Apply induction on d. The case of $d = 1$ is obvious, and that of $d = 2$ was in fact considered in proving Theorem 3. Suppose that the claim is proved for the dimension $d-1$ and check it for d. Represent the set of parameters in the form (16).

Consider the point $\left(\frac{1}{u^{p^1_1(d|d)}}, \dots, \frac{1}{u^{p^1_d(d|d)}}\right) \in \mathbb{R}^d$ and through it draw hyperplanes parallel to the coordinate hyperplanes. Cutting E_u^d by these hyperplanes gives

$$E_u^d \subset \bigcup_{n=1}^{d} E_u^{d-1}(n) \times \left(-\frac{1}{u^{p^1_n(d|d)}}, \frac{1}{u^{p^1_n(d|d)}}\right], \tag{18}$$

where $E_u^{d-1}(n)$ is the $(d-1)$-dimensional "cross-like" set that does not contain x_n, and the variable x_n belongs to $\left(-\frac{1}{u^{p^1_n(d|d)}}, \frac{1}{u^{p^1_n(d|d)}}\right]$. By the inductive assumption, each of the sets $E_u^{d-1}(n)$ lies in the union of $(d-1)!$ intervals:

$$E_u^{d-1}(n) \subset \bigcup_{j=1}^{(d-1)!} \Pi\left(\frac{1}{u^{q^j_{d-1,1}(n)}}, \dots, \frac{1}{u^{q^j_{d-1,d-1}(n)}}\right), \tag{19}$$

where the parameters $q^j_{d-1,k}(n) \in \mathscr{P}^n_{d-1}$ appear in the expression of the function

$$\ell^j_{\mathscr{P}^n_{d-1}}(\widehat{x}_n) = \sum_{k=1, k\ne n}^{d} q^j_{d-1,k}(n) x_k.$$

Combining (18) and (19), we get

$$E_u^d \subset \bigcup_{n=1}^{d} \bigcup_{j=1}^{(d-1)!} \Pi\Bigg(\frac{1}{u^{q^j_{d-1,1}(n)}}, \dots, \underbrace{\frac{1}{u^{p^1_n(d|d)}}}_{n\text{th coordinate}}, \dots, \frac{1}{u^{q^j_{d-1,d-1}(n)}}\Bigg).$$

It remains to observe that $\ell^j_{\mathscr{P}^n_{d-1}}(\widehat{x}_n) + p^1_n(d|d) x_n = \ell^{\widetilde{j}}_{\mathscr{P}_d}(x)$ for some $1 \le \widetilde{j} \le d!$. $\square$

Given $\mathbf{t} \in \mathbb{R}_+^d$, put $\phi_{\mathbf{t}}(x) = \left(\frac{t_1 x_1}{2}, \dots, \frac{t_d x_d}{2}\right)$.

Lemma 6. *Suppose the fulfillment of the power-law estimate (4) of the spectral measure σ_f. Then, for every set of parameters $\mathscr{P}_d$, we have*

$$\sigma_f\big(\phi_{\mathbf{t}}^{-1}(E_u^d)\big) \le \frac{2^{\sigma_{d,1}(\boldsymbol{\alpha})} A}{t_1^{\alpha_1} \cdots t_d^{\alpha_d}} \sum_{j=1}^{d!} \frac{1}{u^{\ell^j_{\mathscr{P}_d}(\boldsymbol{\alpha})}}. \tag{20}$$

Proof. By Lemma 5, we have

$$\phi_{\mathbf{t}}^{-1}(E_u^d) \subset \bigcup_{j=1}^{d!} \phi_t^{-1}\left(\Pi\left(\frac{1}{u^{q^j_{d,1}}}, \dots, \frac{1}{u^{q^j_{d,d}}}\right)\right) = \bigcup_{j=1}^{d!} \Pi\bigg(\frac{2}{t_1 u^{q^j_{d,1}}}, \dots, \frac{2}{t_d u^{q^j_{d,d}}}\bigg).$$

Then

$$\begin{aligned}
\sigma_f\big(\phi_{\mathbf{t}}^{-1}(E_u^d)\big) &\le \sum_{j=1}^{d!} \sigma_f\left(\Pi\bigg(\frac{2}{t_1 u^{q^j_{d,1}}}, \dots, \frac{2}{t_d u^{q^j_{d,d}}}\bigg)\right) \\
&\le A \sum_{j=1}^{d!} \bigg(\frac{2}{t_1 u^{q^j_{d,1}}}\bigg)^{\alpha_1} \cdots \bigg(\frac{2}{t_d u^{q^j_{d,d}}}\bigg)^{\alpha_d} \\
= \frac{2^{\sigma_{d,1}(\boldsymbol{\alpha})} A}{t_1^{\alpha_1} \cdots t_d^{\alpha_d}} &\sum_{j=1}^{d!} u^{-(\alpha_1 q^j_{d,1} + \dots + \alpha_d q^j_{d,d})} = \frac{2^{\sigma_{d,1}(\boldsymbol{\alpha})} A}{t_1^{\alpha_1} \cdots t_d^{\alpha_d}} \sum_{j=1}^{d!} \frac{1}{u^{\ell^j_{\mathscr{P}_d}(\boldsymbol{\alpha})}}. \quad \square
\end{aligned}$$

Let us formulate the main result of this section whose particular case is Theorem 3.

Theorem 4. *If the power-law estimate (4) of the spectral measure σ_f holds then*
(1) *if $\boldsymbol{\alpha} \in \ker P_d^\kappa$, $\kappa \in [0, 2)$, then $\|A_{\mathbf{t}} f\|_2^2 = \mathscr{O}\big(t_1^{-\alpha_1} \cdots t_d^{-\alpha_d}\big)$ as $t_1, \dots, t_d \to \infty$;*
(2) *if $\boldsymbol{\alpha} \in \ker P_d^2$ then $\|A_{\mathbf{t}} f\|_2^2 = \mathscr{O}\big(t_1^{-\alpha_1} \cdots t_d^{-\alpha_d} \ln\big(t_1^{\alpha_1} \cdots t_d^{\alpha_d}\big)\big)$ as $t_1, \dots, t_d \to \infty$;*
(3) *if $\boldsymbol{\alpha} \in \ker P_d^\kappa$, $\kappa > 2$, then $\|A_{\mathbf{t}} f\|_2^2 = \mathscr{O}\big(t_1^{-\frac{2\alpha_1}{\kappa}} \cdots t_d^{-\frac{2\alpha_d}{\kappa}}\big)$ as $t_1, \dots, t_d \to \infty$.*

Proof. Reckoning with inequality (20) in Lemma 6, for any $\varepsilon \in [0, 1)$, we infer

$$\begin{aligned}
\|A_{\mathbf{t}} f\|_2^2 &= 2 \int_0^1 u \sigma_f\left(\left\{x \in \mathbb{R}^d : \left|\frac{2\sin\big(\frac{t_1 x_1}{2}\big)}{t_1 x_1} \cdots \frac{2\sin\big(\frac{t_d x_d}{2}\big)}{t_d x_d}\right| > u\right\}\right) du \\
&\le 2 \int_0^\varepsilon u \|f\|_2^2 \, du + 2 \int_\varepsilon^1 u \sigma_f\left(\phi_{\mathbf{t}}^{-1}\big(E_u^d\big)\right) du \\
&= \varepsilon^2 \|f\|_2^2 + 2 \int_\varepsilon^1 u \frac{2^{\sigma_{d,1}(\boldsymbol{\alpha})} A}{t_1^{\alpha_1} \cdots t_d^{\alpha_d}} \sum_{j=1}^{d!} \frac{1}{u^{\ell^j_{\mathscr{P}_d}(\boldsymbol{\alpha})}} \, du.
\end{aligned}$$

Since $\boldsymbol{\alpha} \in \ker P_d^\kappa$, by Proposition 4, for all $j = 1, \dots, d!$, we have the equalities $\ell^j_{\mathscr{P}_d}(\boldsymbol{\alpha}) = \kappa$. Hence,

$$\|A_t f\|_2^2 \le \varepsilon^2 \|f\|_2^2 + 2d! \frac{2^{\sigma_{d,1}(\boldsymbol{\alpha})} A}{t_1^{\alpha_1} \cdots t_d^{\alpha_d}} \int_\varepsilon^1 u^{1-\kappa} \, du.$$

Then we apply the same arguments as in Theorem 3 for $d = 2$. $\square$

4.4. Neighborhoods of zero in the form of cubes. For studying the power-law convergence of the ergodic averages $A_{\mathbf{t}}f$ and $A_{\mathbf{n}}f$ constructed along d-dimensional cubes (i.e., $t_1 = \cdots = t_d$ and $n_1 = \cdots = n_d$), we can consider the power-law singularity of the spectral measure only for neighborhoods of zero that are d-dimensional cubes. Suppose that

$$\sigma_f((-\varepsilon, \varepsilon]^d) \leq A\varepsilon^{\alpha}$$

for some $A > 0$ and $\alpha \geq 0$ for all $\varepsilon > 0$. In this case, a power-law estimate for the ergodic means with the same exponent α can be obtained only for $\alpha \in [0, 2)$. We demonstrate this again on the example of $\mathbb{R}^d$. Here we use the best embedding of $\phi_{t\cdot\mathbf{1}}^{-1} E_u^d$ into the d-dimensional cube:

$$\phi_{t\cdot\mathbf{1}}^{-1} E_u^d \subset \left(-\frac{2}{tu}, \frac{2}{tu}\right]^d, \quad t > 0,\ u \in (0,1).$$

Then

$$\|A_{t\cdot\mathbf{1}}f\|_2^2 = 2\int_0^1 u\sigma_f\left(\left\{x \in \mathbb{R}^d : \left|\frac{2\sin\left(\frac{tx_1}{2}\right)}{tx_1} \cdots \frac{2\sin\left(\frac{tx_d}{2}\right)}{tx_d}\right| > u\right\}\right) du$$

$$\leq 2\int_0^1 u\sigma_f\left(\phi_{t\cdot\mathbf{1}}^{-1}(E_u^d)\right) du \leq 2\int_0^1 u\sigma_f\left(\left(-\frac{2}{tu}, \frac{2}{tu}\right]^d\right) du \leq 2A\int_0^1 u\left(\frac{2}{tu}\right)^{\alpha} du$$

$$= \frac{2^{1+\alpha}A}{t^{\alpha}(2-\alpha)} = \mathscr{O}(t^{-\alpha}).$$

If we apply Theorem 4(1) for $t_1 = \cdots = t_d$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_d = \alpha$ then we get

$$\|A_{t\cdot\mathbf{1}}f\|_2^2 = \mathscr{O}(t^{-\alpha}) \quad \text{as } t \to \infty.$$

However, here α can take values in the half-interval $[0, \alpha_0)$, where

$$\alpha_0 := \sup_{P_d^{\kappa}(\alpha_1,\ldots,\alpha_d)=0,\kappa\in[0,2)} \alpha = \sup_{P_d^2(\alpha/d,\ldots,\alpha/d)=0} \alpha > 2.$$

Thus, the range of the exponents of the singularity of the spectral measure for which we proved the same power-law rate of convergence shows that to obtain estimates of the convergence rate for averages along cubes, it is also more natural to study the spectral measures of neighborhoods of zero in the form of parallelepipeds (and not cubes). On the other hand, even for parallelepipeds, we have not obtained estimates for the entire possible range of the exponents of the convergence rate of the corresponding ergodic averages. For example, of interest are the exponents close to the maximum possible rate (see [18, 19]). Therefore, maybe, when averaging along parallelepipeds, we have to consider the spectral measures of more “tricky” neighborhoods of zero. This is a topic for further research.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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