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TWO SERIES OF COMPONENTS OF THE MODULI SPACE OF SEMISTABLE REFLEXIVE RANK 2 SHEAVES ON THE PROJECTIVE SPACE

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Abstract—We construct two new infinite series of irreducible components of the moduli space of semistable nonlocally free reflexive rank 2 sheaves on the three-dimensional complex projective space. In the first series the sheaves have an even first Chern class, and in the second series they have an odd one, while the second and third Chern classes can be expressed as polynomials of a special form in three integer variables. We prove the uniqueness of components in these series for the Chern classes given by those polynomials.

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Introduction

Constructing new series of components of equivalence classes of stable or semistable sheaves, as well as isomorphism classes of stable or semistable bundles on $\mathbb{P}^3$, is one of the most promising directions in the study of their moduli spaces (schemes). A few such series have already been constructed and studied. The number-theoretic properties of the series and sheaves (bundles) in these spaces are of particular interest. In particular, these properties are studied in [1] for the Ein components of stable bundles with the first Chern class $c_1 = 0$. The authors of [2] obtain the formula for the dimension of these components for the cases $c_1 = 0$ and $c_1 = -1$. The authors of [3] study these properties for the bundles whose isomorphism classes constitute the Ein components, and give precise formulas for finding the spectra of modified instanton bundles and calculating the dimension of the moduli space of the bundles. The authors of [4] present the formulas for finding the exact number of two types of Vedernikov components of stable bundles with $c_1 = 0$ (as a particular case of the Ein components) and give a criterion for the existence of these components with an arbitrary second Chern class. The authors of [5, 6] construct the infinite series of irreducible components of stable rank 2 bundles with the second Chern class depending quadratically on one and respectively two integer parameters. Moreover, the authors of [7] construct the infinite series of smooth irreducible components of the moduli space of symplectic vector bundles of an arbitrary even rank $2r$ with $r \geq 1$ and, as a particular case, they obtain infinite series of irreducible components of the moduli space of stable rank 2 bundles on $\mathbb{P}^3$. The formulas for calculating the second Chern class of the bundles in this series are complicated polynomial expressions in increasing numbers of integer parameters.

The articles by Jardim, Markushevich, and Tikhomirov [8] and Almeida, Jardim, and Tikhomirov [9] are of particular interest as they provide new infinite series of irreducible components of the moduli space $M(e; n, m)$ of semistable rank 2 sheaves with Chern classes $c_1 = e \in \{-1, 0\}$, $c_2 = n$, and $c_3 = m$ on $\mathbb{P}^3$, whose generic points[1)] are nonlocally free reflexive sheaves. Furthermore, n and m are polynomials of respectively the second and third degree in three integer parameters a, b, and c. In [10], the authors establish the rationality of the irreducible components of the moduli space of stable rank 2 sheaves on $\mathbb{P}^3$ belonging to the infinite subseries of the series of irreducible components which constructed in that

[1)]Henceforth by a *generic point* of an irreducible component we understand a closed point belonging to some dense open subset of this component.

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article and described in [8]. Some examples of the moduli spaces $M(e;n,m)$ with at most one irreducible component whose generic point corresponds to a reflexive sheaf are studied in [11] for the cases $e=0$ and $n \le 3$, in [9] for the cases $e=-1$ and $n \le 2$, in [12] for the sheaves with the maximal values of the class m, and in [13] for the cases $e=0$ and $e=-1$, where n and m are polynomials respectively of the second and third degree in three integer parameters. In this article we construct two new infinite series of irreducible components of the moduli space $M(e;n,m)$ (Theorem 5) and prove uniqueness (Theorem 6) for the components of the moduli of reflexive sheaves with Chern classes e, n, and m in these series.

The article is organized as follows:

Section 1 establishes a few preliminary results on the number of integer solutions to systems of equations of a certain type (Theorems 1–4). Section 2 presents the main results; namely, in Theorem 5 we construct two new infinite series of irreducible components of the moduli space $M(e;n,m)$, while in Theorem 6 we establish uniqueness for the components of the moduli of reflexive sheaves with the appropriate values of Chern classes in these series. We assume the base field k to be algebraically closed of characteristic 0.

1. Preliminary Results

Denote the open subset of $M(e;n,m)$ consisting of stable reflexive sheaves by $\mathscr{R}(e;n,m)$.

Given a tuple $(a,b,c,d) \in \mathbb{Z}^4_{\ge 0}$ with

$$a+b+2=c+d, \tag{1}$$

consider the family of reflexive rank 2 sheaves F obtained as the cokernels of the mappings α whose degeneration locus

$$\delta(\alpha)=\{x \in \mathbb{P}^3 : \alpha(x) \text{ is not injective}\}$$

is 0-dimensional:

$$0 \to a \cdot \mathscr{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} c \cdot \mathscr{O}_{\mathbb{P}^3}(-1) \oplus d \cdot \mathscr{O}_{\mathbb{P}^3} \to F(k) \to 0. \tag{2}$$

Using (1), we can express one of the four parameters in terms of the remaining three, namely:

$$d=a+b+2-c. \tag{3}$$

Henceforth all our expressions depend only on a, b, and c. The validity of (2) imposes the conditions $(a,b) \ne (0,0)$ and $c \le a+b+2$ on the triple $(a,b,c) \in \mathbb{Z}^3_{\ge 0}$.

Straightforward calculations involving (3) show that

$$c_1(F)=e \in \{-1,0\}, \quad c_2(F)=n_e(a,b,c), \quad c_3(F)=m(a,b,c), \tag{4}$$

where the Chern classes $c_2(F)$ and $c_3(F)$ are given below by (5) and (6) for $e=0$, and by (16) and (6) for $e=-1$.

The case of the first Chern class $c_1(F)=0$. Given $(a,b,c) \in \mathbb{Z}^3_{\ge 0}$, put

$$n_0(a,b,c)=\frac{(3a+2b-c)^2}{4}+\frac{9a+4b-c}{2}, \tag{5}$$

$$m(a,b,c)=\frac{(3a+2b-c)^3}{6}+\frac{(3a+2b-c)(9a+4b-c)}{2}+\frac{27a+8b-c}{3}. \tag{6}$$

Moreover, let

$$S_0=\{(a,b,c) \in \mathbb{Z}^3_{\ge 0} : a=c \bmod 2\}.$$

Below we study the properties of the mapping

$$(a,b,c) \mapsto (n_0(a,b,c), m(a,b,c))$$

on S_0. Put

$$k = \frac{3a+2b-c}{2}, \quad l = \frac{9a+4b-c}{2}. \tag{7}$$

Clearly, if $(a,b,c) \in S_0$ then k and l are integers. Moreover,

$$n_0(a,b,c) = k^2 + l, \quad m(a,b,c) = \frac{4}{3}k^3 + 2kl - \frac{4}{3}k + 2l + 2a. \tag{8}$$

Consider the system of equations

$$n_0(a,b,c) = n, \quad m(a,b,c) = m, \tag{9}$$

where n and m are certain integers with $(a,b,c) \in S_0$. We may assume that m is even because otherwise system (9) obviously lacks solutions on S_0. It is easy to deduce from (7) and (8) that all solutions to (9) in S_0 are given by

$$(a,b,c) = (f_0(k), g_0(k), h_0(k)),$$

where $f_0(k)$, $g_0(k)$, and $h_0(k)$ are

$$\begin{aligned} f_0(k) &= \frac{1}{3}k^3 + k^2 + \left(\frac{2}{3} - n\right)k - n + \frac{1}{2}m, \\ g_0(k) &= -k^3 - 4k^2 + (3n-3)k + 4n - \frac{3}{2}m, \\ h_0(k) &= -k^3 - 5k^2 + (3n-6)k + 5n - \frac{3}{2}m, \end{aligned} \tag{10}$$

while the integer parameter k satisfies the system of inequalities

$$f_0(k) \geq 0, \quad g_0(k) \geq 0, \quad h_0(k) \geq 0. \tag{11}$$

Theorem 1. *System* (9) *has at most two solutions* $(a,b,c) \in S_0$*; it has exactly two solutions if and only if*

$$n = t^2 + 4t + 3, \quad m = \frac{4}{3}t^3 + 8t^2 + \frac{44}{3}t + 8, \tag{12}$$

where $t \geq -1$ *is an integer, and in this case* $(a,b,c) \in \{(t+1, 0, t+3), (0, t+1, 0)\}$.

Proof. Suppose that (11) has two solutions $k = t$ and $k = t+s$, where t and s are some integers and $s \geq 1$. Show that $s = 1$.

Indeed, $3f_0(t+s) + g_0(t) \geq 0$ and $3f_0(t+s) + h_0(t+s) \geq 0$. Solving each of these inequalities for n, we arrive at

$$t^2 + 2(s+1)t + s^2 + 2s \leq n \leq t^2 + \frac{3s^2+6s-1}{3s-1}t + \frac{s^3+3s^2+2s}{3s-1}$$

where the difference between the right- and left-hand sides equals

$$-\frac{(s-1)((3s+1)t + 2s^2 + 4s)}{3s-1}.$$

Assume now that $s \geq 2$. Then $(3s+1)t + 2s^2 + 4s \leq 0$, whence

$$t \leq -\frac{2s^2+4s}{3s+1}. \tag{13}$$

Moreover, $3f_0(t) + g_0(t+s) \geq 0$ and $3f_0(t+s) + h_0(t) \geq 0$ which yields

$$t^2 + \frac{3s^2+8s+1}{3s+1}t + \frac{s^3+4s^2+3s}{3s+1} \leq n \leq t^2 + \frac{3s^2+6s-4}{3s-2}t + \frac{s^3+3s^2+2s}{3s-2}.$$

Here the difference between the right- and left-hand sides equals

$$\frac{(3s^2+7s-2)t+8s^2+8s}{(3s+1)(3s-2)}.$$

Consequently,

$$(3s^2+7s-2)t+8s^2+8s \geq 0,$$

and so

$$t \geq -\frac{8s^2+8s}{3s^2+7s-2}. \tag{14}$$

However, (13) and (14) are inconsistent for an arbitrary $s \geq 2$ because

$$-\frac{2s^2+4s}{3s+1}+\frac{8s^2+8s}{3s^2+7s-2}=-\frac{2s(3s^3+s^2-4s-8)}{(3s+1)(3s^2+7s-2)}<0.$$

Thus, $s=1$ and (11) has precisely two solutions $k=t$ and $k=t+1$. Then the first equality in (12) is valid. Next, $f_0(t+1) \geq 0$, $g_0(t) \geq 0$ imply that

$$\frac{4}{3}t^3+8t^2+\frac{44}{3}t+8 \leq m \leq \frac{4}{3}t^3+8t^2+\frac{44}{3}t+8,$$

whence we obtain the second equality in (12). Finally, inserting $k=t$ into (10) yields

$$(a,b,c)=(t+1,0,t+3),$$

while $k=t+1$ yields $(a,b,c)=(0,t+1,0)$. The condition $t \geq -1$ is necessary and sufficient for both solutions to lie in S_0. $\square$

Observe that for arbitrary n and m system (11) has a bounded solution set for k which we can find by using the method of intervals. The endpoints of the intervals comprising the solution set are the roots of cubic polynomials (10) which can be found explicitly.

Example. (a) Take $(n,m)=(4,6)$. Then $k \in [-0.629,-0.279]$ and there are no integer solutions.

(b) If $(n,m)=(7,26)$ then $k \in [0.766,1.405]$, whence $k=1$ and $(a,b,c)=(1,2,5)$.

(c) Take $(n,m)=(8,32)$. This pair of values of n and m is of the form (12) with $t=1$, and the corresponding (a,b,c) are described in Theorem 1.

We are particularly interested in solving (9) in

$$S_0^* = \{(a,b,c) \in S_0 : (a,b) \neq (0,0),\ c \leq a+b+2\} \subset S_0$$

for arbitrary positive integers n and m. Here we can avoid solving (11) and obtain the explicit formulas for the solution to (9).

Exclude from consideration all pairs (n,m) resulting from (12) with $t \geq 0$: for each of them the possible solutions (a,b,c) are presented in Theorem 1, and they all lie in S_0^*. Assume also that m is even. In particular, we may assume henceforth that $n+1$ is not a complete square. This allows us to uniquely express n as $n=K^2+L$, where the integers K and L satisfy $K \geq 0$ and $2K \leq L \leq 4K+3$. Indeed,

$$(K+1)^2 < n+1 < (K+2)^2,$$

whence $K=[\sqrt{n+1}]-1$, and so $L=n-([\sqrt{n+1}]-1)^2$. Define the segment $I(n)$ as

$$I(n)=[\max\{0,-3K+L-2\}, \min\{(1/3)(L-K),(2/3)(L-2K)\}].$$

This segment always contains integers. Indeed, this can fail only when $B<A$, where

$$A=\max\{0,-3K+L-2\}, \quad B=\min\{(1/3)(L-K),(2/3)(L-2K)\}.$$

Since $L \geq 2K$, we have $B \geq 0$. Therefore, $B < A$ implies that $A = -3K + L - 2 > 0$, and so $L > 3K + 2$. In this case

$$\frac{1}{3}(L - K) < \frac{2}{3}(L - 2K),$$

meaning that $B = \frac{1}{3}(L - K)$. However,

$$\frac{1}{3}(L - K) < -3K + L - 2$$

is equivalent to $L > 4K + 3$, which is impossible.

REMARK. We can show that the length of $I(n)$ is at most $\frac{2}{3}(K + 1) \sim \frac{2}{3}\sqrt{n}$ and this upper bound is attained at $n = K^2 + 3K + 2$.

The following theorem presents a criterion for the solvability of (9) in S_0^*, as well as gives an explicit formula for the solution to (9) which is unique by Theorem 1.

Theorem 2. *System* (9) *is solvable in* S_0^* *if and only if* $M \in I(n)$, *where*

$$M = -\frac{2}{3}K^3 - KL + \frac{2}{3}K - L + \frac{1}{2}m.$$

Under this condition, the unique solution to (9) *in* S_0^* *is*

$$(a, b, c) = (M, -3M - K + L, -3M - 4K + 2L). \tag{15}$$

PROOF. If $(a, b, c) \in S_0^*$ then (7) satisfy the restrictions $k \geq 0$ and $l \geq 1$. Moreover, $2k \leq l \leq 4k + 3$. Indeed, the left inequality is equivalent to $3a + c \geq 0$, while the right one, to $c \leq a + \frac{4}{3}b + 2$, which follows from $c \leq a + b + 2$.

The above implies that every solution $(a, b, c) \in S_0^*$ of the first equation in (9) must satisfy the system

$$\frac{3a + 2b - c}{2} = K, \quad \frac{9a + 4b - c}{2} = L,$$

which yields $b = -3a - K + L$ and $c = -3a - 4K + 2L$. If $a = b = 0$ then $L = K$, and so $K \geq 2K$, whence $K = 0 = L$, which is impossible. Therefore, the condition $(a, b) \neq (0, 0)$ is met automatically. The same applies to the conditions $a \equiv c \pmod 2$. Also, the inequalities $a \geq 0$, $b \geq 0$, $c \geq 0$, and $c \leq a + b + 2$ hold if and only if $a \in I(n)$. Thus,

$$(a, b, c) = (a, -3a - K + L, -3a - 4K + 2L),$$

where a is one of the integers in $I(n)$. It remains to solve the equation

$$m(a, -3a - K + L, -3a - 4K + 2L) = m$$

for a in $I(n)$. This equation is of the form

$$\frac{4}{3}K^3 + 2KL - \frac{4}{3}K + 2L + 2a = m,$$

whence we find that $a = M$. Thus, (9) is either not solvable in S_0^* or has the unique solution (15) provided that $M \in I(n)$. $\square$

EXAMPLE. (a) Take $(n, m) = (6, 24)$. Then $K = 1$, $L = 5$, $I(n) = [0, \frac{4}{3}]$, and $M = 2$. Since $2 \notin [0, \frac{4}{3}]$, system (9) is not solvable in S_0^*.

(b) If $(n, m) = (13, 66)$ then $K = 2$, $L = 9$, $I(n) = [1, \frac{7}{3}]$, and $M = 2$. Here $2 \in [1, \frac{7}{3}]$, and so the unique solution to (9) in S_0^* is $(a, b, c) = (2, 1, 4)$.

Remark. For an arbitrary fixed positive integer n all positive integer values of m for which (9) is solvable in S_0^* can be expressed as

$$\frac{4}{3}K^3 + 2KL - \frac{4}{3}K + 2L + 2a$$

with a ranging over all integers on $I(n)$.

The case of the first Chern class $c_1(F) = -1$. Here we have similar results and present them without technical details. Put

$$n_{-1}(a,b,c) = \frac{(3a+2b-c+1)^2}{4} + 3a + b \tag{16}$$

and consider the mapping $(a,b,c) \mapsto (n_{-1}(a,b,c), m(a,b,c))$ on the set

$$S_{-1} = \{(a,b,c) \in \mathbb{Z}^3_{\geq 0} : a \neq c \bmod 2\}.$$

Moreover, put $k = \frac{3a+2b-c+1}{2}$ and $l = 3a+b$. If $(a,b,c) \in S_{-1}$ then k and l are integers. Furthermore, $n_{-1}(a,b,c) = k^2 + l$ and $m(a,b,c) = \frac{4}{3}k^3 + 2kl - \frac{1}{3}k + l + 2a$.

Consider the system of equations

$$n_{-1}(a,b,c) = n, \quad m(a,b,c) = m \tag{17}$$

on S_{-1}. Here we may assume that the integers n and m are of the same parity. All solutions to (17) in S_{-1} can be expressed as

$$(a,b,c) = (f_{-1}(k), g_{-1}(k), h_{-1}(k))$$

where $f_{-1}(k)$, $g_{-1}(k)$, and $h_{-1}(k)$ are

$$f_{-1}(k) = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \left(\frac{1}{6} - n\right)k - \frac{1}{2}n + \frac{1}{2}m,$$

$$g_{-1}(k) = -k^3 - \frac{5}{2}k^2 + \left(3n - \frac{1}{2}\right)k + \frac{5}{2}n - \frac{3}{2}m,$$

$$h_{-1}(k) = -k^3 - \frac{7}{2}k^2 + \left(3n - \frac{5}{2}\right)k + \frac{7}{2}n - \frac{3}{2}m + 1,$$

while the integer parameter k satisfies the system of inequalities

$$f_{-1}(k) \geq 0, \quad g_{-1}(k) \geq 0, \quad h_{-1}(k) \geq 0. \tag{18}$$

Theorem 3. *System* (17) *has at most one solution* $(a,b,c) \in S_{-1}$.

Proof. Suppose that system (18) has solutions $k = t$ and $k = t+s$, where t and s are some integers and $s \geq 1$. Then $s = 1$.

Indeed, $3f_{-1}(t+s) + g_{-1}(t) \geq 0$ and $3f_{-1}(t+s) + h_{-1}(t+s) \geq 0$. This implies the two-sided inequality

$$t^2 + (2s+1)t + s^2 + s - \frac{1}{2} \leq n \leq t^2 + \frac{3s^2+3s}{3s-1}\,t + \frac{s^3 + \frac{3}{2}s^2 + \frac{1}{2}s}{3s-1}$$

in which the difference between right- and left-hand sides equals

$$-\frac{(s-1)((3s+1)t + 2s^2 + \frac{5}{2}s - \frac{1}{2})}{3s-1}.$$

Furthermore, we can apply the same method as in the proof of Theorem 1 to exclude the case $s \geq 2$.

However, for $s = 1$ we obtain

$$n = t^2 + 3t + \frac{3}{2},$$

which is impossible either, because n is an integer. $\square$

As above, we are mostly interested in solving (17) in

$$S^*_{-1} = \{(a,b,c) \in S_{-1} : (a,b) \neq (0,0),\ c \leq a+b+2\} \subset S_{-1}$$

for arbitrary positive integers n and m. Here we can also indicate an explicit formula for the solution to (17) which is unique by Theorem 3.

First, represent n as $n = K^2 + L$, where K and L are integers with $K \geq 0$ and $K \leq L < 3K+2$. We have

$$(2K+1)^2 \leq 4n+1 < (2K+3)^2,$$

which yields

$$K = \left[\frac{1}{2}(\sqrt{4n+1} - 1)\right] \quad \text{and} \quad L = n - \left[\frac{1}{2}(\sqrt{4n+1} - 1)\right]^2.$$

Define the segment $J(n)$ as

$$J(n) = [\max\{0, -2K+L-1\}, \min\{(1/3)L, (1/3)(2L-2K+1)\}].$$

This segment always contains integers, while its length is at most $\frac{1}{3}(2K+1) \sim \frac{2}{3}\sqrt{n}$ and reaches this upper bound at $n = K^2 + 2K + 1$.

Theorem 4. *System* (17) *is solvable in* S^*_{-1} *if and only if* $M \in J(n)$, *where*

$$M = -\frac{2}{3}K^3 - KL + \frac{1}{6}K - \frac{1}{2}L + \frac{1}{2}m.$$

Under this condition the unique solution to (17) *in* S^*_{-1} *is*

$$(a,b,c) = (M, -3M+L, -3M-2K+2L+1). \tag{19}$$

Proof. If $(a,b,c) \in S^*_{-1}$ then $k \geq 0$ and $l \geq 1$ with $k \leq l < 3k+2$. Hence, every solution $(a,b,c) \in S^*_{-1}$ to the first equation in (17) must satisfy

$$\frac{3a+2b-c+1}{2} = K, \quad 3a+b = L,$$

whence we find $b = -3a + L$ and $c = -3a - 2K + 2L + 1$. Furthermore, we can show that

$$(a,b,c) = (a, -3a+L, -3a-2K+2L+1),$$

where a is an integer on $J(n)$. It remains to solve the equation

$$m(a, -3a+L, -3a-2K+2L+1) = m$$

for a on $J(n)$. Simplifying, we obtain

$$\frac{4}{3}K^3 + 2KL - \frac{1}{3}K + L + 2a = m,$$

whence $a = M$. Finally, system (17) is either not solvable or has a unique solution (19) provided that $M \in J(n)$. $\square$

Remark. For an arbitrary fixed positive integer n all positive integer values of m for which (17) is solvable in S^*_{-1} can be expressed as

$$\frac{4}{3}K^3 + 2KL - \frac{1}{3}K + L + 2a,$$

where a runs over all integers on $J(n)$.

2. New Series of Components of the Moduli Space of Semistable Reflexive Sheaves

We now construct and study two new series of irreducible components of the space $M(e;n,m)$ whose generic point is a stable reflexive nonlocally free sheaf: one series for $c_1=0$ and the other for $c_1=-1$.

Put $G_{(a,b)} = a\cdot\mathscr{O}_{\mathbb{P}^3}(-3)\oplus b\cdot\mathscr{O}_{\mathbb{P}^3}(-2)$ and $E_{(c,d)} = c\cdot\mathscr{O}_{\mathbb{P}^3}(-1)\oplus d\cdot\mathscr{O}_{\mathbb{P}^3}$.

Theorem 5. (i) *In the case $c_1(F)=0$, the family in* (2) *constitutes a generically smooth irreducible component $\mathscr{S}(a,b,c,d)$ of $\mathscr{R}(c_1,c_2,c_3)$ of expected dimension $8c_2(F)-3$ for $k:=(3a+2b-c)/2\in\mathbb{Z}_{\geq 0}$, while the second and third Chern classes $c_2(F)=n_0(a,b,c)$ and $c_3(F)=m(a,b,c)$ of the sheaves F are determined by* (5) *and* (6) *respectively.*

(ii) *In the case $c_1(F)=-1$, the family in* (2) *constitutes a generically smooth irreducible component $\mathscr{S}(a,b,c,d)$ of $\mathscr{R}(c_1,c_2,c_3)$ of expected dimension $8c_2(F)-5$ for $k:=(3a+2b-c+1)/2\in\mathbb{Z}_{\geq 0}$, while the second and third Chern classes $c_2(F)=n_{-1}(a,b,c)$ and $c_3(F)=m(a,b,c)$ of the sheaves F are determined by* (16) *and* (6) *respectively.*

More precisely, $\widetilde{\mathscr{S}}(a,b,c,d)$ is an open subset of $\mathrm{Hom}(G_{(a,b)},E_{(c,d)})$ consisting of monomorphisms with 0-dimensional degeneration loci; then

$$\mathscr{S}(a,b,c,d)=\widetilde{\mathscr{S}}(a,b,c,d)/((\mathrm{Aut}(G_{(a,b)})\times\mathrm{Aut}(E_{(c,d)}))/\mathbb{C}^*).$$

Proof. Given a tuple $(a,b,c,d)\in\mathbb{Z}^4_{\geq 0}$ satisfying (1), consider the morphism

$$\alpha: G_{(a,b)}\to E_{(c,d)}.$$

If the degeneration locus $\delta(\alpha)$ is 0-dimensional then the cokernel of α is a reflexive rank 2 sheaf on $\mathbb{P}^3$, which we normalize in order to include it into (2).

(i) Consider the case $e=c_1(F)=0$. The dimension of the family of reflexive rank 2 sheaves constructed in the short exact sequence (2) can be found as

$$\dim\mathscr{S}(a,b,c,d)=\dim\mathrm{Hom}(G_{(a,b)},E_{(c,d)})-\dim\mathrm{Aut}(G_{(a,b)})-\dim\mathrm{Aut}(E_{(c,d)})+1.$$

We have

$$\dim\mathrm{Hom}(G_{(a,b)},E_{(c,d)})=10ac+20ad+4bc+10bd,$$

$$\dim\mathrm{Aut}(G_{(a,b)})=a^2+4ab+b^2,\quad \dim\mathrm{Aut}(E_{(c,d)})=c^2+4cd+d^2.$$

Hence, substituting $d=a+b+2-c$ (cp. (1)), we obtain

$$\dim\mathscr{S}(a,b,c,d)=18a^2+8b^2+2c^2+24ab-8bc-12ac+36a+16b-4c-3.$$

On the other hand, taking (4) and (5) into account, we have

$$\begin{aligned}8c_2(F)-3&=8\left(\frac{(3a+2b-c)^2}{4}+\frac{9a+4b-c}{2}\right)-3\\&=18a^2+8b^2+2c^2+24ab-8bc-12ac+36a+16b-4c-3.\end{aligned}$$

It is clear now that the dimension of our family equals precisely $8c_2(F)-3$. Furthermore, observe that each F defined in (2) satisfies $\mathrm{H}^0(F(-1))=0$, and therefore F is always stable. Consequently, to obtain the equalities $\dim\mathrm{Ext}^1(F,F)=8c_2(F)-3$ we only have to verify that $\dim\mathrm{Ext}^2(F,F)=0$; by analogy with [8], according to deformation theory this would imply that this family is an open subset of a component. Indeed, applying the functor $\mathrm{Hom}(\cdot,F(k))$ to (2) yields

$$\mathrm{Ext}^1(G_{(a,b)},F(k))\to\mathrm{Ext}^2(F,F)\to\mathrm{Ext}^2(E_{(c,d)},F(k)).$$

The group on the left vanishes because $\mathrm{H}^1(F(t)) = 0$ for each $t \in \mathbb{Z}$, while the group on the right vanishes because

$$\mathrm{H}^2(F(k)) = \mathrm{H}^2(F(k+1)) = 0;$$

all these equalities follow from (2). Eventually the family of sheaves in (2) yields a generically smooth irreducible component of the expected dimension in the moduli space of stable reflexive rank 2 sheaves on $\mathbb{P}^3$ with the appropriate Chern classes.

(ii) In the case $e = c_1(F) = -1$ the arguments and calculations are similar to case (i), but take into account (4) and (16). $\square$

Theorem 6. *For all pairs of integers $n \geq 1$ and $m \geq 0$ and the values $e \in \{-1, 0\}$ the series of Theorem 5 include at most one component of the moduli space $M(e; n, m)$.*

Proof. This result is a direct consequence of Theorems 1 and 3. It remains to separately consider the cases of two solutions $(a, b, c) \in \{(t+1, 0, t+3), (0, t+1, 0)\}$ to (9) which are mentioned in Theorem 1.

For the solution $(a, b, c) = (t + 1, 0, t + 3)$, we have

$$k = \frac{3a + 2b - c}{2} = t$$

and $d = a + b + 2 - c = 0$. Therefore, (2) becomes

$$0 \to (t+1) \cdot \mathscr{O}_{\mathbb{P}^3}(-3) \xrightarrow{\alpha} (t+3) \cdot \mathscr{O}_{\mathbb{P}^3}(-1) \to F(t) \to 0.$$

For the solution $(a, b, c) = (0, t + 1, 0)$, we have

$$k = \frac{3a + 2b - c}{2} = t + 1$$

and $d = a + b + 2 - c = t + 3$. Thus, (2) becomes

$$0 \to (t+1) \cdot \mathscr{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} (t+3) \cdot \mathscr{O}_{\mathbb{P}^3} \to F(t+1) \to 0.$$

In the second case we end up with the same sequence as in the first one up to a twist by $\mathscr{O}_{\mathbb{P}^3}(1)$, and so the same family of sheaves. $\square$

Remark. It is not difficult to see that the two constructed infinite series of irreducible components of the moduli space of semistable nonlocally free reflexive rank 2 sheaves on $\mathbb{P}^3$ have intersections with the similar series in [8, 9]. Namely, for $c = 0$, the structures of sheaves in the new series coincide with the structures of sheaves in the previously constructed series; therefore, the corresponding components coincide as well.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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