ISSN 0037–4466, Siberian Mathematical Journal, 2024, Vol. 65, No. 1, pp. 106–117. © Pleiades Publishing, Ltd., 2024.
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BIRMAN–HILDEN BUNDLES. I

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UDC 515.145.2:515.148

Abstract—A topological fibered space is a Birman–Hilden space whenever in each isotopic pair of its fiber-preserving (taking each fiber to a fiber) self-homeomorphisms the homeomorphisms are also fiber-isotopic (isotopic through fiber-preserving homeomorphisms). We present a series of sufficient conditions for a fiber bundle over the circle to be a Birman–Hilden space.

DOI: 10.1134/S0037446624010117

Keywords: fiber bundle, fibering, fiber-preserving, fiberwise, locally trivial bundle, fiber-preserving self-homeomorphism, mapping class group, isotopy, homotopy, homotopy equivalence, manifold

1. Introduction

This article develops the theory of Birman–Hilden bundles. A topological fibered space is a *Birman–Hilden space* whenever in each isotopic pair of its fiber-preserving self-homeomorphisms the homeomorphisms are also fiber-isotopic (i.e., isotopic through fiber-preserving homeomorphisms). Throughout the article, by a *fiber-preserving* mapping we mean a mapping carrying each fiber into some fiber, not necessarily the same, while by an *isotopy* of self-homeomorphisms we mean an isotopy in the class of self-homeomorphisms rather than in the class of embeddings.

A *Birman–Hilden bundle* is a bundle whose total space is a Birman–Hilden space. If a fibered space (bundle) is a Birman–Hilden space (bundle) then we say also that it *has the Birman–Hilden property* or *lies in the Birman–Hilden class*.

It is natural to consider the membership in the Birman–Hilden class in the context of the family of problems about the location of the subspaces of mappings preserving a certain structure or possessing some other additional properties in the space of mappings of a more general nature of some objects of a more general form (cf. the Smale conjecture) and interpret it as the question of injectivity on the level of π_0 of the identical embedding of the space of the subgroup of fiber-preserving self-homeomorphisms of a fibered space into the group of all its self-homeomorphisms or, equivalently, to consider the path-connectedness for the subgroup formed by the fiber-preserving self-homeomorphisms isotopic to the identity mapping.

The membership in the Birman–Hilden class was studied in [1–16] for the case of branched coverings of surfaces; for a survey and additional references on this topic, see [14]. This problem is studied in [17, 18] for the case of Seifert fibrations, as well as for the case of coverings of three-dimensional manifolds. In knot theory and the theory of three-dimensional manifolds the problem of membership in the Birman–Hilden class arises for the three-dimensional manifolds fibered over the circle.

The present article establishes a series of theorems about sufficient conditions for membership in the Birman–Hilden class for locally trivial bundles over the circle. The available properties of the mapping spaces of manifolds guarantee that the sufficient conditions of this paper apply to the vast families of fibered manifolds. In particular, the obtained results imply that for $n \in \{1, 2, 3\}$ all connected compact n-dimensional manifolds locally trivially fibered over the circle have the Birman–Hilden property, including nonorientable ones and manifolds with nonempty boundary. Since deriving the corollaries we are interested from the sufficient conditions presented below requires considering many subcases and invoking a substantial part of the theory of mapping spaces of manifolds, we relegate the proofs of these corollaries to a separate article.

Original article submitted August 3, 2023; revised November 27, 2023; accepted November 28, 2023.

To state the main results of this article, we introduce some notation. Given a topological space X, denote the group of all self-homeomorphisms of X endowed with the compact-open topology by $Homeo(X)$. Denote the component of $Homeo(X)$ that contains the identity mapping id_X by $Homeo_1(X)$. Denote the space of continuous mappings $X \to X$ homotopic to the identity endowed with the compact-open topology by $Map_1(X, X)$.

Theorem 1. *Suppose that a path-connected topological space X admits no homotopic but not isotopic self-homeomorphisms, meaning that $Map_1(X, X) \cap Homeo(X) = Homeo_1(X)$, and that either the group $Homeo_1(X)$ is simply-connected or the inclusion $Homeo_1(X) \subset Map_1(X, X)$ induces an isomorphism of the fundamental groups. Then every locally trivial bundle over the circle with fiber X has the Birman–Hilden property.*

Theorem 1 admits refinements and generalizations in several directions. The condition "either the group $Homeo_1(X)$ is simply-connected or the inclusion $Homeo_1(X) \subset Map_1(X, X)$ induces an isomorphism of the fundamental groups," as well as the similar conditions in the theorems below, can be relaxed to a bulkier condition in terms of the injectivity of the mappings of conjugacy classes of the fundamental groups and their HNN-extensions induced by the inclusion $Homeo_1(X) \subset Map_1(X, X)$. In particular, Theorem 1 is supplemented by the case that $\pi_1(Homeo_1(X))$ has exactly two conjugacy classes; however, we do not pursue this direction in this article. Of greater interest for us is the generalization of Theorem 1 refining the case of pairs (of space and subspace).

Given a subspace Z of X, denote by $Map(X, X; Z)$ the space of continuous mappings $X \to X$ carrying Z to Z endowed with the compact-open topology; and by $Map_1(X, X; Z)$, the path-connected component of the identity. Say that a subspace Z of X is *h-invariant* whenever

(i) $f(Z) = Z$ for every $f \in Homeo(X)$;

(ii) for every locally trivial bundle $p : E \to S^1$ over the circle with fiber X, each $h \in Homeo_1(E)$ satisfies $h(\overline{Z}) = \overline{Z}$, where $\overline{Z}$ is the subbundle of E corresponding to Z; for instance, the boundary of a manifold is h-invariant by the Invariance of Domain Theorem.

Theorem 2. *Consider a path-connected space X with an h-invariant (possibly empty) subspace Z. Suppose that*

$$Map_1(X, X; Z) \cap Homeo(X) = Homeo_1(X),$$

meaning that X admits no nonisotopic self-homeomorphisms homotopic in the class of mappings carrying Z to Z, and either $Homeo_1(X)$ is simply-connected or the inclusion $Homeo_1(X) \subset Map_1(X, X; Z)$ induces an isomorphism of the fundamental groups. Then every locally trivial bundle over the circle with fiber X has the Birman–Hilden property.

Theorem 1 is a particular case of Theorem 2 with $Z = \varnothing$.

In order to state another version of some sufficient conditions for the membership in the Birman–Hilden class, we need more notation and definitions. Given a subspace Z of X, denote by $Map(X, X; [Z])$ the subspace of $Map(X, X; Z)$ consisting of the mappings that are the identity on Z, by $Homeo(X; [Z])$ the subgroup $Homeo(X) \cap Map(X, X; [Z])$, and by $Map_1(X, X; [Z])$ and $Homeo_1(X; [Z])$ the path-connected components of the identity in $Map(X, X; [Z])$ and $Homeo(X; [Z])$ respectively.

Definition. Given a fibered space E, denote by $Fib(E)$ the subgroup of fiber-preserving self-homeomorphisms in $Homeo(E)$ and the component of $Fib(E)$ containing the identity mapping id_E by $Fib_1(E)$. Say that a bundle $p : E \to B$ has the *epimorphism property* whenever the inclusion $Fib_1(E) \subset Homeo_1(E)$ induces an epimorphism on the level of fundamental groups.

Theorem 3. *Consider a locally trivial bundle $p : E \to S^1$ over the circle with fiber X, where X is a connected compact manifold with nonempty boundary ∂X. Suppose that*

(1) *X admits no pair of self-homeomorphisms related by a homotopy that is the identity on the boundary but not related by an isotopy that is the identity on the boundary; i.e.,*

$$Map_1(X, X; [\partial X]) \cap Homeo(X; [\partial X]) = Homeo_1(X; [\partial X]);$$

(2) *either the group* $Homeo_1(X; [\partial X])$ *is simply-connected or the inclusion* $Homeo_1(X; [\partial X]) \subset Map_1(X, X; [\partial X])$ *induces an isomorphism of the fundamental groups;*

(3) *the restriction of the bundle* p *to each connected component of the boundary* ∂E *has the Birman–Hilden property and the epimorphism property.*

Then p *has the Birman–Hilden property.*

The rest of the article aims to proving Theorems 2 and 3. Section 2 establishes a few auxiliary lemmas. Section 3 proves Theorem 2. Section 4 provides a proposition about extending a fiber-preserving isotopy of the proof of Theorem 3. Section 5 contains the proof of Theorem 3.

2. Fiberwise Homotopies and Homotopic Sections

Definition. Call a homotopy, and in particular an isotopy, of a mapping from a space into the total space of a bundle *fiberwise* whenever its projection to the base of the bundle is the identical homotopy. Call a fiber-preserving self-homeomorphism of a fibered space *fiberwise* whenever it takes each fiber to the same fiber. Call a homotopy, and in particular an isotopy, of a mapping of a space into the total space of a bundle *special* whenever the projection to the base of the restriction of this homotopy to each point of the mapped space is a contractible loop.

Lemma 1. *Two continuous mappings of a topological space into the total space of a locally trivial bundle over the circle are fiberwise homotopic if and only if they are related by a special homotopy.*

Proof. Every fiberwise homotopy is special by definition. To verify the converse, observe that we can represent every locally trivial bundle over the circle as the mapping torus of some self-homeomorphism f of the fiber F of the bundle:

$$E = \frac{F \times [0,1]}{(x,1) \sim (f(x),0)}.$$

This representation induces the one-dimensional locally trivial foliation ξ on the total space of the bundle with each leaf covering the base. Projecting each special homotopy path to the fiber F' containing the endpoints of the path along the fibers of ξ, we take the special homotopy to a fiberwise one. □

Remark. We can justify Lemma 1 even for not necessarily locally trivial bundles by using the "covering homotopy axiom"; see [19, 11.7. Second covering homotopy theorem]; however, this path is less convenient for working with subbundles; see Step 1.3 in the proof of Theorem 2.

Lemma 2. *Two sections of a locally trivial bundle over the circle are isotopic in the class of sections if and only if they are homotopic in the class of continuous mappings.*

Proof. All sections isotopic in the class of sections are homotopic in the class of continuous mappings. Let us prove the converse.

If two sections are homotopic then the projection to the base of a homotopy relating them yields a homotopy of the identity mapping of the base, which amounts to a loop in $Map_1(S^1, S^1)$. A homotopy of the identity of a space associates to each point of the space a loop in that space. For loops in the circle, as mappings from the circle into the circle, the choice of the orientation determines the degree of the mapping, in the one-dimensional case also called the index of the mapping. The index takes integer values and depends on the point continuously. Consequently, all loops of a prescribed homotopy have the same index. This implies that the specified homotopic sections are also related by a special homotopy; for instance, we obtain such homotopy as the composition of the specified homotopy with the power (inverse to the index) of the homotopy "rotating" the curve of the section "along itself," which is in turn the composition of the section with a complete rotation of the base. By Lemma 1, the sections related by a special homotopy are fiberwise homotopic. The fiberwise homotopy of sections is an isotopy in the class of sections. □

Remark. Like Lemma 1, Lemma 2 admits a proof, even for not necessarily locally trivial bundles, via the "covering homotopy axiom"; see [19, 11.7. Second covering homotopy theorem]. This alternative

method of proof generalizes to the case that the base of the bundle is a space B with the following properties (cf. the properties of fibers in Theorems 1–3):

- the inclusion $Homeo_1(B) \subset Map_1(B, B)$ induces an epimorphism of the fundamental groups.

Remark. It makes sense to regard Lemma 2 as some generalization of the available result simplified to one strand braids that closed braids in the solid torus are isotopic if and only if they represent the same conjugacy class of the braid group; see [20; 21, Theorem 1; 22, Proposition 10.16; 23, Theorem 2.1].

Lemma 3. *Consider a topological space X, a subspace Z of X, and a self-homeomorphism $f : X \to X$ with $f(Z) = Z$. Take the locally trivial bundle $\mathscr{X}_f$ over the circle with fiber X corresponding to the self-homeomorphism f, i.e., the bundle whose total space is the mapping torus for f,*

$$\mathscr{X}_f = \frac{X \times [0,1]}{(x,1) \sim (f(x),0)},$$

as well as the subbundle[1)] $\mathscr{Z}_f$ of $\mathscr{X}_f$ corresponding to the subspace Z. Assume that both Z and X are path-connected. If the inclusion $Z \subset X$ induces an epimorphism or a monomorphism of the fundamental groups then so does the inclusion $\mathscr{Z}_f \subset \mathscr{X}_f$.

Proof. This is straightforward from the noncommutative version of the so-called 4-Lemmas applied to a commutative diagram consisting of two exact sequences of bundles. $\square$

Remark. Lemma 3 generalizes to the case of a bundle and a subbundle over an arbitrary normal compact path-connected base.

Lemma 4. *Under the hypotheses of Lemma 3, if the inclusion $Z \subset X$ induces an isomorphism of the fundamental groups then two sections of the subbundle $\mathscr{Z}_f$ are isotopic in the class of sections of $\mathscr{Z}_f$ if and only if they are isotopic in the class of sections of $\mathscr{X}_f$.*

Proof. By Lemma 3, the inclusion $\mathscr{Z}_f \subset \mathscr{X}_f$ induces an isomorphism of the fundamental groups. If two sections of $\mathscr{Z}_f$ are isotopic in the class of sections of $\mathscr{X}_f$ then they represent the same conjugacy class in $\pi_1(\mathscr{X}_f)$, and so in $\pi_1(\mathscr{Z}_f)$ as well. Hence, Lemma 2 implies that these sections are isotopic in the class of sections of $\mathscr{Z}_f$. $\square$

3. Proof of Theorem 2

Consider a locally trivial bundle $p : E \to S^1$ with fiber X and a fiber-preserving self-homeomorphism $h : E \to E$ isotopic to the identity mapping id_E. We need to show that under the assumptions of the theorem h is fiber-isotopic to the identity.

Step 1.1. *Reduction to the case of a fiberwise self-homeomorphism.*

Each fiber-preserving self-homeomorphism of the total space of a locally trivial bundle induces a self-homeomorphism of the base[2)]. Denote the self-homeomorphism of the base S^1 induced by the prescribed self-homeomorphism h by h_B. Take a section $q : S^1 \to E$ of p that exists by the assumption that the fibers are path-connected; see [24, Theorem 7.1] for instance. Applying to q an isotopy carrying id_E to h and projecting the resulting isotopy of the curve to the base, we obtain a homotopy between the identity mapping id_B of the base and h_B. The classical constructions imply that homotopic self-homeomorphisms of the circle are isotopic. Lift the isotopy between h_B and id_B to a fiber-preserving isotopy of E; the existence of such lifting for locally trivial bundles follows, for instance, from [19, 11.3. First covering homotopy theorem]. This yields a fiber-preserving isotopy relating h to a self-homeomorphism h' giving the identity mapping of the base.

1) As usual, we sometimes call the total space of a bundle just a bundle for brevity.

2) Indeed, the definition of the direct product topology implies that a locally trivial bundle is an open mapping. This implies that the self-bijection on the base induced by a fiber-preserving self-homeomorphism of the total space of a locally trivial bundle carries open sets into open sets, i.e., it is a self-homeomorphism.

Therefore, the situation reduces to the case of a self-homeomorphism inducing the identity mapping on the base: Since h and h' are fiber-isotopic, in order to complete the proof of Theorem 2 it suffices to show that h' and id_E are fiber-isotopic.

STEP 1.2. *Reduction to the case of a self-homeomorphism related to* id_E *by a special*[3] *isotopy.*

For every isotopy $E \times [0,1] \to E$ relating id_E and a fiberwise self-homeomorphism the projection of the path of each point of E to the base is a loop. Moreover, by the path-connectedness assumption the free homotopy type of this loop and its index are independent of the choice of the point; we discuss this argument in greater detail in the proof of Lemma 2. Lift a complete turn of the circle to a fiber-preserving isotopy $E \times [0,1] \to E$; see [19, 11.3. First covering homotopy theorem]. Supplementing h' with a due number of such turns, we obtain a self-homeomorphism $h'' : E \to E$ which is fiber-isotopic to h' and related to id_E by an isotopy such that the projections of all paths to S^1 are index 0 loops, i.e., a special isotopy.

STEP 1.3. *Reduction to the case of a self-homeomorphism related to* id_E *by a fiberwise homotopy.*

Since the self-homeomorphism h'' in the previous step and the identity self-homeomorphism id_E are related by a special isotopy, by Lemma 1 these self-homeomorphisms are also related by a fiberwise homotopy. Furthermore, the construction of the proof of Lemma 1 in view of the assumed invariance of Z implies that the fiberwise homotopy relating h'' and id_E can be chosen so that at each moment in each fiber X' this homotopy carries the subset $Z' \subset X'$ corresponding to $Z \subset X$ into Z' because the foliation ξ in the construction of the proof of Lemma 1 is such that each of its leaves is contained either in the subbundle or in its complement.

STEP 2. *Transition to the induced bundle with fiber* $Map_1(X, X; Z)$*.*

Express E, which is a locally trivial bundle over the circle, as the mapping torus of some self-homeomorphism $f : X \to X$ of the fiber X:

$$E = \frac{X \times [0,1]}{(x,1) \sim (f(x),0)}.$$

Associated to f, there is an automorphism A_f of the monoid $Map_1(X, X; Z)$ carrying each mapping $m \in Map_1(X, X; Z)$ into $f \circ m \circ f^{-1}$. Consider the locally trivial bundle over the circle with fiber $Map_1(X, X; Z)$ defined by the automorphism A_f:

$$\mathscr{E} := \frac{Map_1(X, X; Z) \times [0,1]}{(m,1) \sim (f \circ m \circ f^{-1}, 0)}.$$

Since A_f carries the subspace $Homeo_1(X) \subset Map_1(X, X; Z)$ into itself, some subbundle of $\mathscr{E}$ corresponds to the subspace $Homeo_1(X)$, and we denote it by $\mathscr{H}_1$. Consider also the subbundle $\mathscr{H}'$ with fiber $Map_1(X, X; Z) \cap Homeo(X)$, which is well-defined because we assume that Z is h-invariant. The hypothesis that $Map_1(X, X; Z) \cap Homeo(X) = Homeo_1(X)$ yields $\mathscr{H}' = \mathscr{H}_1$.

The construction provides the natural bijection between the sections of the bundle $\mathscr{E}$ and the continuous mappings $E \to E$ carrying each fiber X' to the same fiber by a mapping related to the identity mapping $\mathrm{id}_{X'}$ by a homotopy (in the fiber) at each moment carrying the subspace $Z' \subset X'$ corresponding to the invariant subspace $Z \subset X$ into Z'. In particular, since h'' and id_E are related by a homotopy of this form, associated to the self-homeomorphism h'' there is some section of $\mathscr{E}$ that we denote by $\gamma_{h''}$, while corresponding to a homotopy of this form there is an isotopy in the class of sections between the section $\gamma_{h''}$ and the *"trivial"* section γ_0. By the trivial section we understand a section consisting of the points of the fibers of $\mathscr{E}$ corresponding to the identity mappings of the appropriate fibers of E.

[3] For the definition of a special isotopy, see Section 2.

Since the points in $\gamma_{h''}$ are self-homeomorphisms of the corresponding fibers of E, the section $\gamma_{h''}$ lies in the subbundle $\mathscr{H}'$, which coincides with $\mathscr{H}_1$ as we explained already. In order to show that h'' and id_E are related by a fiberwise isotopy, it suffices to verify that $\gamma_{h''}$ and γ_0 are isotopic in the class of sections of $\mathscr{H}_1$.

In the case that $Homeo_1(X)$ is simply-connected, $\gamma_{h''}$ and γ_0 are isotopic because in this case two arbitrary sections of $\mathscr{H}_1$ are isotopic. (Two arbitrary sections of a locally trivial bundle are isotopic when the base is the circle and the fiber is simply-connected, see [24, Theorem 7.1] for instance: as the base we consider $S^1 \times [0,1]$, while as a partial section, the union of sections over $S^1 \times \{0\}$ and over $S^1 \times \{1\}$.)

In the case that the inclusion $Homeo_1(X) \subset Map_1(X, X; Z)$ induces an isomorphism of the fundamental groups, the existence of an isotopy between the sections $\gamma_{h''}$ and γ_0 in the class of sections of $\mathscr{H}_1$ follows from their isotopy in the class of sections of $\mathscr{E}$ by Lemma 4.

Thus, we showed that if a fiber-preserving self-homeomorphism h of a locally trivial bundle E over the circle with fiber X is isotopic to the identity then it is fiber-isotopic to a self-homeomorphism h'' which is fiber-isotopic to the identity. This demonstrates that every locally trivial bundle over the circle with fiber X has the Birman–Hilden property. The proof of Theorem 2 is complete. $\square$

Appendix. For the convenience of references, we will distinguish the statement whose particular case arises on Step 2 of the proof of Theorem 2 above.

Proposition 1. *Consider a locally trivial bundle $p : E \to S^1$ with fiber X and a fiberwise self-homeomorphism $h : E \to E$. Take a normal subgroup $G(X)$ of $Homeo(X)$, for instance, $G(X) = Homeo_1(X)$ or $G(X) = Homeo_1(X, [Z])$, where Z is some h-invariant subspace of X. If $G(X)$ is simply-connected, while in each fiber X' of p the restriction $h|_{X'}$ of h to X' lies[4)] in $G(X')$, then h and id_E are related by a fiberwise isotopy whose restriction to each fiber X' lies in $G(X')$.*

Proof. The claim is justified by the construction of Step 2 in the proof of Theorem 2 above: Given a bundle p, we construct the corresponding bundle $\mathscr{H} \to S^1$ with fiber $Homeo(X)$. The latter has a subbundle $\mathscr{G}$ with fiber $G(X)$. The self-homeomorphisms h and id_E correspond to sections of $\mathscr{G}$. Since the fiber of $G(X)$ is simply-connected, these sections are isotopic in $\mathscr{G}$. For every isotopy between sections in $\mathscr{G}$ there is an associated fiberwise isotopy between id_E and h, as required. $\square$

4. Extension of a Fiber-Preserving Isotopy

In order to prove Theorem 3, we need some version of the Isotopy Extension Theorem generalized to the case of fibered spaces. Proposition 2 to be proved in Section 4 amounts to such generalization. The literature offers several versions of the Isotopy Extension Theorem for manifolds and their submanifolds and subsets, which allow for a variety of generalizations to fibered spaces. Several natural approaches are also known to proving such assertions. Note that in the version we present as Proposition 2 both the level of generalization and the method of proof are chosen somewhat arbitrarily. On the one hand, more special statements with shorter proofs would suffice. For instance, in the case that the base is the circle, we have a version of proof using the mapping torus, whereas if we confine ourselves to the case that the base is a manifold then we can give a shorter proof using the classical result [25, Corollary 1.3] that every isotopy of a compact manifold decomposes as a product of isotopies with the support of each lying in one of the elements of a prescribed open cover. On the other hand, this proposition also generalizes to larger classes of submanifolds, to not necessarily compact bases (for instance, to normal locally compact Lindelöf[5)] bases; see C_σ-spaces in [19], the theorem [19, 11.3. First covering homotopy theorem] and the method of its proof) and so on, but we avoid these generalizations here, guided by the thought that we do not need such generality to prove Theorem 3.

[4)] Since we assume that $G(X)$ is a normal subgroup of $Homeo(X)$, the subgroup $G(X')$ and accordingly the containment of $g \in Homeo(X')$ in $G(X')$ are well-defined regardless of the choice of a homeomorphism between X' and X.

[5)] A space is *Lindelöf* or *finally-compact* whenever each open cover has an at most countable subcover.

Proposition 2 (extension of a fiber-preserving isotopy). *Consider a compact manifold F with nonempty boundary ∂F, a normal compact space B, a locally trivial bundle $p : E \to B$ with fiber F, and the subbundle $E_\partial \subset E$ corresponding to ∂F. Then every fiber-preserving isotopy of the identity mapping of E_∂ extends to a fiber-preserving isotopy of the identity mapping of E.*

Proof. To simplify checking the properties of the mappings in this proof, it is convenient to interpret the isotopies of every space W under consideration as the self-homeomorphisms of the product $W \times [0, 1]$ preserving the second coordinate as in [26, 27].

Moreover, since here we are exclusively interested in the isotopies of identity mappings, in this proof we mean by isotopies precisely: the self-homeomorphisms $W \times [0, 1] \to W \times [0, 1]$ preserving the second coordinate and equal to the identity on $W \times \{0\}$.

Furthermore, by the *product* of isotopies $\tau : W \times [0, 1] \to W \times [0, 1]$ and $\rho : W \times [0, 1] \to W \times [0, 1]$ we understand their composition as self-homeomorphisms of $W \times [0, 1]$, i.e., the product in the sense of the group $Homeo(W \times [0, 1])$. To avoid confusion, we will denote this product by $\tau \star \rho$; i.e.,

$$\tau \star \rho = \tau \circ \rho.$$

The *inverse isotopy* is understood accordingly as the inverse in $Homeo(W \times [0, 1])$.

Denote the set of all isotopies of the identity mapping of a space W by $\Lambda_1(W)$ (the latter is a group with the operation described above).

By *extensible* isotopies we understand the fiber-preserving isotopies in $\Lambda_1(E_\partial)$ which extend to fiber-preserving isotopies in $\Lambda_1(E)$.

The product of extensible isotopies is an extensible isotopy because the product of extensions extends the product. Therefore, in order to justify the claim, it suffices to show that each fiber-preserving isotopy τ in $\Lambda_1(E_\partial)$ decomposes as a product of extensible ones.

A fiber-preserving isotopy of the total space of a bundle induces an isotopy of its base. Denote by $\overline{\tau}$ the isotopy of the base B induced by a fiber-preserving isotopy τ. By the classical constructions of bundle theory, see [19, 11.3. First covering homotopy theorem] for instance, each isotopy of the identity mapping of the base of a locally trivial bundle with a normal compact base lifts to a fiber-preserving isotopy of the identity mapping of the total space, so that $\Lambda_1(E)$ contains a fiber-preserving isotopy $\widetilde{\tau}$ which induces $\overline{\tau}$. The original isotopy τ decomposes as the product of the restriction of $\widetilde{\tau}$ to E_∂, which is extensible by construction, and some fiberwise isotopy $\dot{\tau} \in \Lambda_1(E_\partial)$. Therefore, the proof reduces to the case of a fiberwise isotopy $\dot{\tau}$.

In the remaining part of the proof we verify firstly that fiberwise isotopies are extensible in the case of direct products and then deduce from that the general case of a fiberwise isotopy $\dot{\tau}$ using the normality of the base and decomposing $\dot{\tau}$ as a product of fiberwise isotopies with "sufficiently small" supports whose extensibility follows directly from that for direct products.

Therefore, we will prove firstly that the fiberwise isotopies $\ddot{\tau} \in \Lambda_1(B' \times \partial F)$ extend from $B' \times \partial F$ to $B' \times F$ in the case of direct products. (This holds for an arbitrary base B', but we only need the case $B' \subset B$.) The classical result [28] about a collar neighborhood of the boundary of a metrizable manifold implies that ∂F has an open neighborhood $U(\partial F)$ in F and a homeomorphism

$$h : \partial F \times [0, 2) \to U(\partial F)$$

carries $\partial F \times \{0\}$ into ∂F. Use h to identify $U(\partial F)$ with the product $\partial F \times [0, 2)$, thereby introducing coordinates on $U(\partial F)$, and the neighborhood $B' \times U(\partial F)$ with the product $B' \times \partial F \times [0, 2)$. Given a fiberwise isotopy $\ddot{\tau}$ in $\Lambda_1(B' \times \partial F)$, define the mapping

$$\widehat{\tau} : B' \times \partial F \times [0, 2) \times [0, 1] \to B' \times \partial F \times [0, 2) \times [0, 1]$$

as

$$\widehat{\tau}(b, x, r, t) = \begin{cases} (b, x, r, t) & \text{for } t \leq r, \\ \big(b, \ddot{\tau}^b_{t-r}(x), r, t\big) & \text{for } t \geq r, \end{cases} \tag{1}$$

where b, x, r, and t are coordinates in B', ∂F, $[0,2)$, and $[0,1]$ respectively, and $\ddot{\tau}^b$ denotes the restriction of the isotopy $\ddot{\tau}$ to the fiber $\{b\} \times \partial F$. It is clear from the formula that $\widehat{\tau}$ is continuous, bijective, and has a continuous inverse, whence we conclude that $\widehat{\tau}$ is a fiberwise isotopy in $\Lambda_1(B' \times \partial F \times [0,2))$ extending $\ddot{\tau}$. The isotopy $\widehat{\tau}$ is the identity on $B' \times \partial F \times [1,2)$ and supplementing it with the identity isotopy on $B' \times (F \setminus U(\partial F))$, we obtain some fiberwise isotopy $\widehat{\tau}^+$ in $\Lambda_1(B' \times F)$ extending $\ddot{\tau}$. It is clear from the defining formula that the projection of the support $\operatorname{supp}(\widehat{\tau}^+)$ of the resulting extension to the base B' coincides with the projection of the support $\operatorname{supp}(\ddot{\tau})$ of the original isotopy, which we will use below.

Return to the fiberwise isotopy $\dot{\tau}$ on E_∂. Our goal is to decompose $\dot{\tau}$ as a product of isotopies whose supports are “small” in a certain sense and whose extensibility is easy to establish. Since the base B is compact, there is a finite open cover $\{U_1, \dots, U_k\}$ of B such that over each U_i the bundle p is trivial, while by normality there are open subcovers $\{U_1', \dots, U_k'\}$ and $\{U_1'', \dots, U_k''\}$ of B such that U_i includes $\operatorname{clos}(U_i')$, while U_i' includes $\operatorname{clos}(U_i'')$ for all i; see the so-called Contraction Lemma as one of the equivalent defining properties of a normal space in [29, p. 446]. Moreover, by the Urysohn Lemma for normal spaces (see also [29, p. 446]) there is a collection of functions $\{\varphi_1, \dots, \varphi_k\}$, with $\varphi_i : B \to [0,1]$, such that $\varphi_i(U_i'') = \{1\}$ and $\varphi_i(B \setminus U_i') = \{0\}$.

Express the isotopy $\dot{\tau}$ as a product of isotopies with the projections of supports contained in the supports of functions in the tuple $\{\varphi_1, \dots, \varphi_k\}$. To this end, introduce the concept of a *partial isotopy* defined from an isotopy and a function on the space. Given an isotopy $\rho \in \Lambda_1(E_\partial)$ and a continuous function $\widetilde{\varphi} : E_\partial \to [0,1]$, define $\rho^{\widetilde{\varphi}} : E_\partial \times [0,1] \to E_\partial \times [0,1]$ by putting

$$\rho^{\widetilde{\varphi}}(e,t) = (\operatorname{pr}_{E_\partial}(\rho(e, \widetilde{\varphi}(e) \cdot t)), t),$$

where $\operatorname{pr}_{E_\partial}$ stands for the projection $E_\partial \times [0,1] \to E_\partial$. Equivalently,

$$\rho_t^{\widetilde{\varphi}}(e) = \rho_{\widetilde{\varphi}(e) \cdot t}(e).$$

The mapping $\rho^{\widetilde{\varphi}}$ is continuous because its first projection is a composition of continuous functions, including the product $(e,t) \mapsto \widetilde{\varphi}(p(e)) \cdot t$. Thus, we may regard $\rho^{\widetilde{\varphi}}$ as a homotopy. If the isotopy ρ is fiberwise, while $\widetilde{\varphi}$ is constant on each fiber, then it is not difficult to see that $\rho^{\widetilde{\varphi}}$ is also bijective, while the inverse mapping $E_\partial \times [0,1] \to E_\partial \times [0,1]$ can be described as

$$(\rho^{\widetilde{\varphi}})^{-1}(e,t) = (\operatorname{pr}_{E_\partial}(\rho^{-1}(e, \widetilde{\varphi}(e) \cdot t)), t).$$

This shows that in the specified case $(\rho^{\widetilde{\varphi}})^{-1}$ is continuous, whence we conclude that for a fiberwise isotopy ρ and a function $\widetilde{\varphi}$ constant on each fiber the homotopy $\rho^{\widetilde{\varphi}}$ is a fiberwise isotopy.

The formula of this definition implies directly that the fiberwise isotopy ρ and the function $\widetilde{\varphi}$ constant on each fiber satisfy

$$\operatorname{supp}(\rho^{\widetilde{\varphi}}) \subset \operatorname{supp}(\rho) \cap \operatorname{supp}(\widetilde{\varphi}), \tag{2}$$

$$\operatorname{supp}((\rho \star \rho^{\widetilde{\varphi}})^{-1}) \subset \operatorname{supp}(\rho) \setminus \widetilde{\varphi}^{-1}(1). \tag{3}$$

Return to the isotopy $\dot{\tau}$ and the tuple of functions $\{\varphi_1, \dots, \varphi_k\}$. Define new functions $\widetilde{\varphi}_i : E_\partial \to [0,1]$ as $\widetilde{\varphi}_i(e) = \varphi_i(p(e))$. Since the isotopy $\dot{\tau}$ is fiberwise, while the functions $\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_k$ by definition are constant on the fibers, we have the well-defined isotopies

$$\rho_{(0)} := \dot{\tau}, \quad \rho_{(1)} := \dot{\tau} \star (\dot{\tau}^{\widetilde{\varphi}_1})^{-1}, \quad \rho_{(i)} := \rho_{(i-1)} \star \big(\rho_{(i-1)}^{\widetilde{\varphi}_i}\big)^{-1}. \tag{4}$$

They yield the decomposition

$$\dot{\tau} = (\dot{\tau} \star (\dot{\tau}^{\widetilde{\varphi}_1})^{-1}) \star \dot{\tau}^{\widetilde{\varphi}_1} = \rho_{(1)} \star \rho_{(0)}^{\widetilde{\varphi}_1} = \rho_{(2)} \star \rho_{(1)}^{\widetilde{\varphi}_2} \star \rho_{(0)}^{\widetilde{\varphi}_1} =$$
$$\dots = \rho_{(k)} \star \rho_{(k-1)}^{\widetilde{\varphi}_k} \star \dots \star \rho_{(1)}^{\widetilde{\varphi}_2} \star \rho_{(0)}^{\widetilde{\varphi}_1}.$$

By (3), we infer from (4) that

$$\operatorname{supp}(\rho_{(i)}) \subset \operatorname{supp}(\rho_{(i-1)}) \setminus \widetilde{\varphi}_i^{-1}(1) = \operatorname{supp}(\rho_{(i-1)}) \setminus p^{-1}(U_i'').$$

Therefore, since $\{U_1'', \ldots, U_k''\}$ is a cover of the base, this implies that $\operatorname{supp}(\rho_{(k)}) = \varnothing$, so that $\rho_{(k)}$ is the identity (trivial) isotopy, and so

$$\dot{\tau} = \rho_{(k-1)}^{\widetilde{\varphi}_k} \star \cdots \star \rho_{(1)}^{\widetilde{\varphi}_2} \star \rho_{(0)}^{\widetilde{\varphi}_1}.$$

From (2) we find that the condition

$$\operatorname{supp}\bigl(\rho_{(i-1)}^{\widetilde{\varphi}_i}\bigr) \subset \operatorname{supp}(\widetilde{\varphi}_i) \subset p^{-1}(U_i')$$

holds for each $i \in \{1, \ldots, k\}$. Hence, $\dot{\tau}$ decomposes as the product of fiberwise isotopies $\chi_{(i)} = \rho_{(i-1)}^{\widetilde{\varphi}_i}$. For each of them the closure of support lies in a set of the form $p^{-1}(U_i)$, on which the bundle has the structure of a direct product. It remains to show that the isotopies $\chi_{(i)}$ are extensible.

Take the restriction $\ddot{\chi}_{(i)}$ of $\chi_{(i)}$ to $E_\partial \cap p^{-1}(U_i)$. Since on $p^{-1}(U_i)$ the bundle has the structure of a direct product, we can use (1) to extend the isotopy $\ddot{\chi}_{(i)}$ to $p^{-1}(U_i)$ so that the projection to U_i of the support $\operatorname{supp}(\widehat{\chi}_{(i)})$ of $\widehat{\chi}_{(i)}$ coincides with the projection to U_i of the support $\operatorname{supp}(\ddot{\chi}_{(i)})$ of the isotopy $\ddot{\chi}_{(i)}$ itself:

$$p(\operatorname{supp}(\widehat{\chi}_{(i)})) = p(\operatorname{supp}(\ddot{\chi}_{(i)})) \ \subset\ U_i'.$$

Then

$$\operatorname{supp}(\widehat{\chi}_{(i)}) \subset p^{-1}(U_i'),$$

and so

$$\operatorname{clos}(\operatorname{supp}(\widehat{\chi}_{(i)})) \subset \operatorname{clos}\left(p^{-1}(U_i')\right) = p^{-1}\left(\operatorname{clos}(U_i')\right) \subset p^{-1}(U_i).$$

Take the mapping $\widetilde{\chi}_{(i)}$ coinciding with $\widehat{\chi}_{(i)}$ on $p^{-1}(U_i) \times [0,1]$ and the identity on $(E \setminus p^{-1}(U_i)) \times [0,1]$. Then $\widetilde{\chi}_{(i)}$ is an isotopy since its restrictions to the elements of the two-element open cover

$$\{p^{-1}(U_i) \times [0,1],\ (E \setminus \operatorname{clos}(\operatorname{supp}(\widetilde{\chi}_{(i)}))) \times [0,1]\}$$

are continuous. Thus, $\widetilde{\chi}_{(i)}$ is a fiberwise isotopy in $\Lambda_1(E)$ extending

$$\chi_{(i)} = \rho_{(i-1)}^{\widetilde{\varphi}_i}.$$

This completes the proof of Proposition 2. □

5. Proof of Theorem 3

Let us establish Theorem 3 as a particular case of the following:

Proposition 3. *Consider a path-connected space X, a nonempty h-invariant subspace Z of X, a locally trivial bundle $p : E \to S^1$ with fiber X, and the subbundle $\overline{Z}$ of E corresponding to Z. Suppose that*

(1) $Map_1(X, X; [Z]) \cap Homeo(X; [Z]) = Homeo_1(X; [Z])$, meaning that X admits no pair of self-homeomorphisms related by a homotopy that is the identity on Z but not related by an isotopy that is the identity on Z;

(2) either the group $Homeo_1(X; [Z])$ is simply-connected or the inclusion $Homeo_1(X; [Z]) \subset Map_1$ $(X, X; [Z])$ induces an isomorphism of the fundamental groups.

Suppose also that

(C1) the pair $(E, \overline{Z})$ is a Borsuk pair;

(C2) the natural projection $Fib_1(E) \to Fib_1(\overline{Z})$ is surjective and induces an epimorphism of the fundamental groupoids;

(C3) *the number of path-connected components of $\overline{Z}$ is finite, each of these components is compact and as a subbundle of $\overline{Z}$ has the Birman–Hilden and epimorphism properties.*

Then p has the Birman–Hilden property.

PROOF OF PROPOSITION 3. We need to show that an arbitrary fiber-preserving self-homeomorphism $h : E \to E$ isotopic to the identity is fiber-isotopic to the identity.

STEP 1.1. *Reduction to the case of a homeomorphism $h' : E \to E$ whose restriction to $\overline{Z}$ is the identity and which is related to id_E by an isotopy whose restriction to $\overline{Z}$ is a contractible loop in $Homeo_1(\overline{Z})$.*

Take a path $\widetilde{\lambda}$ in $Homeo_1(E)$ from id_E to h and the corresponding path λ in $Homeo_1(\overline{Z})$ from $\mathrm{id}_{\overline{Z}}$ to $h|_{\overline{Z}}$. By (C3), for each connected component $\overline{Z}^c$ of $\overline{Z}$ we have the Birman–Hilden property

$$Homeo_1(\overline{Z}^c) \cap Fib(\overline{Z}^c) = Fib_1(\overline{Z}^c)$$

and the homomorphism $\pi_1(Fib_1(\overline{Z}^c)) \to \pi_1(Homeo_1(\overline{Z}^c))$ induced by the inclusion is surjective by the epimorphism property, so that we can extend the restriction λ^c of λ to $\overline{Z}^c$ using a fiber-preserving isotopy to a contractible loop λ_+^c in $Homeo_1(\overline{Z}^c)$. Since the fiber is path-connected, while h is fiber-preserving, we infer that for every isotopy from id_E to h, given two points x and y of E with $p(x) = p(y)$, and in particular, given two points x and y in distinct connected components of $\overline{Z}$ but lying in the same fiber, meaning that $p(x) = p(y)$, the projections of the paths of these points to the base S^1 are related by a homotopy with fixed endpoints. By condition (C3) the number of components is finite and they are compact, so it is not difficult to deduce that we can match the extensions λ_+^c for different components $\overline{Z}^c$ of $\overline{Z}$ in the sense of projections to the base so that the path λ also extends to a contractible loop λ_+ in $Homeo_1(\overline{Z})$ by a fiber-preserving isotopy. By (C2) we can extend the path $\widetilde{\lambda}$ by a fiber-preserving isotopy to a path $\widetilde{\lambda}_+$ in $Homeo_1(E)$ whose restriction to $\overline{Z}$ is a loop homotopic to λ_+, i.e., a contractible loop. Denote by h' the endpoint of the path $\widetilde{\lambda}_+$. Then by construction the restriction $h'|_{\overline{Z}}$ of the self-homeomorphism h' to $\overline{Z}$ is the identity and h' is related to id_E by an isotopy whose restriction to $\overline{Z}$ yields a loop λ_+ contractible in $Homeo_1(\overline{Z})$. Since h and h' are related by a fiber-preserving isotopy, to demonstrate that h and id_E are fiber-isotopic, it suffices to verify that so are h' and id_E.

STEP 1.2. *Transition from an isotopy between h' and id_E to a homotopy between the same h' and id_E which is the identity on $\overline{Z}$.*

Take an isotopy $\tau : E \times [0, 1] \to E$ with $\tau_0 = \mathrm{id}_E$ and $\tau_1 = h'$ whose restriction to $\overline{Z}$ yields a contractible loop in $Homeo_1(\overline{Z})$; see Step 1.1. The condition of contractibility means that, given the restriction $\tau|_{\overline{Z}}$ of τ as a loop in $Homeo_1(\overline{Z})$, there is a homotopy with fixed endpoints contracting this loop to a point, i.e., there is a continuous mapping

$$\rho : \overline{Z} \times [0, 1] \times [0, 1] \to \overline{Z}$$

such that

(1) the restriction of ρ to $\overline{Z} \times [0, 1] \times \{0\}$ coincides with the restriction of τ to $\overline{Z} \times [0, 1]$;

(2) denoting by $\rho_{s,t}$ the restriction of ρ to $\overline{Z} \times \{s\} \times \{t\}$, we obtain

$$\rho_{0,r} = \rho_{r,1} = \rho_{1,r} = \mathrm{id}_{\overline{Z}}$$

for each $r \in [0, 1]$.

Since $(E, \overline{Z})$ is a Borsuk pair by condition (C1), so is $(E \times [0, 1], \overline{Z} \times [0, 1])$; the implication can be seen, for instance, by applying the available criterion stating that a pair (S, T) is a Borsuk pair if and only if the subspace $(S \times \{0\}) \cup (T \times [0, 1])$ is a retract of $S \times [0, 1]$; see [30, p. 14] for instance. This implies that, given the isotopy τ regarded as a mapping from $E \times [0, 1]$ to E, there is a homotopy $\kappa : E \times [0, 1] \times [0, 1] \to E$ such that

(1) the restriction of κ to $E \times [0, 1] \times \{0\}$ coincides with τ;

(2) the restriction of κ to $\overline{Z} \times [0,1] \times [0,1]$ coincides with ρ.

Denoting by $\kappa_{s,t}$ the restriction of κ to $E \times \{s\} \times \{t\}$, define the homotopy $\tau' : E \times [0,3] \to E$ by going along three sides of the square as

$$\tau'_t = \begin{cases} \kappa_{0,t} & \text{for } t \in [0,1], \\ \kappa_{t-1,1} & \text{for } t \in [1,2], \\ \kappa_{1,3-t} & \text{for } t \in [2,3]. \end{cases}$$

By construction, τ' relates h' to id_E and is the identity on $\overline{Z}$.

Step 1.3. *Transition to a homotopy between h' and id_E which not only is the identity on $\overline{Z}$ but also is fiberwise, meaning that it keeps each point in the same fiber at all times.* (Cf. the construction of Step 1.3 in the proof of Theorem 2.)

Since h' carries fibers to fibers, $\overline{Z}$ is nonempty (because we assume that Z is nonempty), while the fiber X is path-connected, we infer that the homotopy carrying id_E to h' which is the identity on $\overline{Z}$ is special. Hence, Lemma 1 implies that id_E and h' are related through a fiberwise homotopy as well. Moreover, it is clear from the construction of the proof of Lemma 1 that we can choose the fiberwise homotopy carrying id_E to h' to be the identity on $\overline{Z}$.

Step 2. *The induced bundle with fiber $Map_1(X, X; [Z])$.*

Steps 1.1–1.3 reduce the setup to the case of a homeomorphism $h' : E \to E$ whose restriction to $\overline{Z}$ is the identity and which is related to id_E by a fiberwise homotopy that is the identity on $\overline{Z}$. (Cf. the proof of Theorem 2, where the setup reduces to the case of a self-homeomorphism $h'' : E \to E$ related to id_E by a fiberwise homotopy at each moment carrying the subspace $Z' \subset X'$ corresponding to the invariant subspace $Z \subset X$ into Z'.)

The final step of the proof of Proposition 3 repeats the construction of the final Step 2 of the proof of Theorem 2 with the monoid $Map_1(X, X; Z)$ and the group $Homeo_1(X; Z)$ replaced by $Map_1(X, X; [Z])$ and $Homeo_1(X; [Z])$ respectively: from the given bundle p we construct the induced bundle with fiber $Map_1(X, X; [Z])$, and so forth. $\square$

Proof of Theorem 3. Theorem 3 is a particular case of Proposition 3. Let us indicate the results implying that the hypotheses of Proposition 3 hold in Theorem 3, omitting the conditions that are explicit in the statement of Theorem 3.

- The property that the boundary of the manifold is an h-invariant subspace of it follows from the Invariance of Domain Theorem.
- The property that the pair consisting of the manifold and its boundary is a Borsuk pair, which is condition (C1) in Proposition 3, follows, for instance, from the result [28] about a collar neighborhood of the boundary; see also [30, Example 0.15].
- The property that in Theorem 3 the natural projection $Fib_1(E) \to Fib_1(\partial E)$ is surjective and induces an epimorphism of the fundamental groupoids, which is condition (C2) in Proposition 3, follows from Proposition 2, as every fiber-preserving isotopy of the boundary of a manifold with boundary which is locally trivially fibered over the circle extends to a fiber-preserving isotopy of the whole manifold. $\square$

Acknowledgments. The author is grateful to Yu.S. Belousov, I.A. Dynnikov, S.S. Podkorytov, and E.A. Fominykh for useful discussions. The author is also grateful to the referee for valuable comments.

FUNDING

The study was supported by the Russian Science Foundation grant no. 22–11–00299, https://rscf.ru/en/project/22-11-00299/.

CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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