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THE RIESZ–ZYGMUND SUMS OF FOURIER–CHEBYSHEV RATIONAL INTEGRAL OPERATORS AND THEIR APPROXIMATION PROPERTIES

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Abstract—Studying the approximation properties of a certain Riesz–Zygmund sum of Fourier–Chebyshev rational integral operators with constraints on the number of geometrically distinct poles, we obtain an integral expression of the operators. We find upper bounds for pointwise and uniform approximations to the function $|x|^s$ with $s \in (0, 2)$ on the segment $[-1, 1]$, an asymptotic expression for the majorant of uniform approximations, and the optimal values of the parameter of the approximant providing the greatest decrease rate of the majorant. We separately study the approximation properties of the Riesz–Zygmund sums for Fourier–Chebyshev polynomial series, establish an asymptotic expression for the Lebesgue constants, and estimate approximations to $f \in H^{(\gamma)}[-1, 1]$ and $\gamma \in (0, 1]$ as well as pointwise and uniform approximations to the function $|x|^s$ with $s \in (0, 2)$.

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Introduction

Consider the Fourier series $\sum_{n=0}^{+\infty} a_n(f)\varphi_n(x)$ of some function f with respect to some orthogonal system $\{\varphi_n(x)\}_{n=0}^{+\infty}$. The expressions

$$R_n^{\lambda,\delta}(f,x) = \sum_{k=0}^{n}\left(1-\left(\frac{k}{n+1}\right)^{\lambda}\right)^{\delta} a_k(f)\varphi_k(x), \quad \delta,\lambda>0,\ n\in\mathbb{N}, \tag{1}$$

are *Riesz sums* of orthogonal Fourier series [1, 2].

Riesz's summation method has wide application in the theory of Dirichlet series and analytic number theory [3], as well as in the theory of Fourier series [4]. The relations of Riesz's method with Cesàro's summation methods and the methods of discrete Riesz means of various orders are studied in [5, 6].

In (1) put $\delta = 1$ and $\lambda = 2$. In this case the sums of the form

$$R_n^{2,1}(f,x) = \sum_{k=0}^{n}\left(1-\left(\frac{k}{n+1}\right)^{2}\right) a_k(f)\varphi_k(x), \quad n=0,1,\dots, \tag{2}$$

coincide with the normal Zygmund means $Z_{2n}^2(f,x)$ (see [7–9] for instance) and consequently they are called *Riesz–Zygmund means*. They enjoy a series of interesting properties. Chikina used Riesz–Zygmund means to estimate approximations to functions of bounded p-variation in the L_p space in [10]. The approximation properties of Riesz–Zygmund means for Fourier series with respect to Vilenkin's multiplicative system in L_p are studied in [11].

It is not difficult to see from (2) that

$$R_n^{2,1}(f,x) = \frac{1}{(n+1)^2}\sum_{k=0}^{n}(2k+1)s_k(f,x),$$

where $s_k(f,x) = \sum_{k=0}^{n} a_k(f)\varphi_k(x)$ for $n \in \mathbb{N}$ are partial sums of orthogonal Fourier series.

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Among the methods of rational approximation, some series of operators are distinguished that are analogous to the well-known periodic polynomial operators of Fourier, Fejér, Jackson, and de la Vallée Poussin [12–14]. The Riesz–Zygmund means of rational Fourier series with respect to a certain orthogonal system of Chebyshev–Markov rational fractions with two geometrically distinct poles were constructed and studied in [15]; in particular, there were found some estimates for uniform rational approximations to the function $|x|^s$ with $s \in (0, 2)$ and the parameter values ensuring the best estimates by this method.

In 1979 Rovba introduced [16] the Fourier–Chebyshev rational integral operator. Given a set $\{a_k\}_{k=1}^n$ of numbers, where a_k are either real with $|a_k| < 1$ or pairwise complex conjugate, on the set of functions $f(x)$ summable with the weight $1/\sqrt{1-x^2}$ on the segment consider the rational integral operator associated to the system of Chebyshev–Markov rational functions [16]:

$$s_n(f, x) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(\cos v) \frac{\sin \lambda_n(v, u)}{\sin \frac{v-u}{2}}\, dv, \quad x = \cos u, \tag{3}$$

where

$$\lambda_n(v, u) = \int\limits_u^v \lambda_n(y)\, dy,$$

$$\lambda_n(y) = \frac{1}{2} + \sum_{k=1}^n \frac{1 - |z_k|^2}{1 + 2|z_k| \cos(y - \arg z_k) + |z_k|^2}, \quad z_k = \frac{a_k}{1 + \sqrt{1 - a_k^2}}, \ |\alpha_k| < 1.$$

The operator $s_n : f \to \mathbb{R}_n(A)$, where $\mathbb{R}_n(A)$ is the set of rational functions of the form

$$\frac{p_n(x)}{\prod\limits_{k=1}^n (1 + a_k x)}, \quad p_n(x) \in \mathbb{P}_n, \ a_k = \frac{2z_k}{1 + z_k^2}, \ k = 1, 2, \dots, n,$$

while A is the set of parameters $\{z_1, z_2, \dots, z_n\}$ and $s_n(1, x) \equiv 1$. In particular, if we put $a_k = 0$ and $k = 1, \dots, n$ then $s_n(f, x)$ amounts to the partial sums of Fourier–Chebyshev polynomial series.

Operators (3) have found wide application in rational approximation [17–19]. Some classes of functions on $[-1, 1]$ discovered with their use reflect the specific features of rational approximation. Approximations to the Markov function on the segment $[-1, 1]$ by the Abel–Poisson sums for operators (3) were studied in [20]. A similar problem for Fejér sums was solved in [21].

It is of interest to introduce Riesz–Zygmund sums for operators (3) with a fixed number of geometrically distinct poles in the extended complex plane. This article establishes an integral expression for the introduced operator and some approximation properties of the corresponding polynomial analog.

The problem of approximating $|x|$ on $[-1, 1]$ has a rich history stemming from the twentieth century, when the polynomial approximation of this nonsmooth function drew the interests of Lebesgue, Jackson, and Bernstein [22]. Newman's article [23] on the rational approximation of $|x|$ on $[-1, 1]$ provided a new impetus in this direction. The subject has developed in many articles [24, 25]. The most complete result is due to Stahl [26].

The studies of approximation to $|x|^s$ with $s > 0$ also go back to Bernstein [27]. Many available articles deal with both the best approximations to this function [28–31] and particular approximation methods [32–34].

As a separate problem, this article addresses approximations to $|x|^s$ with $s \in (0, 2)$ by Riesz–Zygmund sums for operators (3). We obtain the appropriate estimates for uniform rational approximations and establish that the Riesz–Zygmund sums under study for a certain choice of parameters also yield uniform approximations better in the sense of order than the corresponding polynomial analogs.

1. The Riesz–Zygmund Sums for Rational Integral Operators

Given a positive integer $q \in (0, n)$, denote by A_q the subset of parameters in A such that among the numbers $z_1, z_2, \dots, z_n$, exactly q are distinct and the multiplicity of each parameter equals m with $n = mq$. Consider the sums

$$R_{n,q}(f, x) = \frac{1}{(m+1)^2} \sum_{k=0}^{m} (2k+1) s_{kq}(f, x), \quad x \in [-1, 1],\ m \in \mathbb{N} \cup \{0\}, \tag{4}$$

where $s_{kq}(f, \cdot)$ for $k = 1, 2, \dots, m$ is operator (3) of order kq. It is natural to call (4) the Riesz–Zygmund sums for (3).

It follows from (4) that $R_{n,q} : f \to \mathbb{R}_n(A_q)$ where $\mathbb{R}_n(A_q)$ is the set of rational functions of the form

$$\frac{\pi_n(x)}{\left(\prod\limits_{k=1}^{q} (1 + a_k x)\right)^m}, \quad \pi_n(x) \in \mathbb{P}_n,\ z_k = \frac{a_k}{1 + \sqrt{1 - a_k^2}},\ k = 1, 2, \dots, q,\ n = mq;$$

furthermore, $R_{n,q}(1, x) \equiv 1$.

Thus, we will discuss approximation by rational functions with q geometrically distinct poles in the extended complex plane, each of multiplicity m. For the first time approximation with constraints on the number of geometrically distinct poles was studied by Lungu [35, 36].

Theorem 1. *Riesz–Zygmund rational functions* (4) *admit the integral expression*

$$R_{n,q}(f, x) = \frac{1}{8\pi(m+1)^2} \int_{-\pi}^{\pi} f(\cos v) K_{n,q}(u, v)\, dv, \quad x \in [-1, 1],\ x = \cos u, \tag{5}$$

where

$$K_{n,q}(u, v) = \frac{1}{\sin \frac{v-u}{2} \sin^2 \frac{\lambda_q}{2}} \left(\sin\left(\lambda_q - \frac{v-u}{2}\right) - \sin t \right.$$
$$\left. + (2m+3) \sin\left(\frac{v-u}{2} + m\lambda_q\right) - (2m+1) \sin\left(\frac{v-u}{2} + (m+1)\lambda_q\right) \right),$$

$$\lambda_q = \lambda_q(u, v) = \int_u^v \sum_{k=1}^{q} \frac{1 - |z_k|^2}{1 + 2|z_k| \cos(y - \arg z_k) + |z_k|^2}\, dy.$$

Proof. It is known [16] that (3) can be expressed as

$$s_n(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\cos v) \left(\zeta \frac{\omega_n(\zeta)}{\omega_n(\xi)} - \xi \frac{\omega_n(\xi)}{\omega_n(\zeta)} \right) \frac{dv}{\zeta - \xi}, \quad \xi = e^{iu},\ \zeta = e^{iv},\ x = \cos u,$$

where

$$\omega_n(\zeta) = \prod_{k=1}^{n} \frac{\zeta + z_k}{1 + z_k \zeta}, \quad z_k = \frac{a_k}{1 + \sqrt{1 - a_k^2}},\ |z_k| < 1.$$

In the case of q distinct poles the last integral expression, whose image is a rational function of order kq, for $k = 0, 1, 2, \dots$, is of the form

$$s_{kq}(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\cos v) \left(\zeta \left(\frac{\omega_q(\zeta)}{\omega_q(\xi)} \right)^k - \xi \left(\frac{\omega_q(\xi)}{\omega_q(\zeta)} \right)^k \right) \frac{dv}{\zeta - \xi}, \quad k = 0, 1, 2, \dots.$$

Insert this expression into (4) and switch the order of summation and integration. This yields

$$R_{n,q}(f,x) = \frac{1}{2\pi(m+1)^2} \int_{-\pi}^{\pi} f(\cos v) K_{n,q}(u,v)\, dv, \quad x \in [-1,1],\ x = \cos u,$$

where

$$K_{n,q}(u,v) = \left(\sqrt{\frac{\zeta}{\xi}} \sum_{k=0}^{m} (2k+1) \left(\frac{\omega_q(\zeta)}{\omega_q(\xi)} \right)^k - \sqrt{\frac{\xi}{\zeta}} \sum_{k=0}^{m} (2k+1) \left(\frac{\omega_q(\xi)}{\omega_q(\zeta)} \right)^k \right) \frac{1}{2i \sin \frac{v-u}{2}}.$$

Observe that

$$\frac{\omega_q(\zeta)}{\omega_q(\xi)} = e^{i\lambda_q(u,v)},$$

where $\lambda_q(u,v)$ is defined in the statement of this theorem.

In order to arrive at (5), it suffices to observe that in $K_{n,q}(u,v)$ the expression in parentheses is the difference of two complex conjugate expressions and to apply the available equality

$$\sum_{k=0}^{m} (2k+1) q^k = \frac{1 + q - (2m+3)q^{m+1} + (2m+1)q^{m+2}}{(1-q)^2}, \quad q \neq 1. \tag{6}$$

The proof of Theorem 1 complete. $\square$

When $z_k = 0$ for $k = 1, 2, \dots, q$, denote $R_{n,q}(f,x)$ by $R_n(f,x)$. The last quantities amount to the Riesz–Zygmund sums for Fourier series with respect to the Chebyshev polynomial system of the first kind. This yields the following corollary.

Corollary 1. *We have the integral expression*

$$R_n(f,x) = \frac{1}{8\pi(n+1)^2} \int_{-\pi}^{\pi} f(\cos v) K_n(u,v)\, dv, \quad x = \cos u,\ x \in [-1,1], \tag{7}$$

where

$$K_n(u,v) = \frac{(2n+3)\sin(2n+1)\frac{v-u}{2} - (2n+1)\sin(2n+3)\frac{v-u}{2}}{\sin^3 \frac{v-u}{2}}.$$

The integral expression (7) appears in [15], where it was obtained as a particular case of Riesz–Zygmund sums for Fourier series with respect to the orthogonal system of Chebyshev–Markov rational functions with two geometrically distinct poles.

2. Studying Riesz–Zygmund Sums in the Polynomial Case

Let us elucidate the asymptotic behavior of the Lebesgue constant of the operator (7) as $n \to \infty$, i.e., the expression

$$L_n = \frac{1}{8\pi(n+1)^2} \int_{-\pi}^{\pi} \left| \frac{(2n+3)\sin(2n+1)(t/2) - (2n+1)\sin(2n+3)(t/2)}{\sin^3(t/2)} \right| dt,\ n \in \mathbb{N}.$$

Theorem 2. *We have the asymptotic equality*

$$L_n = \frac{2}{\pi} \int_{0}^{x_1} \frac{\sin u}{u}\, du + \frac{4}{\pi} \int_{x_1}^{+\infty} \frac{|\sin u - u\cos u|}{u^3}\, du + O\left(\frac{1}{n+1}\right) \quad \text{as } n \to \infty, \tag{8}$$

where $x_1 = 4.493\dots$ is the first root of the equation $\psi(u) = \sin u - u \cos u = 0$ on the interval $(0, +\infty)$.

PROOF. Since the integrand is an even function, write down

$$L_n = \frac{1}{4\pi(n+1)^2} \int\limits_0^{\pi} \frac{|(2n+3)\sin(2n+1)(t/2) - (2n+1)\sin(2n+3)(t/2)|}{\sin^3(t/2)}\, dt.$$

Straightforward transformations reduce the integral on the right-hand side to

$$L_n = \frac{1}{2\pi(n+1)^2} \int\limits_0^{\pi} \frac{|\sin(n+1)t\cos t/2 - 2(n+1)\cos(n+1)t\sin t/2|}{\sin^3 t/2}\, dt.$$

Using the easy asymptotic equality

$$\frac{1}{\sin^3 u} - \frac{1}{u^3} = O\left(\frac{1}{u}\right) \quad \text{as } u \to 0,$$

we conclude that

$$L_n = \frac{4}{\pi(n+1)^2} \int\limits_0^{\pi} \frac{|\sin(n+1)t\cos t/2 - 2(n+1)\cos(n+1)t\sin t/2|}{t^3}\, dt + O\left(\frac{1}{n+1}\right).$$

In the integral on the right-hand side change the variable $(n+1)t \mapsto u$. Then

$$L_n = \frac{4}{\pi} \int\limits_0^{(n+1)\pi} \frac{\left|\sin u \cos \frac{u}{2(n+1)} - 2(n+1)\cos u \sin \frac{u}{2(n+1)}\right|}{u^3}\, du + O\left(\frac{1}{n+1}\right).$$

Since uniformly in $u \in (0, (n+1)\pi)$ as $n \in \mathbb{N}$ we have

$$\cos\frac{u}{2(n+1)} = 1 - O\left(\left(\frac{u}{2(n+1)}\right)^2\right),$$

$$\sin\frac{u}{2(n+1)} = \frac{u}{2(n+1)} - O\left(\left(\frac{u}{2(n+1)}\right)^3\right),$$

appreciating that $|u \pm v| = |u| + \theta|v|$, with $\theta \in [-1,1]$, we obtain

$$\left|\sin u \cos\frac{u}{2(n+1)} - 2(n+1)\cos u \sin\frac{u}{2(n+1)}\right| = |\sin u - u\cos u| + \frac{\theta_1 u^3}{(n+1)^2},$$

where θ_1 is some constant independent of n and u.

The last equality implies that

$$L_n = \frac{4}{\pi} \int\limits_0^{(n+1)\pi} \frac{|\sin u - u\cos u|}{u^3}\, du + O\left(\frac{1}{n+1}\right) \quad \text{as } n \to \infty.$$

The function $\psi(u) = \sin u - u\cos u$ has exactly n zeros on $(0, (n+1)\pi)$. Denote them by $x_1, x_2, \ldots, x_n$. Then

$$L_n = \frac{4}{\pi}\left[\int\limits_0^{x_1} \frac{\sin u - u\cos u}{u^3}\, du + \int\limits_{x_1}^{(n+1)\pi} \frac{|\sin u - u\cos u|}{u^3}\, du\right] + O\left(\frac{1}{n+1}\right) \quad \text{as } n \to \infty.$$

Since
$$\int \frac{\sin u - u\cos u}{u^3}\,du = \frac{1}{2}\int \frac{\sin u}{u}\,du - \frac{\sin u - u\cos u}{2u^2} + C,$$
from the last asymptotic equality we infer that
$$L_n = \frac{2}{\pi}\int\limits_0^{x_1} \frac{\sin u}{u}\,du + \frac{4}{\pi}\int\limits_{x_1}^{(n+1)\pi} \frac{|\sin u - u\cos u|}{u^3}\,du + O\left(\frac{1}{n+1}\right) \quad \text{as } n\to\infty.$$
The first integral on the right-hand side exists. Considering that
$$\int\limits_{x_1}^{(n+1)\pi} \frac{|\sin u - u\cos u|}{u^3}\,du \le \int\limits_{x_1}^{(n+1)\pi} \frac{du}{u^3} + \int\limits_{x_1}^{(n+1)\pi} \frac{du}{u^2} = \frac{1}{2x_1^2} + \frac{1}{x_1} + O\left(\frac{1}{n+1}\right),$$
we conclude that the second integral exists as well as $n\to\infty$. Consequently, we arrive at (8). The proof of Theorem 2 is complete. $\square$

Remark 1. Theorem 2 implies that the Lebesgue constants of Riesz–Zygmund sums for Fourier–Chebyshev polynomial series are bounded. The latter in turn means that the Riesz–Zygmund sums under study converge uniformly for every $f\in C[-1,1]$. Theorem 2 is an algebraic analog of a result of Stepanets [37, p. 261]. $\square$

Consider the classes $H^{(\gamma)}[-1,1]$, with $\gamma\in(0,1]$, of $f(x)$ satisfying the Lipschitz condition with exponent γ and constant 1, meaning the condition
$$|f(x_1)-f(x_2)| \le |x_1-x_2|^\gamma, \quad x_1,x_2\in[-1,1].$$

Theorem 3. *If f belongs to $H^{(\gamma)}[-1,1]$ with $\gamma\in(0,1]$ then*
$$|f(x)-R_n(f,x)| \le \frac{2}{\pi}\left(\frac{\sqrt{1-x^2}}{n+1}\right)^{\gamma} c_1(\gamma) + O\left(\frac{(\sqrt{1-x^2})^\gamma}{n+1}\right) + \delta_n^{(\gamma)}(x) \tag{9}$$
if $\gamma\in(0,1)$ and
$$|f(x)-R_n(f,x)| \le \frac{8}{\pi^2}\frac{\sqrt{1-x^2}\ln(n+1)}{n+1} + O\left(\frac{\sqrt{1-x^2}}{n+1}\right) + \delta_n^{(1)}(x) \tag{10}$$
if $\gamma=1$, where
$$\delta_n^{(\gamma)}(x) = \begin{cases} \frac{2^{2-\gamma}c_2(\gamma)|x|^\gamma}{(n+1)^{2\gamma}} + O\big(\frac{|x|^\gamma}{n+1}\big), & \gamma\in(0,1/2),\\ \frac{2^{\frac{5}{2}}}{\pi^2}\frac{\sqrt{|x|}\ln(n+1)}{n+1} + O\big(\frac{\sqrt{|x|}}{n+1}\big), & \gamma=1/2,\\ \frac{2^{2-\gamma}|x|^\gamma}{\pi^{2(1-\gamma)}(2\gamma-1)(n+1)} + O\big(\frac{|x|^\gamma}{(n+1)^{2\gamma}}\big), & \gamma\in(1/2,1], \end{cases}$$
$$c_1(\gamma) = \frac{1}{2-\gamma}\int\limits_0^{x_1} u^{\gamma-1}\sin u\,du + \int\limits_{x_1}^{+\infty} \frac{|\sin u - u\cos u|}{u^{3-\gamma}}\,du,$$
$$c_2(\gamma) = \frac{1}{2-2\gamma}\int\limits_0^{x_1} u^{2\gamma-1}\sin u\,du + \int\limits_{x_1}^{+\infty} \frac{|\sin u - u\cos u|}{u^{3-2\gamma}}\,du,$$
while $x_1 = 4.493\ldots$ is the first root of the equation $\psi(u)=\sin u - u\cos u = 0$ on the interval $(0,+\infty)$.

PROOF. Since $R_n(\cdot,\cdot)$ is exact on the constants, from (7) we find that

$$f(x) - R_n(f,x) = \frac{1}{8\pi(n+1)^2}\int_{-\pi}^{\pi}[f(\cos u) - f(\cos(u+t))]$$
$$\times \frac{(2n+3)\sin(2n+1)(t/2) - (2n+1)\sin(2n+3)(t/2)}{\sin^3(t/2)}\,dt,$$

where $x = \cos u$ and $x \in [-1,1]$.

Splitting the integral on the right-hand side into two integrals over $[-\pi,0]$ and $[0,\pi]$ and changing the variable $t \mapsto -t$ in the first one, we obtain

$$f(x) - R_n(f,x) = I_n(+u) + I_n(-u), \tag{11}$$

where

$$I_n(\pm u) = \frac{1}{4\pi(n+1)^2}\int_0^{\pi}[f(\cos u) - f(\cos(u\pm t))]$$
$$\times \frac{\sin(n+1)t\cos(t/2) - 2(n+1)\cos(n+1)t\sin(t/2)}{\sin^3(t/2)}\,dt.$$

Accounting for

$$|f(\cos u) - f(\cos(u\pm t))| \le 2^{\gamma}\left[|\sin u|^{\gamma}\sin^{\gamma}\frac{t}{2} + |\cos u|^{\gamma}\sin^{2\gamma}\frac{t}{2}\right],$$

we see that

$$|I_n(\pm u)| \le \frac{2^{\gamma-2}}{\pi(n+1)^2}[|\sin u|^{\gamma}J_1 + |\cos u|^{\gamma}J_2], \quad n \in \mathbb{N}, \tag{12}$$

where

$$J_1 = \int_0^{\pi}\frac{|\sin(n+1)t\cos(t/2) - 2(n+1)\cos(n+1)t\sin(t/2)|}{\sin^{3-\gamma}(t/2)}\,dt,$$

$$J_2 = \int_0^{\pi}\frac{|\sin(n+1)t\cos(t/2) - 2(n+1)\cos(n+1)t\sin(t/2)|}{\sin^{3-2\gamma}(t/2)}\,dt.$$

Inspect each integral separately. Using in J_1 the easy asymptotic equality

$$\frac{1}{\sin^{3-\gamma}(t/2)} - \frac{1}{(t/2)^{3-\gamma}} = O\left(\frac{1}{t^{1-\gamma}}\right) \quad \text{as } t \to 0,\ \gamma \in (0,1],$$

we obtain

$$J_1 = 2^{3-\gamma}\int_0^{\pi}\frac{|\sin(n+1)t\cos(t/2) - 2(n+1)\cos(n+1)t\sin(t/2)|}{t^{3-\gamma}}\,dt$$
$$+O\left(\int_0^{\pi}\frac{|\sin(n+1)t\cos(t/2) - 2(n+1)\cos(n+1)t\sin(t/2)|}{t^{1-\gamma}}\,dt\right) \quad \text{as } n \to \infty.$$

In the integral on the right-hand side change the variable as $(n+1)t \mapsto u$. Then

$$J_1 = 2^{3-\gamma}(n+1)^{2-\gamma} \int\limits_0^{(n+1)\pi} \frac{\left|\sin u \cos \frac{u}{2(n+1)} - 2(n+1)\cos u \sin \frac{u}{2(n+1)}\right|}{u^{3-\gamma}}\, du + O(n+1).$$

The last estimate implies obviously that

$$J_1 = 2^{3-\gamma}(n+1)^{2-\gamma} \int\limits_0^{(n+1)\pi} \frac{|\sin u - u\cos u|}{u^{3-\gamma}}\, du + O(n+1) \quad \text{as } n \to \infty. \tag{13}$$

Assume that $\gamma \in (0,1)$. Acting as in the proof of Theorem 2, we find that

$$J_1 = 2^{3-\gamma}(n+1)^{2-\gamma} c_1(\gamma) + O(n+1) \quad \text{as } n \to \infty, \tag{14}$$

where $c_1(\gamma)$ is defined in the statement of the theorem.

Assume now that $\gamma = 1$. Then (13) yields

$$J_1 = 4(n+1)\Bigg[1 - \cos x_1 + \sum_{k=1}^{n-1}(-1)^k \int\limits_{x_k}^{x_{k+1}} \frac{\sin u - u\cos u}{u^2}\, du$$

$$+(-1)^n \int\limits_{x_n}^{(n+1)\pi} \frac{\sin u - u\cos u}{u^2}\, du\Bigg] + O(n+1) \quad \text{as } n \to \infty.$$

Calculating the integral on the right-hand side, we arrive at the asymptotic equality

$$J_1 = 4(n+1)\left[1 + 2\sum_{k=1}^{n}(-1)^k \cos x_k\right] + O(n+1) \quad \text{as } n \to \infty. \tag{15}$$

It is known [38, p. 30] that the roots x_k, for $k = 1, 2, \ldots, n$, of the equation $\psi(u) = \sin u - u\cos u = 0$ or $\operatorname{tg} u = u$, which is the same, have the asymptotic expansion

$$x_k = \frac{\pi(2k+1)}{2} - \frac{2}{\pi(2k+1)} + o\left(\frac{1}{k}\right) \quad \text{as } k \to \infty.$$

Furthermore, in (15) we obtain

$$J_1 = 4(n+1)\left[1 + 2\sum_{k=1}^{n} \sin\frac{2}{\pi(2k+1)} + c\right] + O(n+1) \quad \text{as } n \to \infty,$$

where c is some positive constant.

Using the inequality $\sin\theta \le \theta$ for $\theta \ge 0$, we find that

$$J_1 \le \frac{16(n+1)}{\pi} \sum_{k=1}^{n} \frac{1}{2k+1} + O(n+1) \quad \text{as } n \to \infty.$$

This implies that

$$J_1 \le \frac{8}{\pi}(n+1)\ln(n+1) + O(n+1) \quad \text{as } n \to \infty. \tag{16}$$

Combining (14) and (16), as $n \to \infty$ we obtain

$$J_1 \le \begin{cases} 2^{3-\gamma}(n+1)^{2-\gamma} c_1(\gamma) + O(n+1), & \gamma \in (0,1), \\ \frac{8}{\pi}(n+1)\ln(n+1) + O(n+1), & \gamma = 1. \end{cases} \tag{17}$$

Inspect the integral J_2; see (12). Arguing as for the integral J_1, we arrive at

$$J_2 = 2^{3-2\gamma}(n+1)^{2-2\gamma} \int_0^{(n+1)\pi} \frac{|\sin u - u\cos u|}{u^{3-2\gamma}}\,du + O(n+1).$$

If $\gamma \in (0,1/2)$ then

$$J_2 = 2^{3-2\gamma}(n+1)^{2-2\gamma} c_2(\gamma) + O(n+1), \tag{18}$$

where the constant $c_2(\gamma)$ is defined in the statement of the theorem.

If $\gamma = 1/2$ then

$$J_2 \le \frac{8}{\pi}(n+1)\ln(n+1) + O(n+1) \quad \text{as } n \to \infty. \tag{19}$$

Assume that $\gamma \in (1/2, 1]$. Then

$$J_2 = 2^{3-2\gamma}(n+1)^{2-2\gamma}\left[\int_0^{x_1} \frac{\sin u - u\cos u}{u^{3-2\gamma}}\,du + \int_{x_1}^{(n+1)\pi} \frac{|\sin u - u\cos u|}{u^{3-2\gamma}}\,du\right] + O(n+1) \quad \text{as } n \to \infty.$$

Taking the equality

$$\int_0^{x_1} \frac{\sin u - u\cos u}{u^{3-2\gamma}}\,du = \frac{1}{2-2\gamma}\int_0^{x_1} u^{2\gamma-1}\sin u\,du$$

into account, as well as the easy inequality

$$\int_{x_1}^{(n+1)\pi} \frac{|\sin u - u\cos u|}{u^{3-2\gamma}}\,du$$

$$\le \frac{(\pi(n+1))^{2\gamma-1}}{2\gamma-1} - \frac{x_1^{2\gamma-1}}{2\gamma-1} + \frac{1}{(2-2\gamma)x_1^{2-2\gamma}} - \frac{1}{(2-2\gamma)(\pi(n+1))^{2-2\gamma}},$$

we arrive at the estimate

$$J_2 \le \frac{2^{3-2\gamma}\pi^{2\gamma-1}(n+1)}{2\gamma-1} + O((n+1)^{2-2\gamma}) \quad \text{as } n \to \infty. \tag{20}$$

Combining (18), (19), and (20), as $n \to \infty$ we obtain

$$J_2 \le \begin{cases} 2^{3-2\gamma}(n+1)^{2-2\gamma} c_2(\gamma) + O(n+1), & \gamma \in (0,1/2), \\ \frac{8}{\pi}(n+1)\ln(n+1) + O(n+1), & \gamma = 1/2, \\ \frac{2^{3-2\gamma}\pi^{2\gamma-1}(n+1)}{2\gamma-1} + O((n+1)^{2-2\gamma}), & \gamma \in (1/2,1]. \end{cases} \tag{21}$$

Inserting (17) and (21) into (12) and considering (11), we obtain (9) and (10). The proof of Theorem 3 is complete. □

Remark 2. Theorem 3 implies that approximations to $f \in H^{(\gamma)}[-1,1]$ with $\gamma \in (0,1]$ by the Riesz–Zygmund sums for Fourier–Chebyshev series depend substantially on the location of x on $[-1,1]$; furthermore, approximations at the endpoints decrease faster than on the segment in general.

2. Rational Approximations to $|x|^s$

Let us study approximations to $|x|^s$, where $s \in (0,2)$, on $[-1,1]$ by sums (4) with a fixed number of geometrically distinct poles in the extended complex plane. Since the function in question has a power singularity at $x = 0$ and is even, we have to make a special choice of parameters of the rational approximant. As above, take an arbitrary positive integer $q \in (0,n)$. Consider the set A_{2q} of $2n$ parameters of the form

$$z_k = i\alpha_k,\ z_{k+q} = -i\alpha_k, \quad k = 1,2,\dots,q,\ n = mq.$$

In other words, we will discuss approximation by the rational functions of the form

$$r_{2n}(x) = \frac{\pi_n(x^2)}{\Big(\prod\limits_{k=1}^{q}(1+a_k^2x^2)\Big)^m}, \quad a_k = \frac{2z_k}{1+z_k^2},$$

where $\pi_n(x^2)$ is an even polynomial of degree at most $2n$.

Put

$$\varepsilon_{2n,2q}(x, A_{2q}) = |x|^s - R_{2n,2q}(|\cdot|^s, x), \quad x \in [-1,1],$$

$$\varepsilon_{2n,2q}(A_{2q}) = \||x|^s - R_{2n,2q}(|\cdot|^s, x)\|_{C[-1,1]}, \quad n \in \mathbb{N}.$$

Theorem 4. *For approximations to $|x|^s$ with $s \in (0,2)$ on $[-1,1]$ by the Riesz–Zygmund sums* (4) *we have*

(1) *the estimate for pointwise approximations*

$$|\varepsilon_{2n,2q}(x, A_{2q})| \le \frac{2^{2-s}}{(m+1)^2\pi}\sin\frac{\pi s}{2}\int\limits_0^1 \frac{(1-t^2)^s t^{1-s}}{\sqrt{1+2t^2\cos 2u+t^4}}$$
$$\times \frac{1+|\chi_{2q}(t)| - (2m+3)|\chi_{2q}(t)|^{m+1} + (2m+1)|\chi_{2q}(t)|^{m+2}}{(1-|\chi_{2q}(t)|)^2}\,dt, \tag{22}$$

$x = \cos u$, $x \in [-1,1]$, where

$$\chi_{2q}(t) = \prod_{k=1}^{q}\frac{t^2-\alpha_k^2}{1-\alpha_k^2t^2};$$

(2) *the estimate for uniform approximations*

$$\varepsilon_{2n,2q}(A_{2q}) \le \varepsilon^*_{2n,2q}(A_{2q}), \quad n \in \mathbb{N}, \tag{23}$$

where

$$\varepsilon^*_{2n,2q}(A_{2q}) = \frac{2^{2-s}}{(m+1)^2\pi}\sin\frac{\pi s}{2}$$
$$\times \int\limits_0^1 (1-t^2)^{s-1}t^{1-s}\frac{1+|\chi_{2q}(t)| - (2m+3)|\chi_{2q}(t)|^{m+1} + (2m+1)|\chi_{2q}(t)|^{m+2}}{(1-|\chi_{2q}(t)|)^2}\,dt. \tag{24}$$

Proof. Using the accuracy of Riesz–Zygmund sums on the constants, we find from (4) that

$$\varepsilon_{2n,2q}(x, A_{2q}) = \frac{1}{(m+1)^2}\sum_{k=0}^{m}(2k+1)\delta_{2k,2q}(x, A_{2q}), \quad x \in [-1,1], \tag{25}$$

where $\delta_{2k,2q}(x, A_{2q})$ for $k = 0, \dots, m$ amount to the approximations to $|x|^s$ on $[-1, 1]$ by (3). It is known [17] that we have the integral expression

$$\delta_{2n}(x, A) = \frac{(-1)^n 2^{1-s}}{\pi} \sin\frac{\pi s}{2} \int\limits_0^1 (1-t^2)^s t^{1-s} \left[\frac{\xi^2 \omega_{2n}(\xi)}{1+t^2\xi^2} + \frac{\overline{\omega_{2n}(\xi)}}{\xi^2+t^2}\right] \chi_{2n}(t)\, dt,$$

where

$$\omega_{2n}(\xi) = \prod_{k=1}^{n} \frac{\xi^2+\alpha_k^2}{1+\alpha_k^2\xi^2}, \quad \chi_{2n}(t) = \prod_{k=1}^{n} \frac{t^2-\alpha_k^2}{1-\alpha_k^2 t^2}, \quad \xi = e^{i\theta},\ x = \cos\theta.$$

With our constraints on the parameters of the approximant, the last expression becomes

$$\delta_{2m,2q}(x, A_{2q}) = \frac{(-1)^m 2^{1-s}}{\pi} \sin\frac{\pi s}{2} \int\limits_0^1 (1-t^2)^s t^{1-s} \left[\frac{\xi^2 \omega_{2q}^m(\xi)}{1+t^2\xi^2} + \frac{\overline{\omega_{2q}^m(\xi)}}{\xi^2+t^2}\right] \chi_{2q}^m(t)\, dt, \quad m \in \mathbb{N}.$$

Inserting the last integral expression into (25) and switching the order of summation and integration, we obtain

$$\varepsilon_{2n,2q}(x, A_{2q}) = \frac{2^{1-s}}{(m+1)^2\pi} \sin\frac{\pi s}{2}$$
$$\times \int\limits_0^1 (1-t^2)^s t^{1-s} \sum_{k=0}^{m} (-1)^k (2k+1) \chi_{2q}^k(t) \left[\frac{\xi^2 \omega_{2q}^k(\xi)}{1+t^2\xi^2} + \frac{\overline{\omega_{2q}^k(\xi)}}{\xi^2+t^2}\right] dt.$$

The terms in square brackets are complex conjugates of each other, and so their sum is real. Straightforward rearrangements yield

$$\frac{\xi^2 \omega_{2q}^k(\xi)}{1+t^2\xi^2} + \frac{\overline{\omega_{2q}^k(\xi)}}{\xi^2+t^2} = \frac{2\cos\psi_{2qk}(x,t)}{\sqrt{1+2t^2\cos 2u+t^4}}, \quad k = 0, 1, 2, \dots, m,$$

where

$$\psi_{2qk}(x,t) = \arg\frac{\xi^2 \omega_{2q}^k(\xi)}{1+t^2\xi^2}, \quad k = 0, 1, 2, \dots, m.$$

In result,

$$\varepsilon_{2n,2q}(x, A_{2q}) = \frac{2^{2-s}}{(m+1)^2\pi} \sin\frac{\pi s}{2}$$
$$\times \int\limits_0^1 \frac{(1-t^2)^s t^{1-s}}{\sqrt{1+2t^2\cos 2u+t^4}} \sum_{k=0}^{m} (-1)^k (2k+1) \chi_{2q}^k(t) \cos\psi_{2qk}(x,t)\, dt. \tag{26}$$

From the last expression, we arrive at

$$|\varepsilon_{2n,2q}(x, A_{2q})| \le \frac{2^{2-s}}{(m+1)^2\pi} \sin\frac{\pi s}{2} \int\limits_0^1 \frac{(1-t^2)^s t^{1-s}}{\sqrt{1+2t^2\cos 2u+t^4}} \sum_{k=0}^{m} (2k+1) |\chi_{2q}(t)|^k\, dt.$$

Applying (6) to the sum in the integrand, we justify the first claim of Theorem 4, i.e., the estimate in (22).

From (22) we obtain (23) by observing that

$$\sqrt{1+2t^2\cos 2u+t^4} \ge 1-t^2, \quad t \in [0,1],\ u \in \mathbb{R}.$$

The proof of Theorem 4 is complete. $\square$

Put $\alpha_k = 0$ for $k = 1, 2, \dots, q$ in Theorem 4. Then $\varepsilon_{2n,2}(x, O) = \varepsilon_{2n}^{(0)}(x)$ and $\varepsilon_{2n,2}(O) = \varepsilon_{2n}^{(0)}$ amount respectively to the pointwise and uniform approximations of $|x|^s$ with $s \in (0, 2)$ on $[-1, 1]$ by the Riesz–Zygmund sums for Fourier–Chebyshev polynomial series. In this case Theorem 4 implies

Corollary 2. *For approximations to $|x|^s$ with $s \in (0,2)$ on $[-1,1]$ by the Riesz–Zygmund sums for Fourier–Chebyshev polynomial series we have*

(1) *the pointwise estimate*

$$|\varepsilon_{2n}^{(0)}(x)| \le \frac{2^{2-s}}{\pi(n+1)^2}\sin\frac{\pi s}{2}$$

$$\times \int_0^1 \frac{(1-t^2)^s t^{1-s}}{\sqrt{1+2t^2\cos 2u + t^4}} \frac{1+t^2-(2n+3)t^{2n+2}+(2n+1)t^{2n+4}}{(1-t^2)^2}\,dt, \quad x\in[-1,1]; \tag{27}$$

(2) *the integral expression*

$$\varepsilon_{2n}^{(0)} = \frac{2^{2-s}}{\pi(n+1)^2}\sin\frac{\pi s}{2}\int_0^1 (1-t^2)^{s-1}t^{1-s}\frac{1+t^2-(2n+3)t^{2n+2}+(2n+1)t^{2n+4}}{(1-t^2)^2}\,dt, \quad n\in\mathbb{N}. \tag{28}$$

The inequality in (27) *is sharp, as the equality is attained for $x=0$.*

Proof. We obtain (27) directly from (22) by putting $\alpha_k = 0$ for $k = 1,2,\dots,q$. Let us show that (27) is accurate. To this end, put $x=0$ in (27). Then

$$|\varepsilon_{2n}(0)| \le \frac{2^{2-s}}{\pi(n+1)^2}\sin\frac{\pi s}{2}\int_0^1 (1-t^2)^{s-1}t^{1-s}\frac{1+t^2-(2n+3)t^{2n+2}+(2n+1)t^{2n+4}}{(1-t^2)^2}\,dt. \tag{29}$$

On the other hand, putting $\alpha_k = 0$ for $k=1,2,\dots,q$ in (26) yields

$$\varepsilon_{2n}(x) = \frac{2^{2-s}}{\pi(n+1)^2}\sin\frac{\pi s}{2}\int_0^1 \frac{(1-t^2)^s t^{1-s}}{\sqrt{1+2t^2\cos 2u+t^4}}$$

$$\times\sum_{k=0}^{n}(-1)^k(2k+1)\cos\psi_{2k}(x,t)t^{2k}\,dt, \quad x\in[-1,1],\ s\in(0,2),$$

where

$$\psi_{2k}(x,t) = \arg\frac{\xi^2}{1+\xi^2t^2} + 2ku, \quad \xi = e^{iu},\ x=\cos u.$$

Inserting $x=0$ in the last relation, we verify that the inequality in (29) becomes an equality.

From (23) with $\alpha = 0$, appreciating that the estimate in the polynomial case is attainable for $x=0$, we obtain (28). The proof of Corollary 2 is complete. $\square$

3. Asymptotics for the Majorant of Uniform Approximation

Let us find an asymptotic expression for (24) as $m\to\infty$. To this end, in the integral on the right change the variable of integration as $t^2 = (1-u)/(1+u)$, and so $dt = -du/((1+u)^{\frac{3}{2}}(1-u)^{\frac{1}{2}})$. Then

$$\varepsilon_{2n,2q}^{*}(A_{2q}) = \frac{2}{(m+1)^2\pi}\sin\frac{\pi s}{2}\int_0^1 \mu_s(u)G_n(|\pi_q(u)|)\,du, \tag{30}$$

where

$$G_n(y) = \frac{1+y-(2m+3)y^{m+1}+(2m+1)y^{m+2}}{(1-y)^2},$$

$$\mu_s(u) = \frac{u^{s-1}}{(1+u)(1-u^2)^{\frac{s}{2}}}, \quad \pi_q(u) = \prod_{k=1}^{q}\frac{\beta_k-u}{\beta_k+u}, \quad \beta_k = \frac{1-\alpha_k^2}{1+\alpha_k^2},\ k=1,2,\dots,q.$$

Observe that in the case under consideration for each $n \in \mathbb{N}$ we can choose the corresponding tuple of parameters $(\beta_1, \beta_2, \ldots, \beta_q)$, i.e., $\beta_k = \beta_k(n)$ for $k = 1, 2, \ldots, q$. Impose the conditions

$$\lim_{m\to\infty} m\beta_k = \infty, \quad k = 1, 2, \ldots, q, \; n = mq; \tag{31}$$

without loss of generality we may also assume that the parameters β_k, $k = 1, 2, \ldots, q$, are ordered as

$$0 < \beta_q \le \beta_{q-1} \le \cdots \le \beta_1 \le 1.$$

Theorem 5. *For $\varepsilon^*_{2n,2q}(A_{2q})$ as $m \to \infty$ we have the asymptotic equalities*

$$\varepsilon^*_{2n,2q}(A_{2q}) \sim \frac{1}{\pi}\sin\frac{\pi s}{2}\begin{cases} \dfrac{2^{2-s}\Gamma(s)}{(1-s)(2-s)\Big(\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\Big)^s (m+1)^s} + \Phi_m^{(s)}(A_{2q}), & s \in (0,1), \\ \dfrac{2\ln(m+1)}{(m+1)\sum\limits_{j=1}^{q}\frac{1}{\beta_j}} + \Phi_m^{(1)}(A_{2q}), & s = 1, \\ \dfrac{2^{2-s}\Gamma(s)}{(2-s)\Big(\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\Big)^s (m+1)^s} + \Phi_m^{(s)}(A_{2q}), & s \in (1,2). \end{cases} \tag{32}$$

with

$$\Phi_m^{(s)}(A_{2q}) = \frac{2}{(m+1)^2}\left[\sum_{j=1}^{q-1}\int\limits_{\beta_{j+1}}^{\beta_j}\mu_s(u)\frac{1+|\pi_q(u)|}{(1-|\pi_q(u)|)^2}\,du + \int\limits_{\beta_1}^{1}\mu_s(u)\frac{1+|\pi_q(u)|}{(1-|\pi_q(u)|)^2}\,du\right], \tag{33}$$

where $\Gamma(s)$ is Euler's gamma function and $n = mq$.

Proof. Express (30) as

$$\varepsilon^*_{2n,2q}(A_{2q}) = \frac{2}{(m+1)^2\pi}\sin\frac{\pi s}{2}\big[I_n^{(1)}(A_{2q}) + I_n^{(2)}(A_{2q}) + I_n^{(3)}(A_{2q})\big], \quad m \in \mathbb{N}, \tag{34}$$

where

$$I_n^{(1)}(A_{2q}) = \int\limits_0^{\beta_q}\mu_s(u)G_n(\pi_q(u))\,du, \quad I_n^{(2)}(A_{2q}) = \sum_{j=1}^{q-1}\int\limits_{\beta_{j+1}}^{\beta_j}\mu_s(u)G_n(|\pi_q(u)|)\,du,$$

$$I_n^{(3)}(A_{2q}) = \int\limits_{\beta_1}^{1}\mu_s(u)G_n(|\pi_q(u)|)\,du.$$

Let us separately study the asymptotic behavior as $m \to \infty$ of each of the three quantities. For $I_n^{(1)}(A_{2q})$ apply the method for studying the asymptotic behavior of integrals which was proposed in [39, p. 375]. Differentiate the integral on the right-hand side with respect to m twice. Then

$$\begin{aligned}\frac{\partial I_n^{(1)}(A_{2q})}{\partial m} = \int\limits_0^{\beta_q}\frac{\mu_s(u)}{(1-\pi_q(u))^2}\big(&-2\pi_q^{m+1}(u) - (2m+3)\pi_q^{m+1}(u)\ln\pi_q(u) \\ &+2\pi_q^{m+2}(u) + (2m+1)\pi_q^{m+2}(u)\ln\pi_q(u)\big)\,du,\end{aligned} \tag{35}$$

$$\frac{\partial^2 I_n^{(1)}(A_{2q})}{\partial m^2} = -4\int\limits_0^{\beta_q}\mu_s(u)\frac{\ln\pi_q(u)}{1-\pi_q(u)}e^{(m+1)S(u)}\,du$$

$$-(2m+3)\int\limits_0^{\beta_q} \mu_s(u)\left(\frac{\ln \pi_q(u)}{1-\pi_q(u)}\right)^2 e^{(m+1)S(u)}\,du$$

$$+(2m+1)\int\limits_0^{\beta_q} \mu_s(u)\left(\frac{\ln \pi_q(u)}{1-\pi_q(u)}\right)^2 e^{(m+2)S(u)}\,du, \quad S(u)=\sum_{k=1}^{q}\ln\frac{\beta_k-u}{\beta_k+u}. \tag{36}$$

Assume that $s \in (0,1]$. Inspect the asymptotic behavior of the integrals on the right-hand side as $m \to \infty$ using Laplace's method [40, 41]. The function $S(u)$ decreases on $[0,\beta_q]$, and so reaches its maximum at $u=0$. Expanding $S(u)$ into its Taylor series in a neighborhood of $u=0$ and appreciating that

$$\mu_s(u) \sim u^{s-1}, \qquad \frac{\ln \pi_q(u)}{1-\pi_q(u)} \sim -1 \quad \text{as } u \to 0,$$

for some small $\varepsilon > 0$ as $m \to \infty$ in (48) we see that

$$\frac{\partial^2 I_n^{(1)}(A_{2q})}{\partial m^2} \sim (2m+1)\int\limits_0^{\varepsilon} u^{s-1}\exp\left[-2(m+2)u\sum_{k=1}^{q}\frac{1}{\beta_k}\right]du$$

$$-(2m-1)\int\limits_0^{\varepsilon} u^{s-1}\exp\left[-2(m+1)u\sum_{k=1}^{q}\frac{1}{\beta_k}\right]du.$$

Changing the variables in the integrals on the right-hand side respectively as $2(m+2)\big(\sum_{k=1}^{q} 1/\beta_k\big)u \mapsto t$ and $2(m+1)\big(\sum_{k=1}^{q} 1/\beta_k\big)u \mapsto t$ yields

$$\frac{\partial^2 I_n^{(1)}(A_{2q})}{\partial m^2} \sim \frac{2\Gamma(s)}{\left(2(m+1)\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\right)^s} \quad \text{as } m \to \infty.$$

In order to arrive at asymptotics for $I_n^{(1)}(A_{2q})$, integrate the right- and left-hand sides of the last asymptotic equality with respect to m twice. We have

$$I_n^{(1)}(A_{2q}) \sim \begin{cases} \dfrac{2\Gamma(s)(m+1)^{2-s}}{(1-s)(2-s)\left(2\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\right)^s}, & s \in (0,1), \\ \dfrac{(m+1)\ln(m+1)}{\sum\limits_{j=1}^{q}\frac{1}{\beta_j}}, & s = 1. \end{cases} \tag{37}$$

Assume that $s \in (1,2)$. Then it suffices to differentiate the integral in question with respect to m once. Furthermore, (35) yield

$$\frac{\partial I_n^{(1)}(A_{2q})}{\partial m} = -2\int\limits_0^{\beta_q} \mu_s(u)\frac{\pi_q^{m+1}(u)\,du}{1-\pi_q(u)} - (2m+3)\int\limits_0^{\beta_q} \mu_s(u)\frac{\ln \pi_q(u)}{1-\pi_q(u)}\,\frac{\pi_q^{m+1}(u)\,du}{1-\pi_q(u)}$$

$$+(2m+1)\int\limits_0^{\beta_q} \mu_s(u)\frac{\ln \pi_q(u)}{1-\pi_q(u)}\,\frac{\pi_q^{m+2}(u)\,du}{1-\pi_q(u)}.$$

Using Laplace's method again to study the asymptotic behavior of integrals on the right-hand side, we obtain

$$\frac{\partial I_n^{(1)}(A_{2q})}{\partial m} \sim \frac{(2m+1)\Gamma(s)}{\left(2(m+1)\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\right)^s}, \quad s \in (1,2), \quad \text{as } m \to \infty.$$

Integrating the last asymptotic equality with respect to m, we arrive at

$$I_n^{(1)}(A_{2q}) \sim \frac{2\Gamma(s)(m+1)^{2-s}}{(2-s)\left(2\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\right)^s}, \quad s \in (1,2), \quad \text{as } m \to \infty. \tag{38}$$

Equalities (37) and (38) show that, as $m \to \infty$,

$$I_n^{(1)}(A) \sim \begin{cases} \frac{2\Gamma(s)(m+1)^{2-s}}{(1-s)(2-s)\left(2\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\right)^s}, & s \in (0,1), \\ \frac{(m+1)\ln(m+1)}{\sum\limits_{j=1}^{q}\frac{1}{\beta_j}}, & s = 1, \\ \frac{2\Gamma(s)(m+1)^{2-s}}{(2-s)\left(2\sum\limits_{k=1}^{q}\frac{1}{\beta_k}\right)^s}, & s \in (1,2). \end{cases} \tag{39}$$

Consider $I_n^{(2)}(A)$. Split each of the $q-1$ integrals appearing in its definition into the three integrals as

$$I_n^{(2)}(A_{2q}) = \sum_{j=1}^{q-1}\int_{\beta_{j+1}}^{\beta_j} \mu_s(u)\frac{1+|\pi_q(u)|}{(1-|\pi_q(u)|)^2}\,du + \delta_m(A_{2q}),$$

where

$$\delta_m(A_{2q}) = -2(m+1)\sum_{j=1}^{q-1}\int_{\beta_{j+1}}^{\beta_j} \mu_s(u)\frac{|\pi_q(u)|^{m+1}}{1-|\pi_q(u)|}\,du$$

$$-\sum_{j=1}^{q-1}\int_{\beta_{j+1}}^{\beta_j} \mu_s(u)\frac{(1+|\pi_q(u)|)|\pi_q(u)|^{m+1}}{(1-|\pi_q(u)|)^2}\,du.$$

The first term in $I_n^{(2)}(A_{2q})$ is independent of m. Since

$$|\pi_q(u)| = \prod_{k=1}^{j}\frac{\beta_k-u}{\beta_k+u}\prod_{k=j+1}^{q}\frac{u-\beta_k}{\beta_k+u} \le \prod_{k=1}^{j}\frac{1-\beta_{j+1}/\beta_k}{1+\beta_{j+1}/\beta_k}\prod_{k=j+1}^{q}\frac{1-\beta_k/\beta_j}{1+\beta_k/\beta_j}, \quad u \in [\beta_{j+1},\beta_j],$$

we conclude that for constant β_k, where $k=1,2,\ldots,q$, we have $|\pi_q(u)| \le d < 1$ for $u \in [\beta_{j+1},\beta_j]$, where $j=1,2,\ldots,q-1$. Consequently, $\delta_m(A_{2q})$ decreases as a geometric progression as $m \to \infty$. However, if $\beta_k = \beta_k(m)$ for $k=1,2,\ldots,q$ then it is not difficult to establish that $|\pi_q(u)| \le e^{-4\frac{\beta_{j+1}}{\beta_j}}$. If (31) is fulfilled then obviously $|\pi_q(u)|^{m+1} \to 0$ as $m \to \infty$. Hence, we conclude that

$$I_n^{(2)}(A_{2q}) \sim \sum_{j=1}^{q-1}\int_{\beta_{j+1}}^{\beta_j} \mu_s(u)\frac{1+|\pi_q(u)|}{(1-|\pi_q(u)|)^2}\,du \quad \text{as } m \to \infty. \tag{40}$$

Arguing similarly for $I_n^{(3)}(A)$, we find that

$$I_n^{(3)}(A_{2q}) \sim \int\limits_{\beta_1}^{1} \mu_s(u) \frac{1+|\pi_q(u)|}{(1-|\pi_q(u)|)^2}\, du \quad \text{as } m \to \infty. \tag{41}$$

Basing on the asymptotic relations (39)–(41), from (34) we obtain the asymptotic equality (32), which completes the proof of Theorem 5. $\square$

In Theorem 5 put $\beta_j = 1$ for $j = 1, 2, \dots, q$. The quantity $\varepsilon^*_{2n,2}(O) = \varepsilon^{(0)}_{2n}$ amounts to a uniform approximation to $|x|^s$ with $s \in (0, 2)$ by the Riesz–Zygmund sums for Fourier–Chebyshev polynomial series. Furthermore, Theorem 5 implies the following:

Corollary 3. *The uniform approximations to $|x|^s$ with $s \in (0, 2)$ by series (28) as $n \to \infty$ satisfy*

$$\varepsilon^{(0)}_{2n} \sim \frac{1}{\pi} \sin \frac{\pi s}{2} \begin{cases} \frac{2^{2-s}\Gamma(s)}{(1-s)(2-s)(n+1)^s}, & s \in (0,1), \\ \frac{2\ln(n+1)}{(n+1)}, & s = 1, \\ \frac{2^{2-s}\Gamma(s)}{(2-s)(n+1)^s}, & s \in (1,2), \end{cases}$$

where $\Gamma(s)$ is Euler's gamma function.

4. The Best Estimate for Approximations by Rational Riesz–Zygmund Sums

It is of interest to minimize the right-hand side of (32) by choosing the tuple $\{\beta_1^*, \beta_2^*, \dots, \beta_q^*\}$ optimal for this problem, which means seeking the best estimate for the uniform approximation to $|x|^s$ with $s \in (0, 2)$ by the rational Riesz–Zygmund sums. Put

$$\varepsilon_{2n,2q} = \inf_{A_{2q}} \varepsilon_{2n,2q}(A_{2q}), \quad \varepsilon^*_{2n,2q} = \inf_{A_{2q}} \varepsilon^*_{2n,2q}(A_{2q}).$$

Note the obvious inequality resulting from (23): $\varepsilon_{2n,2q} \le \varepsilon^*_{2n,2q}$, $n \in \mathbb{N}$. In view of the last estimate, below we discuss asymptotics for the majorant of uniform approximations.

Theorem 6. *As $n \to \infty$ we have the asymptotics*

$$\varepsilon^*_{2n,2q} \sim \frac{1}{\pi} \sin \frac{\pi s}{2} q^{2(1-\frac{2-s}{2+s}(\frac{2-s}{2})^{q-1})} \begin{cases} \frac{\mu(s,q)}{(n+1)^{2(1-\frac{2-s}{2+s}(\frac{2-s}{2})^{q-1})}}, & s \in (0,1), \\ \mu(1,q)\frac{[\ln(n+1)]^{\frac{2}{3}(\frac{1}{2})^{q-1}}}{(n+1)^{2-\frac{2}{3}(\frac{1}{2})^{q-1}}}, & s = 1, \\ \frac{\mu(s,q)}{(n+1)^{2(1-\frac{2-s}{2+s}(\frac{2-s}{2})^{q-1})}}, & s \in (1,2), \end{cases} \tag{42}$$

where

$$\mu(s,q) = \begin{cases} \frac{(2+s)[c_1(s)]^{\frac{s}{2+s}} 2^{\frac{2}{s}\frac{2-s}{2+s}(1-\frac{s^2-2s+2}{2}(\frac{2-s}{2})^{q-2})}[\Gamma(s)]^{\frac{2}{2+s}(\frac{2-s}{2})^{q-1}}}{s^{1-\frac{2}{2+s}(\frac{2-s}{2})^{q-1}}(2-s)^{\frac{2}{2+s}(\frac{2}{s}-(\frac{2-s}{2})^{q-1})}(1-s)^{\frac{2}{2+s}(\frac{2-s}{2})^{q-1}}}, & s \in (0,1), \\ 3(4-\pi)^{\frac{1}{3}} 2^{\frac{1}{3}(1-(\frac{1}{2})^{q-2})}, & s = 1, \\ \frac{(2+s)[c_1(s)]^{\frac{s}{2+s}} 2^{\frac{2}{s}\frac{2-s}{2+s}(1-\frac{s^2-2s+2}{2}(\frac{2-s}{2})^{q-2})}[\Gamma(s)]^{\frac{2}{2+s}(\frac{2-s}{2})^{q-1}}}{s^{1-\frac{2}{2+s}(\frac{2-s}{2})^{q-1}}(2-s)^{\frac{2}{2+s}(\frac{2}{s}-(\frac{2-s}{2})^{q-1})}}, & s \in (1,2), \end{cases}$$

$$c_1(s) = \int\limits_0^1 \frac{u^{s+1}\, du}{(1+u)(1-u^2)^{\frac{s}{2}}} = \frac{1}{s}\Gamma\left(1-\frac{s}{2}\right)\left[\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}+\frac{s}{2}\right) - \Gamma\left(1+\frac{s}{2}\right)\right], \quad s \in (0,2).$$

PROOF. Let us cheek (32). It is obvious that for constant β_k, where $k = 1, 2, \ldots, q$, the order in these relations does not differ from the polynomial one. Suppose that $\beta_k = \beta_k(m) \to 0$ and $\beta_{k+1} = o(\beta_k)$ as $m \to \infty$, with the fulfillment of (31). In this case it is not difficult to see that as $m \to \infty$ we have the asymptotics

$$\sum_{k=1}^{q} \frac{1}{\beta_k} \sim \frac{1}{\beta_q},$$

$$1 - \prod_{k=1}^{j} \frac{\beta_k - u}{\beta_k + u} \prod_{k=j+1}^{q} \frac{u - \beta_k}{u + \beta_k} \sim \frac{2u}{\beta_j}, \quad j = 1, 2, \ldots, q-1, \ u \in [\beta_{j+1}, \beta_j],$$

$$1 - \prod_{k=1}^{q} \frac{u - \beta_k}{u + \beta_k} \sim \frac{2\beta_1}{u}, \quad u \in [\beta_1, 1].$$

Furthermore, from (33) we find that

$$\Phi_m^{(s)}(A_{2q}) \sim \frac{1}{(m+1)^2} \left[\frac{1}{2-s} \sum_{j=1}^{q-1} \frac{\beta_j^2}{\beta_{j+1}^{2-s}} + \frac{c_1(s)}{\beta_1^2} \right], \quad s \in (0, 2),$$

where $c_1(s)$ is defined in the statement of this theorem.

The asymptotic equalities in (32) as $m \to \infty$ become

$$\varepsilon_{2n,2q}^{*}(A_{2q}) \sim \frac{1}{\pi(m+1)^2} \sin\frac{\pi s}{2} \Psi^{(s)}(A_{2q}), \tag{43}$$

where

$$\Psi^{(s)}(A_{2q}) = c_q(s, m)\beta_q^s + \frac{1}{2-s} \sum_{j=1}^{q-1} \frac{\beta_j^2}{\beta_{j+1}^{2-s}} + \frac{c_1(s)}{\beta_1^2},$$

$$c_q(s, m) = \begin{cases} \frac{2^{2-s}\Gamma(s)(m+1)^{2-s}}{(1-s)(2-s)}, & s \in (0, 1), \\ 2(m+1)\ln(m+1), & s = 1, \\ \frac{2^{2-s}\Gamma(s)(m+1)^{2-s}}{2-s}, & s \in (1, 2). \end{cases}$$

For each $s \in (0, 2)$ the right-hand side of (43) is a function of $(\beta_1, \beta_2, \ldots, \beta_q)$ continuous at each point of the q-dimensional cube $[\delta, 1]^q$, where $\delta = \delta(n) > 0$ is some quantity depending on n and bounding the parameter set $(\beta_1, \beta_2, \ldots, \beta_q)$ on the left for every n. According to Weierstrass's Theorem the right-hand side of (43) has a strict minimum at some $\beta^* = (\beta_1^*, \beta_2^*, \ldots, \beta_q^*)$. Furthermore, since $\beta_k = 1$ for $k = 1, \ldots, q$ correspond to the polynomial case, while when $\beta_k(n) \to 0$ as $n \to \infty$ the right-hand side of (43) grows boundedly and rather fast, we may assume that β^* is an interior point of the cube $[\delta, 1]^q$. In order to find the optimal tuple β^*, solve the extremal problem

$$\Psi^{(s)}(A_{2q}) = c_q(s, m)\beta_q^s + \frac{\beta_{q-1}^2}{(2-s)\beta_q^{2-s}} + \cdots + \frac{\beta_2^2}{(2-s)\beta_3^{1-s}} + \frac{\beta_1^2}{(2-s)\beta_2^{1-s}} + \frac{c_1(s)}{\beta_1^2} \underset{A_{2q}}{\longrightarrow} \inf. \tag{44}$$

The function $\Psi^{(s)}(A_{2q})$ of $(\beta_1, \beta_2, \ldots, \beta_q)$ is continuously differentiable on $(0, 1)^q$. It is natural to seek the minimum point of the function where the necessary condition for extremum holds: $\partial\Psi^{(s)}(A_{2q})/\partial\beta_k = 0$ for $k = 1, 2, \ldots, q$. Straightforward calculations lead to the system of equations

$$\begin{cases} s c_q(s, m)\beta_q^{s-1} - \frac{\beta_{q-1}^2}{\beta_q^{3-s}} = 0, \\ \frac{2}{2-s} \frac{\beta_{q-1}}{\beta_q^{2-s}} - \frac{\beta_{q-2}^2}{\beta_{q-1}^{3-s}} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{2}{2-s} \frac{\beta_2}{\beta_3^{2-s}} - \frac{\beta_1^2}{\beta_2^{3-s}} = 0, \\ \frac{2}{2-s} \frac{\beta_1}{\beta_2^{2-s}} - \frac{2c_1(s)}{\beta_1^3} = 0, \end{cases} \tag{45}$$

from which we successively determine

$$\frac{\beta_{q-1}^2}{\beta_q^{2-s}} = sc_q(s,m)\beta_q^s,\ \frac{\beta_{q-2}^2}{\beta_{q-1}^{2-s}} = \frac{2}{2-s}sc_q(s,m)\beta_q^s,\ \frac{\beta_{q-3}^2}{\beta_{q-2}^{2-s}} = \left(\frac{2}{2-s}\right)^2 sc_q(s,m)\beta_q^s,$$
$$\dots,\frac{\beta_1^2}{\beta_2^{2-s}} = \left(\frac{2}{2-s}\right)^{q-2} sc_q(s,m)\beta_q^s,\ \frac{2c_1(s)}{\beta_1^2} = \left(\frac{2}{2-s}\right)^{q-1} sc_q(s,m)\beta_q^s.$$

Inserting the last relations into (44), we find that for the optimal tuple of parameters of $\Psi^{(s)}(A_{2q})$ becomes

$$\begin{aligned}\Psi^{(s)}(A_{2q}^*) &= c_q(s,m)\beta_q^s + \frac{1}{2-s}sc_q(s,m)\beta_q^s + \frac{1}{2-s}\frac{2}{2-s}sc_q(s,m)\beta_q^s + \\ &\cdots + \frac{1}{2-s}\left(\frac{2}{2-s}\right)^{q-2} sc_q(s,m)\beta_q^s + \frac{1}{2}\left(\frac{2}{2-s}\right)^{q-1} sc_q(s,m)\beta_q^s \\ &= \frac{2+s}{2}\left(\frac{2}{2-s}\right)^{q-1} c_q(s,m)\beta_q^s = \frac{(2+s)c_1(s)}{s\beta_1^{*2}}. \end{aligned} \tag{46}$$

It remains to find β_1^*. To this end, inspect (45) once again. We successively determine that

$$\begin{aligned}\left(\frac{\beta_{q-1}}{\beta_q}\right)^2 &= sc_q(s,m), \\ \left(\frac{\beta_{q-2}}{\beta_{q-1}}\right)^2 &= \frac{2}{2-s}(sc_q(s,m))^{\frac{2-s}{2}}, \\ \left(\frac{\beta_{q-3}}{\beta_{q-2}}\right)^2 &= \frac{2}{2-s}\left(\frac{2}{2-s}(sc_q(s,m))^{\frac{2-s}{2}}\right)^{\frac{2-s}{2}}, \\ &\cdots\cdots\cdots \\ \left(\frac{\beta_1}{\beta_2}\right)^2 &= \left(\frac{2}{2-s}\right)^{\frac{2}{s}(1-(\frac{2-s}{2})^{q-2})} (sc_q(s,m))^{(\frac{2-s}{2})^{q-2}}, \\ \beta_2 &= \left(\frac{\beta_1^4}{(2-s)c_1(s)}\right)^{\frac{1}{2-s}}. \end{aligned}$$

The last two equations here yield

$$\frac{1}{\beta_1^{*2}} = \left(\frac{(sc_q(s,m))^{(\frac{2-s}{2})^{q-2}}}{((2-s)c_1(s))^{\frac{2}{2-s}}}\left(\frac{2}{2-s}\right)^{\frac{2}{s}(1-(\frac{2-s}{2})^{q-2})}\right)^{\frac{2-s}{2+s}}.$$

For the found β_1^* in (46) we have

$$\Psi^{(s)}(A_{2q}^*) = \frac{(2+s)[c_1(s)]^{\frac{s}{2+s}}[c_q(s,m)]^{\frac{2}{2+s}(\frac{2-s}{2})^{q-1}} 2^{\frac{2}{s}\frac{2-s}{2+s}(1-(\frac{2-s}{2})^{q-2})}}{s^{1-\frac{2}{2+s}(\frac{2-s}{2})^{q-1}}(2-s)^{\frac{4}{s(2+s)}(1-(\frac{2-s}{2})^{q-1})}}, \quad s \in (0,2).$$

Returning to the original values of $c_1(s)$ and $c_q(s,m)$ and recalling that $n = mq$, from the last equality and (43) we obtain (42). The proof of Theorem 6 is complete. $\square$

In Theorem 6 put $q = 1$, so that the approximant has two geometrically distinct poles in the extended complex plane. In this case we obtain the following

Corollary 4. *As $n \to \infty$ we have the asymptotics*

$$\varepsilon^*_{2n,2} \sim \frac{1}{\pi} \sin \frac{\pi s}{2} \begin{cases} (2+s)\Big(\frac{2^{1-s}\Gamma(s)[c_1(s)]^{\frac{s}{2}}}{s^{\frac{s}{2}}(2-s)^{\frac{s}{2}}(1-s)}\Big)^{\frac{2}{2+s}} \frac{1}{(n+1)^{\frac{4s}{2+s}}}, & s \in (0,1), \\ 3\sqrt[3]{\frac{4-\pi}{2}}\frac{\ln^{\frac{2}{3}}(n+1)}{(n+1)^{\frac{4}{3}}}, & s = 1, \\ (2+s)\Big(\frac{2^{1-s}\Gamma(s)[c_1(s)]^{\frac{s}{2}}}{s^{\frac{s}{2}}(2-s)^{\frac{s}{2}}}\Big)^{\frac{2}{2+s}} \frac{1}{(n+1)^{\frac{4s}{2+s}}}, & s \in (1,2). \end{cases}$$

Some results with a similar order were obtained in [15] by studying approximations to $|x|^s$ with $s \in (0,2)$ by the Riesz–Zygmund sums for Fourier series with respect to the orthogonal system of Chebyshev–Markov rational fractions with two geometrically distinct poles.

Remark 3. Comparing the results of Theorem 6 and Corollary 3, we conclude that for an arbitrary value of $s \in (0,2)$ by making a special choice of parameters of the approximant we can attain the rate of uniform rational approximation by Riesz–Zygmund sums greater than for the corresponding polynomial analogs. This result is valid, in particular, in the case of two geometrically distinct poles of the rational approximant.

Put

$$\varepsilon_{n,q}(x^\gamma) = \inf_{A_q} \|x^\gamma - R_{n,q}(x^\gamma, x)\|_{C[0,1]}, \quad n \in \mathbb{N},$$

where $R_{n,q}(x^\gamma, x)$ are the rational Riesz–Zygmund sums of order n with q geometrically distinct poles constructed for the function x^γ with $\gamma \in (0,1)$ on the segment $[0,1]$.

It is known [31] that the best uniform rational approximation satisfies

$$\mathfrak{R}_{2n}(|x|^{2\alpha}, [-1,1]) = \mathfrak{R}_n(x^\alpha, [0,1]), \quad n \in \mathbb{N}.$$

Arguing appropriately in our case, we obtain the following

Corollary 5. *As $n \to \infty$ we have $\varepsilon_{n,q}(x^\gamma) \le \varepsilon^*_{n,q}(x^\gamma)$, where*

$$\varepsilon^*_{n,q}(x^\gamma) \sim \frac{\sin \pi\gamma}{\pi} q^{2(1-\frac{(1-\gamma)^q}{1+\gamma})} \begin{cases} \frac{\mu(\gamma,q)}{(n+1)^{2(1-\frac{(1-\gamma)^q}{1+\gamma})}}, & \gamma \in (0,\frac{1}{2}), \\ \mu(1/2,q)\frac{[\ln(n+1)]^{\frac{2}{3}(\frac{1}{2})^{q-1}}}{(n+1)^{2-\frac{2}{3}(\frac{1}{2})^{q-1}}}, & \gamma = \frac{1}{2}, \\ \frac{\mu(\gamma,q)}{(n+1)^{2(1-\frac{(1-\gamma)^q}{1+\gamma})}}, & \gamma \in (\frac{1}{2},1), \end{cases}$$

$$\mu(\gamma,q) = \begin{cases} \frac{(1+\gamma)[c_2(\gamma)]^{\frac{\gamma}{1+\gamma}}[\Gamma(2\gamma)]^{\frac{1}{1+\gamma}(1-\gamma)^{q-1}}}{2^{\frac{\gamma+(2\gamma^2-4\gamma+1)(1-\gamma)^{q-1}}{\gamma(1+\gamma)}}\gamma^{1-\frac{(1-\gamma)^{q-1}}{1+\gamma}}(1-\gamma)^{\frac{1}{1+\gamma}(\frac{1}{\gamma}-(1-\gamma)^{q-1})}(1-2\gamma)^{\frac{(1-\gamma)^{q-1}}{1+\gamma}}}, & \gamma \in (0,\frac{1}{2}), \\ 3(4-\pi)^{\frac{1}{3}}2^{\frac{1}{3}(1-(\frac{1}{2})^{q-2})}, & \gamma = 1, \\ \frac{(1+\gamma)[c_2(\gamma)]^{\frac{\gamma}{1+\gamma}}[\Gamma(2\gamma)]^{\frac{1}{1+\gamma}(1-\gamma)^{q-1}}}{2^{\frac{\gamma+(2\gamma^2-4\gamma+1)(1-\gamma)^{q-1}}{\gamma(1+\gamma)}}\gamma^{1-\frac{(1-\gamma)^{q-1}}{1+\gamma}}(1-\gamma)^{\frac{1}{1+\gamma}(\frac{1}{\gamma}-(1-\gamma)^{q-1})}}, & \gamma \in (\frac{1}{2},1), \end{cases}$$

$$c_2(\gamma) = \int\limits_0^1 \frac{u^{2\gamma+1}\,du}{(1+u)(1-u^2)^\gamma} = \frac{\Gamma(1-\gamma)}{s}\left[\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}+\gamma\right) - \Gamma(1+\gamma)\right], \quad \gamma \in (0,1).$$

Conclusion

This article studies the approximation properties of the Riesz–Zygmund sums for Fourier–Chebyshev rational integral operators with a fixed number of geometrically distinct poles. We have found an integral expression for the rational approximation method in question.

Studying approximations to $|x|^s$ with $s \in (0, 2)$ on $[-1, 1]$ by Riesz–Zygmund sums, we established estimates for pointwise and uniform rational approximations and found an asymptotic estimate for the majorant of uniform approximations depending on the parameter of the approximant. For each $s \in (0, 2)$ we found the optimal values of parameters which ensure the smallest majorant of uniform approximations.

Also studying the approximation properties of the Riesz–Zygmund sums for Fourier–Chebyshev polynomial series, we found asymptotics for the Lebesgue constant and established estimates for approximations to $f \in H^{(\gamma)}[-1, 1]$ with $\gamma \in (0, 1]$. We obtained sharp asymptotics for uniform approximation to $|x|^s$ with $s \in (0, 2)$.

The results of this article imply that for a special choice of parameters of the approximant the decrease rate of uniform rational approximations turns out significantly above its polynomial analogs.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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