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STRUCTURE OF THE VARIETY OF ALTERNATIVE ALGEBRAS WITH THE LIE-NILPOTENCY IDENTITY OF DEGREE 5

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Abstract—We construct an additive basis for a relatively free alternative algebra of Lie-nilpotent degree 5, describe the associative center and core of this algebra, and find the T-generators of the full center. Also, we give some asymptotic estimate for the codimension of the T-ideal generated by a commutator of degree 5 in a free alternative algebra, and find a finite-dimensional superalgebra that generates the variety of alternative algebras with the Lie-nilpotency of the selfadjoint operator of degree 5.

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Introduction

It was in [1, 2] that the study began of the structure of a relatively free associative algebra with the Lie-nilpotency identity of degree 5 over a field of characteristic zero. In particular, there was described the core of the algebra (the greatest ideal included in the center), and classified the proper central polynomials of the algebra.

More detailed results were obtained in [3, 4] on the structure of this algebra over a scalar ring that contains $\frac{1}{6}$. In particular, there was constructed the additive basis of the algebra. Also, the core was described, the center was studied, and an asymptotic was found of the sequence of codimensions of the ideal $T^{(5)}$ of a free associative algebra.

This article extends these results to the variety of the alternative algebras Lie-nilpotent of degree 5. Note that similar results were obtained previously for the metabelian alternative algebras in [5].

The article consists of six sections, and addresses the relatively free alternative algebra $A^{(5)}$ satisfying the Lie-nilpotency identity of degree 5.

Section 1 contains the main notations and identities of alternative algebras.

In Section 2, we prove two lemmas on commutators and the product theorem for $A^{(5)}$. The author proved the product theorem in a general form for the alternative and Jordan algebras in [6, 7].

Section 3 presents the construction of an auxiliary superalgebra $S^{(5)}$ used in disproving a series of identity relations.

Section 4 gives an additive basis for the associator ideal D of $A^{(5)}$, and contains an asymptotic of the multilinear part $P_n \cap D$ of D.

Section 5 presents some decomposition of the variety $\mathrm{Alt}^{(5)}$ of alternative algebras with the Lie-nilpotency identity of degree 5 into the union of three subvarieties, as well as a finite-dimensional superalgebra whose Grassmann envelope generates $\mathrm{Alt}^{(5)}$.

Section 6 deals with the central elements in $A^{(5)}$. In particular, we prove that the core $Z^*(A^{(5)})$ of $A^{(5)}$ is generated by a weak Hall element $[[a, b]^2, b]$, and the core $Z^*(A^{(5)})$ has zero intersection with the associator ideal $D(A^{(5)})$. Furthermore, we point out the generators as a T-subspace of $Z(A^{(5)})$.

1. The Main Notions

In what follows, the term "algebra" stands for a linear algebra, usually unital, over an infinite domain Φ contains $\frac{1}{6}$.

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An algebra is *alternative* provided that it enjoys the identities

$$x^2y = x(xy), \quad xy^2 = (xy)y.$$

By the Artin Theorem, an algebra is alternative if and only if its every 2-generated subalgebra is associative (see [8, 9]).

If a, b, and c are some elements in A, then we put

$[a, b] = ab - ba$ for the commutator of a and b;

$a \circ b = ab + ba$ for the symmetric product of a and b;

$(a, b, c) = (ab)c - a(bc)$ for the associator of a, b, and c.

In the sequel, we use the skew-symmetry in all variables of the associator of an alternative algebra without further explanation.

Recall the following identities that hold in every alternative algebra (see [8, 9]):

$$[xy, z] = x[y, z] + [x, z]y + 3(x, y, z), \tag{1.1}$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 6(x, y, z), \tag{1.2}$$

$$(x^2, y, z) = x \circ (x, y, z) = (x, x \circ y, z), \tag{1.3}$$

$$(x, y, yz) = (x, y, z)y, \quad (xy, y, z) = y(x, y, z). \tag{1.4}$$

Let A be an alternative algebra, and let $A^{(+)}$ and $A^{(-)}$ be the adjoint algebras with respect to the "symmetrized" product $x.y = \frac{1}{2}x \circ y$ and the commutator $[x, y]$. It is known (see [9]) that $A^{(+)}$ is a special Jordan algebra, and $A^{(-)}$ is a Malcev algebra. In particular, $A^{(-)}$ enjoys the *Sagle identity* [10]

$$[x, y, z, t] + [y, z, t, x] + [z, t, x, y] + [t, x, y, z] = [x, z, [y, t]]. \tag{1.5}$$

As usual, if the location of parentheses is not specified then we assume that the latter is right-normed, for example, $[x, y, z, t] = [[[x, y], z], t]$.

Every alternative algebra A enjoys the identities (see [9, 11])

$$4(a, b, c)^{(+)} = 2(b, a, c) + [b, [a, c]], \tag{1.6}$$

$$2[(x, y, z), t] = ([x, y], z, t) + ([y, z], x, t) + ([z, x], y, t), \tag{1.7}$$

$$(xy, z, t) + (x, y, [z, t]) = x(y, z, t) + (x, z, t)y, \tag{1.8}$$

where $(a, b, c)^{(+)}$ is the associator in the Jordan algebra $A^{(+)}$. Note that (1.8) implies that

$$([x, y], z, t) + 2(x, y, [z, t]) = [x, (y, z, t)] + [(x, z, t), y]. \tag{1.9}$$

Following [6], an alternative algebra A is *Lie-nilpotent of degree* n if $A^{(-)}$ is nilpotent of index n; i.e., each multilinear commutator word in $x_1, x_2, \dots, x_n$ is an identity of A, and there is a shorter commutator word nonzero in A.

As usual, denote by $D_a : x \to [x, a]$ the "commutation" of an algebra A defined by an element a. Put $R_{a,b} : x \to (x, a, b)$. By (1.1) and (1.3), D_a and $R_{a,b}$ are derivations of $A^{(+)}$. Using the Leibniz rule for derivations, we get

$$(x \circ y)D_aD_b = x \circ (yD_aD_b) + (xD_aD_b) \circ y + (xD_{\overline{a}}) \circ (yD_{\overline{b}}), \tag{1.10}$$

where the bar over a and b means symmetrization over them.

Unless otherwise stated, we will use the notations

$$U = U(A) = [A, A], \quad A' = \mathrm{idl}_A(U), \quad V = V(A) = (A, A, A), \quad D(A) = \mathrm{idl}_A(V),$$

where $\mathrm{idl}_A(X)$ stands for an ideal of A generated by a set X.

Also, we will use the following notation for the centers of an algebra A:

$K(A) = \{k \in A \mid (\forall a \in A)[k, a] = 0\}$ for the commutative center,

$N_{\mathrm{Ass}}(A) = \{n \in A \mid (\forall a, b \in A)(n, a, b) = 0\}$ for the associative center,

$Z(A) = K(A) \cap N_{\mathrm{Ass}}(A)$ for the (full) center,

$C^*(A)$ is the greatest ideal of A which lies in the center $C(A)$, the ideal $C^*(A)$ is a *C-core*.

Now, $K(A) = Z(A)$ by (1.2), and $Z^*(A) \subseteq \mathrm{Ann}(A')$ by (1.1). Clearly, $A' = UA$.

2. Preliminary Results

2.1. Auxiliary lemmas. Agree on the following notations: $F_{\mathrm{Alt}}[X]$ is the free alternative (unital) algebra over a set $X = \{x_1, x_2, \dots\}$ of free generators; $[x_1, \dots, x_n]$ is a right-normed commutator of degree $n \geq 2$; i.e., $[x_1, x_2] = x_1x_2 - x_2x_1$ and $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$ by induction.

In what follows, we denote by $\mathrm{Alt}^{(5)}$ the variety of alternative algebras with the identity

$$[x_1, x_2, \dots, x_5] = 0 \tag{2.1}$$

which is the *Lie-nilpotency identity of degree* 5.

Denote by $U^{(n)}$ a T-subspace spanned by the right-normed commutator $[x_1, x_2, \dots, x_n]$ with $n \leq 5$.

Lemma 2.1. *We have*

$$[x_1, x_2, x_3, [x_4, x_5]] = 0$$

in $\mathrm{Alt}^{(5)}$. *In other words, each commutator of degree* ≤ 5 *is a linear combination of right-normed commutators.*

Proof. Proceed through the sequence of items:

1^0. By (1.2), $(a, b, c) \in U^{(3)}$.

2^0. By the Sagle identity (1.5), $[x, y, [z, t]]$ is a linear combination of right-normed commutators.

3^0. It follows from 2^0 and (1.2) that $([a, b], c, d), [(a, b, c), d] \in U^{(4)}$.

4^0. Prove that $([a, b], c, [x, y]) \in U^{(5)}$. Note first that

$$([a, b], a, [x, y]) = [(b, a, [x, y]), a] \in [U^{(4)}, a] \subseteq U^{(5)}$$

by (1.4). It means that $([a, b], c, [x, y])$ is skew-symmetric in all variables modulo $U^{(5)}$. Then modulo $U^{(5)}$ we have

$$([a, b], c, [x, y]) \equiv -([x, b], c, [a, y]) \equiv ([x, y], c, [a, b]) = -([a, b], c, [x, y]),$$

whence $2([a, b], c, [x, y]) \in U^{(5)}$, and we get the required claim.

5^0. Let $z = [a, b]$. Then

$$\begin{aligned} 0 \equiv [x, (y, z, t)] + [(x, z, t), y] \quad (\text{by } 3^0) \quad &= ([x, y], z, t) + 2(x, y, [z, t]) \\ (\text{by } (1.9)) \quad &\equiv 2(x, y, [a, b, t]) \quad (\text{by } 4^0). \end{aligned}$$

From here and by (1.2) we have $[U^{(3)}, U^{(2)}] \subseteq U^{(5)}$. The lemma is proved.

Remark. Using the additive basis for the free alternative Grassmann algebra from [12], it is easy to understand that the commutator of degree 6 of the form $[x_1, x_2, x_3, x_4, [x_5, x_6]]$ is not a linear combination of right-normed commutators of degree 6.

Denote by $T^{(n)}(A)$ the T-ideal of A generated by all commutators of degree n. Here and in the sequel, $A^{(n)}$ stands for the quotient-algebra $A/T^{(n)}(A)$.

Given subsets Y and Z of an algebra A, let $Y * Z$ stand for the subspace spanned by the products yz and zy, where $y \in Y$ and $z \in Z$.

The next two lemmas were proved in [6] (see Lemmas 2.1 and 3.2):

Lemma 2.2. *Given* $x, y, z \in A$, *we have*

$$[A, x] * [U^{(3)}, x] + [U^{(2)}, x] * (A, x, y) + [A, x] * (U^{(2)}, x, y) + (A, x, y)(A, x, z) \subseteq T^{(5)}.$$

Lemma 2.3. *Given $x, y, z, t \in A = A^{(5)}$, we have*

$$[x,y]^2 \in N_{\mathrm{Ass}}(A), \quad [[x,y]^2, z] \in Z(A), \quad [[x,y]^2, [z,t]] = 0.$$

2.2. The product theorem for $A^{(5)}$. The product theorem for alternative algebras was proved in [6]. In general speaking, it differs from the product theorem for the associative algebras and has the form $\mathrm{idl}_A((A,A,A)^{(+)}) * T^{(3)}(A) = 0$ for $A = A^{(5)}$. Essentially, the only particular case will be indicated in this section, when the product theorem for associative algebras is carried over verbatim to alternative algebras. More precisely, we have

Lemma 2.4. $(T^{(3)})^2 = 0$ *in* $A = A^{(5)}$.

Proof. Proceed through the sequence of items:

1^0. $(V(A), A, A) = 0 = (A, U, U)$ by (1.2) and Lemma 2.1.

2^0. Prove that $f = 0$, where $f := (a,b,c) \circ (x,y,z)$.

Note first that f is skew-symmetric in all variables. Indeed, by (1.3) and 1^0

$$(a,b,c) \circ (a,y,z) = (a \circ (a,y,z), b, c) - a \circ ((a,y,z), b, c) = ((a^2, y, z), b, c) = 0.$$

Consequentially interchanging a and x, b and y, c and z, we get

$$\begin{aligned} f = (a,b,c) \circ (x,y,z) &= -(x,b,c) \circ (a,y,z) = (x,y,c) \circ (a,b,z) \\ &= -(x,y,z) \circ (a,b,c) = -f, \end{aligned}$$

whence $2f = 0$ and $f = 0$.

3^0. $(a,b,c) \circ [x,y,z] = 0$ by the product theorem for alternative algebras [6], 2^0, and (1.6).

4^0. By analogy with 3^0 we get $[x,y,z] \circ [a,b,c] = 0$. Hence, $U^{(3)}U^{(3)} = 0$. Since $U^{(3)} \subseteq N_{\mathrm{Ass}}(A)$ and $T^{(3)} = U^{(3)}A = AU^{(3)}$; therefore, $(T^{(3)})^2 = 0$. The lemma is proved.

2.3. Some identities in $A^{(5)}$.

Lemma 2.5. *The ideal $T^{(3)} = T^{(3)}(A)$ of $A = A^{(5)}$ possesses the properties*

$$[[T^{(3)}, A], A] = 0 = [T^{(3)}, [A, A]].$$

If $t \in T^{(3)}$ then t and $w := t[y_1, z_1] \dots [y_n, z_n]$ with $n \geq 1$ belong to the associative core of A, and w is skew-symmetric in $y_1, z_1, \dots, y_n, z_n$.

Proof. Take $u \in U^{(3)}$ and $x, a, b \in A$. Then by Lemma 2.2 and (1.10) we consequentially get

$$[u,a] \circ [x,a] = 0, \quad [u, \overline{a}] \circ [x, \overline{b}] = 0,$$

$$(u \circ x) D_a D_b = u \circ (x D_a D_b) + (u D_a D_b) \circ x + (u D_{\overline{a}}) \circ (x D_{\overline{b}}) = 0.$$

Verify that $(T^{(3)}(A), A, A) = 0$. Firstly, we have $(U^{(3)}, A, A) = 0$. Secondly, $(U^{(3)}A, A, A) = 0$ by (1.8) and Lemma 2.4. The assertions on $T^{(3)}$ follow from the above-proven relations.

Moreover, it is clear that the placement of parentheses on w is immaterial. Since $[a,b][a,c] \in T^{(3)}$, the assertion about w holds. The lemma is proved.

During the proof of Lemma 2.5, it was noted that the placement of parentheses on w is immaterial. Throughout the article this remark will be used without further specification.

Lemma 2.6. *We have*

$$[(x,y,z)\cdot[z,t],a]=([x,z],y,z)\cdot[a,t]$$

in $A^{(5)}$.

Proof. Notice firstly that

$$([x,z],x,a)\cdot[z,t]=0. \tag{2.2}$$

Indeed, by Lemma 2.2

$$([x,z],x,a)\cdot[z,t]=-([x,z],x,z)\cdot[a,t]=0.$$

Linearizing (2.2) by x, we obtain

$$\{([x,z],y,a)+([y,z],x,a)\}\cdot[z,t]=0. \tag{2.3}$$

By Lemma 2.4 and (1.1)

$$\begin{gathered}2[(x,y,z)\cdot[z,t],a]=2[(x,y,z),a]\cdot[z,t]\\ =\{([x,y],z,a)+([y,z],x,a)+([z,x],y,a)\}\cdot[z,t]\quad\text{(by (1.7))}\\ =\{([y,z],x,a)+([z,x],y,a)\}\cdot[z,t]\quad\text{(by Lemma 2.2)}\\ =-2([x,z],y,a)\cdot[z,t]\quad\text{(by (2.3))}\quad=2([x,z],y,z)\cdot[a,t]\quad\text{(by Lemma 2.2).}\end{gathered}$$

After reduction by 2 we get the required identity. The lemma is proved.

Lemma 2.7. *We have*

$$[(a,b,x)\cdot[b,y],a]=0$$

in $A^{(5)}$.

Proof. By (1.1) and Lemma 2.4

$$\begin{gathered}[(a,b,x)\cdot[b,y],a]=[(a,b,x),a]\cdot[b,y]=(a,b,[a,x])\cdot[b,y]\quad\text{(by (1.4))}\\ =0\quad\text{(by Lemma 2.2).}\end{gathered}$$

The lemma is proved.

3. The Auxiliary Superalgebra $S = S^{(5)}$

Construct the auxiliary superalgebra $S=S^{(5)}$ which is the extension of the 6-dimensional ideal with zero product and the 2-dimensional superalgebra $\Phi[\sqrt{1}]$.

Consider the superalgebra $S=S_0\oplus S_1$, where the even part S_0 has a basis $1,a,b,c$, and the odd part S_1 possesses a basis x,a',b',c'. Denote the subspace spanned by a, b, c, a', b', and c' by R. Define the product on S by the following multiplication table on assuming that 1 is the unity of S, $x^2=1$, and $R^2=0$:

$$a\cdot x=a',\quad b\cdot x=b',\quad c\cdot x=c',\quad a'\cdot x=a+b,\quad b'\cdot x=b,\quad c'\cdot x=c,$$

$$x\cdot a=a'-c',\quad x\cdot a'=a+2b-c,\quad x\cdot b=b',\quad x\cdot b'=b,\quad x\cdot c=3b'-c',\quad x\cdot c'=3b-c.$$

Lemma 3.1. *The superalgebra S is alternative.*

Proof. To verify the alternativity it suffices to compute nonzero associators in basis elements. Each of such associators contains two elements x and $r\in R$. If r coincides with one of the elements b, b', c, and c'; then

$$(x,x,r)=(x,r,x)=(r,x,x)=0.$$

Consider the remaining possibilities $r=a$ and $r=a'$:

1) $(x,a,x)=(xa)x-x(ax)=(a'-c')x-xa'=a'x-c'x-xa'=(a+b)-c-(a+2b-c)=-b$,

$$(a,x,x)=(ax)x-a=a'x-a=(a+b)-a=b,$$

$$(x,x,a)=a-x(xa)=a-x(a'-c')=a-xa'+xc'=a-(a+2b-c)+(3b-c)=b;$$

2) $(x,a',x)=(xa')x-x(a'x)=(a+2b-c)x-x(a+b)=ax+2bx-cx-xa-xb=a'+2b'-c'-(a'-c')-b'=b'$,

$$(a',x,x)=(a'x)x-a'=(a+b)x-a'=bx=b',$$

$$\begin{aligned}(x,x,a')&=a'-x(xa')=a'-x(a+2b-c)=a'-xa-2xb+xc\\&=a'-(a'-c')-2b'+(3b'-c')=b'.\end{aligned}$$

We see that for each of the cases 1 and 2 the superalternativity identities hold

$$(p,q,r)+(-1)^{|p||q|}(q,p,r)=0,\quad (p,q,r)+(-1)^{|q||r|}(p,r,q)=0,$$

where $|p|$ stands for the parity of p. The lemma is proved.

Lemma 3.2. *The superalgebra S is Lie-nilpotent of degree* 5.

Proof. Define the supercommutator $[a,b]_s=ab-(-1)^{|a||b|}ba$ of homogeneous elements a and b. By induction, we define the a right-normed commutator of an arbitrary degree. Prove that S satisfies the identity

$$[x_1,x_2,\dots,x_5]_s=0, \tag{3.1}$$

where the right-normed supercommutator stands on the left-hand side.

Let $D(S)$ be the associator ideal of S, and let P be a vector space spanned by b and b'. Note that $[[S,S]_s,S]_s\subseteq D(S)$. Using the computations from Lemma 3.1 and the multiplication table of S, we see that $D(S)=P$. Now, $[b',x]_s=2b$ and $[b,x]_s=0$, whence (3.1) holds.

It is easy that $[[[a',x]_s,x]_s,x]_s=6b\neq 0$. The lemma is proved.

Let G be the associative Grassmann algebra with 1 and the set of standard generators $\xi_1,\xi_2,\dots$; let $G=G_0\oplus G_1$ be the standard grading of G. Denote the Grassmann envelope of the auxiliary superalgebra S by $G(S)=(G_0\otimes S_0)\oplus(G_1\otimes S_1)$.

Lemma 3.3. $([a',x]_s,x,x)=2b$ *in S. In particular,*

$$h_1:=([x_1,x_2],x_3,x_4)[x_5,x_6]\dots[x_{2k-1},x_{2k}]\neq 0$$

in $G(S)$ for every $k\geq 3$. Furthermore, the following two elements in $G(S)$ which are of degree 2 in x_1 and multilinear in the remaining variables are nonzero:

$$h_2:=(x_1,x_2,x_3)[x_1,x_4][x_5,x_6]\dots[x_{2k-1},x_{2k}]\neq 0,$$

$$h_3:=([x_1,x_2],x_1,x_3)[x_4,x_5]\dots[x_{2k},x_{2k+1}]\neq 0.$$

Proof. Indeed, from the multiplication table of S and the computations of Lemma 3.1, we have

$$[a',x]_s=a'x+xa'=2a+3b-c,$$

$$([a',x]_s,x,x)=2(a,x,x)+3(b,x,x)-(c,x,x)=2b.$$

Hence, $([a',x]_s,x,x)\neq 0$. Whence it is immediate that $h_1\neq 0$. If $h_2=0$; then, linearizing by the substitution $x_1\to[y,z]$, we get a nonzero element of the form h_1.

If $h_3=0$; then, linearizing this equality, we obtain

$$\{([y,x_2],z,x_3)+([z,x_2],y,x_3)\}[x_4,x_5]\dots[x_{2k},x_{2k+1}]=0.$$

Putting $y=\xi_1\otimes a'$, $z=\xi_2\otimes x$, and $x_i=\xi_{i+1}\otimes x$, with $i=\overline{2,2k+1}$, we get $([a',x]_s,x,x)=0$, which is impossible. The lemma is proved.

4. An Additive Basis for the Ideal $D(A^{(5)})$

In [13], an additive basis was constructed for the associator ideal $D(A^{(4)})$ of the relatively free alternative algebra $A^{(4)}$ satisfying the Lie-nilpotency identity of degree 4. The aim of Section 4 is to construct an additive basis for $D(A^{(5)})$. To this end, it suffices to construct an additive basis for the vector space $P_n(A^{(5)}) \cap D(A^{(5)})$, where $P_n(A^{(5)})$ is a subspace of $A^{(5)}$ consisting of the multilinear polynomials that depend on the variables in $X_n = \{x_1, \dots, x_n\}$.

4.1. Proper polynomials of odd degree. A polynomial $f \in P_n(A^{(5)})$ is *proper* provided that f vanishes under the substitution $x_i = 1$ for all $i = \overline{1, n}$.

Let $n = 3 + 2k$, $k \geq 1$, and $X_n = \{p, q, r, y_i, z_i, \; i = \overline{1, k}\}$. Put

$$f_n(p, q, r) = (p, q, r)w, \quad F_n = \operatorname{span}\langle f_n(p, q, r)\rangle, \tag{4.1}$$

where $w = [y_1, z_1][y_2, z_2] \dots [y_k, z_k]$ and $\operatorname{span}\langle Y \rangle$ stands for the subspace spanned by Y.

It is easy to understand that every proper polynomial of odd degree $n \geq 5$ in $D(A^{(5)})$ belongs to F_n.

It was proved in [5] that the vector space of multilinear elements from F_n in an alternative metabelian algebra is generated by the elements

$$f_n = (x_1, x_2, x_3)[x_4, x_5] \dots [x_{n-1}, x_n],$$

$$f_n^{\mathrm{Id}+(2,4)} = (x_1, \overline{x_2}, x_3)[\overline{x_4}, x_5] \dots [x_{n-1}, x_n], \quad f_n^{\mathrm{Id}+(3,i)} \quad (i = \overline{4, n}).$$

To prove this fact the identity $(x, y, z)[y, z] = 0$ and the skew-symmetry of $f_n(p, q, r)$ in the variables from the product of commutators were used; and the metabelian identity was not used. By the identity $(x, y, z) \circ [y, z] = 0$ [9] and Lemma 2.5, it follows that these properties hold for $A^{(5)}$.

Lemma 4.1. $\dim_\Phi(P_n \cap F_n) = n - 1$ *if* $n = 2k + 3$, $k \geq 1$.

Proof. It suffices to prove the linear independence of f_n, $f_n^{\mathrm{Id}+(2,4)}$, and $f_n^{\mathrm{Id}+(3,i)}$, with $i = \overline{4, n}$. Assume that

$$\alpha f_n + \rho f_n^{\mathrm{Id}+(2,4)} + \sum_{i \geq 4} \chi_i f_n^{\mathrm{Id}+(3,i)} = 0 \tag{4.2}$$

for some scalars α, ρ, and χ_i. Consider the Grassmann envelope $G(S)$ of S. Inserting $x_1 = 1 \otimes a$ and $x_i = \xi_i \otimes x$, $i \geq 2$, in (4.2), we obtain $\alpha = 0$ by Lemma 3.3. Then

$$\rho f_n^{\mathrm{Id}+(2,4)} + \sum_{i \geq 4} \chi_i f_n^{\mathrm{Id}+(3,i)} = 0. \tag{4.3}$$

Let $i_0 \geq 5$ be a fixed index. Putting $x_{i_0} = u \in U$ in (4.3), we get $\chi_{i_0}(x_1, x_2, u)[x_4 x_5] \dots [x_{n-1} x_n] = 0$, whence $\chi_{i_0} = 0$ by Lemma 3.3, and $\rho f_n^{\mathrm{Id}+(2,4)} + \chi_4 f_n^{\mathrm{Id}+(3,4)} = 0$. Putting $x_1 = x_2 = t$ in this equality, we get $\rho(t, x_4, x_3)[t, x_5] \cdots = 0$. Then $\rho = 0$ and $\chi_4 f_n^{\mathrm{Id}+(3,4)} = 0$. By analogy, $\chi_4 = 0$. Thus, we demonstrated that $\alpha = 0$, $\rho = 0$, and all $\chi_i = 0$ when $i \geq 4$. The lemma is proved.

4.2. Proper polynomials of even degree. Let $n = 4 + 2k$, $k \geq 0$, and $X_n = \{p, q, r, s, y_i, z_i, \; i = \overline{1, k}\}$. Put

$$g_n(p, q, r, s) = ([p, q], r, s)w, \quad G_n = \operatorname{span}\langle g_n(p, q, r, s)\rangle, \tag{4.4}$$

where $w = [y_1, z_1][y_2, z_2] \dots [y_k, z_k]$. It is easy to understand that every proper polynomial in $D(A^{(5)})$ of even degree $n \geq 4$ belongs to G_n. Prove that the vector space $P_n \cap G_n$ of multilinear polynomials in G_n is spanned by the elements of the form

$$g_n(j) = ([x_1, x_j], x_{i_1}, x_{i_2})[x_{i_3}, x_{i_4}] \dots [x_{i_{n-1}}, x_{i_n}],$$

$$g_n' = ([x_2, x_3], x_1, x_4)[x_5, x_6] \dots [x_{n-1}, x_n],$$

where $\{i_1, \dots, i_n, j\} = \{2, 3, \dots, n\}$, $i_1 < i_2 < \cdots < i_n$.

Denote the subspace that is spanned by $g_n(j)$ by P. Note that $([a,b],x,y)[z,t]$ is skew-symmetric in x, y, z, and t by Lemma 2.2. Hence, if $([p,q],r,s)v$ is distinct from the elements of the form $g_n(j), g'_n$, where v is a product of commutators; then we may assume that $([p,q],r,s) = ([p,q],x_1,x_m)$ with $m \in 2,3$. Since $([a,b],a,b) = 0$ in alternative algebras; therefore, its linearization

$$([p,q],x_1,x_m) + ([x_1,q],p,x_m) + ([p,x_m],x_1,q) + ([x_1,x_m],p,q) = 0 \tag{4.5}$$

holds. Hence, modulo P, $([p,q],x_1,x_m)v$ is equivalent up to a sign with one of the elements $([p,x_2],x_1,x_3)v$ or $([p,x_3],x_1,x_2)v$. Each of these elements is equivalent modulo P to $\pm g'_n$ by the same argument. Thus, the claim is proved.

Lemma 4.2. $\dim_\Phi(P_n \cap G_n) = n$ *if* $n = 2k+4$, $k \geq 0$.

Proof. Prove the linear independence of $g_n(j)$ and g'_n, where $j = \overline{2,n}$. Assume that $\sum_{j\geq 2} \alpha_j g_n(j) + \rho g'_n = 0$ for some scalars α_j and ρ. Prove first that $\alpha_j = 0$ for all $j \geq 4$.

Since all cases are considered analogously, we assume that $j = 4$. Note that $g_n(j)$ with $j \neq 4$ and g'_n are Jordan derivations in x_4. Then the equality

$$\alpha_4(a,x_2,x_3)[x_1,a][x_5,x_6]\dots[x_{n-1},x_n] = 0$$

must be satisfied. Linearizing it by the substitution $a \to u \in U$, we get $\alpha_4 = 0$. In the same way, we show that $\alpha_j = 0$ for $j = 5,\dots,n$.

Hence,

$$\{\alpha_2([x_1,x_2],x_3,x_4) + \rho([x_2,x_3],x_1,x_4) - \alpha_3([x_3,x_1],x_2,x_4)\}w = 0, \tag{4.6}$$

where $w = [x_5,x_6]\dots[x_{n-1},x_n]$. Interchanging in this equality x_1, x_2, x_3 twice by cycle and adding the three so-obtained equalities termwise, we get

$$\delta\{([x_1,x_2],x_3,x_4) + ([x_2,x_3],x_1,x_4) + ([x_3,x_1],x_2,x_4)\}w = 0,$$

where $\delta = \alpha_2 + \rho - \alpha_3$. From here $\delta[(x_1,x_2,x_3),x_4]w = 0$ by (1.7). If $[(x_1,x_2,x_3),x_4]w = 0$ then $h := [([x_1,x_2],x_3,x_4)w$ is skew-symmetric in all variables and $h = 0$ by (1.9), which contradicts Lemma 3.3. Therefore, $\delta = 0$ and $\rho = \alpha_3 - \alpha_2$. Then by (4.6) and (4.5)

$$\begin{aligned}
0 &= \{\alpha_2([x_1,x_2],x_3,x_4) + (\alpha_3-\alpha_2)([x_2,x_3],x_1,x_4) - \alpha_3([x_3,x_1],x_2,x_4)\}w \\
&= \{\alpha_2\{([x_1,x_2],x_3,x_4) + ([x_3,x_2],x_1,x_4)\}w \\
&\quad +\alpha_3\{([x_2,x_3],x_1,x_4) + ([x_1,x_3],x_2,x_4)\}w \\
&= -\alpha_2\{([x_1,x_4],x_3,x_2) + ([x_3,x_4],x_1,x_2)\}w \\
&\quad +\alpha_3\{([x_2,x_3],x_1,x_4) + ([x_1,x_3],x_2,x_4)\}w.
\end{aligned}$$

Since three among the last four terms are Jordan derivations in x_2; therefore, $\alpha_3([x_2,x_3],x_1,x_4)w$ possesses this property, whence $\alpha_3(t,x_1,x_4)[t,x_3]w = 0$. Then $\alpha_3 = 0$. Analogously, $\alpha_2 = 0$. Thus, the lemma is proved.

Remark. Let a variety $\mathfrak{M}$ be unitarily closed, and let $B = F_{\mathfrak{M}}[X]$ be a relatively free algebra (with unity 1) in $\mathfrak{M}$. Recall that the infinity condition of the scalars domain Φ provides that every T-ideal of B is homogeneous and so invariant under the operators $\Delta(y)$ [9]. An arbitrary polynomial (not necessarily multilinear) is *proper* provided that it is annihilated by all operators in $\Delta(1)$.

A *standard monomial* over X is a right-normed product $x_1^{k_1}\cdots x_l^{k_l}$, where $k_i \geq 0$ for all $i = \overline{1,l}$.

Following [14], recall that

(a) every element in a T-ideal T of $C = F_{\mathrm{Alt}}[X]$ is a linear combination of the elements of the form gv, where g is a proper polynomial in T, and v is a standard monomial over X;

(b) every proper polynomial g of A is linearly expressed by the products $\pi = t_1 \dots t_m$ of values of the terms t_i, $i = \overline{1,m}$, in the signature $\Sigma_0 = \{c\}$, where $c(x,y) = [x,y]$ is a commutator;

(c) if $g \in D(C)$ then g is a linear combination of the suitable products of the form $\pi = t_1 \dots t_m$, each of which contains a term t_i in the signature $\Sigma_1 = \{c, d\}$, where $d(x, y, z) = (x, y, z)$ is an associator containing the symbol d.

By (1.2) the term $d(x, y, z)$ in C is linearly expressed by some terms in the signature $\Sigma_0 = \{c\}$.

It follows from the results of Subsection 2.1 that every proper polynomial of the ideal $D(A^{(5)})$ is a linear combination of homogeneous polynomials of the shape (4.1) and (4.4). If a polynomial of this type is of degree ≥ 3 in some variable then it is zero. Thus, it is easy to find an additive basis for the associator ideal $D(A^{(5)})$. However, we will not need it in a general form, and so we does not write it out.

4.3. Growth of dimension of $P_n(A) \cap D(A)$.

Proposition 4.1. *Let $A = A^{(5)}$ and $d_n = \dim_\Phi P_n(A) \cap D(A)$. Then*

$$d_n \sim (n-1) \cdot 2^{n-1}.$$

Proof. Let $X_n = Y \cup Z$ and $Y \cap Z = \varnothing$, where $Y = \{y_1, \dots, y_k\}$ and $Z = \{z_1, \dots, z_l\}$. A standard monomial over $Z = \{z_1, \dots, z_l\}$ is a right-normed product of variables $z_1, \dots, z_l$ under condition that these variables are ordered in the ascending order of indices. Denote by $F_k(Y)$ and $G_k(Y)$ the images of $P_k \cap F_k$ and $P_k \cap G_k$ under the isotonic maps $X_k \longrightarrow Y$ and $x_i \mapsto y_i$, with $i = \overline{1,k}$. Note that each element in $P_n(A) \cap D(A)$ is a linear combination of some elements of the form $f(y_1, \dots, y_k) \cdot v(z_1, \dots, z_l)$, where $k + l = n$, $f \in F_k \cup G_k$, and $v(z_1, \dots, z_l)$ is a right monomial over Z.

We need the two well-known equalities (see [15]):

$$k\binom{n}{k} = n\binom{n-1}{k-1}, \quad \sum_k \binom{n+1}{2k+1} = 2^n.$$

By Lemmas 4.1 and 4.2, for a suitable sequence $\theta_n = o(2^n)$ (which is infinitesimal with respect to 2^n as $n \to \infty$) with $n \geq 5$ we have

$$d_n = \theta_n + \sum_k 2k \left\{ \binom{n}{2k} + \binom{n}{2k+1} \right\}$$

(since F_k and G_k are not defined for small values of k)

$$= \theta_n + \sum_k 2k\binom{n+1}{2k+1} = \theta_n + \sum_k (2k+1)\binom{n+1}{2k+1} - \sum_k \binom{n+1}{2k+1}$$
$$= \theta_n + (n+1)\sum_k \binom{n}{2k} - 2^n = \theta_n + (n+1)2^{n-1} - 2^n = \theta_n + (n-1)2^{n-1},$$

as required.

From [4, Theorem 3.1] and Proposition 4.1 we infer

Theorem 1. *Let $A = A^{(5)}$ and $c_n = \dim_\Phi P_n(A)$. Then the asymptotic holds $c_n \sim n^2 \cdot 2^{n-2}$.*

5. Components of the Variety $\mathrm{Alt}^{(5)}$

Denote by $\mathrm{Ass}^{(5)}$ the variety of associative Lie-nilpotent algebras of degree 5; and by $\mathrm{var}(A)$, the variety generated by A.

Lemmas 4.1 and 4.2 yield

Proposition 5.1. $\mathrm{Alt}^{(5)}$ *is the union of* $\mathrm{Ass}^{(5)}$ *and* $\mathrm{var}(G(S))$.

$\mathrm{Ass}^{(5)}$ may be also decomposed into a union of two components. For their description, let us introduce two auxiliary algebras.

5.1. The superalgebra $V = V^{(5)}$. Let $V = V_0 \oplus V_1$ be an associative superalgebra with unity 1 in $\mathrm{Ass}^{(5)}$, and let V be generated by an even element r and odd element x, satisfying the relations

$$r^2 \in Z(V), \quad x^2 = 1, \quad r \circ x = r^4 = 0.$$

Show that V is of dimension 8 over Φ. Note that V is spanned by $r^k x^\varepsilon$, where $0 \le k \le 3$ and $\varepsilon \in \{0, 1\}$. Denote by Δ an algebra over Φ with the basis 1 and ρ, on assuming that $\rho^2 = 0$. Verify that V is a free Δ-module generated by 1, r, x, and rx. First, consider the algebra M over Δ with the specified basis and the following multiplication table:

1) 1 is the unity,
2) $r \cdot r = \rho 1$, $x \cdot r = -rx$, $rx \cdot r = -\rho x$,
3) $1 \cdot x = x$, $r \cdot x = rx$, $x \cdot x = 1$, $rx \cdot x = r$,
4) $1 \cdot rx = rx$, $r \cdot rx = \rho x$, $x \cdot rx = -r$, $rx \cdot rx = -\rho 1$.

It is easy to see that for the associative and commutative ring $\Delta = \Phi[\rho]$ the algebra M is associative. Indeed, the operators of right multiplication by r, x, and rx in the basis $1, r, x, rx$ are as follows:

$$R(r) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -\rho & 0 \end{pmatrix}, \quad R(x) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$R(rx) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \rho & 0 \\ 0 & -1 & 0 & 0 \\ -\rho & 0 & 0 & 0 \end{pmatrix}.$$

Using these operators, it is easy to verify the associativity of M; for example,

$$R(x)R(rx) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \rho & 0 \\ 0 & -1 & 0 & 0 \\ -\rho & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \rho & 0 \end{pmatrix} = -R(r) = R(x \cdot rx),$$

$$R(rx)^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \rho & 0 \\ 0 & -1 & 0 & 0 \\ -\rho & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 \\ 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & -\rho \end{pmatrix} = R(rx \cdot rx).$$

We have the grading

$$V_0 = \mathrm{span}(1, r, \rho, \rho r), \quad V_1 = \mathrm{span}(x, rx, \rho x, \rho rx)$$

of the Φ-algebra V. Prove that V satisfies (3.1). First,

$$[x, r]_s = [x, r] = x \cdot r - r \cdot x = -2rx, \quad [x, x]_s = 2, \quad [x, rx]_s = xrx + rx^2 = -r + r = 0,$$
$$[rx, r]_s = [rx, r] = r[x, r] = -2\rho x, \quad [rx, rx]_s = 2rxrx = -2\rho 1.$$

Hence, $V^{s(2)} = [V, V]_s \subseteq \mathrm{span}(1, rx) + \widehat{\rho}V$, where $\widehat{\rho} = \Phi \cdot \rho$. Then

$$V^{s(3)} = [V^{s(2)}, V]_s \subseteq [rx, V]_s + \widehat{\rho}V^{s(2)} \subseteq \widehat{\rho} + \widehat{\rho}x + \widehat{\rho}rx,$$
$$V^{s(4)} = [V^{s(3)}, V]_s \subseteq \widehat{\rho}[\mathrm{span}(1, x, rx), V]_s = \widehat{\rho}[\mathrm{span}(x, rx), V]_s \subseteq \widehat{\rho} + \widehat{\rho}rx,$$
$$V^{s(5)} = [V^{s(4)}, V]_s = 0.$$

Show that for every n there are $x_0 \in V_0$, $x_1, y_1, z_1, \dots, y_n$, and $z_n \in V_1$ such that

$$[x_1, x_0, x_0, x_0][y_1, z_1]_s \cdots [y_n, z_n]_s \neq 0. \tag{5.1}$$

We have

$$[x, r] = -2rx, \quad [[x, r], r] = [-2rx, r] = 4\rho x,$$
$$[[[x, r], r], r] = [4\rho x, r] = 4\rho[x, r] = -8\rho rx.$$

Since $[x, x]_s = 2$ in V, (5.1) is proved.

5.2. The algebra $W = W^{(5)}$. Let $F_6^{(5)}$ be the relatively free unital associative algebra of rank 6 with identity (2.1), let I be its ideal generated by the monomials of degree 7, and let $W = F_6^{(5)}/I$ be the quotient algebra. Notice that W is finite-dimensional.

An additive basis of the free algebra in $\mathrm{Ass}^{(5)}$ was constructed in [4]. The model algebra was used to verify the linear independence of the specified system of polynomials. However, the algebras $G(V)$ and W may be used instead of this model algebra. Therefore, the following holds:

Proposition 5.2. $\mathrm{Ass}^{(5)}$ *is the union of the varieties* $\mathrm{var}(G(V))$ *and* $\mathrm{var}(W)$.

5.3. On finite-dimensional superalgebras that generate a given variety. Kemer proved in [16] that every variety of associative algebras over a field of characteristic 0 is generated by the Grassmann envelope of a finite-dimensional superalgebra.

Definition. Let $\mathfrak{M}$ be a variety of algebras. We say that a *superalgebra M generates* $\mathfrak{M}$ provided that $\mathfrak{M}$ is generated by its Grassmann envelope $G(M)$.

Theorem 2. *The variety* $\mathrm{Alt}^{(5)}$ *of alternative algebras that are Lie-nilpotent of degree* 5 *is generated by a finite-dimensional superalgebra.*

The algebra $V \oplus W \oplus S$ may be taken as such superalgebra.

It follows from Proposition 5.2 that $\mathrm{Ass}^{(5)}$ is generated by the finite-dimensional superalgebra $V \oplus W$.

Gordienko [17] found a finite-dimensional superalgebra that generates $\mathrm{Ass}^{(4)}$.

6. Central Elements in $A^{(5)}$

As before, denote the free algebra of countable rank in $\mathrm{Alt}^{(5)}$ by $A^{(5)}$.

6.1. The associative center of $A^{(5)}$.

Theorem 3. *The associative center $N_{\mathrm{Ass}}(A)$ of $A = A^{(5)}$ coincides with the ideal $T^{(3)}(A)$; consequently, $N_{\mathrm{Ass}}(A)$ coincides with its associative core $N^*_{\mathrm{Ass}}(A)$.*

Proof. By Lemma 2.5, $T^{(3)}(A) \subseteq N_{\mathrm{Ass}}(A)$. To finish the proof it suffices to notice that

$$w = [y_1, z_1][y_2, z_2] \dots [y_{k+1}, z_{k+1}]$$

does not belong to $N_{\mathrm{Ass}}(A)$. Indeed,

$$([a', x]_s[x, x]_s^k, x, x) = 2^k([a', x]_s, x, x) \neq 0$$

in S by Lemma 3.3, which was required to prove.

6.2. The central core of $A^{(5)}$. The homogeneous polynomials presents linear combinations of some polynomials of the form (4.1) and (4.4) are *regular D-elements.*

Denote the set of variables in X on which f depends by $\mathrm{vr}(f)$.

Lemma 6.1. *The associator ideal $D(A)$ of $A = A^{(5)}$ has zero intersection with $Z^*(A)$; i.e., $D(A) \cap Z^*(A) = 0$.*

Proof. For the sake of completeness, we present the necessary arguments on repeating the arguments from Proposition 5.2 of [4]. Let f be a homogeneous element in $D(A)$. Then $f = \sum_{v_i} p(v_i)v_i$, where the summation is over the standard monomials v_i, while $p(v_i)$ are regular D-elements, and $\mathrm{vr}(f) = \mathrm{vr}(p(v_i)) \cup \mathrm{vr}(v_i)$. Using partial derivations by the variables in $\mathrm{vr}(v_i)$, we get $p(v_i) \in Z^*(A)$. Then $p(v_i)[x, y] = 0$ for all $x, y \in X$. Let $\mathrm{vr}(f) < x < y$. If $p(v_i) \neq 0$ then $p(v_i)[x, y]$ may be considered as a nonzero multilinear polynomial, which contradicts Section 4 (the additive basis of the associator ideal). Hence, each $p(v_i)$ is 0, and $f = 0$. The lemma is proved.

By [3, the corollary of the theorem] and Lemma 6.1, the following holds:

Theorem 4. *The central core $Z^*(A)$ of A is generated as a T-subspace by the weak Hall element $[[x_1, x_2]^2, x_2]$.*

6.3. The full center of $A^{(5)}$. The aim of this section is the proof of the following theorem.

Theorem 5. *The full center $Z(A)$ of $A = A^{(5)}$ is generated as a T-space by the elements*

$$[x_1, x_2, x_3, x_4], \quad [[x_1, x_2, x_3] \cdot x_4, x_5], \quad [[x_1, x_2]^2, x_3]. \tag{6.1}$$

We precede the proof of this theorem with two lemmas. Denote the T-space of A generated by (6.1) by Z_0. Note that elements (6.1) are central by Lemmas 2.3–2.5. Using the results of [4], it suffices to understand that every central element from the associator ideal $D(A)$ belongs to Z_0.

Recall that a vector space spanned by f_n, $f_n^{\mathrm{Id}+(2,4)}$, and $f_n^{\mathrm{Id}+(3,i)}$ $(i = \overline{4, n})$ was denoted by F_n in Subsection 4.1.

Lemma 6.2. *The vector space F_n does not contain nonzero elements in the center $Z(A)$ of $A = A^{(5)}$.*

Proof. Notice first that F_n is spanned by

$$f_n = (x_1, x_2, x_3)[x_4, x_5] \dots [x_{n-1}, x_n], \quad f_n(4) = f_n^{\mathrm{Id}+(2,4)},$$
$$f_n(i) = (x_1, x_2, \overline{x_3})[\overline{x_i}, x_4][x_{j_1}, x_{j_2}] \dots [x_{j_{n-6}}, x_{j_{n-5}}], \quad 5 \le i \le n,$$

where $j_1 < \dots < j_{n-5}$ and $\{j_1, \dots, j_{n-5}\} \cup \{1, 2, 3, 4, i\} = \{1, 2, \dots, n\}$.

Indeed, $f_n^{\mathrm{Id}+(3,i)}$ and $f_n(i)$ with $5 \le i \le n$ coincide up to a sign $\pm$ by Lemma 2.5.

Consider an element of the form $k = \alpha f_n + \beta f_n(4) + \sum_{i=5}^n \gamma_i f_n(i)$, where $\alpha, \beta, \gamma_i \in \Phi$. Assume that $k \in K(A)$. Then its value k' at $x_1 = x_2 = z \in X$ also belongs to $K(A)$. Only the second summand for k in the specified element is nonzero; consequently, $k' = \beta f_n(4)|_{x_1=x_2=z} \in K(A)$. It follows that $\beta[(z, x_4, x_3)[z, x_5]v, a] = 0$, where v is a product of commutators in which none of the variables z, x_3, x_4, x_5, and a appears. By Lemmas 2.4–2.6

$$0 = \beta[(z, x_4, x_3)[z, x_5]v, a] = \beta[(z, x_4, x_3)[z, x_5], a]v = \beta([x_3, z], x_4, z)[x_5, a]v,$$

whence $\beta = 0$ by Lemma 3.3, and $k = \alpha f_n + \sum_{i=5}^n \gamma_i f_n(i)$. Since $k \in K(A)$, $[k, x_1] = 0$. Note that if v is a commutator word then $(a, b, x)[b, y]v = (a, b, x)[b, yv]$ by (1.1) and Lemmas 2.4–2.6, whence $[f_n(i), x_1] = 0$ by Lemma 2.7. Then

$$0 = \alpha[f_n, x_1] = \alpha[(x_1, x_2, x_3)w, x_1] = \alpha(x_1, [x_1, x_2], x_3)w$$

for $w = [x_4, x_5] \dots [x_{n-1}, x_n]$. By Lemma 3.3 $\alpha = 0$ and $k = \sum_{i=5}^n \gamma_i f_n(i)$. Choose a fixed index $i_0 \ge 5$ and consider the equality $[k, x_{i_0}] = 0$. Since $[t[a, b], a] = [t, a][a, b] = 0$ for every $t \in T^{(3)}(A)$ by Lemmas 2.5 and 2.2; therefore,

$$0 = \sum_{i=5}^n \gamma_i [f_n(i), x_{i_0}] = \gamma_{i_0}[f_n(i_0), x_{i_0}]$$
$$= \gamma_{i_0}[(x_1, x_2, x_{i_0})[x_3, x_4][x_{j_1}, x_{j_2}] \dots [x_{j_{n-6}}, x_{j_{n-5}}], x_{i_0}],$$

where the last element is multilinear in all variables but x_{i_0}, relative to which it is of degree 2. By analogy to the previous one, from here we get $\gamma_{i_0} = 0$. The lemma is proved.

In what follows, we also need the vector space G_n of Subsection 4.2. Note that it is central; i.e., $G_n \subseteq Z(A)$.

Lemma 6.3. *Let a be a standard monomial, and let g be a polynomial of the form*

$$g = wu_1 \cdots u_s, \tag{6.2}$$

where $w = [v, x]$, $v \in (X, X, X)$, $u_1, \dots, u_s \in U^{(2)}$, $x \in X$. Then there exist some elements f_i of the form (4.1) *and some standard monomials $a_i \in A$ such that*

$$ga + \sum_i f_i a_i \in [D(A), A].$$

Proof. Using Theorem 3, transform ga:

$$ga = ([v, x]u_1 \cdots u_s)a = [v, x](u_1 \cdots u_s a) = [v \cdot u_1 \cdots u_s a, x] - vu_1 \cdots u_s[a, x].$$

The first summand $z = [v \cdot u_1 \cdots u_s a, x]$ lies in $[D(A), A]$, and the second $vu_1 \cdots u_s[a, x]$ may be written as $\sum f_i a_i$ by Lemma 4.1 using the induction on degree of a, where f_i are some elements of the form (4.1), and a_i are some standard monomials. The lemma is proved.

We can now finish the proof of Theorem 5. It was demonstrated in [4] that the center of a free associative algebra, which is Lie-nilpotent of degree 5, is generated as a T-space by the elements (6.1). Therefore, it suffices to consider $f \in Z(A) \cap D(A)$.

As noticed above (see the Remark at the end of Subsection 4.2), an arbitrary polynomial in $D(A)$ may be written as $\sum_i f_i a_i$, where f_i are proper polynomials in $D(A)$, and a_i are some standard monomials; it is easy to obtain this representation by induction on degree of f as well.

Note that f_i are of the form (4.1) or (4.4). Every polynomial of the form (4.4) is a linear combination of some elements of the form (6.2) by (1.4) and (1.9).

If $f \in D(A) \cap Z(A)$ then f may be written as

$$f \equiv \sum_i f_i a_i \tag{6.3}$$

modulo $[D(A), A]$ by Lemma 6.3, where f_i are linear combinations of some elements of the form (4.1), and a_i are some pairwise different standard monomials.

Prove by contradiction that all $f_i = 0$. Clearly, $[D(A), A]$ is invariant under the substitutions $x = 1$ for every $x \in X$.

Choose an element a_m of maximal degree among the standard monomials a_i from (6.3) and substitute $x = 1$ in it for all x appearing in a_m. Then $f_i \in [D(A), A] \subseteq Z(A)$, a contradiction with Lemma 6.2. Thus, we proved that $D(A) \cap Z(A) \subseteq [D(A), A] \subseteq Z_0$. Theorem 5 is proved.

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CONFLICT OF INTEREST

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