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ADMISSIBLE INFERENCE RULES OF MODAL WCP-LOGICS

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Abstract—We study admissible rules for the extensions of the modal logics S4 and GL with the weak co-covering property and describe some explicit independent basis for the admissible rules of these logics. The resulting basis consists of an infinite sequence of rules in compact and simple form.

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1. Introduction

The modern applications of logic in computer science and artificial intelligence need some language adapted to describing various dynamical systems. The language of nonclassical logics, for instance modal or temporal, plays this role successfully. However, the facts and assertions in this language are originally described using the formulas describing models in general, but incapable of expressing variable conditions and assumptions. We can model these conditions and assumptions by various versions of the concept of logical implication. One distinguishing feature of the approach we propose to logical implication is that we study the latter in terms of inference rules or sequents rather than just formulas or propositions.

The formalism describing the properties of models by way of formulas is deeply developed, widely extended, and thoroughly presented in the research literature. The formation serves as a basis for representing and studying human thinking. However, formulas describe only stable static phenomena; each assertion only fixes a fact but is unable to capture variable conditions. Thus, to study (structural) inference rules or sequents, expressions consisting of the premises that are prescribed collections of assumptions and conclusions provides a finer and more expressive tool for modeling thinking and calculations. The premise of an inference rule expresses the current information given in the form of assumptions, while its application gives some conclusion or fact that we can infer from the assumptions. Inference rules also enable us to model the standard situation in studying logical implication: Given some assumptions or facts, what do they imply; what is their consistent corollary?

The concept of a (structural) inference rule generalizes obviously that of a formula: An arbitrary formula can be regarded as a structural inference rule without any premise or assumptions. However, admissible inference rules turned out much stronger than the usual structural rules: Owing to Harrop's example of 1960 in [1], we know that even the intuitionistic logic Int is not structurally complete, meaning that Int contains admissible but undeductive inference rules; i.e., rules not expressible by formulas. Owing to the examples by Mints in [2] and Port in [3], this applies also to a large class of modal logics.

In 1955 Lorenzen introduced the concept of an admissible (structural) inference rule in [4]. In an arbitrary logic, those inference rules are admissible that preserve the set of theorems of the logic. It is clear that every deductible rule is admissible in the specified logic, but the converse fails in general as shown by the examples of Harrop, Mints, and Port. We can also conclude directly from the definition that the set of all inference rules admissible in some logic λ constitutes the *greatest* class of inference rules with which we can extend the axiomatic system of this logic without changing the set of provable theorems. Moreover, admissible rules considerably strengthen the deductive system of a given logic.

The history of studying admissible rules stems from 1975, when the Fridman problem appeared in [5] of the existence of an algorithmic criterion for the admissibility of rules in the intuitionistic logic Int. In the classical logic the admissibility question is solved trivially: The admissible rules are only the

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deductible inference rules. Admissible but nondeductible inference rules exist in the case of nonclassical logics. In the mid 1970s Mints obtained sufficient conditions for the deducibility of rules of a particular form in [2]. In 1984 Rybakov found a positive solution to the Fridman problem on the existence of an algorithm recognizing admissible inference rules in the intuitionistic logic Int in [6]. Some criterion for the admissibility of inference rules was later stated for a broad class of modal and superintuitionistic logics in [7].

Another method for describing all admissible rules in a logic goes back to the Kuznetsov problem of the existence of a finite basis for the admissible inference rules in Int. Possessing a basis for the admissible rules, we can deduce from it all remaining rules. Tsitkin obtained the first positive result in [8], finding a basis for all quasicharacteristic inference rules admissible in Int. Originally the studies of bases for admissible inference rules in nonstandard logics focused on the most important individual logics like the logic of provability GL or the systems S4 and S5, as well as on generalizing the available methods and obtaining the general technique applicable not only to particular individual logics but to the whole subclasses of logics which contain the most interesting and important logics. In general the Kuznetsov problem is solved negatively not only for Int [9], but also for the majority of other basic logics. Rybakov showed in [7, Chapter 4] that the logics Int, KC, K4, S4, and many basic others lack any finite basis for the admissible rules in finitely many variables.

With the negative solution to the Kuznetsov problem for many basic nonclassical logics, by the end of the twentieth century the basis for admissible inference rules could be obtained only from the available algorithmic criterion for admissibility; see [7, Chapter 3.5]. However, this computationally complicated criterion is inapplicable to describing such basis in any easily graspable form. Thus, it becomes an urgent problem to give some explicit description of an easily graspable basis for all admissible inference rules at least for the main basic logics, as well as for those "strong" tabular logics that admit a finite basis for admissible rules, see [10, 11]. The first step in this direction was made in 2000: Some recursive basis was obtained for the admissible rules of the intuitionistic logic Int consisting of rules in a semireduced form in [12]. Later Iemhoff obtained an explicit basis for the admissible rules of the logic Int and its extensions with the weak co-covering property and the disjunctive property in [14] and a few subsequent articles. Rybakov constructed an explicit basis for all admissible rules of the logic S4 in [15]. An explicit independent basis was obtained in [16, 17] for the admissible rules of the extensions of S4 which inherit the admissible rules of S4, i.e., possessing the (strong) co-covering property. Fedorishin obtained an explicit basis for the admissible rules of the logic GL in [18]. The presence of the weak co-covering property and the disjunctive property of the logic were key in the proofs of those results.

This article continues studying bases for the admissible rules of modal logics. Using the technique of [15, 18], we describe some explicit independent basis for the admissible inference rules in the extensions of the logics S4 and GL with the weak co-covering property WCP-logics), but without the disjunctive property. We decrease the number of variables on which the rules forming bases in S4 and GL depend (cp. [15, 18]) and avoid the disjunctive property of the logic. Thus, we managed to considerably extend the class of logics for which some explicit independent basis for admissible rules is described.

2. Definitions and Preliminary Results

We begin by briefly recalling the needed definitions and results. We recommend [7] as a detailed introduction into the subject.

A *language of modal logics* consists of a countable set of propositional variables $p_1, \dots, p_n, \dots$, the logical connectives $\neg, \wedge, \vee, \rightarrow$ of classical logic, and the unary modal operator $\Box$. The operator $\Diamond$, also used below, is defined as $\Diamond\alpha = \neg\Box\neg\alpha$. A *normal modal logic* is a set L of modal formulas containing all propositional tautologies, the scheme of axioms $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$, and closed under substitutions, the separation rule $\alpha, \alpha \rightarrow \beta \vdash \beta$ and the necessity $\alpha \vdash \Box\alpha$. Denote the minimal modal logic by K. The extension of K by the scheme $\Box\alpha \rightarrow \Box\Box\alpha$ is denoted by $K4$; the extension of $K4$ by the scheme $\Box\alpha \rightarrow \alpha$ produces the logic S4. If L is a normal modal logic then for a formula $\alpha \in L$ we write $\vdash_L \alpha$ or $L \vdash \alpha$, but if the logic L is either fixed or clear from context then we write $\vdash \alpha$ for simplicity. Henceforth we

consider only the logics extending S4 or GL.

A *frame* is a pair $\mathscr{F} := \langle F, R\rangle$, where F is a nonempty set and R is a binary relation on F. Meaningfully, F represents the set of all "possible" worlds, while R is the relation of passage from one world to the other. Henceforth we denote both the frame itself and its underlying set by the same letter, for instance $\mathscr{F}$. Since we consider the logics extending S4 (or GL), the attainability relation R on the frames is assumed reflexive (or irreflexive) and transitive.

A *model* is a triple $\mathscr{M} := \langle F, R, V\rangle$, where $\langle F, R\rangle$ is a frame, and the *evaluation* V is a mapping from the set of propositional variables into the power set 2^F of F. The evaluation V associates to each variable the set $V(p)$ of worlds in which the variable p is true. Express the truth of p at $x \in F$ in the specified evaluation V as $(F, x) \models_V p$. When the underlying set or frame is clear from the context, we indicate the truth of a variable as $x \models_V p$.

The *truth of a formula* α on $x \in \mathscr{F}$ for a specified evaluation V is defined inductively as follows:

- $x \models_V p \iff x \in V(p)$;
- $x \models_V \neg\alpha \iff x \not\models_V \alpha$;
- $x \models_V \alpha \vee \beta \iff x \models_V \alpha$ or $x \models_V \beta$;
- $x \models_V \alpha \wedge \beta \iff x \models_V \alpha$ and $x \models_V \beta$;
- $x \models_V \alpha \to \beta \iff x \not\models_V \alpha$ or $x \models_V \beta$;
- $x \models_V \Box\alpha \iff \forall y \in \mathscr{F}(xRy \Longrightarrow y \models_V \alpha)$;
- $x \models_V \Diamond\alpha \iff \exists y \in \mathscr{F}(xRy \& y \models_V \alpha)$.

A formula A is *true on a model* $\mathscr{M} = \langle F, R, V\rangle$, in symbols $\mathscr{M} \models A$ or $\mathscr{F} \models_V A$, whenever A is true at each element of $\mathscr{M}$ for the evaluation V. A formula A is *true on a frame* $\mathscr{F}$, in symbols $\mathscr{F} \models A$, whenever A is true on an arbitrary model $\mathscr{M}$ generated by $\mathscr{F}$, i.e., A is true for arbitrary evaluation on $\mathscr{F}$.

A subset $\mathscr{X}$ of a model $\mathscr{M}$ is *definable* whenever there exists a formula α such that $\forall z \in \mathscr{M}[z \models_V \alpha \iff z \in \mathscr{X}]$. Accordingly, $z \in \mathscr{M}$ is *definable* whenever so is the set $\{z\}$. The evaluation V is *definable* in $\mathscr{M}$ provided that $V(p)$ is definable for every p in the domain of V.

A model $\mathscr{M} = \langle F, R, V\rangle$ is *adequate* for some logic L or an *L-model*, whenever every formula provable in L is true on $\mathscr{M}$. Accordingly, a frame $\langle F, R\rangle$ is *adequate* for some logic L whenever all provable formulas of L are true on $\langle F, R\rangle$. A class K of frames is *characteristic* for some logic L whenever every frame in K is adequate for L and for each formula not provable in L there is a frame in K on which this formula is refuted. For a given class $\mathscr{K}$ of frames the *logic $L(\mathscr{K})$ generated by $\mathscr{K}$* is the set of all formulas true on all frames in $\mathscr{K}$. In this case we say that $L(\mathscr{K})$ is *generated by* the class $\mathscr{K}$ of frames.

A modal logic L is *decidable* whenever for each formula there is an algorithm enabling us to establish its provability in this logic. A modal logic L is *residually finite* whenever for each formula α not provable in L there exists a finite frame (or finite algebra) adequate for L on which α is not true. A modal logic L has the *disjunctive property* whenever for all formulas α and β the provability in L of $\Box\alpha \vee \Box\beta$ implies the provability in L of either $\Box\alpha$ or $\Box\beta$.

Recall that, given a frame $\mathscr{F} = \langle F, R\rangle$, a set $C \subseteq \mathscr{F}$ is a *cluster* whenever

(1) xRy for all x and y in C;

(2) $(xRy \& yRx) \Longrightarrow y \in C$ for all $x \in C$ and $y \in F$.

A cluster is *proper* if $|C| > 1$; otherwise, it is a *singleton*. Given $a \in F$, denote the cluster generated by a by $C(a)$.

Every set of pairwise R-incomparable clusters of a frame F is an *antichain*. An antichain $\mathscr{A}$ is *nontrivial* whenever $\mathscr{A}$ consists of at least two distinct clusters, and otherwise $\mathscr{A}$ is *trivial*. Given $a \in \mathscr{F}$, put $a^R = \{z \mid aRz\}$ and $a^{<R} = a^R \setminus C(a)$ and say that *a generates* the subframe a^R of F *as the root*. Call $\mathscr{F}$ a *root frame* whenever $\forall b \in \mathscr{F} aRb$ for some $a \in \mathscr{F}$. This element a is also called the *root* of $\mathscr{F}$. The set $X^R := \bigcup\{z^R \mid z \in X\}$ is the *open subframe* generated by X. The concepts of a root model, submodel, and open submodel are defined similarly.

Say that a frame $\mathscr{F}$ is an *L-frame* whenever all theorems of the logic L are true on $\mathscr{F}$ for arbitrary evaluations of the variables, i.e., $\mathscr{F}$ is adequate for L. Accordingly, the set $L(\mathscr{F})$ of formulas true on $\mathscr{F}$ is the logic generated by $\mathscr{F}$.

A cluster $C(a)$ in F is a *co-covering* for a set or antichain $X \subseteq F$ whenever $a^R \backslash C(a) = X^R$. Say that an element a is a *co-covering* for $X \subseteq F$ provided that the singleton cluster $C(a)$ is a co-covering for X. Henceforth we understand by a *co-covering* a singleton cluster i.e. a co-covering. Refer as an L-*co-covering* to every co-covering generating L-frame as the root.

Refer as the *depth* of an element x of a model or frame $\mathscr{F}$ to the maximal number of clusters in the chains of clusters beginning with $C(x)$ which contain x. Denote the set of all elements of depth at most n of a frame F by $S_{\leq n}(\mathscr{F})$; while the set of elements of depth n, by $S_n(\mathscr{F})$.

Given a frame $\mathscr{F}$, an evaluation V, and an inference rule $r := \alpha_1, \dots, \alpha_k/\beta$, say that r *is true on* $\mathscr{F}$ for V, writing $\mathscr{F} \models_V r$, if as soon as $\forall x \in \mathscr{F}\, \forall i\ (x \models_V \alpha_i)$ then $\forall x \in \mathscr{F}\ (x \models_V \beta)$. A rule r is true on $\mathscr{F}$ provided that r is true on $\mathscr{F}$ for every evaluation V, in symbols $\mathscr{F} \models r$.

An inference rule $\alpha_1(p_1, \dots, p_n), \dots, \alpha_k(p_1, \dots, p_n)/\beta(p_1, \dots, p_n)$ is *admissible* in some logic λ, in symbols $r \in \mathrm{Ad}(\lambda)$, whenever $(\forall j\, \alpha_j(\delta_1, \dots, \delta_n) \in \lambda)$ implies that $\beta(\delta_1, \dots, \delta_n) \in \lambda$ for all formulas $\delta_1, \dots, \delta_n$. Given an inference rule r, denote the premise of r by $\mathrm{Pr}(r)$.

The admissible rules of a propositional modal (superintuitionistic) logic L admit an algebraic description: There are associated quasi-identities true on the countable rank free algebra $\mathfrak{F}_w(L)$ of the variety $\mathrm{Var}(L)$ of algebras corresponding to this logic, and so we have the following

Proposition 2.1 [1, Chapter 3]. *The inference rule*

$$r = \{\alpha_1(p_1, \dots, p_n), \dots, \alpha_k(p_1, \dots, p_n)/\beta(p_1, \dots, p_n)\}$$

is admissible in some logic L if and only if the quasi-identity

$$r^* = \{\alpha_1(p_1, \dots, p_n) = 1 \& \cdots \& \alpha_k(p_1, \dots, p_n) = 1 \Longrightarrow \beta(p_1, \dots, p_n) = 1\}$$

is true on the countable rank free algebra $\mathfrak{F}_w(L)$ of the variety $\mathrm{Var}(L)$.

A rule r is a *corollary* to $r_1, \dots, r_k$ in a logic λ whenever the conclusion of r can be deduced from the premise of r using the theorems of λ, the rules $r_1, \dots, r_k$, and the postulated inference rules of λ. Refer to the set $\mathrm{Ad}^*(\lambda)$ of admissible rules of λ as a *basis for the admissible rules* if for every admissible rule r there are $r_1, \dots, r_k \in \mathrm{Ad}^*(\lambda)$ such that r can be deduced from $r_1, \dots, r_k$ in λ.

Proposition 2.2 [3, Chapters 3.5 and 4.1]. *A set $r_1, \dots, r_k$ of rules constitutes a basis for the admissible inference rules in a logic λ if and only if $r_1^*, \dots, r_k^*$ is a basis for the quasi-identities of $\mathfrak{F}_w(\lambda)$.*

A model $\langle F, R, V\rangle$, where $V : P_n \to 2^F$ and $P_n = \{p_1, p_2, \dots, p_n\}$, is n-*characteristic* for some logic L if and only if $\alpha \in L \iff \langle F, R, V\rangle \models \alpha$ for every formula $\alpha(p_1, \dots, p_n)$ in $p_1, \dots, p_n$.

Our study essentially uses the structure of an n-characteristic model for the residually finite logics extending the logic S4 or GL: With its use we describe the admissible inference rules in these logics. Following [7, Chapter 3], we describe the construction of this model. Take a residually finite logic λ extending S4 and a set $P_n = \{p_1, p_2, \dots, p_n\}$ of propositional variables. The first slice $S_1(C_n(\lambda))$ of this model consists of the set of clusters pairwise nonisomorphic as models with respect to all possible evaluations V of the variables in P_n, and all elements of each cluster have distinct values. Assuming that $S_{\leq m}(C_n(\lambda))$ is already constructed, we obtain the slice $S_{m+1}(C_n(\lambda))$ of depth $m + 1$ as follows: Choose an arbitrary antichain $\mathscr{X} \subset S_{\leq m}(C_n(\lambda))$ of clusters which contains at least one cluster of depth m and adjoin from below a copy of each cluster C from $S_1(C_n(\lambda))$ as a co-covering for $\mathscr{X}$ provided that:

(i) $C^R = \mathscr{X}^R \cup \{C\}$ is a λ-frame;

(ii) if $\mathscr{X} = \{C_1\}$ then C is not isomorphic to a submodel of C_1.

Continuing this process, we eventually obtain the model $Ch_n(\lambda)$. For the extensions of the logic GL such model is constructed similarly, but we use only irreflexive elements. Let us state the properties of the resulting model in the following statements:

Proposition 2.3 [7, Chapter 3]. *For every residually finite logic λ extending S4 or GL the model $C_n(\lambda)$ is n-characteristic and each element of $C_n(\lambda)$ is definable.*

Proposition 2.4 [7, Chapter 3]. *For every residually finite logic λ extending S4 or GL, an inference rule r is admissible in λ if and only if r is true on the frame $C_n(\lambda)$ for every n and arbitrary definable evaluation of variables.*

In this study we also need the reduced form of modal inference rules. Say that a rule R is in *reduced form* whenever $R := \bigvee_{1\le j\le m} \phi_j/\Box x_0$, where $\phi_j := \bigwedge_{0\le i\le k} x_i^{a_i} \wedge \bigwedge_{0\le i\le k} \Diamond x_i^{b_i}$ with $a_i, b_i \in \{0,1\}$, where $x^0 := x$ and $x^1 := \neg x$. Given a term ϕ_j of the premise of a rule in reduced form, put

$$\theta_1(\phi_j) := \{x_j : 0 \le j \le k, a_j = 0\}, \quad \theta_2(\phi_j) := \{x_j : 0 \le j \le k, b_j = 0\},$$

$$\theta_3(\phi_j) := \{x_j : 0 \le j \le k, a_j = 1\}, \quad \theta_4(\phi_j) := \{x_j : 0 \le j \le k, b_j = 1\}.$$

Proposition 2.5 [7]. *Given a modal inference rule R, there exists a rule $rf(R)$ in reduced form equivalent to R with respect to truth on (GL-) S4-algebras and (GL-) S4-frames; moreover, R and $rf(R)$ are simultaneously deductible or admissible in an arbitrary modal logic extending S4 (or GL).*

3. A Basis for Admissible Inference Rules in WCP-Logics over S4

Say that some logic λ extending the logic S4 has a *weak co-covering property* over S4 whenever, given a finite root λ-frame $\mathscr{F}$ and a nontrivial antichain $\mathscr{X}$ of clusters in $\mathscr{F}$, the frame $\mathscr{F}_1$ obtained by adjoining a singleton reflexive co-covering as the root to the frame $\bigcup_{c\in\mathscr{X}^R} c^R$ is a λ-frame as well. Call the logics with this property the WCP-*logics* over S4.

Given $i, j, n \in N$ with $n > 1$ and $1 \le i, j \le n$, define the formulas

$$\pi_i := p_i \wedge \bigwedge_{j\ne i} \neg p_j; \quad A_n := \bigwedge_{1\le i\le n} \Diamond \pi_i;$$

$$A_{n,1} := \Box\Big[\bigwedge_{1\le i\le n} (p_i \to \neg\Diamond q)\Big]; \quad B := q \vee \neg\Diamond q.$$

Also define the sequence of inference rules

$$\mathscr{R}_1 := \frac{\Diamond p \wedge \Diamond\neg p}{p \wedge \neg p}; \quad \mathscr{R}_n := \frac{\Box\big(A_{n,1} \wedge \neg(A_n \wedge B)\big)}{\Box\neg A_n}.$$

Lemma 1. *The rule $\mathscr{R}_1$ is admissible in every residually finite modal logic $\lambda \supseteq$ S4.*

Proof. The claim is almost obvious. By the construction of the n-characteristic model $C_k(\lambda)$, its first slice contains degenerate singleton clusters. For each evaluation of a variable p on the elements of the first slice generating degenerate clusters, either p or $\neg p$ is true. Accordingly, on such elements the premise of the rule $\mathscr{R}_1$ does not hold, which implies that this inference rule is admissible in λ. $\Box$

Observe also that the premise of $\mathscr{R}_1$ is satisfiable (for instance, on proper clusters), but not unifiable, meaning that it does not become a theorem of the logic upon any substitution. Thus, such a rule is useless in the proof; we refer to the rules with nonunifiable premise as *passive*. The rule $\mathscr{R}_1$ constitutes a basis for passive inference rules; see [13, Theorem 3.4].

Theorem 3.1. *The rules $\mathscr{R}_n$, for $n > 1$, are admissible in every residually finite logic λ extending S4 and possessing the weak co-covering property.*

Proof. Suppose that for some n the inference rule $\mathscr{R}_n$ is not admissible in λ. Then by Proposition 2.4 there exists a definable evaluation V of the variables of $\mathscr{R}_n$ for which $\mathscr{R}_n$ is refuted on some k-characteristic model $C_k(\lambda)$. Hence,

$$C_k(\lambda) \Vdash_V \Box(A_{n,1} \wedge \neg(A_n \wedge B))\& C_k(\lambda) \nVdash_V \Box\neg A_n. \tag{1}$$

Consequently, there exists $a \in C_k(\lambda)$ with $a \nVdash_V \Box\neg A_n$, which yields $\exists a_1 : aRa_1\&a_1 \Vdash_V A_n$. Then there are $b_1, \dots, b_n \in C_k(\lambda)$ such that $a_1Rb_i\&b_i \Vdash_V \pi_i$. By the weak co-covering property, there exists

a reflexive element $b \in C_k(\lambda)$ which is a co-covering for the set $\{C(b_1), C(b_2), \dots, C(b_n)\}$ of R-minimal clusters generated by $b_1, \dots, b_n$.

The choice of b implies that $b \Vdash_V A_n$. By (1) we have $b \Vdash_V A_{n,1}$. Since b is a co-covering for $\{b_1, \dots, b_n\}$, it is easy to verify that the formula B holds on b for the evaluation V. Indeed, $b_i \Vdash_V p_i$ and by (1) we have $b_i \Vdash_V A_{n,1}$, which yields $b_i \Vdash_V \neg\Diamond q$ for all $i \leq n$. Hence, we infer that $b \models_V q$ or $b \models_V \neg q$ implies $b \models_V \neg\Diamond q$. Thus, $b \Vdash_V A_n \wedge B$ holds, which contradicts $b \Vdash_V \Box\neg(A_n \wedge B)$ by (1).

If all elements $b_1, \dots, b_n \in Ch_k(\lambda)$ lie in the same cluster then we can treat each of them as a covering for the remaining ones, and the formula $(A_n \wedge B)$ also holds on it, which contradicts the assumption. $\Box$

Consider some logic λ extending S4 and satisfying the following

(I) λ is residually finite;

(II) it has weak co-covering property;

(III) a rule r in reduced form is inadmissible in λ if and only if there exists a λ-model $\mathscr{M} = \langle F, V\rangle$ such that

(i) $\forall x \in F x \models_V \bigvee \phi_j$;

(ii) $\exists y \in \mathscr{F} y \not\models_V \Box x_0$;

(iii) $\forall \mathscr{D} \subseteq F \exists e \in F e \models_V \phi_e, \phi_e \in \mathrm{Pr}(r), \& \theta_2(\phi_e) = \theta_1(\phi_e) \cup \bigcup_{z\in\mathscr{D}}(\theta_1(\phi_z) \cup \theta_2(\phi_z))$.

Condition (III) is an analog of admissibility criterion for a rule in reduced form. For the logics K4, S4, Grz, and GL the corresponding theorems can be found in [7, Chapter 3.9] for instance. If the premise of a rule r is true on some model $\mathscr{M} = \langle F, V\rangle$ then only one formula ϕ_j from the premise holds on each element of this model. Indeed, the definition of ϕ_j yields $\theta_1(\phi_j) \cup \theta_3(\phi_j) = \theta_2(\phi_j) \cup \theta_4(\phi_j) = \mathrm{Var}(r) = \{x_0, \dots, x_k\}$. For a prescribed evaluation, the sets $\theta_i(\phi_j)$ are uniquely determined for each element of the model.

Recall the definition of wrapping algebra; see Definitions 2.5.1 and 2.5.3 of [7]. Given a frame $\mathscr{F} = \langle F, R\rangle$, the modal algebra $\mathscr{F}^+$ such that

(1) $\langle 2^F, \wedge, \vee, \to, \neg, \bot, \top\rangle$ is the Boolean algebra of all subsets of F,

(2) $\forall X \subseteq F(\Box X = \{a : \forall y(aRy) \Longrightarrow (y \in X)\})$

is *associated* to $\mathscr{F}$.

Similarly, for a model $\mathscr{M} = \langle F, R, V\rangle$ the algebra $\mathscr{M}^+$ generated by the set of elements $\{V(p)\}$ of $\langle F, R\rangle^+$ is *associated* to $\mathscr{M}$.

Verify now that all admissible inference rules of every logic λ satisfying conditions (I)–(III) can be deduced in this logic from the tuple $\{\mathscr{R}_n\}$ of rules.

Theorem 3.2. *Suppose that a modal logic λ extending S4 satisfies conditions* (I)–(III). *Then every admissible rule r of λ in reduced form can be deduced from the rules $\{\mathscr{R}_n,\ n \in N\}$.*

Proof. Assume on the contrary that some rule r admissible in λ cannot be deduced from $\{\mathscr{R}_n, n \in N\}$. Then by Theorem 1.4.11 of [7] there is a λ-algebra $\mathscr{B} \in \mathrm{Var}(\lambda)$ which distinguishes these rules: $\forall n \mathscr{B} \models \mathscr{R}_n$ but $\mathscr{B} \not\models r$.

Let us establish an auxiliary statement. Consider the modal algebra

$$\mathscr{A} := \mathscr{F}^+(V(x_0), V(x_1), \dots, V(x_k))) \in \mathrm{Var}(\lambda)$$

generated by the set $(V(x_0), V(x_1), \dots, V(x_k)) \subseteq \mathscr{F}$ of subsets of the wrapping algebra $\mathscr{F}^+$, where $\mathscr{F}$ is a given reflexive and transitive λ-frame. Take an inference rule r in reduced form.

Lemma 2. *If r is admissible in some logic λ satisfying conditions* (I)–(III) *and is refuted on the algebra $\mathscr{A} \in \mathrm{Var}(\lambda)$ then for some n the rule $\mathscr{R}_n$ is also refuted on $\mathscr{A}$.*

Proof. Take a rule r in reduced form admissible in some logic λ:

$$r := \bigvee_{1\leq j\leq t} \phi_j / \Box x_0,$$

where

$$\phi_j := \bigwedge_{0\le i\le k} x_i^{a_i} \wedge \bigwedge_{0\le i\le k} \Diamond x_i^{b_i}, \quad a_i, b_i \in \{0,1\},\ x^0 := x,\ x^1 := \neg x.$$

Suppose that r is refuted on the algebra $\mathscr{A} \in \mathrm{Var}(\lambda)$ for some evaluation $V(x_i) := \mathscr{Y}_i \in \mathscr{A}$. Since r is refuted on $\mathscr{A}$, it follows that

$$\mathscr{F} \models_V \bigvee_{1\le j\le t} \phi_j; \exists\, b \in \mathscr{F} : b \not\models_V \Box x_0.$$

Consider the algebra $(b^R)^+$ generated the by frame b^R and its subalgebra

$$\mathscr{B} := (b^R)^+(V(x_0), \ldots, V(x_k))$$

generated by the set of elements $V(x_0), \ldots, V(x_k)$. Since r is refuted on b^R, we see that $\mathscr{B} \not\models_V r$.

Now apply condition (III) met by λ. Since r is admissible in λ and conditions (i), (ii) hold on b^R, we infer that condition (iii) is violated:

$$\begin{aligned} &\exists \mathscr{G} \subseteq b^R \forall e \in F : e \models_V \phi_e \& \phi_e \in \mathrm{Pr}(r) \\ &\Longrightarrow \theta_2(\phi_e) \neq \theta_1(\phi_e) \cup \bigcup \{(\theta_1(\phi_z) \cup \theta_2(\phi_z)) : z \in \mathscr{G} \& z \models_V \phi_z \& \phi_z \in \mathrm{Pr}(r)\}. \end{aligned} \tag{$*$}$$

Take an antichain X consisting of R-minimal elements of $\mathscr{G}$, i.e., $X^R = \mathscr{G}^R \cup X$. Then X is nontrivial and lacks co-coverings in $\mathscr{F}$. Otherwise, if there is $c \in b^R$ with $c^R = X^R$ or $c^R = X^R \cup \{c\}$ then condition $(*)$ is violated for the formula $\phi_c \in \mathrm{Pr}(r) : c \models_V \phi_c$.

Consider two sets $\mathscr{X}$ and $\mathscr{Y}$ consisting of all disjunctive terms of the premise of r with nonempty truth sets in b^R and X^R respectively:

$$\mathscr{X} := \{\phi_j \mid \exists c \in b^R : c \models_V \phi_j\}, \quad \mathscr{Y} := \{\phi_j \mid \exists e \in X^R : e \models_V \phi_j\}.$$

Define the set $\mathscr{D}$ of disjuncts of the premise of r true on the elements of the antichain $X \subset b^R$, i.e.,

$$\mathscr{D} := \{\phi_s \mid \exists e \in X e \models_V \phi_s\},$$

and the set of disjuncts $\mathfrak{B}$ true on $\mathscr{F}$:

$$\mathfrak{B} := \{\phi_s \mid \exists e \in \mathscr{F} e \models_V \phi_s \& \phi_s \in \mathrm{Pr}(r)\}.$$

Since the fixed antichain $X \subset b^R$ lacks a co-covering in $\mathscr{F}$, for all $\phi_j \in \mathfrak{B}$ we find that

$$\theta_2(\phi_j) \neq \theta_1(\phi_j) \cup \bigcup_{\phi\in\mathscr{D}} (\theta_1(\phi) \cup \theta_2(\phi)). \tag{2}$$

Denote by n the cardinality of $\mathscr{D}$, i.e., $n := |\mathscr{D}|$. Since the antichain $X \subset b^R$ is nontrivial, it is clear that $n > 1$. Put

$$P_V := \mathrm{Var}(r) = \{x_0, \ldots, x_k\};$$

$$P_T := \mathrm{Var}\Big(\bigcup_{x\in\mathscr{D}} (\theta_1(\phi_x) \cup \theta_2(\phi_x))\Big) = \{p \mid \exists c \in X : c \models_V p \vee c \models_V \Diamond p\}.$$

Fix a bijective correspondence f between $\{p_1, p_2, \ldots, p_n\}$ and $\mathscr{D}$. Extend the evaluation V to $\mathscr{B}$ from the variables of r to the variables of $\mathscr{R}_n$ as

$$V(p_i) := V(f(p_i)) \quad \& \quad V(q) := V(P_V - P_T). \tag{3}$$

Proposition 3.3. *The rule $\mathscr{R}_n$ is refuted on the algebra $\mathscr{B}$ for the evaluation V.*

Proof. Since the element b belongs to the R-minimal cluster for the frame generating $\mathscr{B}$ and $\forall x \in X$ we have $x \models_V \pi_i$ for some i, it follows that

$$b \models_V \bigwedge_{1 \le i \le n} \Diamond \pi_i,$$

i.e., $b \models_V A_n$. Hence, we conclude that $\mathscr{B} \not\models_V \Box\neg A_n$.

Take $c \in b^R$ and suppose that $c \models_V p_i$. Then $c \models_V f(p_i)$ and therefore $c \models_V \phi_i$ for $\phi_i \in \mathscr{D}$. Hence, by the choice of P_T, we infer from $c \models_V \phi_i$ for $\phi_i \in \mathscr{D}$ that $c \models_V \neg\Diamond q$. Therefore, $c \models_V A_{n,1}$, which yields $b \models_V \Box A_{n,1}$ because $c \in b^R$ is arbitrary.

Suppose that $c \models_V A_n$. Moreover, suppose that $c \models_V \phi_c$ for $\phi_c \in \mathscr{Z}$. Then $c \models_V A_n \iff c \models_V \Diamond\phi_j$ for all $\phi_j \in \mathscr{D}$. Consequently, $c \models_V \Diamond\phi_j, \forall\phi_j \in \mathscr{D}$. Hence, $P_T \subseteq \theta_2(\phi_c)$ and $c \not\models_V q$ by the choice of P_T. Suppose also that $c \models_V \neg\Diamond q$. Then $\forall\, e \in c^R e \not\models q$. This implies that $c^R \cap V(P_V - P_T) = \varnothing$. Thus, $\theta_2(\phi_c) \subseteq P_T$.

Combining the above, we obtain $\theta_2(\phi_c) = P_T$; i.e.,

$$\theta_2(\phi_c) = \theta_1(\phi_c) \cup \bigcup_{\phi \in \mathscr{D}} (\theta_1(\phi) \cup \theta_2(\phi))$$

and $\phi_c \in \mathscr{Z}$, which contradicts (2). $\Box$

Extend the evaluation V on the algebra $\mathscr{A}$ with the variable q of $\mathscr{R}_n$ so that to refute $\mathscr{R}_n$ on $\mathscr{A}$.

Lemma 3. $\mathscr{A} \not\models_V \mathscr{R}_n$.

Proof. The nontrivial antichain $X \subset b^R$ fixed above is a set of clusters: $X = \{C_1, C_2, \ldots, C_n\}$. Since the premise of the rule is true on the whole frame $\mathscr{F}$, for each $t \in \mathscr{F}$ there is a unique formula ϕ_t of the premise such that $t \models_V \phi_t$ holds.

On the frame $\mathscr{F} \setminus b^R$ extend the evaluation of the variables of $\mathscr{R}_n$ as follows: Put

$$X^{-R} = \{x : xRC_1 \& xRC_2 \& \cdots \& xRC_n\}$$

and

$$\begin{aligned} V(q) &:= \{y \in \mathscr{F} \setminus X^R : y \notin X^{-R} \& \exists x \in X^{-R}(xRy)\} \\ &= V\Big(\neg \bigwedge_{\phi_j \in \mathscr{D}} \Diamond\phi_j \wedge \neg\Box\Big(\bigvee_{\phi_j \in \mathscr{Y}} \phi_i\Big) \wedge \bigvee\Big\{\phi_y : \exists\phi_z\Big(z \models_V \bigwedge_{\phi_j \in \mathscr{D}} \Diamond\phi_j \Longrightarrow z \models_V \Diamond\phi_y\Big)\Big\}\Big). \end{aligned}$$

Verify that for this evaluation V the rule $\mathscr{R}_n$ is refuted on $\mathscr{F}$.

The definition of V implies directly that

$$\forall e \in X^R e \models_V \neg\Diamond q; \quad \forall x \in \mathscr{F}(x \models_V p_i \iff x \in X).$$

By the definition of the evaluation of p_i and q, it is obvious that $\forall x \in \mathscr{F}$ we have $x \not\models_V \bigvee_{1 \le i \le n} p_i \wedge \Diamond q$. Considering that

$$\Box A_{n,1} = \Box \bigwedge_{1,n} (p_i \to \neg\Diamond q) = \Box \bigwedge_{1,n} (\neg p_i \vee \neg\Diamond q) = \Box\neg\Big[\bigvee_{1,n} (p_i \wedge \Diamond q)\Big],$$

we obtain $\forall x \in \mathscr{F}(x \models_V \Box A_{n,1})$.

Suppose that $z \models_V A_n$ for some element $z \in \mathscr{F}$, i.e., the antichain $X = \{C_1, C_2, \ldots, C_n\}$ is R-attainable from z and the cluster $C(z)$ is not a co-covering for X; hence, $z \in \mathscr{X}^{-R}$. Consequently, $z \models_V \neg q$ holds by the definition of $V(q)$. Moreover, there is $y \in \mathscr{F}$ with $y \notin X^{-R}$, $y \notin X^R$, and zRy.

Indeed, if the cluster $C(z)$ is an immediate R-predecessor for X, meaning that the depth $d(C(z))$ of $C(z)$ is $\max_{i\in X} d(i) + 1$, then there must exist at least one element y with zRy such that $C(y) \cup \mathscr{X}$ forms an antichain for which $C(z)$ is a co-covering. (There may be several $y_1, \dots, y_k$ with this property: $C(y_1) \cup \dots \cup C(y_k) \cup \mathscr{X}$ forms an antichain.) Then y satisfies $y \notin \mathscr{X}^{-R}$ and $y \notin \mathscr{X}^R$, as well as zRy.

If the cluster $C(z)$ is not an immediate R-predecessor for X then from $C(z)$ either some R-predecessor $C(z_1)$ for X is attainable or some elements $z_1, z_2, \dots, z_k$ are R-attainable, which are immediate R-predecessors for subsets of X, and $X \subseteq z_1^R \cup \dots \cup z_k^R$ holds. In the first case, as above, we infer that there exists y with the required properties. In the second case as y we can take z_1, for instance, also with the required properties.

Therefore, the required $y \in \mathscr{F}$ exists and satisfies $y \notin \mathscr{X}^{-R}$, $y \notin \mathscr{X}^R$, and zRy. In this event $y \models_V q$. Hence, $z \models_V \neg q \wedge \Diamond q$, which yields $z \models_V \neg(A_n \wedge B)$.

We showed that for this definition of evaluation the premise of the rule is true at every element of $\mathscr{F}$. Since b is an R-predecessor of the antichain $\mathscr{X} = \{C_1, C_2, \dots, C_n\}$ and $\forall x \in C_i$ we have $x \models_V p_i$ and $x \not\models p_j$ for $i \neq j$, we infer that $b \models_V \bigwedge_{1\leq i\leq n} \Diamond\pi_i$ is satisfied; i.e., $b \models_V A_n$. Hence, $b \not\models_V \Box\neg A_n$, which proves that $\mathscr{R}_n$ is refutable on $\mathscr{F}$ for this evaluation V.

The proof of Lemma 3 is complete. $\square$

Thus, as soon as a rule r admissible in the logic λ is refuted on $\mathscr{A}$, so does one of the rules $\mathscr{R}_n$ of the collection $\{\mathscr{R}_n,\, n \in N\}$. This completes the proof of Theorem 3.2. $\square$

Theorem 3.4. *If a modal logic* λ ($\supseteq S4$) *meets conditions* (I)–(III) *then the set* $\{\mathscr{R}_n,\, n \in N,\, n > 1\}$ *of rules constitutes an independent basis for the admissible inference rules of* λ.

Proof. Theorems 3.1 and 3.2 imply that $\{\mathscr{R}_n,\, n \in N\}$ constitutes a basis for the admissible rules. Let us verify its independence.

As [19] shows, the finite set $\{\mathscr{R}_n,\, n \in N,\, n \leq L\}$ of rules constitutes a basis for the admissible rules of the residually finite logics of width L. Therefore, consider only the WCP-logics of unbounded width.

Fix an arbitrary positive integer $n > 1$. Define some λ-frame $\mathscr{F}$ distinguishing the rules $\mathscr{R}_n$ as follows. The first slice consists of the unique reflexive element: $S_1(\mathscr{F}) := \{a_0\}$ with a_0Ra_0. The second slice of this frame amounts to an antichain of t reflexive elements: $S_2(\mathscr{F}) := \{a_1, a_2, \dots, a_t\}$ for $\forall i \leq t(a_iRa_0 \& a_iRa_i)$, where $t > n$. Choose the smallest t for which the antichain $\{a_1, a_2, \dots, a_t\}$ of all elements in the second slice has a λ-co-covering b_0. If this value of t fails to exist then the logic is of finite width, which is impossible by assumption. Fix a nontrivial antichain $X = \{a_1, a_2, \dots, a_n\} \subseteq S_2(\mathscr{F})$.

To construct the third slice, we choose all antichains, including the trivial ones, distinct from the fixed antichain $\{a_1, a_2, \dots, a_n\}$, and to each of them adjoin from below a reflexive element as a λ-co-covering; i.e., when this element generates a λ-frame as the root. Assume that the attainability relation is transitive. Observe that if some co-covering $b_0 \in S_3(\mathscr{F})$ for the antichain $\{a_1, a_2, \dots, a_t\}$ generates a λ-frame as the root then the co-covering for each of its subsets also generates a λ-frame.

Suppose now that $S_{\leq k}(\mathscr{F})$ of depth at most k, where $3 \leq k$, is already constructed. Construct the slice $S_{k+1}(\mathscr{F})$ of depth $k+1$ as follows. Choose all antichains of clusters $\mathscr{X} \subset S_{\leq k}(\mathscr{F})$, including the trivial ones, which contain at least one cluster of depth k and are distinct from the antichain $\{a_1, a_2, \dots, a_n\}$. Then adjoin from below to each of them a singleton cluster when it generates a λ-frame as the root. Continuing this procedure for subsequent slices, we obtain a λ-frame $\mathscr{F}$ in which every antichain distinct from $\{a_1, a_2, \dots, a_n\}$ has a λ-co-covering, and this antichain itself has an R-predecessor.

Put $X^{-R} = \{z \mid zRa_1 \& zRa_2 \& \cdots \& zRa_n\}$; i.e., X^{-R} is the set of elements from which the whole fixed antichain $\{a_1, a_2, \dots, a_n\}$ is attainable (its lower cone). Define on $\mathscr{F}$ the evaluation V refuting the rule $\mathscr{R}_n$ as

$$V(p_i) := a_i, a_i \in S_2(\mathscr{F}),\ 1 \leq i \leq n; \quad \forall i > nV(p_i) = \varnothing;$$

$$V(q) := \{a_{n+1}, \dots, a_t\} \cup (S_3(\mathscr{F}) \setminus X^{-R}).$$

Observe that the premise of $\mathscr{R}_1$ does not hold at the element a_0 of the first slice; i.e., the rule is true on $\mathscr{F}$. By analogy with the proof of Lemma 3, it is easy to show that $\mathscr{R}_n$ is refuted on $\mathscr{F}$ for this

evaluation.

Indeed, by the definition of V we have $\forall a_i \in X a_i \models_V p_i \& a_i \not\models_V p_j$ for $i \neq j$, and so $a_i \models_V \pi_i$. Then the co-covering of the antichain of all elements $\{a_1, a_2, \dots, a_t\}$ of the second slice, which is b_0, satisfies $b_0 \not\models_V \Box\neg A_n$; i.e., the conclusion of the rule is refuted for this evaluation.

Verify that the premise is true on the whole frame $\mathscr{F}$. The definition of $V(q)$ yields $\forall i \leq n\, a_i \not\models_V \neg\Diamond q$. Then $\forall x \in \mathscr{F}\ x \not\models_V \bigvee_{1\leq i\leq n} p_i \wedge \Diamond q$ obviously holds. Since

$$\Box A_{n,1} = \Box \bigwedge_{1,n}(p_i \to \neg\Diamond q) = \Box \bigwedge_{1,n}(\neg p_i \vee \neg\Diamond q) = \Box\neg\Big[\bigvee_{1,n}(p_i \wedge \Diamond q)\Big],$$

we infer that $\forall x \in \mathscr{F} x \models_V \Box A_{n,1}$.

Now take $z \in \mathscr{F}$ with $z \models_V A_n$. Then $z \in X^{-R}$ and the definition of $V(q)$ yields $z \not\models_V q$. Since the cluster $C(z)$ is not a co-covering for X, no $C(z)$ does exist in $\mathscr{F}$, but the whole antichain is attainable from $C(z)$, it follows that there is $y \in \mathscr{F}$ with $y \notin X^R$ and $y \notin X^{-R} \& zRy$.

Indeed, if z belongs to $S_3(\mathscr{F})$ and is an R-predecessor for the fixed antichain X, then at least one of the elements $\{a_{n+1}, \dots, a_t\} \subseteq S_2(\mathscr{F} \setminus X)$ is also attainable from z. Then by the definition of $V(q)$ we conclude that $z \models_V \Diamond q$, and so $z \models_V \neg q \wedge \Diamond q$, which implies the truth of $z \models_V \neg(A_n \wedge B)$.

However, if the depth of z is strictly greater than 3; i.e., the cluster $C(z)$ is not an immediate R-predecessor of X, then either there is some cluster $z_1 \in S_3(\mathscr{F})$ which is R-attainable from z_1 and is an immediate R-predecessor of X, but not a co-covering, or there are elements $z_1, z_2, \dots, z_k \in S_3(\mathscr{F} \setminus X^{-R})$, where $k > 1$, which are R-attainable from it and are immediate R-predecessors of a proper subset of X, and we have $X \subseteq z_1^R \cup \dots \cup z_k^R$. In the first case there is $y \in \mathscr{F}$ with $z_1 R y$ and the required properties $y \notin X^R$ and $y \notin X^{-R} \& zRy$; then zRy by transitivity. In the second case we can take z_1 as such element, which also has the required properties. Once again by the definition of $V(q)$ we conclude that $z \models_V \Diamond q$, which implies the truth of $z \models_V \neg(A_n \wedge B)$.

Suppose now that for $k \neq n$ the rule $\mathscr{R}_k$ is refuted on $\mathscr{F}$; i.e., there is an element $c \in \mathscr{F}$ satisfying $c \not\models_V \Box\neg A_k$. For $k > n$ this is impossible because $c \models_V \Diamond p_j$ with $j > n$ cannot hold, and therefore the conclusion of the rule $\mathscr{R}_k$ is valid for this evaluation.

Consider the case $k < n$. Then there are $a_1, \dots, a_k \in S_2(\mathscr{F})$ such that $cRa_j \& a_j \models_V \pi_j$. By the construction of $\mathscr{F}$ the antichain $\{a_1, \dots, a_k\}$ has a co-covering z in $\mathscr{F}$ because it is distinct from the antichain $\{a_1, a_2, \dots, a_n\}$ fixed above and the logic has the weak co-covering property. It is easy to verify as in the proof of Theorem 3.2 that on this co-covering z the premise of the rule $\mathscr{R}_k$ does not hold. Since z is a co-covering for the antichain $\{a_1, \dots, a_k\}$, where $k < n$ and $a_j \models_V \pi_j$; we have $z \models_V A_k$. Since $a_i \models_V \neg\Diamond q$, it is easy to verify that the truth or falsehood of q on z implies the satisfiability of the formula B of the premise of the rule, which yields $z \models_V (A_n \wedge B)$. Consequently, the premise of the rule is refuted on z, which also implies the truth of $\mathscr{R}_k$. Thus, $\mathscr{R}_k$ is true on $\mathscr{F}$ for $k \neq n$. $\Box$

From this we directly obtain the following:

Corollary 1. *The set $\{\mathscr{R}_n, n \in N\}$ of rules constitutes an independent basis for the admissible inference rules of the logics S4, S4.1, S4.2, Grz, and Grz.2.*

The definitions (axiomatics) of these logics and descriptions of the characteristic classes of frames can be found in [7, Chapter 2] for instance. The residual finiteness of these logics was proved in Theorems 2.6.12, 2.6.25, and 2.8.11 of [7]. In order to verify the remaining hypotheses of the theorem, it is important to note that each nontrivial antichain of elements of an n-characteristic model, or its components in the case of the logics S4.2 and Grz.2, has a reflexive co-covering. Consequently, the adjunction of a reflexive element as a co-covering to an arbitrary antichain of elements of an arbitrary root frame adequate for the logics on this list also produces a frame adequate to the logic, i.e., these logics have the weak co-coverings property. Condition (3), which is an admissibility criterion for rules in reduced form, for the logics S4 and Grz was justified in Theorems 3.9.6 and 3.9.9 of [7]. For the remaining logics the presence of all possible co-coverings enables us to reproduce the proof of this criterion in the simpler situations; i.e., condition (III) holds for these logics as well.

4. A Basis for Admissible Inference Rules in WCP-Logics over GL

Say that some logic λ extending the logic GL has *weak co-covering property* over GL whenever, given a finite root λ-frame $\mathscr{F}$ and a nontrivial antichain $\mathscr{X}$ of clusters in $\mathscr{F}$, the frame $\mathscr{F}_1$ obtained by adjoining a singleton irreflexive co-covering as the root to the frame $\bigcup_{c\in(\mathscr{X}^R\cup\mathscr{X})} c^R$ is also a λ-frame. Call the logics with this property the WCP-*logics* over GL.

Put $\Box_0\alpha := \alpha \wedge \Box\alpha$ and $\Diamond_0 := \alpha \vee \Diamond\alpha$. Given $i, j, n \in N$ with $n > 1$ and $1 \leq i, j \leq n$, define the formulas

$$\pi_i := p_i \wedge \bigwedge_{j\neq i} \neg p_j; \quad A_n := \bigwedge_{1\leq i\leq n} \Diamond\pi_i;$$

$$A^{ir}_{n,1} := \Box_0\Big[\bigwedge_{1\leq i\leq n}(p_i \to \neg\Diamond_0 q)\Big]; \quad B^{ir} := \neg\Diamond q.$$

Given $n \in N$ with $n > 1$, also define the sequence of inference rules

$$\mathscr{R}^{ir}_n := \frac{\Box_0\big(A^{ir}_{n,1} \wedge \neg(A_n \wedge B^{ir})\big)}{\Box_0\neg A_n}.$$

Consider some logic λ extending GL and enjoying the following:

(a) λ is residually finite;

(b) has the weak co-covering property over GL;

(c) a rule r in reduced form is inadmissible in λ if and only if there exists a λ-model $\mathscr{M} = \langle F, V\rangle$ such that:

(i) $\forall x \in F x \models_V \bigvee \phi_j$;

(ii) $\exists y \in \mathscr{F} y \not\models_V \Box x_0$;

(iii) $\forall \mathscr{D} \subseteq F \exists e \in F e \models_V \phi_e, \phi_e \in \Pr(r), \& \theta_2(\phi_e) = \bigcup_{z\in\mathscr{D}}(\theta_1(\phi_z) \cup \theta_2(\phi_z))$.

Verify that the sequence $\{\mathscr{R}^{ir}_n,\ n \in N,\ n > 1\}$ of inference rules constitutes an independent basis for the admissible rules of the logic λ extending GL and satisfying conditions (a)–(c). Since the proof essentially repeats that for the extensions of S4 in the simpler case, below we just outline it wherever that should not hamper understanding.

Theorem 4.1. *The rules $\mathscr{R}^{ir}_n$ for $n > 1$ are admissible in every residually finite logic λ extending GL and satisfying the weak co-covering property over GL.*

Proof. Demonstration repeats essentially that of Theorem 3.1. If the inference rule $\mathscr{R}^{ir}_n$ is inadmissible in λ; i.e., there exists $a \in Ch_k(\lambda)$ with $a \not\Vdash_V \Box_0\neg A_n$, then there are $b_1, \dots, b_n \in Ch_k(\lambda)$ satisfying $aRb_i \& b_i \Vdash_V \pi_i$. By the weak co-covering property, there exists an irreflexive element $b \in Ch_k(\lambda)$ which is a co-covering for the set $\{b_1, \dots, b_n\}$. The choice of b yields $b \Vdash_V A_n$. By the truth of the premise of the rule, we have $b \Vdash_V A^{ir}_{n,1}$. Since b is a co-covering for $\{b_1, \dots, b_n\}$; it is easy to verify that $(A_n \wedge B^{ir})$ holds on b for the evaluation V, which is a contradiction. $\Box$

Theorem 4.2. *If a modal logic λ ($\supseteq$ GL) meets conditions* (a)–(c) *then every admissible rule r of λ in reduced form can be deduced from the rules $\{\mathscr{R}^{ir}_n,\ n \in N,\ n > 1\}$.*

Proof. Demonstration is similar to that of Theorem 3.2. Hence, in this case we outline the argument, omitting some details. Assume on the contrary that some rule r admissible in λ cannot be deduced from the rules $\mathscr{R}^{ir}_n$, where $n \in N$ with $n > 1$. Then there is a λ-algebra $\mathscr{B} \in \mathrm{Var}(\lambda)$ that distinguishes these rules: $\forall n \mathscr{B} \models \mathscr{R}^{ir}_n, \mathscr{B} \not\models r$.

Consider the modal algebra

$$\mathscr{A} := \mathscr{F}^+(V(x_0), V(x_1), \dots, V(x_k))) \in \mathrm{Var}(\lambda)$$

generated by the set of subsets $(V(x_0), V(x_1), \dots, V(x_k)) \subseteq \mathscr{F}$ of the wrapping algebra $\mathscr{F}^+$ defined above where $\mathscr{F}$ is a given irreflexive and transitive λ-frame.

Lemma 4. *If a rule r in reduced form is admissible in some logic λ satisfying conditions* (a)–(c) *and refuted on the algebra $\mathscr{A} \in \mathrm{Var}(\lambda)$ then for some n the rule $\mathscr{R}_n^{ir}$ is also refuted on $\mathscr{A}$.*

Proof. Take a rule r in reduced form refuted on the algebra $\mathscr{A} \in \mathrm{Var}(\lambda)$ for some evaluation $V(x_i) := \mathscr{Y}_i \in \mathscr{A}$. Since r is refuted on $\mathscr{A}$, it follows that

$$\mathscr{F} \models_V \bigvee_{1 \le j \le t} \phi_j; \exists\, b \in \mathscr{F} : b \not\models_V \Box x_0.$$

As in the reflexive case, only one formula from the premise holds on each element $x \in \mathscr{F}$. Consider the algebra $(b^R)^+$ generated by the frame b^R and its subalgebra $\mathscr{B} := (b^R)^+(V(x_0), \dots, V(x_k))$ generated by the set $V(x_0), \dots, V(x_k)$. Since r is refuted on b^R, it follows that $\mathscr{B} \not\models_V r$.

Property (c) satisfied by λ shows that condition (iii) is violated on b^R:

$$\exists \mathscr{G} \subseteq b^R \forall e \in F : e \models_V \phi_e \& \phi_e \in \Pr(r)$$
$$\Longrightarrow \theta_2(\phi_e) \neq \bigcup \{(\theta_1(\phi_z) \cup \theta_2(\phi_z)) : z \in \mathscr{G} \& z \models_V \phi_z \& \phi_z \in \Pr(r)\}. \tag{$**$}$$

Take some antichain X consisting of R-minimal elements of $\mathscr{G}$, i.e., $X^R = \mathscr{G}^R \cup X$. Then X is nontrivial and lacks a co-covering in $\mathscr{F}$. Otherwise, if there is $c \in b^R$ for which either $c^R = X^R \cup X$ or $c^R = X^R \cup \{c\}$ then condition $(**)$ is not satisfied for the formula $\phi_c \in \Pr(r) : c \models_V \phi_c$.

As above, define the sets of disjuncts of the premise of the rule:

$$\mathscr{Z} := \{\phi_j \mid \exists c \in b^R \cup b : c \models_V \phi_j\}; \quad \mathscr{Y} := \{\phi_j \mid \exists e \in X^R \cup X : e \models_V \phi_j\};$$
$$\mathscr{D} := \{\phi_s \mid \exists e \in X e \models_V \phi_s\}; \quad \mathfrak{B} := \{\phi_s \mid \exists e \in \mathscr{F} e \models_V \phi_s \& \phi_s \in \Pr(r)\}.$$

Since the fixed antichain $X \subset b^R$ lacks a co-covering in $\mathscr{F}$, we infer that $\forall\, \phi_j \in \mathfrak{B}$ we have

$$\theta_2(\phi_j) \neq \bigcup_{\phi \in \mathscr{D}} (\theta_1(\phi) \cup \theta_2(\phi)). \tag{4}$$

As in the proof of Proposition 3.3 above, extend the evaluation V to the algebra $\mathscr{B}$ from the variables of r to the variables of $\mathscr{R}_n^{ir}$ as

$$V(p_i) := V(f(p_i)) \quad \& \quad V(q) := V(P_V - P_T). \tag{5}$$

Repeating the proof of Proposition 3.3 almost verbatim, we can justify the following:

Proposition 4.3. *The rule $\mathscr{R}_n^{ir}$ is refuted on the algebra $\mathscr{B}$ for the evaluation V.*

Extend the evaluation of the variable q so that the premise of the rule R_n^{ir} becomes true on $\mathscr{F} \setminus b^R$:

$$V(q) := \{y \in \mathscr{F} \setminus X^R : y \notin X^{-R} \& \exists x \in X^{-R}(xRy)\}$$
$$= V\Big(\neg \bigwedge_{\phi_j \in \mathscr{D}} \Diamond \phi_j \wedge \neg \Box_0 \Big(\bigvee_{\phi_j \in \mathscr{Y}} \phi_i \Big) \wedge \bigvee \Big\{ \phi_y : \exists \phi_z (z \models_V \bigwedge_{\phi_j \in \mathscr{D}} \Diamond \phi_j \Longrightarrow z \models_V \Diamond \phi_y \Big) \Big\} \Big).$$

Since b is an R-predecessor of the antichain X and $\forall e \in X e \models_V \pi_i$, we infer that $b \not\models_V \Box_0 \neg A_n$; i.e., the conclusion of the rule is refuted on $\mathscr{A}$. It remains to show by analogy with the proof of Lemma 3 that the following holds:

Proposition 4.4. *With the evaluation thus defined, the premise of R_n^{ir} is true on the algebra $\mathscr{A}$.*

Proof. Take $c \in \mathscr{F}$ with $c \models_V p_i$. Then $c \models_V f(p_i)$, which implies that $c \models_V \phi_i$ for $\phi_i \in \mathscr{D}$. In particular, $c \in X$. Then

$$c \not\models_V \bigwedge_{\phi_j \in \mathscr{D}} \Diamond \phi_j \& c \models_V \Box_0 \Big(\bigvee_{\phi_j \in \mathscr{Y}} \phi_i \Big)$$

by the choice of $\mathscr{D}$ and $\mathscr{Y}$, i.e., $c \not\models_V q$. If $e \in c^R$ then

$$e \models_V \Box_0 \Big(\bigvee_{\phi_j \in \mathscr{Y}} \phi_i \Big)$$

and so $e \not\models_V q$. Thus, $c \models_V \neg\Diamond_0 q$ and therefore $c \models_V A^{ir}_{n,1}$. For the remaining elements $c \notin X$ we have $\forall i\, c \not\models_V p_i$, i.e., $c \models_V A^{ir}_{n,1}$.

Now take $c \in \mathscr{F}$ with $c \models_V A_n$. Then the definition of evaluation yields

$$c \models_V \bigwedge_{\phi_j \in \mathscr{D}} \Diamond\phi_j,$$

whence $c \not\models_V q$ and $X \subseteq c^R$. Since c is not a co-covering of the antichain X and $X \subseteq c^R$, it follows that c is an R-predecessor of some element z co-covering a proper subset of X, and therefore

$$z \not\models_V \Box_0 \Big(\bigvee_{\phi_j \in \mathscr{Y}} \phi_i \Big) \& z \not\models_V \bigwedge_{\phi_j \in \mathscr{D}} \Diamond\phi_j,$$

or an R-predecessor of some element e, an R-maximal predecessor of X, from which some element $z \notin X^R$ is attainable; i.e.,

$$e \models_V \bigwedge_{\phi_j \in \mathscr{D}} \Diamond\phi_j \& \exists z \in e^R z \not\models_V \Box_0 \Big(\bigvee_{\phi_j \in \mathscr{X}} \phi_i \Big) \& z \not\models_V \bigwedge_{\phi_j \in \mathscr{D}} \Diamond\phi_j.$$

In both cases the definition of $V(q)$ yields $z \models_V q$. Therefore, $c \models_V \neg q \& c \models_V \Diamond q$ holds, i.e., $c \not\models_V B$.

Thus, we showed that for this definition of evaluation the premise of the rule is true on $\mathscr{A}$, which proves that $\mathscr{R}^{ir}_n$ is refutable on the algebra $\mathscr{A}$ for this evaluation V.

The proof of Proposition 4.4 is complete. $\Box$

Thus, as soon as a rule r admissible in the logic λ is refuted on $\mathscr{A}$, then so is one of the rules $\mathscr{R}^{ir}_n$, with $n \in N$. This completes the proof of Theorem 4.2. $\Box$

Theorem 4.5. *If a modal logic λ ($\supseteq GL$) meets conditions* (a)–(c) *then the set* $\{\mathscr{R}^{ir}_n,\ n \in N,\ n > 1\}$ *of rules constitutes an independent basis for the admissible inference rules of λ.*

PROOF. Theorems 4.1 and 4.2 imply that the set $\{\mathscr{R}^{ir}_n,\ n \in N\}$ of rules constitutes a basis for the admissible rules.

The proof of its independence almost completely repeats the proof of Theorem 3.4 with the difference that in the construction of the frame $\mathscr{F}_n$ distinguishing the rules we use irreflexive elements instead of reflexive ones. $\Box$

This directly implies the following corollary.

Corollary 2. *The set of rules $\{\mathscr{R}^{ir}_n,\ n \in N,\ n > 1\}$ constitutes an independent basis for the admissible inference rules of the logic GL.*

The definition (axiomatics) of this logic and a description of the characteristic classes of frames can be found in [7, Chapter 2] for instance. The residual finiteness was proved in Theorem 2.6.20 of [7]. In order to verify the remaining hypotheses of theorem, it is important to note that each nontrivial antichain of elements of an n-characteristic model has an irreflexive co-covering. Consequently, the adjunction of an irreflexive element as a co-covering to an arbitrary antichain of elements of an arbitrary root frame adequate to the logic GL also produces a frame adequate to the logic, i.e., this logic has the weak co-covering property. Condition (c), which is an admissibility criterion for rules in reduced form, for the logic GL was justified in Theorem 3.9.12 of [7].

5. Conclusion

This article presents explicit independent bases for the admissible inference rules of a broad class of extensions of the logics S4 and GL with the weak co-covering property. It is clear from the proofs that residual finiteness and the weak co-covering property are necessary, so far we have been unable to relax them. It seems that the only way to strengthen these results is to establish condition (3) for all extensions of the logics K4, S4, GL, and so on. The presence of an (explicit) finite basis of both intransitive logics, like K and T for instance, and logics without the weak co-covering property remains an open question.

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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