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ADMISSIBILITY AND UNIFICATION IN THE MODAL LOGICS RELATED TO S4.2

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Abstract—We study unification and admissibility for an infinite class of modal logics. Conditions superimposed to these logics are to be decidable, Kripke complete, and generated by the classes of rooted frames possessing the greatest clusters of states (in particular, these logics extend modal logic S4.2). Given such logic L and each formula α unifiable in L, we construct a unifier σ for α in L, where σ verifies admissibility in L of arbitrary inference rules α/β with a switched-modality conclusions β (i.e., σ solves the admissibility problem for such rules).

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1. Introduction

We study the unification problem and its usage for the admissibility problem in modal logics. As is well known, the unification problem was originated in computer science (it decides whether two given terms may be transformed into semantically equal ones). Later this problem were naturally generalized to considering semantically equivalent terms. To that time the area worked already well in automated deduction and verification of information (cf. [1–7]).

Usual semantic unification is, in fact, a particular case of the more complicated substitution problem (whether a formula can be made a theorem after replacing a part of variables, keeping the same value for coefficients–parameters). This problem was studied and solved (cf. [8–10] for the intuitionistic logic and modal logics S4 and Grz (but only to determine if a solution exits and to compute a particular one if yes).

The unification in the intuitionistic logic and propositional modal logics over K4 was studied by Ghilardi [11–15] (on using the ideas of projective algebras and the technique based at projective formulas for constructing the finite complete sets of unifiers). These all mentioned techniques brought some new instruments useful for studying the admissibility problem. In particular, these were later utilized and used by many researchers (cf., e.g., [16–23]).

Another area of interest in this direction was connected with the temporal logic (which is very active in information sciences and applications, cf., e.g., [24–29].

The solution for the admissibility problem for the linear temporal logic LTL itself was found in [30], the basis for the admissible rules of LTL was obtained in [31] (earlier the case of LTL with no Until was solved in [32]; the case of the linear temporal logic with future and past follows easily because we may model in this logic the universal modality (cf. [33]). The solution for the unifiability problem in LTL was found in [34], and the solution for unification with parameters in basic modal and intuitionistic logics was also suggested (cf., e.g., [35]).

However, the author in [34] states that not all formulas unifiable in LTL are projective. Though, to find an efficient technique based on projectivity for a good fragment of LTL seemed tempting. More motivation comes from Dzik and Wojtylak [36], where it was shown that any formula unifiable in the linear modal logic S4.3 is projective, which gives a hint that similar might be in a good fragment of LTL (at [36], it was fairly observed that ideas similar to projectivity for linear modal and intuitionistic logics were suggested already in [37, 38].

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In this paper we study unification and admissibility for an infinite class of modal logics L related in a sense to modal logic S4.2. These logics are assumed to have properties: to be decidable, be Kripke complete, and be generated by the classes of rooted frames possessing the greatest clusters of states (all these L extend modal logic S4.2). Given such logic L and a formula A, unifiable in L, we construct a unifier σ for A in L with the following property: This unifier σ verifies the admissibility in L of an arbitrary inference rule A/B with switch-modalized conclusion (that is the formula B has form $B = \Box\Diamond C$ or $B = \Diamond\Box C$). Thus, this unifier σ solves the admissibility problem for any such rule.

2. Definitions and Modal Logics

We will very briefly recall some general information needed in this paper (for history of modal logics or application of the linear temporal logic in computer science cf. [39]; actually we use here a rather restricted set of technical instruments). The language of any modal logic uses the language of Boolean logic extended by modal operations $\Diamond$ (possible) and $\Box$ (necessary). The modal formulas are built up from a set *Prop* of atomic (synonymously, propositional) letters and are closed under the applications of Boolean and modal operations.

Standard semantics for modal logics consists of frames $\mathscr{F}$ (sets W of possible states with binary accessibility relations R). Models are obtained from the frames by introducing valuations V for some chosen sets of propositional letters *Prop*. That is, V assigns truth values to letters from *Prop*. For any $p \in Prop$, $V(p) \subseteq W$, $V(p)$ is the set of all w from W where p is true (with respect to V). The triple $\mathscr{M} := \langle W, R, V\rangle$ is said to be a *Kripke model*.

Given a Kripke model $\mathscr{M}$, the truth values can be extended from the propositions of *Prop* to the arbitrary formulas constructed from these propositions as follows:

$\forall p \in Prop\ (M, a) \Vdash_V p \ \Leftrightarrow a \in W \wedge\ a \in V(p)$;
$(M, a) \Vdash_V (\varphi \wedge \psi) \ \Leftrightarrow (M, a) \Vdash_V \varphi \ \wedge\ (M, a) \Vdash_V \psi$;
$(M, a) \Vdash_V (\varphi \vee \psi) \ \Leftrightarrow (M, a) \Vdash_V \varphi \ \vee\ (M, a) \Vdash_V \psi$;
$(M, a) \Vdash_V \neg\varphi \ \Leftrightarrow not[(M, a) \Vdash_V \varphi]$;
$(M, a) \Vdash_V \Diamond\varphi \ \Leftrightarrow \exists b[(a\ R\ b) \& (M, b) \Vdash_V \varphi]$;
$(M, a) \Vdash_V \Box\varphi \ \Leftrightarrow \forall b[(a\ R\ b) \Rightarrow (M, b) \Vdash_V \varphi]$.

For a Kripke model $\mathscr{M} := \langle W, R, V\rangle$ and a formula φ with letters from the domain of V, φ *is true in* $\mathscr{M}$ (denotation $\mathscr{M} \Vdash \varphi$) if, for each state b from W, the formula φ is true at b (i.e., $(\mathscr{M}, b) \Vdash_V \varphi$). For a frame $\mathscr{F} := \langle W, R\rangle$, we say that a formula φ *is true at* $\mathscr{F}$, (and we will write $F \Vdash \varphi$) if φ is true at any model based at $\mathscr{F}$.

Definition 2.1. Given a class of frames K, the *modal logic generated by* K is the set $L(K)$ of all modal formulas true at every frame from K introducing any possible designation of propositional variables (the notation $L = L(K)$).

The logic $L(K)$ itself is *decidable* if for each formula we may decide if $\varphi \in L(K)$. A formula φ is *satisfiable* in $L(K)$ if there is a model $\mathscr{M}$ constructed on a frame from K, where φ is true at some state of $\mathscr{M}$.

The logic $L(K)$ *is decidable with respect to satisfiability* if a given satisfiable formula φ, we can effectively construct some model (finite and of computable size) for φ.

3. Unifiability and Preliminary Information

In this section we shortly survey some necessary notation and facts. Let For be the set of all formulas in the modal language, and let P be a set of letters. A substitution for P is a mapping $\varepsilon : P \longrightarrow For$. Any such substitution ε can be extended to the set of all formulas in letters from P by $\varepsilon(\varphi(x_1, \ldots, x_n))$ $:= \varphi(\varepsilon(x_1), \ldots, \varepsilon(x_n))$.

Definition 3.1. A formula φ is *unifiable* in a logic L if there is a substitution ε (which is called a *unifier* for φ) such that $\varepsilon(\varphi) \in L$.

We will use the following notation: Given formulas ε_1 and ε_2, the notation $\varepsilon_1 \equiv \varepsilon_2$ means $(\varepsilon_1 \to \varepsilon_2) \wedge (\varepsilon_2 \to \varepsilon_1)$.

Definition 3.2. A unifier ε (for a formula φ in a logic L) is *more general than a unifier* ε_1 iff there is a substitution δ such that $[\varepsilon_1(x) \equiv \delta(\varepsilon(x))] \in L$ for any letter x.

If a logic L is decidable and Kripke-complete, then check the unifiability itself a formula in L is (theoretically, not computationally) is usually a simple task: It suffices to use only ground substitutions: the mappings of variable-letters in the set $\{\bot, \top\}$ (thus here the decidability of the letterless fragment is enough). But checking the unifiability of a formula including computing a unifier is computationally not an easy task at all (exponential).

Definition 3.3. A set of unifiers CU for a given formula φ in a logic L is a *complete set of unifiers*, if the following holds: Given a unifier σ for φ in L, there is a unifier σ_1 from CU where σ_1 is more general than σ.

Finding the complete set of unifiers is very useful, for example, for the verification admissibility of inference rules (this technique was invented and developed by Ghilardi (cf. [11–14]). It uses the notion of projective formula.

Definition 3.4. A formula φ is said to be *projective* in a logic L if the following holds. There is a substitution σ (which is called a projective substitution) which is an unifier for φ such that $\Box\varphi \to [x_i \equiv \sigma(x_i)] \in L$ for any letter x_i from φ.

Now we recall a known result about linear temporal logic LTL in light of our definition for formulas to be projective. Logic LTL is generated by the frame based at all natural numbers with logical operations $Next$ to be next number and the logical operation $Until$, but we will need here only modal operation $\Box$ which as well known is definable by usage $Until$.

Example 3.5 (cf., e.g., [34]). The formula $\varphi = \Box(\Box x \vee (\neg x \wedge Next\Box x))$ is unifiable in LTL but not projective.

Proof. The substitution $x \mapsto \top$ is an obvious unifier for φ. Suppose now that φ is projective and π is a corresponding projective unifier. Consider the run N_V (starting from 0: $|N_V| := \{0, 1, 2, \dots\}$):

$$\overset{x}{\circ} \xrightarrow{N} \overset{\neg x}{\circ} \xrightarrow{N} \overset{\Box x}{\circ} \xrightarrow{N} \overset{\Box x}{\circ} \cdots .$$

Since $(N_V, 1) \Vdash_V \Box\varphi$; therefore, $(N_V, 1) \Vdash_V x \leftrightarrow \pi(x)$. Thus, we have either $(N_V, 0) \Vdash_V \pi(x)$, or $(N_V, 0) \Vdash_V \neg\pi(x)$, or $(N_V, 0) \Vdash_V \neg\Box\pi(x)$ and, at the same time, $(N_V, 0) \Vdash_V \neg Next\Box\pi(x)$. Thus $(N_V, 0) \Vdash_V \neg\pi(\varphi)$; hence, π cannot be an φ-unifier, a contradiction. $\Box$

An inference rule $\varphi_1, \dots \varphi_n/\psi$ is said to be not admissible in a logic L iff there is a unifier for $\varphi_1 \wedge \cdots \wedge \varphi_n$ which is not a unifier for ψ (otherwise we say the rule is admissible). We say then that the substitution σ *disproves admissibility of the rule* $\varphi_1, \dots \varphi_n/\psi$ in L.

4. The Main Results

In fact for every Kripke complete logic L extending S4 a formula φ is unifiable in L iff φ is unifiable in S4. This latter holds iff φ is unifiable in the logic of one element reflexive frame. But how does the structure of all unifiers look like? In our paper we consider only the decidable modal logics generated by classes of Kripke frames with greatest clusters.

Let a frame F to be given. Then we say that F has the *greatest cluster* C if
(0) F has the cluster C, where
(1) $C \subseteq |F|$,
(2) $\forall x, y \in C, xRy$,
(3) $\forall z \in |F| \setminus C, \forall x \in C, zRx$.

Lemma 4.1. *If $\Diamond\Box p \to \Box\Diamond p \in L$ and L is Kripke complete logic with respect to rooted, reflexive and transitive frames; then, for each rooted L-frame $\mathscr{F}$ with a root state s for all $a, b \in \mathscr{F}$, i.e., sRa and sRb, there is $c \in \mathscr{F}$ such that aRc and bRc. Therefore if $\mathscr{F}$ is finite then $\mathscr{F}$ has the greatest cluster.*

Proof. Assume that $\mathscr{F}$ is some rooted L-frame, s is a root of $\mathscr{F}$, $a, b \in \mathscr{F}$, and there is no c such that aRc and bRc. Take the valuation V of p where $V(p) := \{e \mid aRe\}$. Then $(\mathscr{F}, a) \Vdash_V \Box p$ and $(\mathscr{F}, s) \Vdash_V \Diamond\Box p$. But $(\mathscr{F}, b) \Vdash_V \neg\Diamond p$, and so $(\mathscr{F}, s) \nVdash_V \Box\Diamond p$. Consequently,

$$(F, s) \nVdash_V \Diamond\Box p \to \Box\Diamond p;$$

a contradiction. In particular, if $\mathscr{F}$ is finite then it has the greatest cluster. $\Box$

The axiom $\Diamond\Box p \to \Box\Diamond p$ added to the axioms of S4 makes the well-known logic S4.2. Though we will not require the presence of this axiom in our considerations. It is well known that S4.2 has the finite model property, and therefore its every rooted finite frame has the greatest cluster. In fact, there is an infinite family of modal logics above S4.2 which are Kripke complete and are generated by rooted frames with greatest clusters. For example, that are logics of bounded finite width (width n for any n), finite width on depth bigger than a fixed one, logics possessing various combinations of these properties and many others. Therefore our results below will work for infinite families of various logics.

Theorem 4.2. *Let L be a decidable and Kripke complete logic generated by a set K of some (arbitrary) transitive and reflexive frames with greatest clusters; i.e.,*

$$L = L(K) := \{\varphi \mid \forall F \in K, F \Vdash \varphi\}.$$

Given a formula α unifiable in L, we may effectively construct a unifier σ for α in L, where σ may verify admissibility in L arbitrary given inference rule α/β with switch-modalized conclusion (i.e., β has form $\beta = \Box\Diamond\gamma$ or $\beta = \Diamond\Box\gamma$ for some formula γ). That is α/β is not admissible in L iff σ disproves the admissibility of α/β in L.

Proof. Take a unifiable in L formula $\varphi(x_1, \ldots, x_n)$ (unifiability is computable since the presence of the greatest cluster). Then there is a ground unifier, a substitution σ_1, where $\sigma_1(x_i) := g_i$, be its ground unifier: ($g_i \in \{\top, \bot\}$), and $\varphi(\sigma_1(x_i), \ldots, \sigma_1(x_n)) \in L$.

Now for each letter x_i occurring in φ we define the following substitution: We use the notation: $Var(\Box\varphi)$ is the set of all letters from $\Box\varphi$, and $Sub(\Box\varphi)$ is the set of all subformulas of $\Box\varphi$. Given $X \subseteq Var(\Box\varphi)$, we define

$$Let(X) := \bigwedge_{x_i \in Var(\varphi) \cap X} x_i \wedge \bigwedge_{x_i \in Var(\varphi) \& x_i \notin X} \neg x_i.$$

To describe the greatest clusters of frames we need the following formulas, for every $A \subseteq 2^{Var(\varphi)}$,

$$Cl(A) := \Big[\Box\varphi \wedge \Box \bigvee_{X \in A} L(X) \wedge \Box \bigwedge_{X \in A} (\Diamond L(X))\Big].$$

Moreover, for any $A \neq \varnothing$ let we choose and fix a single $X_A \in A$ and define the substitution $\delta_A(x_i) := \top$ if $x_i \in X_A$; otherwise, we let $\delta_A(x_i) := \bot$. (The case when $A = \varnothing$ will not be necessary in our proof.)

Now we are ready to define our substitution. Given $x_i \in Var(\varphi)$, assume that

$$\sigma(x_i) := \Big[\bigvee_{A \subseteq 2^{Var(\varphi)}} [Cl(A) \wedge x_i] \vee [\neg[\Diamond\Box\varphi] \wedge g_i]$$

$$\vee[\neg[\bigvee_{A \subseteq 2^{Var(\varphi)}} Cl(A)] \wedge \bigvee_{A \subseteq 2^{Var(\varphi)}} [\Diamond[Cl(A)] \wedge \delta_A(x_i)]\Big].$$

Note that this substitution, unfortunately, does not allow us to say that the formula φ is a projective formula, because of the presence $\delta_A(x_i)$ above. Moreover, it is impossible to find a projective substitution for any such unifiable formula φ if L does not extend logic S4.3; cf. [36].

Now we first will prove that σ is a unifier for φ in L. Since $L = L(K)$, take a frame $\mathscr{F}$ from K with any valuation V for letters x_i from φ. By definition $\mathscr{F}$ is rooted and has the greatest cluster C. Consider any state a from $\mathscr{F}$.

(I) Assume that there exists $a \in C$ such that $(\mathscr{F}, a) \Vdash_V \neg\Box\varphi$. Then by the definition of σ, it follows that

$$\forall x[[aRx] \Rightarrow [(F, x) \Vdash_V \sigma(x_i) \Leftrightarrow (F, x) \Vdash_V g_i]].$$

Therefore,

$$\forall x[aRx] \Rightarrow (F, x) \Vdash_V \sigma(\Box\varphi).$$

Hence, σ is a unifier for φ at the cluster C, it made φ true at each of its states.

(II) Consider the case when for all $a \in C$, $(\mathscr{F}, a) \Vdash_V \Box\varphi$. Let A to be the collection of all sets of letters x_i from φ valid with respect to the valuation V at some states from C. Then $(\mathscr{F}, a) \Vdash_V Cl(A)$. And this is not a case for any other A_1 and $Cl(A_1)$. Therefore,

$$(\mathscr{F}, a) \Vdash_V \Box\varphi \wedge \Box\Diamond\Box \bigvee_{X \in A} L(X) \wedge \Box\Diamond\Box \bigwedge_{X \in A} (\Diamond L(X)).$$

This formula shows the possible distributions of truth values of letters of the formula φ in states reachable from a by the accessibility relation of the frame. Thus, we have

(1) $\forall x$ if xRa and $(F, x) \Vdash_V Cl(A)$, then $\forall x_i \in Var(\varphi), (F, x) \Vdash_V \sigma(x_i) \iff (F, x) \Vdash_V x_i$,

(2) $\forall \psi \in Sub(\Box\varphi), (F, x) \Vdash_V \sigma(\psi) \iff (F, x) \Vdash_V \psi$.

Moreover, $(\mathscr{F}, x) \Vdash_V Cl(A)$ has to imply $(\mathscr{F}, x) \Vdash_V \Box\varphi$. Therefore,

(3) $\forall x\ [xRa \wedge (F, x) \Vdash_V Cl(A)] \Rightarrow (F, x) \Vdash_V \sigma(\Box\varphi)$.

Hence, σ is a unifier for φ at states x from C, where $(\mathscr{F}, x) \Vdash_V Cl(A)$.

(III) Now we will consider the remaining more complicated and interesting case when $x \in \mathscr{F}$, xRa, $a \in C$, and

$$[xRa \wedge (\mathscr{F}, x) \Vdash_V \neg Cl(A)].$$

As we observed earlier then

$$(\mathscr{F}, x) \Vdash_V \neg Cl(A_1)$$

for any other A_1 because $(\mathscr{F}, a) \Vdash_V Cl(A)$. By the definition of $\sigma(x_i)$ this means that the truth value of $\sigma(x_i)$ with respect to V at each of such x is $\delta_A(x_i)$. Using it and ((2) and (3) from above), we infer that for all such x the truth value of $\sigma(\alpha)$ for each subformula α of φ at x with respect to V is the same as truth value of α with respect to V at the state s_A of C where the truth values of letters x_i are $\delta_A(x_i)$ (cf. the choice of $\delta_A(x_i)$ above). Since $(\mathscr{F}, s_A) \Vdash_V \Box\varphi$, we get $(\mathscr{F}, x) \Vdash_V \Box\sigma(\varphi)$. Thus, as we proved, σ is indeed a unifier for φ.

(IV) We wish now to show that the constructed above unifier σ verifies admissibility in L of any inference rule on the form $\varphi/\Diamond\Box\psi$.

Indeed, a rule $\varphi/\Diamond\Box\psi$ is not admissible in L iff there is a substitution σ_r which is a unifier for φ and not a unifier for $\Diamond\Box\psi$. Then there is a model with a valuation V_1 on a rooted frame $\mathscr{F}$ from K such that $\sigma_r(\varphi)$ is true with respect to V_1 at all states from $\mathscr{F}$ but $\sigma_r(\psi)$ is false at some state from the greatest cluster C from $\mathscr{F}$ with respect to V_1.

Consider the new model on that frame $\mathscr{F}$ with new valuation V, where for each letter x_i from $\varphi/\Diamond\Box\psi$ $V(x_i) := V_1(\sigma_r(x_i)$. Then the following holds:

Lemma 4.3. *The formula φ is true at each state from $\mathscr{F}$ with respect to the valuation V and ψ is false with respect to V at a state from the greatest cluster C.*

The proof is straightforward since we assumed that the formula $\sigma_r(\varphi)$ is true with respect to V_1 at all states from $\mathscr{F}$ and $\sigma_r(\psi)$ is false at a state from the cluster C from the frame $\mathscr{F}$ with respect to V_1.

Consider now the stage of construction in our theorem above for our unifier σ from the valuation V and φ at the model $\mathscr{F}$ with the fixed here valuation V. As it is easy to see

$$\forall x \in C(\mathscr{F}, x) \Vdash_{V_1} \sigma_r(\varphi) \text{ and } \forall x \in C(\mathscr{F}, x) \Vdash_V \varphi \text{ and } \forall x \in C(\mathscr{F}, x) \Vdash_V \sigma(\varphi),$$

$$\exists x \in C(\mathscr{F}, x) \Vdash_{V_1} \neg\sigma_r(\psi), (\exists x \in C(\mathscr{F}, x) \Vdash_V \neg\psi), \exists x \in C(\mathscr{F}, x) \Vdash_V \neg\sigma(\psi).$$

Indeed, if A is the collection of all sets of letters valid at various states of C with respect to the valuation V $\forall x \in C(\mathscr{F}, x) \Vdash_V Cl(A)$ and the valuation for $\sigma(x_i)$ with respect to V coincides with the valuation of x_i themselves with respect to V. Therefore,

$$\exists x \in C(\mathscr{F}, x) \Vdash_V \sigma(\neg\psi).$$

This means that the unifier σ for φ is not a unifier for $\Diamond\Box\psi$. That is we proved that if the rule $\varphi/\Diamond\Box\psi$ is not admissible in L then the effectively constructed in our theorem above unifier σ disproves the admissibility. The case of rules in form $\varphi/\Box\Diamond\psi$ may be proved quite similarly. $\square$

We think that the above results are general and are applicable for for the infinite families of modal logics above S4.2.

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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