ISSN 0037–4466, Siberian Mathematical Journal, 2024, Vol. 65, No. 1, pp. 174–189. © Pleiades Publishing, Ltd., 2024.
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ON THE SEPARABILITY OF ABELIAN SUBGROUPS OF THE FUNDAMENTAL GROUPS OF GRAPHS OF GROUPS. II

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UDC 512.543

Abstract—Consider the fundamental group $\mathfrak{G}$ of an arbitrary graph of groups and some root class $\mathscr{C}$ of groups, i.e., a class containing a nontrivial group and closed under subgroups, extensions, and unrestricted direct products of the form $\prod_{y\in Y} X_y$, where $X, Y \in \mathscr{C}$ and X_y is an isomorphic copy of X for each $y \in Y$. We provide some criterion for the separability by $\mathscr{C}$ of a finitely generated abelian subgroup of $\mathfrak{G}$ valid when the group satisfies an analog of the Baumslag filtration condition. This enables us to describe the $\mathscr{C}$-separable finitely generated abelian subgroups for the fundamental groups of some graphs of groups with central edge subgroups.

DOI: 10.1134/S0037446624010166

Keywords: separability of abelian subgroups, separability of cyclic subgroups, root-class residuality, fundamental group of a graph of groups, tree product

1. Introduction

This is the second part of the article on the separability by root classes of groups of finitely generated abelian subgroups of the fundamental groups of graphs of groups. In the first part [1] we provide some structure theorem for such subgroups and a series of auxiliary statements. Here we apply the results to obtaining two rather general criteria for the separability of the subgroups.

The concept of separable subalgebra, and, in particular, subgroup was introduced by Maltsev [2]. According to his definition, a subgroup Y of a group X is *separable* in X by the *class* $\mathscr{C}$ of groups, or briefly *$\mathscr{C}$-separable* in X, whenever, given $x \in X \setminus Y$, there is a homomorphism σ of X onto a group of class $\mathscr{C}$ satisfying $x\sigma \notin Y\sigma$. The concept of *residuality* is a particular case of separability because the residuality of X by a class $\mathscr{C}$ is equivalent to $\mathscr{C}$-separability of the identity subgroup of X. Recall also that the separability by the class of all finite groups is called *finite* just like residuality.

It is well known that the finite separability of a subgroup Y in a finitely presented group X means the solvability of the algorithmic problem which consists in answering the question whether a given element of X belongs to Y. Apart from this, the separability of certain subgroups often is one of the necessary and/or sufficient conditions for residuality. This happens particularly often for the residuality of various group-theoretic constructions; see [3–12] for instance.

Root classes of groups were introduced by Gruenberg [13]. This concept turned out quite productive for studying the residuality of free constructions of groups (see [6, 8, 11, 12, 14–17]). Thus, studying the separability of subgroups of such constructions by root classes seems totally justifiable, especially because of the above-mentioned relation with residuality conditions.

Recall that, according to one of the equivalent definitions, a class $\mathscr{C}$ of groups containing a nontrivial group is a *root class* whenever $\mathscr{C}$ is closed under subgroups, extensions, and unrestricted direct products of the form $\prod_{y\in Y} X_y$, where $X, Y \in \mathscr{C}$ and X_y is an isomorphic copy of X for each $y \in Y$; see [18]. As examples of a root class, we can take the classes of all finite groups; finite p-groups, with p a prime; periodic $\mathfrak{P}$-groups of finite exponent, with $\mathfrak{P}$ a nonempty set of primes; all solvable groups; and all torsion-free groups. Moreover, it is not difficult to show that if the intersection of a family of root classes of groups includes a nontrivial group then the intersection is a root class too.

Henceforth, given some set $\mathfrak{P}$ of primes, denote the set of all primes outside $\mathfrak{P}$ by $\mathfrak{P}'$. Refer to an integer as a $\mathfrak{P}$-*number* whenever all its prime divisors lie in $\mathfrak{P}$. Recall that a subgroup Y of a group X

Original article submitted March 20, 2023; revised March 20, 2023; accepted August 2, 2023.

is $\mathfrak{P}'$-*isolated* in X whenever for each $x \in X$ and each $q \in \mathfrak{P}'$ from $x^q \in Y$ it follows that $x \in Y$. If the trivial subgroup of X is $\mathfrak{P}'$-isolated then we say that X is $\mathfrak{P}'$-*torsion-free*. If $\mathfrak{P}$ is the set of all primes then every subgroup turns out $\mathfrak{P}'$-isolated.

If the class $\mathscr{C}$ consists of periodic groups then denote by $\mathfrak{P}(\mathscr{C})$ the set of all prime divisors of the orders of elements of the groups in $\mathscr{C}$. In order to simplify the statements below we assume that if $\mathscr{C}$ contains at least one nonperiodic group then $\mathfrak{P}(\mathscr{C})$ is the set of all primes. It is known (cp. Proposition 4.1 below) that, whatever class $\mathscr{C}$ of groups is chosen, the separability of a subgroup by this class implies that it is $\mathfrak{P}(\mathscr{C})'$-isolated. Thus, it makes sense to study the $\mathscr{C}$-separability only of $\mathfrak{P}(\mathscr{C})'$-isolated subgroups, and the generalization of the assertion that all subgroups of some type T are finitely separable means the $\mathscr{C}$-separability of all $\mathfrak{P}(\mathscr{C})'$-isolated subgroups of type T (for instance, cyclic).

The main method for studying the root-class residuality of free constructions of groups is the so-called filtration approach by Baumslag which was first proposed in [19] to study the residual finiteness of a generalized free product of two groups and then extended to other constructions and approximating classes. Kim showed how similar arguments can be applied to prove the finite separability of all cyclic subgroups of a generalized free product of two groups [20] and the HNN-extension with one stable letter [21]. In [22–24] these ideas were adapted to studying the separability by the class of finite p-groups, with p some prime, and extended to the case that not necessarily all subgroups of the free factors or the base group are separable. Rather recently Zhou and Kim have showed in [25, 26] how to apply the filtration approach to proving the finite separability of all finitely generated abelian subgroups of the two above-mentioned constructions. Finally, Baumslag's method for studying residuality was extended in [17] to the case of the residuality by an arbitrary root class of groups and the fundamental group of an arbitrary graph of groups. This article combines all those ideas to obtain a description of the finitely generated abelian subgroups of the fundamental group of a graph of groups separable by a prescribed root class of groups.

When we discuss some group-theoretic construction, the most natural question is as follows: Under which conditions does the construction inherit some property of the constituent groups? The problem we will solve in this article consists in obtaining some criterion for the separability by a root class $\mathscr{C}$ of groups of a $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup of the fundamental group of a graph of groups under the condition that similar criteria are available for all vertex groups of this graph. Our results are stated in Sections 2 and 3, whereas the proofs are given in Sections 4–8.

2. The Main Results

We use the same notation as in [1]. Namely, we assume throughout the article that

(a) Γ is a nonempty undirected connected graph with vertex set $\mathscr{V}$ and edge set $\mathscr{E}$ (loops and multiple edges are allowed);

(b) $\mathscr{T}$ is a fixed maximal tree in Γ with edge set $\mathscr{E}_{\mathscr{T}}$;

(c) $\mathscr{G}(\Gamma)$ is an oriented graph of groups over Γ in which to each vertex $v \in \mathscr{V}$ there is associated a group G_v; while to each edge $e \in \mathscr{E}$, a direction, some group H_e, and injective homomorphisms

$$\varphi_{+e}\colon H_e \to G_{e(1)}, \quad \varphi_{-e}\colon H_e \to G_{e(-1)},$$

where $e(1)$ and $e(-1)$ are the endpoints of e in $\mathscr{G}(\Gamma)$;

(d) $\mathfrak{G}$ is the fundamental group of $\mathscr{G}(\Gamma)$ with the presentation corresponding to $\mathscr{T}$:

$$\left\langle \begin{matrix} G_v\ (v \in \mathscr{V}), \\ t_e\ (e \in \mathscr{E} \setminus \mathscr{E}_{\mathscr{T}}) \end{matrix} \;\middle|\; \begin{matrix} h_e\varphi_{+e} = h_e\varphi_{-e}\ (e \in \mathscr{E}_{\mathscr{T}},\ h_e \in H_e), \\ t_e^{-1}h_e\varphi_{+e}t_e = h_e\varphi_{-e}\ (e \in \mathscr{E} \setminus \mathscr{E}_{\mathscr{T}},\ h_e \in H_e) \end{matrix} \right\rangle.$$

The groups G_v for $v \in \mathscr{V}$ are *vertex groups*, while the subgroups $H_{+e} = H_e\varphi_{+e}$ and $H_{-e} = H_e\varphi_{-e}$, *edge groups*. Given some class $\mathscr{C}$ of groups and some group X, denote by $\mathscr{C}^*(X)$ the family of all normal subgroups of X with the associated quotient groups in $\mathscr{C}$.

The first result of this article is as follows:

Theorem 2.1. *Given a root class $\mathscr{C}$ of groups, suppose that there exists a homomorphism from $\mathfrak{G}$ onto a group of this class acting injectively on all vertex groups. Then each $\mathfrak{P}(\mathscr{C})'$-isolated subgroup of $\mathfrak{G}$ satisfying some nontrivial identity is $\mathscr{C}$-separable; in particular, $\mathfrak{G}$ is residually a $\mathscr{C}$-group.*

Note that sufficient conditions for the existence of a homomorphism in the statement of Theorem 2.1 are found in [6, 8, 11, 12, 16, 27]. The equivalence of the presence of such a homomorphism and the $\mathscr{C}$-residuality of $\mathfrak{G}$ is discussed in [28, 29]. Before we proceed to describe the results on the separability of subgroups of $\mathfrak{G}$ which are obtained in the case that the mentioned homomorphism does not exist, let us present one statement that follows from Theorems 1, 3, and Proposition 2 of [17].

Theorem 2.2. *Consider a root class $\mathscr{C}$ of groups.*

I. *$\mathfrak{G}$ is residually a $\mathscr{C}$-group provided that*

($\mathfrak{i}^1$) $\forall v \in \mathscr{V}$ $\bigcap_{N\in\mathscr{C}^*(\mathfrak{G})}(N \cap G_v) = 1$;

($\mathfrak{ii}^1$) $\forall e \in \mathscr{E},\ \varepsilon = \pm 1 \bigcap_{N\in\mathscr{C}^*(\mathfrak{G})} H_{\varepsilon e}(N \cap G_{e(\varepsilon)}) = H_{\varepsilon e}$.

II. *Assume that assertion $(*)$ holds: for all $v \in \mathscr{V}$ and $M \in \mathscr{C}^*(G_v)$ there exists a subgroup $N \in \mathscr{C}^*(\mathfrak{G})$ such that $N \cap G_v \leq M$. Then $\mathfrak{G}$ is residually a $\mathscr{C}$-group provided that*

($\mathfrak{i}^2$) *all groups G_v for $v \in \mathscr{V}$ are residually $\mathscr{C}$-groups;*

($\mathfrak{ii}^2$) *$H_{\varepsilon e}$ is $\mathscr{C}$-separable in $G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$.*

Note that conditions ($\mathfrak{i}^1$) and ($\mathfrak{ii}^1$) are the weakest of those for which Baumslag's filtration method can be applied, and so they hold whence the residuality of $\mathfrak{G}$ is proved by this method. However, they depend not only on the properties of vertex groups and the included edge subgroups, but also on the structure of $\mathfrak{G}$ as a whole. If assertion $(*)$ holds then the conditions become the more tangible requirements ($\mathfrak{i}^2$) and ($\mathfrak{ii}^2$) unrelated to $\mathfrak{G}$. Furthermore, in view of Proposition 6.1 below, we can express $(*)$ in another form which is easier to prove. Together with that, apparently $(*)$ does not follow from ($\mathfrak{i}^1$) and ($\mathfrak{ii}^1$), and so in general we cannot completely drop the latter. Some more detailed discussion of the hypotheses of Theorem 2.2 appears in [17].

Take a class $\mathscr{C}$ of groups and a pair (X, Y) of subgroups of $\mathfrak{G}$. Consider the collection of conditions:

($\lambda^0_{\mathscr{C}}$) X is a finitely generated abelian subgroup of some group G_v for $v \in \mathscr{V}$, while $Y = 1$;

($\mu^0_{\mathscr{C}}$) X is a finitely generated abelian subgroup of some group $H_{\varepsilon e}$ for $e \in \mathscr{E}$ and $\varepsilon = \pm 1$, while Y is an infinite cyclic subgroup with $Y \cap G_{e(\varepsilon)} = 1$ and $[X, Y] = 1$;

($\lambda^1_{\mathscr{C}}$) condition ($\lambda^0_{\mathscr{C}}$) holds, while X is $\mathfrak{P}(\mathscr{C})'$-isolated in G_v but $\bigcap_{N\in\mathscr{C}^*(\mathfrak{G})} X(N \cap G_v) \neq X$;

($\mu^1_{\mathscr{C}}$) condition ($\mu^0_{\mathscr{C}}$) holds, while X is $\mathfrak{P}(\mathscr{C})'$-isolated in $G_{e(\varepsilon)}$ but $\bigcap_{N\in\mathscr{C}^*(\mathfrak{G})} X(N \cap G_{e(\varepsilon)}) \neq X$;

($\lambda^2_{\mathscr{C}}$) condition ($\lambda^0_{\mathscr{C}}$) holds, while X is $\mathfrak{P}(\mathscr{C})'$-isolated in G_v but is not $\mathscr{C}$-separable in G_v;

($\mu^2_{\mathscr{C}}$) condition ($\mu^0_{\mathscr{C}}$) holds, while X is $\mathfrak{P}(\mathscr{C})'$-isolated in $G_{e(\varepsilon)}$ but is not $\mathscr{C}$-separable in $G_{e(\varepsilon)}$.

Denote by $\mathfrak{D}^k_{\mathscr{C}}(\mathfrak{G})$ for $0 \leq k \leq 2$ the family of all pairs of subgroups of $\mathfrak{G}$ satisfying conditions ($\lambda^k_{\mathscr{C}}$) or ($\mu^k_{\mathscr{C}}$), and put

$$\mathfrak{A}^k_{\mathscr{C}}(\mathfrak{G}) = \big\{XY \mid (X, Y) \in \mathfrak{D}^k_{\mathscr{C}}(\mathfrak{G})\big\}.$$

According to [1], the subgroup A of G is finitely generated abelian if and only if A is conjugate to some subgroup of $\mathfrak{A}^0_{\mathscr{C}}(\mathfrak{G})$. The main result is as follows:

Theorem 2.3. *Consider a root class $\mathscr{C}$ of groups.*

I. *If conditions ($\mathfrak{i}^1$) and ($\mathfrak{ii}^1$) of Theorem 2.2 hold then the $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup of $\mathfrak{G}$ is $\mathscr{C}$-separable in $\mathfrak{G}$ if and only if it is not conjugate to any subgroup of $\mathfrak{A}^1_{\mathscr{C}}(\mathfrak{G})$.*

II. *Assume that assertion $(*)$ holds and so do conditions ($\mathfrak{i}^2$) and ($\mathfrak{ii}^2$) of Theorem 2.2. Then the $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup of $\mathfrak{G}$ is $\mathscr{C}$-separable in $\mathfrak{G}$ if and only if it is not conjugate to any subgroup of $\mathfrak{A}^2_{\mathscr{C}}(\mathfrak{G})$. In particular, if each vertex group has the property that all its $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroups are $\mathscr{C}$-separable then this property holds for $\mathfrak{G}$ as well.*

Theorem 2.3 asserts that if the $\mathscr{C}$-residuality of $\mathfrak{G}$ for some root class $\mathscr{C}$ is established by verifying the hypotheses of Theorem 2.2 (and precisely this happens in many cases) then as a free bonus we

obtain a description for the $\mathscr{C}$-separable finitely generated abelian subgroups of this group. In some cases conditions (i^1) and (ii^1) or (i^2) and (ii^2) are equivalent to the $\mathscr{C}$-residuality of $\mathfrak{G}$ (see [17, 28, 30] for instance) and then the criterion for $\mathscr{C}$-separability of a finitely generated abelian subgroup of this group follows from its $\mathscr{C}$-residuality regardless of the method the latter was proved.

Note that the family $\mathfrak{A}^1_{\mathscr{C}}(\mathfrak{G})$, as well as conditions (i^1) and (ii^1), depends on the structure of $\mathfrak{G}$ as a whole. As for $\mathfrak{A}^2_{\mathscr{C}}(\mathfrak{G})$, in order to describe it we need to know only criteria for the $\mathscr{C}$-separability of $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroups in the vertex groups. Thus, claim II of Theorem 2.3 yields a partial solution to the problem at the end of Section 1.

It is clear that Theorem 2.3 also yields a description of the $\mathscr{C}$-separable cyclic subgroups of $\mathfrak{G}$. Let us state the description explicitly.

Consider two families $\mathfrak{B}^1_{\mathscr{C}}(\mathfrak{G})$ and $\mathfrak{B}^2_{\mathscr{C}}(\mathfrak{G})$ of cyclic subgroups defined as follows:

(1) $Z \in \mathfrak{B}^1_{\mathscr{C}}(\mathfrak{G})$ if and only if Z is a $\mathfrak{P}(\mathscr{C})'$-isolated subgroup of G_v for some $v \in \mathscr{V}$ such that $\bigcap_{N \in \mathscr{C}^*(\mathfrak{G})} Z(N \cap G_v) \neq Z$;

(2) $Z \in \mathfrak{B}^2_{\mathscr{C}}(\mathfrak{G})$ if and only if Z is a $\mathfrak{P}(\mathscr{C})'$-isolated subgroup of G_v for some $v \in \mathscr{V}$ which is not $\mathscr{C}$-separable in this group.

Corollary 2.4. *Consider a root class $\mathscr{C}$ of groups.*

I. *If conditions* (i^1) *and* (ii^1) *of Theorem* 2.2 *hold then a $\mathfrak{P}(\mathscr{C})'$-isolated cyclic subgroup of $\mathfrak{G}$ is $\mathscr{C}$-separable in $\mathfrak{G}$ if and only if it is not conjugate to any subgroup of $\mathfrak{B}^1_{\mathscr{C}}(\mathfrak{G})$.*

II. *Assume that assertion* ($*$) *holds and so do conditions* (i^2) *and* (ii^2) *of Theorem* 2.2. *Then a $\mathfrak{P}(\mathscr{C})'$-isolated cyclic subgroup of $\mathfrak{G}$ is $\mathscr{C}$-separable in this group if and only if it is not conjugate to any subgroup of $\mathfrak{B}^2_{\mathscr{C}}(\mathfrak{G})$. In particular, if each vertex group has the property that all its $\mathfrak{P}(\mathscr{C})'$-isolated cyclic subgroups are $\mathscr{C}$-separable then this is valid for $\mathfrak{G}$ as well.*

Note that claim II of Theorem 2.3 generalizes Theorem 2.10 of [25] and Theorem 3.6 of [26], while Corollary 2.4 generalizes Theorem 2.2 of [21], Theorem 1.1 of [20], Theorems 1 and 2 of [24], and Theorems 2.2.2 and 2.3.2 of [23]. Among those theorems, the first six involve finite separability; and the last two, separability by the class of finite $\mathfrak{P}$-groups, where $\mathfrak{P}$ is an arbitrary set of primes.

3. Some Applications

Let us present a few examples of applications of Theorem 2.3 to the fundamental groups of graphs of groups with central edge subgroups.

Theorem 3.1. *Suppose that the subgroup $H_{\varepsilon e}$ is central in $G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ and $G_{e(\varepsilon)} \neq H_{\varepsilon e}$. If $\mathfrak{G}$ is residually a $\mathscr{C}$-group for some root class $\mathscr{C}$ then a $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup of $\mathfrak{G}$ is $\mathscr{C}$-separable in it if and only if it is not conjugate to any subgroup of $\mathfrak{A}^1_{\mathscr{C}}(\mathfrak{G})$.*

In practice, Theorem 3.1 imposes no restrictions on the graph of groups but the centrality of the edge subgroups. However, its statement involves the family $\mathfrak{A}^1_{\mathscr{C}}(\mathfrak{G})$ of a very complicated structure. Theorem 3.2 and Corollaries 3.3 and 3.4 impose stronger requirements on the graph of groups, but yield some simpler separability criteria that base on $\mathfrak{A}^2_{\mathscr{C}}(\mathfrak{G})$.

Say that a graph $\mathscr{G}(\Gamma)$ of groups is of *type* (t), where $t = \overline{1,2}$, if for every $v \in \mathscr{V}$ the subgroup

$$H_v = \mathrm{sgp}\{H_{\varepsilon e} \mid e \in \mathscr{E},\ \varepsilon = \pm 1,\ v = e(\varepsilon)\}$$

is central in G_v and property (t) that is one of the following holds:

(1) each subgroup H_v for $v \in \mathscr{V}$ amounts to the direct product of its generating subgroups;

(2) the graph Γ is a tree.

Given an arbitrary class $\mathscr{C}$ of groups and some group X with a subgroup Y, say that X is *$\mathscr{C}$-regular with respect to* Y whenever for each subgroup $M \in \mathscr{C}^*(Y)$ there exists a subgroup $N \in \mathscr{C}^*(X)$ with $N \cap Y = M$. Observe that $\mathscr{C}$-regularity is closely related to $\mathscr{C}$-separability and generalizes the concept of a *potent element* introduced in [31].

Theorem 3.2. *Given a root class $\mathscr{C}$ of groups closed under quotients, suppose that $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ and that at least one of the following holds:*

(1) *$\mathscr{G}(\Gamma)$ is a finite graph of groups of type* (1) *and for every $v \in \mathscr{V}$ the group G_v is $\mathscr{C}$-regular with respect to H_v;*

(2) *$\mathscr{G}(\Gamma)$ is a finite graph of groups of type* (2) *and for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ the group $G_{e(\varepsilon)}$ is $\mathscr{C}$-regular with respect to $H_{\varepsilon e}$.*

If $\mathfrak{G}$ is residually a $\mathscr{C}$-group then a $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup of $\mathfrak{G}$ is $\mathscr{C}$-separable in $\mathfrak{G}$ if and only if it is not conjugate to any subgroup of $\mathfrak{A}^2_{\mathscr{C}}(\mathfrak{G})$.

Given some class $\mathscr{C}$ of groups and some abelian group A, refer as the *primary $\mathfrak{P}(\mathscr{C})$-component* of the periodic part of A to the primary component corresponding to a number in $\mathfrak{P}(\mathscr{C})$. Say that a group A is *$\mathscr{C}$-bounded* whenever each primary $\mathfrak{P}(\mathscr{C})$-component of the periodic part of each quotient of A is of finite exponent and has cardinality at most that of some $\mathscr{C}$-group. Call a nilpotent group *$\mathscr{C}$-bounded* whenever it has at least one finite central series with $\mathscr{C}$-bounded abelian factors. Denote the class of $\mathscr{C}$-bounded nilpotent groups by *$\mathscr{C}$-$\mathscr{BN}$*. Note that if $\mathscr{C}$ is a root class then, since G is closed under extensions, the orders of $\mathscr{C}$-groups are not bounded by any integer, and so every finitely generated nilpotent group is $\mathscr{C}$-bounded.

Corollary 3.3. *Given a root class $\mathscr{C}$ of groups which consists of periodic groups, suppose that $\mathscr{G}(\Gamma)$ is a finite graph of groups of type* (1) *or* (2)*, G_v for every $v \in \mathscr{V}$ lies in the class $\mathscr{C}$-$\mathscr{BN}$, and $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$. If $\mathfrak{G}$ is residually a $\mathscr{C}$-group then all its $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroups are $\mathscr{C}$-separable.*

In contrast to Theorem 3.2, Corollary 3.3 does not require $\mathscr{C}$ to be closed under quotients. Propositions 8.2 and 8.4 below give criteria for the $\mathscr{C}$-residuality of $\mathfrak{G}$ in Theorem 3.2 and Corollary 3.3.

Recall that

an abelian group is *bounded* in the sense of Maltsev [2] whenever in each of its quotients all primary components of the periodic part are finite;

a solvable group is *bounded* whenever it has a finite subnormal series with bounded abelian factors.

It is obvious that every polycyclic group is finitely generated, bounded, and solvable. In fact, the converse also holds [32].

Corollary 3.4. *Given a root class $\mathscr{C}$ of groups which consists of periodic groups such that $\mathfrak{P}(\mathscr{C})$ contains all primes, suppose that $\mathscr{G}(\Gamma)$ is a finite graph of groups of type* (1) *or* (2) *and that $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$. If G_v for each $v \in \mathscr{V}$ is a bounded solvable group then all finitely generated abelian subgroups of $\mathfrak{G}$ are $\mathscr{C}$-separable.*

The last statement is a partial generalization of Corollary 3 of [11].

4. The Isolators of Subgroups

Proposition 4.1. *Given an arbitrary class $\mathscr{C}$ of groups, take some group X with a subgroup Y. If Y is $\mathscr{C}$-separable in X then it is $\mathfrak{P}(\mathscr{C})'$-isolated in this group.*

Proof. Proposition 5 of [33] asserts that the claim holds if $\mathscr{C}$ consists of periodic groups. However, if $\mathscr{C}$ contains at least one nonperiodic group then $\mathfrak{P}(\mathscr{C})$ contains all primes, and so every subgroup turns out $\mathfrak{P}(\mathscr{C})'$-isolated. $\square$

Suppose that $\mathscr{C}$ is an arbitrary class of groups. Take some set $\mathfrak{P}$ of primes, some group X, and a subgroup Y of X. It is easy to see that the intersection of an arbitrary number of $\mathscr{C}$-separable ($\mathfrak{P}'$-isolated) subgroups of X is also a $\mathscr{C}$-separable (respectively $\mathfrak{P}'$-isolated) subgroup. Hence, the smallest $\mathscr{C}$-separable and $\mathfrak{P}'$-isolated subgroups including Y are defined. Let us call them the *$\mathscr{C}$-closure* and *$\mathfrak{P}'$-isolator* of Y in X and denote by $\mathscr{C}$-$\mathfrak{Cl}(X,Y)$ and $\mathfrak{P}'$-$\mathfrak{Is}(X,Y)$ respectively. It is easy to see that the $\mathfrak{P}'$-isolator of Y contains the set $\mathfrak{P}'$-$\mathfrak{Rt}(X,Y)$ of elements of X such that $x \in \mathfrak{P}'$-$\mathfrak{Rt}(X,Y)$ if and only if $x^q \in Y$ for some $\mathfrak{P}'$-number q. Proposition 4.1 also implies that $\mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X,Y) \leq \mathscr{C}$-$\mathfrak{Cl}(X,Y)$.

Proposition 4.2 [34, Proposition 4]. *Let $\mathscr{C}$ be an arbitrary class of groups. If X is residually a $\mathscr{C}$-group and Y is a nilpotent subgroup of X of length c then the subgroup $\mathscr{C}$-$\mathfrak{Cl}(X, Y)$ is also a nilpotent group of length c.*

Proposition 4.3 [35, Theorem 4.5]. *Given a set $\mathfrak{P}$ of primes, if X is a locally nilpotent group and Y is a subgroup of X then $\mathfrak{P}'$-$\mathfrak{Is}(X, Y) = \mathfrak{P}'$-$\mathfrak{Rt}(X, Y)$.*

Proposition 4.4. *Suppose that $\mathscr{C}$ is an arbitrary class of groups. If X is residually a $\mathscr{C}$-group and Y is an abelian subgroup of X then*

(1) *the subgroup $\mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, Y)$ is abelian and coincides with the set $\mathfrak{P}(\mathscr{C})'$-$\mathfrak{Rt}(X, Y)$;*

(2) *if Y is a locally cyclic subgroup then so is $\mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, Y)$.*

Proof. As mentioned above,

$$\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(X, Y) \leq \mathscr{C}\text{-}\mathfrak{Cl}(X, Y).$$

Thus, claim (1) follows from Propositions 4.2 and 4.3. Let us verify claim (2).

Lemma. *For $x, y \in X$, if $\langle x\rangle$ stands for the cyclic subgroup generated by x and $y^q \in \langle x\rangle$ for some $\mathfrak{P}(\mathscr{C})'$-number q then the subgroup $\operatorname{sgp}\{x, y\}$ is cyclic.*

Proof. Without loss of generality we may assume that q is the smallest positive $\mathfrak{P}(\mathscr{C})'$-number with $y^q \in \langle x\rangle$. Since $y \in \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, \langle x\rangle)$, claim (1) yields $[x, y] = 1$. Take the numbers k, d, q_1, and k_1 so that $y^q = x^k$ and d is the greatest common divisor of k and q, with $q = dq_1$ and $k = dk_1$. Then d is a $\mathfrak{P}(\mathscr{C})'$-number and $(y^{-q_1}x^{k_1})^d = 1$. Since X is residually a $\mathscr{C}$-group, it is $\mathfrak{P}(\mathscr{C})'$-torsion-free by Proposition 4.1. Consequently, $y^{-q_1}x^{k_1} = 1$ and $y^{q_1} \in \langle x\rangle$. In view of the choice of q this means that $q = q_1$ and $1 = d = ku + qv$ for some integers u and v. Then

$$x = x^{ku+qv} = y^{qu}x^{qv} = (y^u x^v)^q, \quad y = y^{ku+qv} = y^{ku}x^{kv} = (y^u x^v)^k.$$

Therefore, $\operatorname{sgp}\{x, y\}$ is generated by $y^u x^v$. $\square$

Suppose that $x, y \in \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, Y)$. Then, according to claim (1), there exist $\mathfrak{P}(\mathscr{C})'$-numbers q and r such that $x^q, y^r \in Y$. Denoting by z a generator of the subgroup $\operatorname{sgp}\{x^q, y^r\}$ and applying the above lemma to y and z, we find that $\operatorname{sgp}\{z, y\}$ is cyclic and generated by some element z_1. Applying the lemma again, now to z_1 and x, we infer that x and y lie in the cyclic subgroup $\operatorname{sgp}\{z_1, x\}$. $\square$

Proposition 4.5. *Suppose that $\mathscr{C}$ is an arbitrary class of groups. Take a residually $\mathscr{C}$-group X with two abelian subgroups Y and Z satisfying $[Y, Z] = 1$. If Y is a periodic subgroup then*

$$\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(X, YZ) = Y \cdot \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(X, Z).$$

Proof. If $\mathscr{C}$ contains a nonperiodic group then $\mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, YZ) = YZ$ and $\mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, Z) = Z$, and so the required equality holds. Therefore, assume that $\mathscr{C}$ consists of periodic groups and take some $x \in \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, YZ)$. Since X is residually a $\mathscr{C}$-group and the subgroup YZ is abelian, Proposition 4.4 implies that

$$\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(X, YZ) = \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Rt}(X, YZ).$$

Consequently, $x^q = yz$ for some $y \in Y$, $z \in Z$, and a $\mathfrak{P}(\mathscr{C})'$-number q. Since Y is a periodic group and X is residually a $\mathscr{C}$-group, the order r of y is finite, and by Proposition 4.1 it is a $\mathfrak{P}(\mathscr{C})$-number. Since y and z commute, $(x^r)^q = y^r z^r = z^r \in Z$, and so $x^r \in \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, Z)$. Hence, $x^q, x^r \in Y \cdot \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, Z)$ and so $x \in Y \cdot \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(X, Z)$ because q and r are coprime. Thus, we have established the relation

$$\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(X, YZ) \leq Y \cdot \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(X, Z).$$

The inverse inclusion is obvious. $\square$

5. Proof of Theorem 2.1

Proposition 5.1 [34, Proposition 5]. *Given a class $\mathscr{C}$ of groups closed under subgroups and direct products of finitely many factors, take a residually $\mathscr{C}$-group X and its subgroup Y. If Y trivially intersects some subgroup of $\mathscr{C}^*(X)$ then Y is $\mathscr{C}$-separable in X.*

The following is a particular case of Theorem 2.4 of [36]:

Proposition 5.2. *Given a root class $\mathscr{C}$ of groups which consists of periodic groups, consider the class $\mathscr{C}$-$\mathscr{BN}_{\mathfrak{P}(\mathscr{C})}$ of all $\mathfrak{P}(\mathscr{C})'$-torsion-free $\mathscr{C}$-$\mathscr{BN}$-groups. Consider a residually $\mathscr{C}$-$\mathscr{BN}_{\mathfrak{P}(\mathscr{C})}$-group X and $\mathfrak{P}(\mathscr{C})'$-isolated subgroup Y of X. If Y trivially intersects some subgroup of the family $\mathscr{C}$-$\mathscr{BN}^*_{\mathfrak{P}(\mathscr{C})}(X)$ then Y is $\mathscr{C}$-separable in X.*

Proposition 5.3. *Given a root class $\mathscr{C}$ of groups, if X is a free group then each of its $\mathfrak{P}(\mathscr{C})'$-isolated cyclic subgroups is $\mathscr{C}$-separable.*

Proof. Consider a $\mathfrak{P}(\mathscr{C})'$-isolated cyclic subgroup Y of X generated by y and the class $\mathscr{FGN}_0$ of all finitely generated nilpotent torsion-free groups. Since X is residually an $\mathscr{FGN}_0$-group [37], there is a subgroup $M \in \mathscr{FGN}^*_0(X)$ avoiding y. Since the quotient X/M is torsion-free, actually $Y \cap M = 1$, and Proposition 5.1 shows that Y is $\mathscr{FGN}_0$-separable in X. If $\mathscr{C}$ contains at least one nonperiodic group; then it, being closed under subgroups and extensions, contains all polycyclic groups possessing subnormal series with infinite cyclic factors. In particular, $\mathscr{FGN}_0 \subseteq \mathscr{C}$, and so Y turns out a $\mathscr{C}$-separable subgroup. However, if $\mathscr{C}$ consists of periodic groups and $\mathscr{C}$-$\mathscr{BN}_0$ is the class of all torsion-free $\mathscr{C}$-$\mathscr{BN}$-groups then our observations in Section 3 yield $\mathscr{FGN}_0 \subseteq \mathscr{C}$-$\mathscr{BN}_0$, and the $\mathscr{C}$-separability of Y in X is guaranteed by Proposition 5.2. $\square$

Proposition 5.4. *Given a root class $\mathscr{C}$ of groups and a class $\mathscr{D}$ of groups closed under subgroups, if in some group X all $\mathfrak{P}(\mathscr{C})'$-isolated $\mathscr{D}$-subgroups are $\mathscr{C}$-separable then in an arbitrary extension of X by a $\mathscr{C}$-group all $\mathfrak{P}(\mathscr{C})'$-isolated $\mathscr{D}$-subgroups are $\mathscr{C}$-separable.*

Proof. Consider some extension Y of X by a $\mathscr{C}$-group, a $\mathfrak{P}(\mathscr{C})'$-isolated $\mathscr{D}$-subgroup Z of Y, and an arbitrary $y \in Y \setminus Z$. It suffices to find a subgroup $N \in \mathscr{C}^*(Y)$ with $y \notin ZN$. If $y \notin ZX$ then X will do. Assume henceforth that $y \in ZX$ and write $y = zx$ for some $z \in Z$ and $x \in X$.

If $g \in X$ and a prime $q \in \mathfrak{P}(\mathscr{C})'$ satisfy $g^q \in Z \cap X$ then $g \in Z$ because Z is a $\mathfrak{P}(\mathscr{C})'$-isolated subgroup of Y, and so $g \in Z \cap X$. Therefore, the subgroup $Z \cap X$ is $\mathfrak{P}(\mathscr{C})'$-isolated in X, belongs to the class $\mathscr{D}$ because the latter is closed under subgroups, and is $\mathscr{C}$-separable in X by assumptions. Since $y \notin Z$, it follows that $x \notin Z \cap X$, and the above implies that there exists a subgroup $M \in \mathscr{C}^*(X)$ satisfying $x \notin (Z \cap X)M$.

Fix an arbitrary system S of coset representatives of X in Y and put $N = \bigcap_{s\in S} s^{-1}Ms$. Then the subgroup N is normal in Y, while the quotient X/N embeds into the unrestricted direct product $P = \prod_{s\in S} X/s^{-1}Ms$ by Remak's Theorem; see [38, Theorem 4.3.9]. Since all factors of this product are isomorphic to the $\mathscr{C}$-group X/M and are indexed by the elements of the $\mathscr{C}$-group Y/X, the definition of root class yields $P \in \mathscr{C}$. Since $\mathscr{C}$ is closed under subgroups and extensions, this implies that $X/N \in \mathscr{C}$ and $Y/N \in \mathscr{C}$. Thus, $N \in \mathscr{C}^*(Y)$ and $N \leq M$.

Assuming that $y \in ZN$ and $y = z_1u$ for some $z_1 \in Z$ and $u \in N$, we infer that $zx = z_1u$ with $u \in M \leq X$, while $z^{-1}z_1 = xu^{-1} \in Z \cap X$ and $x = (z^{-1}z_1)u \in (Z \cap X)M$ despite the choice of the subgroup M. Therefore, $y \notin ZN$, and N is the required subgroup. $\square$

Proposition 5.5 [29, Proposition 3.4]. *Given a root class $\mathscr{C}$ of groups, if $N \in \mathscr{C}^*(\mathfrak{G})$ with $N \cap H_{\varepsilon e} = 1$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ then there exists a subgroup $M \in \mathscr{C}^*(\mathfrak{G})$ amounting to the free product of some free group and groups which embed into subgroups of the form $N \cap G_v$ for $v \in \mathscr{V}$.*

Proof of Theorem 2.1. Denote by N the kernel of a homomorphism of $\mathfrak{G}$ onto a group of class $\mathscr{C}$ which is injective on all vertex groups; and by $\mathscr{ID}$, the class of groups satisfying some nontrivial identity, not necessarily the same for all groups. Applying Proposition 5.5 to $\mathscr{C}$ and N, we infer that $\mathfrak{G}$ amounts to an extension of some free group M by a $\mathscr{C}$-group. Since every $\mathscr{ID}$-subgroup of a free group is cyclic,

by Proposition 5.3 in M all $\mathfrak{P}(\mathscr{C})'$-isolated $\mathscr{I}\mathscr{D}$-subgroups are $\mathscr{C}$-separable. Since the class $\mathscr{I}\mathscr{D}$ is closed under subgroups, with Proposition 5.4 this implies that in $\mathfrak{G}$ as well all $\mathfrak{P}(\mathscr{C})'$-isolated $\mathscr{I}\mathscr{D}$-subgroups are $\mathscr{C}$-separable. It remains to observe that in an arbitrary extension of a free group by a $\mathscr{C}$-group the trivial subgroup is $\mathfrak{P}(\mathscr{C})'$-isolated and lies in the class $\mathscr{I}\mathscr{D}$. Thus, the $\mathscr{C}$-separability of all $\mathfrak{P}(\mathscr{C})'$-isolated $\mathscr{I}\mathscr{D}$-subgroups implies that $\mathfrak{G}$ is residually a $\mathscr{C}$-group. $\square$

6. On the Fundamental Groups of Graphs of Groups

Assume that for each $v \in \mathscr{V}$ we have chosen some normal subgroup R_v of G_v. As in [17], refer to the family $\mathscr{R} = \{R_v \mid v \in \mathscr{V}\}$ as a *system of compatible normal subgroups* of $\mathfrak{G}$ whenever for every edge $e \in \mathscr{E}$ we have $(R_{e(1)} \cap H_{+e})\varphi_{+e}^{-1} = (R_{e(-1)} \cap H_{-e})\varphi_{-e}^{-1}$. Put

$$\overline{G}_v = G_v/R_v \ (v \in \mathscr{V}), \quad R_e = (R_{e(\pm 1)} \cap H_{\pm e})\varphi_{\pm e}^{-1} \ (e \in \mathscr{E}),$$
$$\overline{H}_e = H_e/R_e \ (e \in \mathscr{E}), \quad \overline{H}_{\varepsilon e} = H_{\varepsilon e}R_{e(\varepsilon)}/R_{e(\varepsilon)} \ (e \in \mathscr{E}, \ \varepsilon = \pm 1).$$

It is easy to see that the mapping $\overline{\varphi}_{\varepsilon e}\colon \overline{H}_e \to \overline{G}_{e(\varepsilon)}$ for $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ carrying the coset hR_e with $h \in H_e$ into $(h\varphi_{\varepsilon e})R_{e(\varepsilon)}$ is well-defined and presents an isomorphism of $\overline{H}_e$ onto the subgroup $\overline{H}_{\varepsilon e}$. Thus, along with the original graph

$$\mathscr{G}(\Gamma) = (\Gamma, \ G_v \ (v \in \mathscr{V}), \ H_e \ (e \in \mathscr{E}), \ \varphi_{\varepsilon e} \ (e \in \mathscr{E}, \ \varepsilon = \pm 1))$$

we may consider the graph of groups

$$\mathscr{G}_{\mathscr{R}}(\Gamma) = (\Gamma, \ \overline{G}_v \ (v \in \mathscr{V}), \ \overline{H}_e \ (e \in \mathscr{E}), \ \overline{\varphi}_{\varepsilon e} \ (e \in \mathscr{E}, \ \varepsilon = \pm 1)),$$

with the same edge directions as in $\mathscr{G}(\Gamma)$.

If the presentation of the fundamental group $\pi_1(\mathscr{G}_{\mathscr{R}}(\Gamma))$ corresponds to a tree $\mathscr{T}$, and we will always assume that this holds, then the identity mapping of the generators of $\mathfrak{G}$ into $\pi_1(\mathscr{G}_{\mathscr{R}}(\Gamma))$ determines a surjective homomorphism that we denote by $\rho_{\mathscr{R}}$. It is not difficult to show that its kernel coincides with the normal closure in $\mathfrak{G}$ of $\bigcup_{v \in \mathscr{V}} R_v$, and $\ker \rho_{\mathscr{R}} \cap G_v = R_v$ for all $v \in \mathscr{V}$.

Given an arbitrary class $\mathscr{C}$ of groups, call a system $\mathscr{R}$ *$\mathscr{C}$-admissible* if there exists a homomorphism of $\pi_1(\mathscr{G}_{\mathscr{R}}(\Gamma))$ onto a group of class $\mathscr{C}$ acting injectively on all vertices of $\overline{G}_v$ for $v \in \mathscr{V}$.

Proposition 6.1 [17, Proposition 2]. *Consider an arbitrary class $\mathscr{C}$ of groups.*

(1) *If N is a normal subgroup of $\mathfrak{G}$ then the family $\{N \cap G_v \mid v \in \mathscr{V}\}$ is a system of compatible normal subgroups of $\mathfrak{G}$. If $N \in \mathscr{C}^*(\mathfrak{G})$ then this system is $\mathscr{C}$-admissible.*

(2) *If $\mathscr{R} = \{R_v \mid v \in \mathscr{V}\}$ is a $\mathscr{C}$-admissible system of compatible normal subgroups of $\mathfrak{G}$ then there is a subgroup $N \in \mathscr{C}^*(\mathfrak{G})$ such that $R_v = N \cap G_v$ for all $v \in \mathscr{V}$.*

Henceforth, if N is a normal subgroup of $\mathfrak{G}$ then we denote the graph $\mathscr{G}_{\mathscr{R}}(\Gamma)$ and the homomorphism $\rho_{\mathscr{R}}$ corresponding to the system $\mathscr{R} = \{N \cap G_v \mid v \in \mathscr{V}\}$ of compatible normal subgroups by $\mathscr{G}_N(\Gamma)$ and ρ_N. Proposition 6.1 and Theorem 2.1 yield

Proposition 6.2. *If $\mathscr{C}$ is a root class of groups and $N \in \mathscr{C}^*(\mathfrak{G})$ then the group $\pi_1(\mathscr{G}_N(\Gamma))$ is residually a $\mathscr{C}$-group and each of its $\mathfrak{P}(\mathscr{C})'$-isolated subgroups satisfying a nontrivial identity is $\mathscr{C}$-separable.*

As in [1], if Δ is a nonempty connected subgraph of Γ; then denote by $\mathscr{G}(\Delta)$ the graph of groups with the same vertex and edges groups, directions and homomorphisms as in the graph $\mathscr{G}(\Gamma)$. Call this subgraph Δ *admissible* whenever the graph $\Delta \cap \mathscr{T}$ is a maximal subtree of Δ. Henceforth, speaking of an admissible subgraph Δ, we assume that the presentation of $\pi_1(\mathscr{G}(\Delta))$ corresponds to the tree $\Delta \cap \mathscr{T}$. It is not difficult to show (see [17, Proposition 1] for instance) that under this assumption the identity mapping of the generators of $\pi_1(\mathscr{G}(\Delta))$ into $\mathfrak{G}$ determines an injective homomorphism, and so we may regard $\pi_1(\mathscr{G}(\Delta))$ as a subgroup of $\mathfrak{G}$.

Proposition 6.3 [17, Proposition 3]. *Given an admissible subgraph Δ of Γ and a presentation of the group $\mathfrak{H} = \pi_1(\mathscr{G}(\Delta))$ corresponding to a tree $\Delta \cap \mathscr{T}$, if N is a normal subgroup of $\mathfrak{G}$ and $M = N \cap \mathfrak{H}$, then the homomorphism ρ_N extends $\rho_M\colon \mathfrak{H} \to \pi_1(\mathscr{G}_M(\Delta))$.*

Proposition 6.4 [1, Proposition 8]. *Given finite subsets $\mathscr{V}' \subseteq \mathscr{V}$, $\mathscr{E}' \subseteq \mathscr{E}$, and $S \subseteq \mathfrak{G}$, there exists an admissible finite subgraph Δ of Γ satisfying $S \subseteq \pi_1(\mathscr{G}(\Delta))$ and containing all vertices of $\mathscr{V}'$ and all edges of $\mathscr{E}'$.*

7. Proof of Theorem 2.3 and Corollary 2.4

Proposition 7.1 [17, Proposition 4]. *Consider a nonempty family Ω of normal subgroups of $\mathfrak{G}$ such that*

(α) $\forall L, M \in \Omega\ \exists N \in \Omega\ N \le L \cap M$;

(β) $\forall v \in \mathscr{V}\ \bigcap_{N\in\Omega}(N \cap G_v) = 1$;

(γ) $\forall e \in \mathscr{E},\ \varepsilon = \pm 1\ \bigcap_{N\in\Omega} H_{\varepsilon e}(N \cap G_{e(\varepsilon)}) = H_{\varepsilon e}$.

Suppose that Γ is a finite graph. If $v \in \mathscr{V}$ and a subgroup $X \le G_v$ satisfy $\bigcap_{N\in\Omega} X(N \cap G_v) = X$ then for each $g \in \mathfrak{G} \setminus X$ there is a subgroup $N \in \Omega$ with $g\rho_N \notin X\rho_N$. In particular, if $g \in \mathfrak{G} \setminus \{1\}$ then $g\rho_N \ne 1$ for some subgroup $N \in \Omega$.

Henceforth, given a nonempty family Ω of subgroups of $\mathfrak{G}$, denote by $\mathfrak{D}_\Omega(\mathfrak{G})$ the family of pairs of subgroups of the same group defined as $(X, Y) \in \mathfrak{D}_\Omega(\mathfrak{G})$ if and only if one of the following holds:

(λ_Ω) X is a finitely generated abelian subgroup of the group G_v for some $v \in \mathscr{V}$, while $Y = 1$, and $\bigcap_{N\in\Omega} X(N \cap G_v) \ne X$;

(μ_Ω) X is a finitely generated abelian subgroup of the group $H_{\varepsilon e}$ for some $e \in \mathscr{E}$ and $\varepsilon = \pm 1$, while Y is an infinite cyclic subgroup satisfying $[X, Y] = 1$ and $Y \cap G_{e(\varepsilon)} = 1$, while $\bigcap_{N\in\Omega} X(N \cap G_{e(\varepsilon)}) \ne X$.

Put $\mathfrak{A}_\Omega(\mathfrak{G}) = \{XY \mid (X, Y) \in \mathfrak{D}_\Omega(\mathfrak{G})\}$.

Proposition 7.2. *Suppose that $\mathscr{C}$ is a root class of groups and take a nonempty subset Ω of the family $\mathscr{C}^*(\mathfrak{G})$ satisfying conditions (α)–(γ) of Proposition 7.1. Consider an admissible subgraph Δ of Γ and put $\mathfrak{H} = \pi_1(\mathscr{G}(\Delta))$ and $\Xi = \{N \cap \mathfrak{H} \mid N \in \Omega\}$. Then*

(1) the family Ξ is nonempty and lies in $\mathscr{C}^*(\mathfrak{H})$; the graph Δ, the group $\mathfrak{H}$, and the family Ξ satisfy conditions (α)–(γ);

(2) if A is a $\mathfrak{P}(\mathscr{C})'$-isolated subgroup of $\mathfrak{G}$ not conjugate to any subgroup of the family $\mathfrak{A}_\Omega(\mathfrak{G})$ and $A \le \mathfrak{H}$ then the subgroup A is $\mathfrak{P}(\mathscr{C})'$-isolated in $\mathfrak{H}$ and is not conjugate to any subgroup of the family $\mathfrak{A}_\Xi(\mathfrak{H})$;

(3) if A is an abelian subgroup of $\mathfrak{H}$, while $g \in \mathfrak{H} \setminus A$, $N \in \Omega$, and $M = N \cap \mathfrak{H}$, then $g\rho_M \notin \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{H}\rho_M, A\rho_M)$ implies that $g\rho_N \notin \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)$.

Proof. (1): It is obvious that Ξ is nonempty and satisfies condition (α). Its definition implies easily that every subgroup X of $\mathfrak{G}$ and every vertex v of Δ satisfy

$$\bigcap_{N\in\Omega} X(N \cap G_v) = \bigcap_{M\in\Xi} X(M \cap G_v).$$

Hence, the graph Δ, the group $\mathfrak{H}$, and the family Ξ satisfy conditions (β) and (γ); moreover, $\mathfrak{D}_\Xi(\mathfrak{H}) \subseteq \mathfrak{D}_\Omega(\mathfrak{G})$ and $\mathfrak{A}_\Xi(\mathfrak{H}) \subseteq \mathfrak{A}_\Omega(\mathfrak{G})$. It remains to observe that if $N \in \Omega$ and $M = N \cap \mathfrak{H}$ then $N \in \mathscr{C}^*(\mathfrak{G})$,

$$\mathfrak{H}/(N \cap \mathfrak{H}) \cong \mathfrak{H}N/N \le \mathfrak{G}/N \in \mathscr{C},$$

and $\mathfrak{H}/(N \cap \mathfrak{H}) \in \mathscr{C}$ because the class $\mathscr{C}$ is closed under subgroups. Consequently, $\Xi \subseteq \mathscr{C}^*(\mathfrak{H})$.

(2): The inclusion $\mathfrak{A}_\Xi(\mathfrak{H}) \subseteq \mathfrak{A}_\Omega(\mathfrak{G})$ established above implies that A is not conjugate to any subgroup of $\mathfrak{A}_\Xi(\mathfrak{H})$. It is obviously $\mathfrak{P}(\mathscr{C})'$-isolated in $\mathfrak{H}$.

(3): By Proposition 6.3, the homomorphism ρ_N extends ρ_M, whence $g\rho_N \notin \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{H}\rho_N, A\rho_N)$. Proposition 4.4 and the $\mathscr{C}$-residuality of $\mathfrak{G}\rho_N$, which holds by Proposition 6.2, imply that the subgroup

$\mathfrak{I}_N = \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)$ is abelian and coincides with the set $\mathfrak{R}_N = \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Rt}(\mathfrak{G}\rho_N, A\rho_N)$. Assuming that $g\rho_N \in \mathfrak{R}_N$, from $g\rho_N \in \mathfrak{H}\rho_N$ and $A\rho_N \leq \mathfrak{H}\rho_N$ we infer that

$$g\rho_N \in \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Rt}(\mathfrak{H}\rho_N, A\rho_N) \leq \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{H}\rho_N, A\rho_N)$$

in contradiction with the above. Thus, $g\rho_N \notin \mathfrak{I}_N$. $\square$

While proving the following claim, we will use without further explanations the concepts and notation of Section 2 of [1]. We refer to Propositions 3, 5, and 6 of [1] as I.3, I.5, and I.6 respectively.

Proposition 7.3. *Given a root class $\mathscr{C}$ of groups, suppose that the graph Γ is finite, consider a nonempty subset Ω of the family $\mathscr{C}^*(\mathfrak{G})$, and suppose that conditions (α)–(γ) of Proposition 7.1 are fulfilled. If A is a $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup of $\mathfrak{G}$ not conjugate to any subgroup of $\mathfrak{A}_\Omega(\mathfrak{G})$ then for every $g \in \mathfrak{G} \setminus A$ there is a subgroup $N \in \Omega$ such that $g\rho_N \notin \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)$.*

Proof. Any subgroup $N \in \Omega$ and an element $u \in \mathfrak{G}$ satisfy

$$\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, (u^{-1}Au)\rho_N) = (u\rho_N)^{-1}\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)(u\rho_N),$$

and so $(u^{-1}gu)\rho_N \notin \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(\mathfrak{G}\rho_N, (u^{-1}Au)\rho_N)$ implies that $g\rho_N \notin \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)$. Therefore, we may replace g and A if need be by their images under some inner automorphism of $\mathfrak{G}$.

The proof goes by induction on the number of edges outside the tree $\mathscr{T}$. Firstly we present the step of induction, and then we will verify the base of induction. Assume that there is at least one edge f outside $\mathscr{T}$ and denote by Δ the graph resulting from Γ by removing f. Then $\mathfrak{G}$ amounts to the HNN-extension of the group $\mathfrak{B} = \pi_1(\mathscr{G}(\Delta))$ with stable letter t_f and associated subgroups H_{+f} and H_{-f}.

Lemma 1. *Given an element x of this HNN-extension and its reduced form $x_0 t_f^{\varepsilon_1} x_1 \dots t_f^{\varepsilon_n} x_n$, there is a subgroup $M \in \Omega$ such that for every subgroup $N \in \Omega$ lying in M the following hold:*

(a) *in the group $\mathfrak{G}\rho_N$ regarded as the HNN-extension of the group $\mathfrak{B}\rho_N$ with stable letter t_f and associated subgroups $H_{\pm f}\rho_N$ the product $x_0\rho_N t_f^{\varepsilon_1} x_1\rho_N \dots t_f^{\varepsilon_n} x_n\rho_N$ is a reduced form for the element $x\rho_N$; in particular, if x is a nonprimitive element then so is $x\rho_N$;*

(b) *if $x_0 = 1$ and $t_f^{\varepsilon_1} x_1 \dots t_f^{\varepsilon_n} x_n$ is a cyclically reduced form for x then $t_f^{\varepsilon_1} x_1\rho_N \dots t_f^{\varepsilon_n} x_n\rho_N$ is a cyclically reduced form for $x\rho_N$.*

Proof. Given $i \in \{0, \dots, n\}$, define the subgroup $L_i \in \Omega$ as follows. If $1 \leq i \leq n-1$ and $\varepsilon_i = -\varepsilon_{i+1}$, and consequently $x_i \notin H_{-\varepsilon_i f}$, then use Proposition 7.1 and find $L_i \in \Omega$ satisfying the condition $x_i\rho_{L_i} \notin H_{-\varepsilon_i f}\rho_{L_i}$. If $x_0 = 1$, $\varepsilon_n = -\varepsilon_1$, and $x_n \notin H_{-\varepsilon_n f}$; then we similarly choose some subgroup $L_n \in \Omega$ such that $x_n\rho_{L_n} \notin H_{-\varepsilon_n f}\rho_{L_n}$. In the remaining cases take as L_i an arbitrary subgroup of the family Ω which is nonempty by assumption.

Put $L = \bigcap_{0 \leq i \leq n} L_i$. Take $N \in \Omega$ and $N \leq L$. Then $\ker\rho_N \leq \ker\rho_{L_i}$ for all $i \in \{0, \dots, n\}$. Hence, the choice of L_i for $0 \leq i \leq n$ implies that if $1 \leq i \leq n-1$ and $\varepsilon_i = -\varepsilon_{i+1}$ then $x_i\rho_N \notin H_{-\varepsilon_i f}\rho_N$; while if $x_0 = 1$, $\varepsilon_n = -\varepsilon_1$, and $x_n \notin H_{-\varepsilon_n f}$, then $x_n\rho_N \notin H_{-\varepsilon_n f}\rho_N$. Thus, every subgroup $M \in \Omega$ lying in L is the required one, and condition (α) guarantees that they exist. $\square$

In view of the remark at the beginning of the proof and Proposition I.5, without loss of generality we may assume that the subgroup A either lies in $\mathfrak{B}$ or decomposes as the direct product $X \times \langle y \rangle$, where y is a nonprimitive cyclically reduced element, while $X \leq H_{+f}$ or $X \leq H_{-f}$. Put $\Xi = \{N \cap \mathfrak{B} \mid N \in \Omega\}$ and consider the three cases:

Case 1. $A \leq \mathfrak{B}$ and $g \in \mathfrak{B}$.

Since Δ is obviously an admissible subgraph of Γ, by claims (1) and (2) of Proposition 7.2 the graph Δ, the group $\mathfrak{B}$, the family Ξ, and the subgroup A satisfy the hypotheses of Proposition 7.3. Since $g \in \mathfrak{B} \setminus A$, with the inductive assumption this implies that there exists a subgroup $M \in \Xi$ satisfying $g\rho_M \notin \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(\mathfrak{B}\rho_M, A\rho_M)$. If a subgroup $N \in \Omega$ satisfies $M = N \cap \mathfrak{B}$ then claim (3) of Proposition 7.2 yields $g\rho_N \notin \mathfrak{P}(\mathscr{C})'$-$\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)$. Consequently, N is the required subgroup.

Case 2. $A \leq \mathfrak{B}$ and $g \notin \mathfrak{B}$.

Since $g \notin \mathfrak{B}$, by Lemma 1 there is a subgroup $N \in \Omega$ with $g\rho_N \notin \mathfrak{B}\rho_N$. Let us verify that $\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N) \le \mathfrak{B}\rho_N$ and so N is the required subgroup.

Indeed, if $\mathscr{C}$ contains nonperiodic groups then

$$\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N) = A\rho_N \le \mathfrak{B}\rho_N.$$

Therefore, assume that $\mathscr{C}$ consists of periodic groups. Since $N \in \Omega \subseteq \mathscr{C}^*(\mathfrak{G})$ and

$$G_v\rho_N \cong G_v/G_v \cap N \cong G_vN/N \le \mathfrak{G}/N$$

for all $v \in \mathscr{V}$, it follows that $H_{+f}\rho_N$ and $H_{-f}\rho_N$ are periodic $\mathfrak{P}(\mathscr{C})$-groups. Proposition 6.2 implies that $\mathfrak{G}\rho_N$ is residually a $\mathscr{C}$-group; consequently, it is $\mathfrak{P}(\mathscr{C})'$-torsion-free. Hence, the subgroups $H_{+f}\rho_N$ and $H_{-f}\rho_N$ are $\mathfrak{P}(\mathscr{C})'$-isolated in $\mathfrak{G}\rho_N$ and in particular

$$\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Rt}(\mathfrak{G}\rho_N, H_{+f}\rho_N) = H_{+f}\rho_N, \quad \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Rt}(\mathfrak{G}\rho_N, H_{-f}\rho_N) = H_{-f}\rho_N.$$

In view of Proposition I.6 this implies that $\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Rt}(\mathfrak{G}\rho_N, A\rho_N) \le \mathfrak{B}\rho_N$. It remains to observe that the subgroup $A\rho_N$ is abelian and Proposition 4.4 yields

$$\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N) = \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Rt}(\mathfrak{G}\rho_N, A\rho_N).$$

Case 3. $A = X \times \langle y \rangle$, where y is a nonprimitive cyclically reduced element, while $X \le H_{+f}$ or $X \le H_{-f}$.

Take the number $\delta = \pm 1$ such that $X \le H_{\delta f}$ and the greatest $\mathfrak{P}(\mathscr{C})'$-divisor r of $\ell(y)$. Since the subgroup A is $\mathfrak{P}(\mathscr{C})'$-isolated in $\mathfrak{G}$ and $g \notin A$, it follows that $g \notin X$ and $g^r \notin A$. In particular, if $\ell(g)r = \ell(y)s$ for some integer $s > 0$; then $y^{-s}g^r, y^{-s}g^{-r} \notin X$. Since y is a cyclically reduced nonprimitive element, by Proposition I.3 so is y^n for every $n > 0$. Hence, $\langle y \rangle \cap G_{f(\delta)} = 1$ and if the subgroup $\overline{X} = \bigcap_{N \in \Omega} X(N \cap G_{f(\delta)})$ is distinct from X then (μ_Ω) holds for the pair $(X, \langle y \rangle)$. However, then $A \in \mathfrak{A}_\Omega(\mathfrak{G})$ in contradiction with the assumption. Thus, $\overline{X} = X$ and by Proposition 7.1 there exist subgroups $L_0, L_1, L_{-1} \in \Omega$ such that $g\rho_{L_0} \notin X\rho_{L_0}$ and if $\ell(g)r = \ell(y)s$ for some integer $s > 0$ then $(y^{-s}g^{\theta r})\rho_{L_\theta} \notin X\rho_{L_\theta}$, where $\theta = \pm 1$.

Using Lemma 1 and condition (α), choose a subgroup $N \in \Omega$ such that $N \le L_{-1} \cap L_0 \cap L_1$, $\ell(g\rho_N) = \ell(g)$, and $\ell(y\rho_N) = \ell(y)$, while $y\rho_N$ is still cyclically reduced. Then $\ker \rho_N \le \ker \rho_{L_i}$ for $i \in \{-1, 0, 1\}$. Thus, $g\rho_N \notin X\rho_N$ and if $\ell(g)r = \ell(y)s$ for some integer $s > 0$ then $(y^{-s}g^{\theta r})\rho_N \notin X\rho_N$, where $\theta = \pm 1$.

Since $y\rho_N$ is a nonprimitive cyclically reduced element, by Proposition I.3 we cannot extract its roots of arbitrarily high degree. Proposition 6.2 shows that $\mathfrak{G}\rho_N$ is residually a $\mathscr{C}$-group, and so Proposition 4.4 implies that the subgroup $\overline{Y}_N = \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, \langle y\rho_N \rangle)$ is cyclic. Denote by z_N its generator such that $z_N^q = y\rho_N$ for some $q > 0$. According to Proposition I.3 then z_N is a nonprimitive cyclically reduced element and $q \mid \ell(y\rho_N) = \ell(y)$. Since q is obviously a $\mathfrak{P}(\mathscr{C})'$-number, this implies that $q \mid r$.

For brevity, denote the subgroup $\mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)$ by $\overline{A}_N$ and suppose that $g\rho_N \in \overline{A}_N$. If $\mathscr{C}$ contains nonperiodic groups then

$$\overline{A}_N = A\rho_N = X\rho_N \cdot \langle y\rho_N \rangle = X\rho_N \cdot \overline{Y}_N.$$

Otherwise, $X\rho_N$ lies in the periodic $\mathfrak{P}(\mathscr{C})$-group $H_{\delta f}\rho_N$ and Proposition 4.5 again implies that $\overline{A}_N = X\rho_N \cdot \overline{Y}_N$. Thus, for some $\xi = \pm 1$, $x \in X$, and $n \ge 0$ we have $(g\rho_N)^\xi = (x\rho_N)z_N^n$.

If $n = 0$ then $g\rho_N \in X\rho_N$ despite the previous argument. Consequently, $n > 0$ and by Proposition I.3 the element $(x\rho_N)^{-1}(g\rho_N)^\xi = z_N^n$ is cyclically reduced and has length $\ell(z_N)n$. At the same time, it is obvious that $\ell((x\rho_N)^{-1}(g\rho_N)^\xi) = \ell(g\rho_N) = \ell(g)$, and so $\ell(g) = \ell(z_N)n$. Since $z_N^q = y\rho_N$ and $r = qk$ for some integer $k > 0$, we see that $z_N^r = (y\rho_N)^k$, and again Proposition I.3 yields $\ell(z_N)r = \ell(y\rho_N)k$. Consequently, $\ell(g)r = \ell(z_N)rn = \ell(y\rho_N)kn = \ell(y)kn$, and by the choice of the subgroup N we have $(y^{-kn}g^{\theta r})\rho_N \notin X\rho_N$, where $\theta = \pm 1$. However, since $\overline{A}_N$ is abelian by Proposition 4.4, from $(g\rho_N)^\xi =$

$(x\rho_N)z_N^n$ we infer that $(g\rho_N)^{\xi r} = z_N^{rn}(x\rho_N)^r = (y\rho_N)^{kn}(x\rho_N)^r$ and $(y\rho_N)^{-kn}(g\rho_N)^{\xi r} \in X\rho_N$. The resulting contradiction shows that $g\rho_N \notin \overline{A}_N$, and so N is the required subgroup.

Thus, the inductive step is complete. Suppose now that Γ is a tree. To prove the claim in this case we induct on the number of the vertices of Γ. If Γ contains only one vertex v, then A is a $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup of G_v and, since $A \notin \mathfrak{A}_\Omega(\mathfrak{G})$, it follows that $\bigcap_{N\in\Omega} AN = A$. Consequently, $g \notin AN$ for some subgroup $N \in \Omega$, which turns out the required subgroup because the mapping ρ_N in this case amounts to the natural homomorphism $\mathfrak{G} \to \mathfrak{G}/N$, while every subgroup of the $\mathscr{C}$-group $\mathfrak{G}/N$ is $\mathfrak{P}(\mathscr{C})'$-isolated by Proposition 4.1.

Assume henceforth that Γ has at least one edge $f \in \mathscr{E}$. Removing f splits Γ into two connected components, and we denote by Δ_ε for $\varepsilon = \pm1$ the one containing the vertex $f(\varepsilon)$. Denote also by $\mathfrak{B}_\varepsilon$ the group $\pi_1(\mathscr{G}(\Delta_\varepsilon))$. Then $\mathfrak{G}$ amounts to the free product of $\mathfrak{B}_1$ and $\mathfrak{B}_{-1}$ with amalgamated subgroups H_{+f} and H_{-f}.

Lemma 2. *Given an element x of the mentioned free product with a reduced form $x_1x_2\dots x_n$, there is a subgroup $M \in \Omega$ such that for every subgroup $N \in \Omega$ lying in M the following hold:*

(a) *in $\mathfrak{G}\rho_N$ regarded as the free product of the groups $\mathfrak{B}_1\rho_N$ and $\mathfrak{B}_{-1}\rho_N$ with amalgamated subgroups $H_{\pm f}\rho_N$ the product $x_1\rho_N x_2\rho_N \dots x_n\rho_N$ is a reduced form for $x\rho_N$; in particular, if x is nonprimitive or cyclically reduced then so is $x\rho_N$;*

(b) *if $x \in \mathfrak{B}_\varepsilon \setminus H_{\varepsilon f}$ for some $\varepsilon = \pm1$ then $x\rho_N \in \mathfrak{B}_\varepsilon\rho_N \setminus H_{\varepsilon f}\rho_N$.*

Proof. If $x \in H_{+f} = H_{-f}$ then the claim is obvious. Therefore, assume that for each $i \in \{1,\dots,n\}$ there exists a number $\varepsilon_i = \pm1$ such that $x_i \in \mathfrak{B}_{\varepsilon_i} \setminus H_{\varepsilon_i f}$ and if $i < n$ then $\varepsilon_i \neq \varepsilon_{i+1}$. Using Proposition 7.1, for every $i \in \{1,\dots,n\}$ we can choose a subgroup $L_i \in \Omega$ with $x_i\rho_{L_i} \in \mathfrak{B}_{\varepsilon_i}\rho_{L_i} \setminus H_{\varepsilon_i f}\rho_{L_i}$. If $N \in \Omega$ and $N \leq L_i$ then $\ker\rho_N \leq \ker\rho_{L_i}$ and consequently $x_i\rho_N \in \mathfrak{B}_{\varepsilon_i}\rho_N \setminus H_{\varepsilon_i f}\rho_N$. Thus, the subgroup $M \in \Omega$ lying in $\bigcap_{1\leq i\leq n} L_i$ is the required one and condition (α) guarantees that it exists. $\square$

As above, in view of the remark at the beginning of the proof and Proposition I.5, without loss of generality we may assume that A either lies in $\mathfrak{B}_\varepsilon$ for some $\varepsilon = \pm1$ or decomposes as the direct product $X \times \langle y\rangle$, where y is a nonprimitive cyclically reduced element and $X \leq H_{+f} = H_{-f}$. Therefore, it suffices to consider the three cases: $A \leq \mathfrak{B}_\varepsilon$ and $g \in \mathfrak{B}_\varepsilon$; $A \leq \mathfrak{B}_\varepsilon$ and $g \notin \mathfrak{B}_\varepsilon$; and $A = X \times \langle y\rangle$, where y is a nonprimitive cyclically reduced element and $X \leq H_{+f} = H_{-f}$.

The arguments in each case repeat word for word those used above to inspect cases (i)–(iii). We should only put $\Xi_\varepsilon = \{N \cap \mathfrak{B}_\varepsilon \mid N \in \Omega\}$, replace the symbols Δ, $\mathfrak{B}$, and Ξ with Δ_ε, $\mathfrak{B}_\varepsilon$, and Ξ_ε respectively, as well as to apply Lemma 2 instead of Lemma 1 in the last two cases. $\square$

Proof of Theorem 2.3.

I. Necessity: For $(X,Y) \in \mathfrak{D}^1_{\mathscr{C}}(\mathfrak{G})$ put $A = XY$ and denote by Z the subgroup of $\mathfrak{G}$ coinciding with G_v if condition $(\lambda^1_{\mathscr{C}})$ holds or with $G_{e(\varepsilon)}$ if condition $(\mu^1_{\mathscr{C}})$ holds. Then the definition of $\mathfrak{D}^1_{\mathscr{C}}(\mathfrak{G})$ implies that $X \leq Z$ and $Y \cap Z = 1$, while $\overline{X} = \bigcap_{N\in\mathscr{C}^*(\mathfrak{G})} X(N \cap Z)$ is distinct from X. Take $g \in \overline{X} \setminus X$. Then $g \in \bigcap_{N\in\mathscr{C}^*(\mathfrak{G})} AN$ and hence g is taken into an element of A by each homomorphism of $\mathfrak{G}$ onto a group of $\mathscr{C}$. If we suppose that $g \in A$ and $g = xy$ with $x \in X$ and $y \in Y$ then, since $g \in \overline{X} \leq Z$, the relations $X \leq Z$ and $Y \cap Z = 1$ imply that $y = 1$ and $g \in X$ despite the choice of g. Consequently, $g \notin A$ and A is not separable in $\mathfrak{G}$ by $\mathscr{C}$.

Thus, all subgroups of the family $\mathfrak{A}^1_{\mathscr{C}}(\mathfrak{G})$ are not $\mathscr{C}$-separable in $\mathfrak{G}$. Conjugate subgroups obviously have the same properties.

Sufficiency: Suppose that A is a $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup of $\mathfrak{G}$ not conjugate to any subgroup of the family $\mathfrak{A}^1_{\mathscr{C}}(\mathfrak{G})$ and take some $g \in \mathfrak{G} \setminus A$. It suffices to find a subgroup $L \in \mathscr{C}^*(\mathfrak{G})$ with $g \notin AL$.

Take a finite generating set S of A. According to Proposition 6.4 applied to $\mathscr{V}' = \varnothing$, $\mathscr{E}' = \varnothing$, and $S \cup \{g\}$, there exists a finite admissible subgraph Δ of Γ such that $S \cup \{g\} \subseteq \pi_1(\mathscr{G}(\Delta))$. Then A is

a $\mathfrak{P}(\mathscr{C})'$-isolated subgroup of $\mathfrak{H} = \pi_1(\mathscr{G}(\Delta))$ and $g \in \mathfrak{H} \setminus A$. Put $\Omega = \mathscr{C}^*(\mathfrak{G})$ and verify that A is not conjugate to any subgroup of the family $\mathfrak{A}_\Omega(\mathfrak{G})$.

Indeed, take $(X, Y) \in \mathfrak{D}_\Omega(\mathfrak{G})$ and either $Z = G_v$ if condition (λ_Ω) holds or $Z = G_{e(\varepsilon)}$ if condition (μ_Ω) holds. If X is $\mathfrak{P}(\mathscr{C})'$-isolated in Z then $XY \in \mathfrak{A}^1_{\mathscr{C}}(\mathfrak{G})$ and the hypotheses of Theorem 2.3 yield $A \not\sim_{\mathfrak{G}} XY$. Otherwise, there exist $z \in Z \setminus X$ and a number $q \in \mathfrak{P}(\mathscr{C})'$ with $z^q \in X$. From $Y \cap Z = 1$ we see that $z \notin XY$, and so the subgroup XY is not $\mathfrak{P}(\mathscr{C})'$-isolated in $\mathfrak{G}$. Consequently, $A \not\sim_{\mathfrak{G}} XY$.

If $M, N \in \mathscr{C}^*(\mathfrak{G})$ then by Remak's Theorem [38, Theorem 4.3.9] the quotient $\mathfrak{G}/M \cap N$ embeds into the direct product of $\mathscr{C}$-groups $\mathfrak{G}/M$ and $\mathfrak{G}/N$ and lies in the class $\mathscr{C}$ because the latter is closed under subgroups and extensions. Since $\mathscr{C}$ is nonempty, the properties mentioned also imply that this class contains the trivial group. Hence, the family $\Omega = \mathscr{C}^*(\mathfrak{G})$ is nonempty and satisfies condition (α) of Proposition 7.1. Since conditions (β) and (γ) of Proposition 7.1 for $\Omega = \mathscr{C}^*(\mathfrak{G})$ coincide with conditions (i^1) and (ii^1) of Theorem 2.2, Proposition 7.2 is applicable to Γ, $\mathfrak{G}$, $\mathscr{C}^*(\mathfrak{G})$, Δ, and A.

By claims (1) and (2) of Proposition 7.2 the graph Δ, the group $\mathfrak{H}$, the family $\Xi = \{N \cap \mathfrak{H} \mid N \in \mathscr{C}^*(\mathfrak{G})\}$, and the subgroup A satisfy the hypotheses of Proposition 7.3. According to the latter, $g \in \mathfrak{H} \setminus A$ implies that for some subgroup $M \in \Xi$ we have $g\rho_M \notin \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{H}\rho_M, A\rho_M)$. Hence, in view of the definition of Ξ, claim (3) of Proposition 7.2 implies the existence of a subgroup $N \in \Omega$ such that $M = N \cap \mathfrak{H}$ and $g\rho_N \notin \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)$.

Since $\mathfrak{G}\rho_N$ is residually a $\mathscr{C}$-group by Proposition 6.2, Proposition 4.4 implies that the subgroup $\mathfrak{I}_N = \mathfrak{P}(\mathscr{C})'\text{-}\mathfrak{Is}(\mathfrak{G}\rho_N, A\rho_N)$ is abelian. Therefore, by Proposition 6.2 again there is a subgroup $L_N \in \mathscr{C}^*(\mathfrak{G}\rho_N)$ with $g\rho_N \notin \mathfrak{I}_N L_N$. Denoting by L the full preimage of L_N under the homomorphism ρ_N, we infer that $L \in \mathscr{C}^*(\mathfrak{G})$. Since $A\rho_N \leq \mathfrak{I}_N$, we find that $g \notin AL$. Thus, L is the required subgroup.

II. Take $v \in \mathscr{V}$ and $X \leq G_v$ and put

$$\overline{X}_1 = \bigcap_{N \in \mathscr{C}^*(\mathfrak{G})} X(N \cap G_v), \quad \overline{X}_2 = \bigcap_{M \in \mathscr{C}^*(G_v)} XM.$$

Then $\overline{X}_1 \leq \overline{X}_2$ by claim $(*)$. On the other hand, if $N \in \mathscr{C}^*(\mathfrak{G})$ then

$$G_v/N \cap G_v \cong G_vN/N \leq \mathfrak{G}/N \in \mathscr{C}.$$

Since the class $\mathscr{C}$ is closed under subgroups, it follows that $N \cap G_v \in \mathscr{C}^*(G_v)$. Hence, $\overline{X}_1 = \overline{X}_2$. It remains to observe that the equality $X = \overline{X}_2$ means that X is a $\mathscr{C}$-separable subgroup of G_v, and therefore conditions (i^1), (ii^1), $(\lambda^1_{\mathscr{C}})$, and $(\mu^1_{\mathscr{C}})$ are equivalent respectively to conditions (i^2), (ii^2), $(\lambda^2_{\mathscr{C}})$, and $(\mu^2_{\mathscr{C}})$. Thus, the claim follows from the first part of the theorem. $\square$

Corollary 2.4 follows from Theorems 2.2 and 2.3 together with

Proposition 7.4. *Given a root class $\mathscr{C}$ of groups, if $\mathfrak{G}$ is residually a $\mathscr{C}$-group then the cyclic subgroup of $\mathfrak{G}$ is conjugate to a subgroup of the family $\mathfrak{B}^k_{\mathscr{C}}(\mathfrak{G})$, where $k = \overline{1, 2}$, if and only if it is conjugate to a subgroup of the family $\mathfrak{A}^k_{\mathscr{C}}(\mathfrak{G})$.*

Proof. Since $\mathfrak{B}^k_{\mathscr{C}}(\mathfrak{G}) \subseteq \mathfrak{A}^k_{\mathscr{C}}(\mathfrak{G})$, we have necessity. Let us verify sufficiency. Take a cyclic subgroup Z of $\mathfrak{G}$ conjugate to a subgroup of the form XY, where $(X, Y) \in \mathfrak{D}^k_{\mathscr{C}}(\mathfrak{G})$. Suppose that the pair (X, Y) satisfies condition $(\mu^k_{\mathscr{C}})$. Then Y is an infinite cyclic subgroup, $X \leq G_{e(\varepsilon)}$ for some $e \in \mathscr{E}$ and $\varepsilon = \pm 1$, while $XY = X \times Y$ and $X \neq \overline{X}_k$, where

$$\overline{X}_1 = \bigcap_{N \in \mathscr{C}^*(\mathfrak{G})} X(N \cap G_{e(\varepsilon)}), \quad \overline{X}_2 = \bigcap_{M \in \mathscr{C}^*(G_{e(\varepsilon)})} XM.$$

As already indicated in the proof of claim II of Theorem 2.3, since the class $\mathscr{C}$ is closed under subgroups, it follows that $\overline{X}_2 \leq \overline{X}_1$, and so $X \neq \overline{X}_1$ for all k. Together with that, being conjugate to Z, the subgroup $XY = X \times Y$ is cyclic. Thus,

$$1 = X \neq \overline{X}_1 = \bigcap_{N \in \mathscr{C}^*(\mathfrak{G})} N \cap G_{e(\varepsilon)} \leq \bigcap_{N \in \mathscr{C}^*(\mathfrak{G})} N$$

in contradiction with the $\mathscr{C}$-residuality of $\mathfrak{G}$. Consequently, (X, Y) satisfies condition $(\lambda^k_{\mathscr{C}})$, and hence $XY \in \mathfrak{B}^k_{\mathscr{C}}(\mathfrak{G})$. $\square$

8. Proofs of Theorems 3.1, 3.2 and Corollaries 3.3, 3.4

Theorem 3.1 follows from claim I of Theorem 2.3 and Proposition 8.1, which combines claim (2) of Theorem 1 of [17] and a particular case of Theorem 2 of [17]. To obtain the latter, in the theorem we should put $w(x,y) = [x,y]$ and $L_{\varepsilon e} = G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$.

Proposition 8.1. *If $\mathscr{C}$ is a root class of groups and $\mathfrak{G}$ is residually a $\mathscr{C}$-group then condition $(\mathfrak{i}^1)$ of Theorem 2.2 holds. If for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ the subgroup $H_{\varepsilon e}$ lies properly in $G_{e(\varepsilon)}$ and is central then condition $(\mathfrak{ii}^1)$ of the same theorem holds as well.*

Proposition 8.2 [11, Proposition 10 and Theorem 4]. *Given a root class $\mathscr{C}$ of groups closed under quotients, suppose that at least one of the following is met:*

(1) *$\mathscr{G}(\Gamma)$ is a finite graph of groups of type* (1) *and for each vertex $v \in \mathscr{V}$ the group G_v is $\mathscr{C}$-regular with respect to the subgroup H_v;*

(2) *$\mathscr{G}(\Gamma)$ is a finite graph of groups of type* (2) *and for any $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ the group $G_{e(\varepsilon)}$ is $\mathscr{C}$-regular with respect to the subgroup $H_{\varepsilon e}$.*

Then

I. *For any $u \in \mathscr{V}$ and $L \in \mathscr{C}^*(G_u)$ there exists a $\mathscr{C}$-admissible system of compatible normal subgroups $\mathscr{R} = \{R_v \mid v \in \mathscr{V}\}$ such that $R_u \leq L$.*

II. *If $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ then the group $\mathfrak{G}$ is residually a $\mathscr{C}$-group if and only if the groups G_v for all $v \in \mathscr{V}$ are residually $\mathscr{C}$-groups and for any $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ the subgroup $H_{\varepsilon e}$ is $\mathscr{C}$-separable in $G_{e(\varepsilon)}$.*

Proof of Theorem 3.2. Combining Proposition 6.1 and claim I of Proposition 8.2, we find that claim $(*)$ of the statement of Theorem 2.2 holds for $\mathfrak{G}$. According to claim II of Proposition 8.2, the $\mathscr{C}$-residuality of $\mathfrak{G}$ implies the fulfillment of conditions $(\mathfrak{i}^2)$ and $(\mathfrak{ii}^2)$ of the same theorem. Thus, the claim being proved follows from claim II of Theorem 2.3. □

Propositions 8.3 and 8.4 given below are particular cases of assertions of [36]; the first one follows from Proposition 6.3 and Theorem 2.2, while the second one, from Theorems 3.5 and 3.6.

Proposition 8.3. *Given a root class $\mathscr{C}$ of groups which consists of periodic groups, consider some $\mathscr{C}$-$\mathscr{BN}$-group X. Then X is $\mathscr{C}$-regular with respect to each central subgroup and every $\mathfrak{P}(\mathscr{C})'$-isolated subgroup of it is $\mathscr{C}$-separable.*

Proposition 8.4. *Given a root class $\mathscr{C}$ of groups which consists of periodic groups, suppose that $\mathscr{G}(\Gamma)$ is a finite graph of groups of type* (1) *or* (2) *and that $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$. If G_v for every $v \in \mathscr{V}$ belongs to the class $\mathscr{C}$-$\mathscr{BN}$ and is $\mathfrak{P}(\mathscr{C})'$-torsion-free then $\mathfrak{G}$ is residually a $\mathscr{C}$-group if and only if $H_{\varepsilon e}$ is $\mathfrak{P}(\mathscr{C})'$-isolated in $G_{e(\varepsilon)}$ for all $e \in \mathscr{E}$ and $\varepsilon = \pm 1$.*

The next assertion combines the particular cases of Propositions 14 and 16 of [11].

Proposition 8.5. *Given a root class $\mathscr{C}$ of groups which consists of periodic groups and closed under quotients, if the set $\mathfrak{P}(\mathscr{C})$ contains all primes then every bounded solvable group is $\mathscr{C}$-regular with respect to each central subgroup and enjoys the property of $\mathscr{C}$-separability of all subgroups.*

Proposition 8.6 [11, Corollary 3]. *Given a root class $\mathscr{C}$ of groups which consists of periodic groups such that the set $\mathfrak{P}(\mathscr{C})$ contains all primes, suppose that $\mathscr{G}(\Gamma)$ is an arbitrary graph of groups of type* (1) *or a finite graph of groups of type* (2). *If the groups G_v for all $v \in \mathscr{V}$ are bounded solvable then $\mathfrak{G}$ is residually a $\mathscr{C}$-group.*

The validity of the next assertion is established in the proof of Proposition 8.7 of [36].

Proposition 8.7. *Given a root class $\mathscr{C}$ of groups which consists of periodic groups, consider the subclass $\mathscr{D}$ of the class of periodic solvable $\mathfrak{P}(\mathscr{C})$-groups of finite exponent comprising all groups of cardinality at most the cardinality of some $\mathscr{C}$-group, not necessarily the same for all groups in $\mathscr{D}$. Then $\mathscr{D}$*

is a root class consisting of periodic groups and closed under quotients, $\mathscr{D} \subseteq \mathscr{C}$, with $\mathfrak{P}(\mathscr{D}) = \mathfrak{P}(\mathscr{C})$ and $\mathscr{D}$-$\mathscr{B}\mathscr{N} = \mathscr{C}$-$\mathscr{B}\mathscr{N}$.

PROOF OF COROLLARY 3.3. According to Proposition 4.1, the absence of $\mathfrak{P}(\mathscr{C})'$-torsion in each vertex group is a necessary condition for the $\mathscr{C}$-residuality of $\mathfrak{G}$. Hence, by Proposition 8.4 in the statement of the corollary we may replace the words "$\mathfrak{G}$ is residually a $\mathscr{C}$-group" with "the groups G_v for all $v \in \mathscr{V}$ are $\mathfrak{P}(\mathscr{C})'$-torsion-free and for any $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ the subgroup $H_{\varepsilon e}$ is $\mathfrak{P}(\mathscr{C})'$-isolated in $G_{e(\varepsilon)}$." Assume that done.

Consider a $\mathfrak{P}(\mathscr{C})'$-isolated finitely generated abelian subgroup A of $\mathfrak{G}$ and the class $\mathscr{D}$ of groups of Proposition 8.7. Then upon replacing $\mathscr{C}$ with $\mathscr{D}$ all hypotheses of the new statement of the corollary remain valid. Under these conditions $\mathfrak{G}$ is residually a $\mathscr{D}$-group by Proposition 8.4. Proposition 8.3 yields $\mathfrak{A}^2_{\mathscr{D}}(\mathfrak{G}) = \varnothing$, and G_v for every $v \in \mathscr{V}$ is $\mathscr{D}$-regular both with respect to H_v and with respect to $H_{\varepsilon e}$ for each $e \in \mathscr{E}$ and $\varepsilon = \pm 1$ such that $v = e(\varepsilon)$. Since the class $\mathscr{D}$ is closed under quotients, Theorem 3.2 now implies that A is $\mathscr{D}$-separable in $\mathfrak{G}$. It remains to observe that Proposition 8.7 implies that $\mathscr{D} \subseteq \mathscr{C}$, and so the subgroup A turns out separable by class $\mathscr{C}$ too. $\square$

PROOF OF COROLLARY 3.4. Use the same strategy as above. Since $\mathfrak{P}(\mathscr{C})$ contains all primes, each subgroup automatically turns out $\mathfrak{P}(\mathscr{C})'$-isolated. Replacing if need be $\mathscr{C}$ with the class $\mathscr{D}$ in the statement of Proposition 8.7, assume now that $\mathscr{C}$ is closed under quotients. This enables us to apply Proposition 8.5, according to which $\mathfrak{A}^2_{\mathscr{C}}(\mathfrak{G}) = \varnothing$ and G_v for every $v \in \mathscr{V}$ is $\mathscr{C}$-regular with respect to each central subgroup. Proposition 8.6 implies that $\mathfrak{G}$ is residually a $\mathscr{C}$-group. Thus, the claim follows from Theorem 3.2. $\square$

FUNDING

The study was supported by the Russian Science Foundation grant no. 22–21–00166, https://rscf.ru/en/project/22-21-00166/.

CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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