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BOOLEAN VALUED ANALYSIS OF BANACH SPACES

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Abstract—We implement the Boolean valued analysis of Banach spaces. The realizations of Banach spaces in a Boolean valued universe are lattice normed spaces. We present the basic techniques of studying these objects as well as the Boolean valued approach to injective Banach lattices.

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Introduction

We implement the ascending-descending program for Banach spaces by considering the structure and properties of the so-called lattice normed spaces, i.e., the vector spaces furnished with some norm taking values in a vector lattice. The most important peculiarities of these spaces are connected with decomposability. Use of decomposability allows us in particular to distinguish a complete Boolean algebra of linear projections in a lattice normed space which is isomorphic to the Boolean algebra of band projections of the norming lattice. Most typical in analysis are the lattice normed spaces of continuous or measurable functions.

The basic properties of lattice normed spaces present some Boolean valued interpretations of the relevant properties of normed spaces. The most fundamental connections are reflected by the three facts:

(i) The internal Banach spaces and external universally complete Banach–Kantorovich spaces are bijective under the procedure of descent from a Boolean valued model.

(ii) Each lattice normed space is realizable as a dense subspace of a Banach space viewed as a vector space over some field, e.g., the rationals, in an appropriate Boolean valued model.

(iii) Each Banach space X is a result of the restricted descent of some Banach space in a Boolean valued model if and only if X includes a complete Boolean algebra of norm one projections which possesses the cyclicity property. In other words, X is a norm complete lattice normed space with a mixed norm.

The structure of the paper is as follows: Section 1 collects the main definitions. Section 2 exhibits basic examples. Section 3 deals with the descents of Banach spaces. Section 4 exposes the properties of Maharam operators. Section 5 presents mixed normed spaces. Section 6 addresses the Boolean valued duality of Banach spaces. Section 7 discusses injective Banach lattices. Section 8 contains some comments and a few remarks on related topics.

Appendix 1 gives the basics of Boolean sets and correspondences. Appendix 2 provides the needed facts on the modified versions of ascending-descending techniques. Appendix 3 collects a few auxiliaries about order bounded operators.

Referring the reader to [1, 2] for the tool kits and notations of Boolean valued analysis, we follow [3] as regards the prerequisites of Boolean algebras. The needed facts of the theory of dominated operators are taken from [4]. Also, we supply some available results with proofs for the reader's convenience.

It is impossible to overview the whole field of applications in principle. We will dwell upon few aspects of the theory that are connected with nonstandard Boolean valued models of set theory. Scott and Takeuti pioneered this track of research (see [5, 6]) which was continued by a scarcity of mathematicians among which most contributions belong to Kusraev, Nishimura, and Ozawa. The Internet opens many

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opportunities to acquaintance with their contributions. Therefore, this paper bases primarily on the articles listed in [1, Chapter 5; 4, Chapter 2]. We distinguish [7] as an easy elementary introduction to Boolean valued analysis.

1. Main Definitions

Function spaces can often be endowed with the natural extra structure by using the elements of some vector lattices whose elements serve as etalons similar to reals. This circumstance is important for studying the properties of these function spaces.

1.1. Consider a vector space X and a real vector lattice E. We assume that all vector lattices are Archimedean. A mapping $|\!\cdot\!| : X \to E^+$ is an *E-valued vector norm* provided that

(a) $|x| = 0 \leftrightarrow x = 0$ for all $x \in X$;

(b) $|\lambda x| = |\lambda||x|$ for all $\lambda \in \mathbb{R}$ and $x \in X$;

(c) $|x + y| \leq |x| + |y|$ for all $x, y \in X$.

A mapping $|\!\cdot\!|$ is a *decomposable norm* or *Kantorovich norm* provided that $|\!\cdot\!|$, together with (a)–(c), satisfies the *decomposition axiom*

(d) for all $e_1, e_2 \in E^+$ and $x \in X$ satisfying $|x| = e_1 + e_2$, there are $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|x_k| = e_k$ with $k := 1, 2$.

If (d) is valid only for disjoint $e_1, e_2 \in E^+$; then the norm $|\!\cdot\!|$ is *disjointly decomposable* or, shortly, *d-decomposable*.

The tuple $(X, |\!\cdot\!|, E)$ as well as (X, E), or $(X, |\!\cdot\!|)$, or even X with other parameters omitted is a *lattice normed space over E* provided that $|\!\cdot\!|$ is an E-valued norm on a vector space X. In this event, E is a *norming lattice* of X. If $|\!\cdot\!|$ is decomposable (or d-decomposable), then we say that so is $(X, |\!\cdot\!|)$.

1.2. If $|x|\wedge|y| = 0$ then $x, y \in X$ are *disjoint* or, more precisely, *metrically disjoint*; in symbols $x \perp y$.

1.2(1). *If $x, y \in X$ are disjoint then*

$$|x + y| = |x| + |y|.$$

Proof. Indeed, $|x| \wedge |y| = 0$ and $|x| \leq |x + y| + |y|$ imply that $|x| \leq |x + y|$ and

$$|x| \leq \big(|x + y| + |y|\big) \wedge |x| \leq |x + y| \wedge |x| \leq |x + y|.$$

By analogy $|y| \leq |x + y|$ and so

$$|x| + |y| = |x| \vee |y| \leq |x + y|. \quad \square$$

1.2(2). *To every two disjoint $e_1, e_2 \in E$ there is at most one pair $x_1, x_2 \in X$ satisfying $x = x_1 + x_2$, $|x_1| = e_1$, and $|x_2| = e_2$.*

Proof. Assume that $|x_1| = |y_1| = e_1$ and $|x_2| = |y_2| = e_2$, while $x = x_1 + x_2 = y_1 + y_2$ and $e_1 \perp e_2$. Then $(x_1 - y_1) \perp (y_2 - x_2)$, and so $|x_1 - y_1| \leq |x_1| + |y_1| = 2e_1$ and $|x_2 - y_2| \leq 2e_2$. By hypothesis

$$0 = |(x_1 - y_1) + (x_2 - y_2)| = |x_1 - y_1| + |x_2 - y_2|,$$

which implies that $x_1 = y_1$ and $x_2 = y_2$. $\square$

As usual, we call $M^\perp := \{x \in X : (\forall\, y \in M)\, x \perp y\}$ the *disjoint complement* of a subset M of X. It is easy that disjoint complements satisfy the usual rules for abstract polars. A *band* of X is a subset M of X such that $M = M^{\perp\perp}$. Let $\mathscr{B}(X)$ stand for the inclusion-ordered set of all bands of X. Say that $K \in \mathscr{B}(X)$ admits a projection provided that $K \oplus K^\perp = X$. The projection $h(\pi)$ on some band K along the band $K^\perp$ is an order projection. Say that X is a *lattice normed space with projections* whenever each band of X admits a projection. For convenience we often write $\mathfrak{B}(X)$ rather than $\mathscr{B}(X)$; and,

abusing the language, we use the terminology of vector lattice theory. However, in the case that X is simultaneously a vector lattice, we have to be careful and avoid confusion.

Speaking about a *Boolean algebra of projections of* X in the sequel, we will bear in mind some set $\mathscr{B}$ of commuting idempotent operators in X. The *bottom* or *zero* of $\mathscr{B}$ is the zero operator and the *top* or *unity* of $\mathscr{B}$ is the identity operator, while the Boolean operations on $\mathscr{B}$ are as follows:

$$\pi \wedge \rho := \pi \circ \rho = \rho \circ \pi, \quad \pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \neg \pi = I_X - \pi \quad (\pi, \rho \in \mathscr{B}).$$

1.3. Given $L \subset E$ and $M \subset X$, put

$$h(L) := \{x \in X : |x| \in L\}, \quad |M| := \{|x| : x \in M\}.$$

Clearly, $|h(L)| \subset L \cap |X|$ and $M \subset h(|M|)$.

1.3(1). *Assume that each nonzero band of $E_0 := |X|^{\perp\perp}$ contains the norm of some nonzero element. Then $\mathscr{B}(X)$ is a complete Boolean algebra and $L \mapsto h(L)$ is an isomorphism between the Boolean algebras $\mathfrak{B}(|X|^{\perp\perp})$ and $\mathscr{B}(X)$.*

Proof. Show first that $h(L^\perp) = h(L)^\perp$ for all $L \in \mathfrak{B}(|X|^{\perp\perp})$. By definition $h(L^\perp) \subset h(L)^\perp$. If $0 \neq x \in h(L)^\perp$ then $|x|$ is disjoint from all members of L of the form $|y|$. Also, $x \notin h(L^\perp)$ implies that $e \leq |x|$ for some $0 < e \in L^+$. However, in this event the band $\{e\}^{\perp\perp}$ has no nonzero elements of the form $|y|$ in contradiction to the hypothesis. The above implies in particular that $h(L) = h(L)^{\perp\perp}$ and so $h(L) \in \mathscr{B}(X)$ for every band L of E_0.

By definition h keeps the intersection of every nonempty set of bands. Now the infimum of an arbitrary family in $\mathscr{B}(X)$ is the intersection of all members of the family since $\left(\bigcup M_\alpha\right)^\perp = \bigcap M_\alpha^\perp$. Thus, h preserves infima and disjoint complements. Assume that $h(L_1) = h(L_2)$ for some bands L_1 and L_2 of E_0. If $|x| \in L_1 \cap L_2^\perp$ then $x \in h(L_1)$ and $x \in h(L_2^\perp) = h(L_2)^\perp$. Consequently, $x = 0$, which implies that the band $L_1 \cap L_2^\perp$ is zero; i.e., $L_1 \subset L_2$. By analogy, $L_2 \subset L_1$. Hence h is injective. Since $M^\perp = h(|M|^\perp)$; therefore, h is surjective. Note also that $h(\{0\}) = \{0\}$ and $h(E_0) = X$. Thus, h is an isomorphism between $\mathfrak{B}(E_0)$ and $\mathscr{B}(X)$. Since the former set is a distributive lattice with top and bottom, so is the latter. We are done on observing that the disjoint complement in $\mathscr{B}(X)$ is the Boolean complement since this is so in $\mathfrak{B}(E_0)$ and h preserves disjoint complements. $\square$

1.4. *Assume that $E_0 := |X|^{\perp\perp}$ is a lattice with the projection property and X is a d-decomposable space. Then X is a space with the projection property. Moreover, there is an isomorphism h from $\mathfrak{P}(E_0)$ to the Boolean algebra of projections $\mathscr{P}(X)$ of X such that*

$$\pi|x| = |h(\pi)x| \quad \left(\pi \in \mathfrak{P}(E_0),\ x \in X\right).$$

Proof. By d-decomposability and 1.2(2) we see that to every $x \in X$ there is a unique pair $x_1, x_2 \in X$ satisfying $x = x_1 + x_2$, $|x_1| = \pi|x|$, and $|x_2| = \pi^\perp|x|$. This means that X is the direct sum of the bands K and $K^\perp$. Let $h(\pi)$ be the projection to $K := h(E_0)$ along $K^\perp$. By the definition of h we have $h(\pi)x \in K = h(\pi E_0)$; i.e., $|h(\pi)x| \in \pi E_0$. Hence, $\pi^\perp|h(\pi)x| = 0$ or $\pi|h(\pi)x| = |h(\pi)x|$. The elements $h(\pi)x$ and $h(\pi^\perp)x$ are disjoint. So, using 1.2(1) we may write

$$\pi|x| = \pi\left(|h(\pi)x| + |h(\pi^\perp)x|\right) = \pi|h(\pi)x|.$$

Consequently,

$$\pi|x| = \pi|h(\pi)x| = |h(\pi)x|.$$

Thus $|h(\pi)x| \in L'$ and $h(\pi)x \in K'$.

No nonzero band $L \in \mathfrak{B}(E_0)$ can be disjoint to $|X|$. So, $|x| \notin L^\perp$ for some $x \in X$. If π is a band projection to L then $\pi|x|$ is other than zero. Since X is d-decomposable, $|x_0| = \pi|x| \in L$ for some $x_0 \in X$. Consequently, we can apply 1.3. If K' is a band of X then $K' = h(L')$ for some $L' \in \mathfrak{B}(E_0)$ by 1.3. Hence,

for $x \in K'$ we have $|h(\pi)x| = \pi|x| \le |x| \in L$. Each band $K \in \mathfrak{B}(X)$ admits the projection π_K along $K^\perp$. Put $\mathscr{P} := \{\pi_K : K \in \mathfrak{B}(X)\}$. Obviously, $\mathscr{P}$ is a complete Boolean algebra of projections. Given an order projection $\rho \in \mathfrak{P}(E_0)$, assign to ρ the projection π_K with $K := h(\rho E_0)$. We will keep the letter h for the resulting mapping to conclude that h is an isomorphism between the Boolean algebras $\mathfrak{P}(E_0)$ and $\mathscr{P}$. $\square$

Identifying $\mathfrak{P}(E_0)$ and $\mathscr{P}(X) := \mathscr{P}$ in the sequel, we will write $\pi|x| = |\pi x|$ for all $x \in X$ and $\pi \in \mathfrak{P}(E_0)$.

1.5. A net $(x_\alpha)_{\alpha\in\mathrm{A}}$ is *bo-convergent* to $x \in X$, in symbols $x = bo\text{-}\lim x_\alpha$, whenever we have a decreasing net $(e_\gamma)_{\gamma\in\Gamma}$ in E such that $\inf_{\gamma\in\Gamma} e_\gamma = 0$ and to every $\gamma \in \Gamma$ there is $\alpha(\gamma) \in \mathrm{A}$ satisfying $|x - x_\alpha| \le e_\gamma$ for all $\alpha \ge \alpha(\gamma)$. Assume that for some $e \in E^+$ the following holds: To every real $\varepsilon > 0$ there is $\alpha(\varepsilon) \in \mathrm{A}$ such that $|x - x_\alpha| \le \varepsilon e$ for all $\alpha \ge \alpha(\varepsilon)$. In this event, (x_α) is *br-convergent* or *convergent with regulator* e to x; in symbols $x = br\text{-}\lim x_\alpha$. A net (x_α) is *bo-Cauchy* or *bo-fundamental*, respectively, *br-Cauchy* or *br-fundamental* provided that $(x_\alpha - x_\beta)_{(\alpha,\beta)\in\mathrm{A}\times\mathrm{A}}$ is *bo*-convergent or, respectively, *br*-convergent to zero; in other words, $(x_\alpha - x_\beta)_{(\alpha,\beta)\in\mathrm{A}\times\mathrm{A}}$ is *bo-vanishing* or *br-vanishing*. A lattice normed space X is *bo-complete* or, respectively, *br-complete* provided that each *bo*-Cauchy or, respectively, *br*-Cauchy net *bo*-converges or, respectively, *br*-converges to some element of X.

Take a family $(x_\xi)_{\xi\in\Xi}$ and arrange the net $(y_\alpha)_{\alpha\in\mathrm{A}}$, where $\mathrm{A} := \mathscr{P}_{\mathrm{fin}}(\Xi)$ is an inclusion-ordered set of all finite subsets of Ξ, and put $y_\alpha := \sum_{\xi\in\alpha} x_\xi$. If there is x satisfying $x := bo\text{-}\lim y_\alpha$ then (x_ξ) is *bo-summable* and x is the *sum* of (x_ξ); in symbols $x = bo\text{-}\sum_{\xi\in\Xi} x_\xi$.

Call $M \subset X$ *norm bounded* provided that $|M|$ is order bounded in E; i.e., there is $e \in E^+$ satisfying $|x| \le e$ for all $x \in M$. Call X *disjointly complete* or *d-complete* provided that every norm bounded subset of pairwise disjoint elements is *bo*-summable.

1.6. A decomposable *bo*-complete lattice normed space is a *Banach–Kantorovich space* or, shortly, a *BK-space*. If a BK-space X is a vector lattice with monotone lattice norm then X is a BK-lattice. Assume that (X, E) is a BK-space and $E = |X|^{\perp\perp}$. By 1.4 we may identify the Boolean algebras $\mathscr{P}(E)$ and $\mathscr{P}(X)$ to conclude that $\pi|x| = |\pi x|$ for all $\pi \in \mathfrak{P}(E)$ and $x \in X$.

1.6(1). *The sum $x := bo\text{-}\sum_{\xi\in\Xi} \pi_\xi x_\xi$ exists for every bounded family $(x_\xi)_{\xi\in\Xi}$ in X and every partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathscr{P}(X)$. Moreover, x is the unique element of X which satisfies the conditions $\pi_\xi x = \pi_\xi x_\xi$ for all $\xi \in \Xi$.*

Proof. If $e := \sup|x_\xi|$ and $y_\gamma := \sum_{\xi\in\gamma} \pi_\xi x_\xi$; then, given $\alpha, \beta \in \mathscr{P}_{\mathrm{fin}}(\Xi)$, we have

$$|y_\alpha - y_\beta| = \left|\sum_{\xi\in\alpha\triangle\beta} \pi_\xi x_\xi\right| \le \left|\sum_{\xi\in\alpha\triangle\beta} \pi_\xi\right| e \le e,$$

where $\alpha\triangle\beta$ is as usual the *symmetric difference* of α and β. Consequently, (y_α) is *bo*-Cauchy, and so $x := bo\text{-}\lim y_\alpha$ exists. $\square$

1.7. The element x in 1.6(1) is often referred to as the *mixing* of (x_ξ) by partition of unity (π_ξ) or, shortly, by (π_ξ) and denoted by $\mathrm{mix}_{\xi\in\Xi}(\pi_\xi x_\xi)$. Thus, a BK-space (X, E) contains the mixings of all norm bounded families of X by all partitions of unity. Also, we conclude that (X, E) is d-complete. As (X, E) is br-complete by definition, we see that (X, E) is d-complete and br-complete simultaneously. Therefore, $E = |X|^{\perp\perp}$ and $E^+ = |X|$; cf. [4, 2.1.7(3)]. Consequently, we may assume that the norming lattice E is a Kantorovich space.

1.7(1). Theorem. *A decomposable lattice normed space X is bo-complete if and only if X is d-complete and br-complete.*

Proof. Cp. [4, Theorem 2.2.3]. We will provide a justification by Boolean valued analysis in 3.6. $\square$

A BK-space X is *universally complete* provided that the norming lattice of X is a universally complete Kantorovich space.

1.7(2). *A BK-space X is universally complete if and only if each family in X has mixings by all partitions of unity in $\mathscr{P}(X)$.*

Proof. Necessity follows from 1.6. As regards sufficiency, cp. [4, 2.2.1]. $\square$

1.8. As a *universal extension* of a lattice normed space (X, E) we mean a universally complete BK-space (Y, mE) together with an isometric linear embedding $\imath : X \to Y$ such that Y coincides with every universally complete subspace of (Y, mE) which includes $\imath X$. Here and elsewhere we let mE stand for the universal completion of the Kantorovich space oE which is the order completion of E. Moreover, we assume that $E \subset mE$. Agree to denote the universal completion of X by mX.

1.9. Let us introduce the concept of dominated operator in a lattice normed space. To this end, consider lattice normed spaces (X, E) and (Y, F) over some vector lattices E and F. A linear operator $T : X \to Y$ is *dominated* provided that there is a positive operator $S : E \to F$ such that

$$\lvert Tx \rvert \le S(\lvert x \rvert) \quad (x \in X).$$

In this event we say that S *dominates* T or S is a *dominant* of T. Let $M(X, Y)$ stand for the set of all dominated operators from X to Y.

If there is a least dominant $\lvert T \rvert$ of an operator T in the ordered vector space $L^{\sim}(E, F)$, then $\lvert T \rvert$ is referred to as the *exact dominant* of T. Thus, $\lvert T \rvert$ is a positive operator from E to F such that

$$\lvert Tx \rvert \le \lvert T \rvert(\lvert x \rvert) \quad (x \in X).$$

1.9(1). *If X is a decomposable lattice normed space and F is a Kantorovich space then each dominated operator $T : X \to Y$ has the exact dominant $\lvert T \rvert$.*

1.9(2). Theorem. *Let X be a decomposable lattice normed space and let Y be a BK-space. Then $M(X, Y)$ is also a BK-space over $L^{\sim}(E, F)$.*

Proof. Cp. [4, Theorem 4.2.6]. $\square$

1.10. We will need orthomorphism algebras in the sequel and so we recall the basics.

Consider a vector lattice E and a vector sublattice $D \subset E$. A linear operator U from D into E is a *stabilizer* provided that $Ux \in \{x\}^{\perp\perp}$ for all $x \in D$. A stabilizer may fail to be regular. A regular stabilizer is an *orthomorphism*.

Denote by $\mathrm{Orth}(E)$ the subspace of the space of the regular operators $L^{\sim}(E)$ which comprises the orthomorphisms with domain E. We also let $\mathscr{Z}(E)$ stand for the order ideal generated by the *identity operator* I_E in $L^{\sim}(E)$. The space $\mathscr{Z}(E)$ is the *center* of E.

We now define the *orthomorphism algebra* $\mathrm{Orth}^{\infty}(E)$ of E as follows: Denote by $\mathfrak{M}$ the collection of all pairs (D, π), where D is an order dense ideal in E and π is an orthomorphism from D to E. Elements (D, π) and (D', π') in $\mathfrak{M}$ are *equivalent* provided that π and π' agree on $D \cap D'$. The factor set of $\mathfrak{M}$ by this equivalence is exactly $\mathrm{Orth}^{\infty}(E)$. Identify every $\pi \in \mathrm{Orth}(E)$ with the corresponding equivalence class in $\mathrm{Orth}^{\infty}(E)$. Then $\mathscr{Z}(E) \subset \mathrm{Orth}(E) \subset \mathrm{Orth}^{\infty}(E)$. The set $\mathrm{Orth}^{\infty}(E)$ is naturally furnished with the structure of an ordered algebra justifying the term *orthomorphism algebra*.

1.10(1). *If E is an Archimedean vector lattice then $\mathrm{Orth}^{\infty}(E)$ is a faithful f-algebra*[1)] *with unity I_E. Moreover, $\mathrm{Orth}(E)$ is an f-subalgebra in $\mathrm{Orth}^{\infty}(E)$ and $\mathscr{Z}(E)$ is an f-subalgebra of bounded elements in $\mathrm{Orth}(E)$.*

1.10(2). *Every Archimedean f-algebra E with unity $\mathbb{1}$ is algebraic and lattice isomorphic to the f-algebra of orthomorphisms. Moreover, the ideal $I(\mathbb{1})$ is mapped onto $\mathscr{Z}(E)$.*

If E is an Archimedean vector lattice then the base of each of the f-algebras $\mathrm{Orth}^{\infty}(E)$, $\mathrm{Orth}(E)$, and $\mathscr{Z}(E)$ is isomorphic to the base of E. If E is a Kantorovich space then $\mathrm{Orth}^{\infty}(E)$ is a universally complete Kantorovich space and $\mathrm{Orth}(E)$ is an order dense ideal of $\mathrm{Orth}^{\infty}(E)$.

[1)] This means that the underlying ring of the poset $\mathrm{Orth}^{\infty}(E)$ is a lattice and multiplication by every positive members of E is a positive band preserving operator; see [8, Definition 2.53].

1.11. We will distinguish the two particular classes of dominated operators:

1.11(1). Assume that $E := \mathbb{R}$ and $Y := F$. Then X is a normed space and $T : X \to F$ is dominated if and only if

$$\{Tx : \ x \in X, \ \|x\| \le 1\}$$

is order bounded in F. The supremum of the above set, denoted by $\lvert T\rvert$, is the *abstract norm* of T. This agrees with the above notation if we identify F and $L^{\sim}(\mathbb{R}, F)$. In this event we also say that T is an *operator with abstract norm.* Let $L_A(X, E)$ stand for the space of operators from X to E with abstract norm.

1.11(2). Assume now that E and F are order dense ideals of the same Kantorovich space. Call $T \in M(X, Y)$ *bounded* provided that $\lvert T\rvert \in \mathrm{Orth}(E, F)$. Let $L_b(X, Y)$ be the space of all bounded operators from X to Y. Clearly, $T \in L_b(X, Y)$ if and only if there is $c \in mE = mF$ satisfying $c \cdot E \subset F$ and $\lvert Tx\rvert \le c\lvert x\rvert$ for all $x \in X$, where we bear in mind the multiplicative structure of mE which is uniquely determined by the choice of unity; cp. [8, Chapter 2].

1.12. Recall that the relative of a Kantorovich space which is called a K_σ*-space* presents a vector lattice that contains joins and meets of nonempty countable sets. Note that if E is a K_σ-space; then, given $0 \le x$ and $u \in E$, the projection $[u] := \pi_u$ on the principal band $\{u\}^{\perp\perp}$ can be calculated as follows:

$$\pi_u x = \sup\{x \wedge (nu) : \ n \in \mathbb{N}\} \quad (0 \le x \in E).$$

Consider a K_σ-space E with unity $\mathbb{1}$. The *trace* of $x \in E$ is the projection of $\mathbb{1}$ onto the band $\{x\}^{\perp\perp}$ which is denoted by e_x. Thus, $e_x := \sup\{\mathbb{1} \wedge (n|x|) : n \in \mathbb{N}\}$. The trace e_x of x serves as a unity of $\{x\}^{\perp\perp}$, as well as a *unit element* in E or a *component* of $\mathbb{1}$; i.e., $e_x \wedge (\mathbb{1} - e_x) = 0$. Given a real λ, let e^x_λ be the trace of the positive part of $\lambda\mathbb{1} - x$; i.e., $e^x_\lambda := e_{(\lambda\mathbb{1}-x)^+}$. The $\mathfrak{E}(\mathbb{1})$-valued mapping $\lambda \mapsto e^x_\lambda$ with $\lambda \in \mathbb{R}$ is the *spectral function* or *characteristic* of x.

1.13. Recall that each unital K_σ-space is realizable as the space of spectral functions. This is done as follows:

A *resolution of unity* in a Boolean algebra B is a mapping $e : \mathbb{R} \to B$ such that
(a) $s \le t \ \to \ e(s) \le e(t)$ for all $s, t \in \mathbb{R}$;
(b) $\bigvee_{t\in\mathbb{R}} e(t) = \mathbb{1}$ and $\bigwedge_{t\in\mathbb{R}} e(t) = \mathbb{0}$;
(c) $\bigvee_{s\in\mathbb{R}, s<t} e(s) = e(t)$ for all $t \in \mathbb{R}$.

Let $\mathfrak{K}(B)$ be the set all resolutions of unity in B. Introduce the order on $\mathfrak{K}(B)$ by the formula

$$e' \le e'' \ \leftrightarrow \ (\forall\, t \in \mathbb{R})(e''(t) \le e'(t)) \quad (e', e'' \in \mathfrak{K}(B)).$$

Assume that B is a σ-algebra and consider a countable dense subfield $\mathbb{P}$ of $\mathbb{R}$. From (a)–(c) it follows that each resolution of unity is uniquely determined from its values on $\mathbb{P}$.

Given $e', e'' \in \mathfrak{K}(B)$, put

$$e : t \mapsto \bigvee\{e'(r) \wedge e''(s) : \ r, s \in \mathbb{P}, \ r + s = t\} \quad (t \in \mathbb{P}),$$
$$e : t \mapsto \bigvee\{e(s) : \ s \in \mathbb{P}, \ s < t\} \quad (t \in \mathbb{R}),$$

which is clearly a resolution of unity in B. Introduce the structure of an abelian group in $\mathfrak{K}(B)$ by letting $e' + e'' := e$. In this event $-e(t) = \bigvee\{\mathbb{1} - e(-s) : \ s \in \mathbb{P}, \ s < t\}$, and the zero $\bar{\mathbb{0}}$ takes the form $\bar{\mathbb{0}}(t) := \mathbb{1}$ in case $t > 0$ and $\bar{\mathbb{0}}(t) := \mathbb{0}$ in case $t \le 0$. Put $\bar{\mathbb{1}}(t) := \mathbb{1}$ in case $t > 1$, and $\bar{\mathbb{1}}(t) := \mathbb{0}$ in case $t \le 1$. Finally, define the multiplication of $e \in \mathfrak{K}(B)$ by a real $\alpha \in \mathbb{R}$ as follows:

$$(\alpha e)(t) := e(t/\alpha) \quad (\alpha > 0, \ t \in \mathbb{R}),$$
$$(\alpha e)(t) := (-e)(-t/\alpha) \quad (\alpha < 0, \ t \in \mathbb{R}).$$

To each $b \in B$ we assign the resolution of unity $\bar{b}$ which is defined by the rule $\bar{b}(t) := \mathbb{1}$ in case $t > 1$, $\bar{b}(t) := b^* := \mathbb{1} - b$ in case $0 < t \le 1$, and $\bar{b}(t) := \mathbb{0}$ in case $t \le 0$.

1.14. *If B is a complete Boolean algebra then $\mathfrak{K}(B)$ with the above operations and order is a universally complete Kantorovich space. The sending of $x \in \mathscr{R}{\downarrow}$ to the spectral function $\lambda \mapsto [\![x < \lambda^\wedge]\!]$ with $\lambda \in \mathbb{R}$ is an isomorphism of the Kantorovich spaces $\mathscr{R}{\downarrow}$ and $\mathfrak{K}(B)$.*

2. Basic Examples

This section provided the basic examples of the spaces of continuous, or weakly continuous, or measurable, or weakly measurable functions which admit natural lattice norms.

2.1. We start with the simplest cases of vector lattices and normed spaces.

2.1(1). If $X = E$ then we take as the vector norm of $x \in X$ the modulus of X; i.e., $\lvert x\rvert := |x| = x \vee (-x)$ for all $x \in X$. This norm is decomposable in view of the Riesz decomposition property [8, 2.1.6]. Indeed, if $|x| = e_1 + e_2$ for some $e_1, e_2, x \in E^+$, then $e_1 + e_2 = x^+ + x^-$ and so we can find $u_1, u_2 \in E^+$ and $v_1, v_2 \in E^+$ such that $e_1 = u_1 + v_1$, $e_2 = u_2 + v_2$, $x^+ = u_1 + u_2$, and $x^- = v_1 + v_2$. Also, $u_k \perp v_l$ for all $k, l := 1, 2$. Put $x_1 := u_1 - v_1$ and $x_2 := u_2 - v_2$. Then $x_1 + x_2 = x^+ - x^- = x$, $|x_1| = u_1 + v_1 = e_1$, and $|x_2| = u_2 + v_2 = e_2$.

2.1(2). If $E = \mathbb{R}$ then X is a normed space. In this event we will use the conventional notation $\|\cdot\|$ and omit any reference to the order structure of the norming lattice.

2.2. Consider the space of continuous functions on some topological space Q and some normed space Y. Let $X := C_b(Q, Y)$ be the space of bounded continuous vector functions from Q to Y. Put $E := C_b(Q, \mathbb{R})$ and define the vector norm $\lvert f\rvert$ of $f \in X$ as follows: $\lvert f\rvert : t \mapsto \|f(t)\|$ for all $t \in Q$. Then $\lvert\cdot\rvert$ is decomposable. Indeed, assume that $\lvert f\rvert = e_1 + e_2$ for some $e_1, e_2 \in E^+$. Determine the vector function $f_1 : Q \to Y$ by the conditions $f_1(t) := e_1(t)f(t)/\|f(t)\|$ if $f(t) \neq 0$ and $f_1(t) := 0$ if $f(t) = 0$. Then $f_1 \in X$ and $f_2 := f - f_1 \in X$. Moreover, $\lvert f_k\rvert = e_k$ for all $k := 1, 2$. The space X is *br*-complete if and only if Y is a Banach space.

2.3. Let Q be an extremally disconnected compact space and let E be an order dense ideal of the universally complete Kantorovich space $C_\infty(Q)$. Denote the collection of equivalence classes of continuous vector functions u each of which has domain $\mathrm{dom}(u)$ a comeager subset of Q and range in a normed space X by $C_\infty(Q, X)$. Recall that a subset M of a topological space is *comeager* provided that the complement of M is meager. In more detail, denote by $\mathscr{C}(Q, X)$ the set of vector functions $u : \mathrm{dom}(u) \to X$ such that $\mathrm{dom}(u)$ is a comeager subset of Q and u is continuous. Define the equivalence $\sim$ on $\mathscr{C}(Q, X)$ as follows: Functions u and v are equivalent provided that they coincide on the intersection of their domains; i.e., $u(t) = v(t)$ for all $t \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$. Denote the factor $\mathscr{C}(Q, X)/{\sim}$ by $C_\infty(Q, X)$.

Note that $C_\infty(Q, X)$ has the natural structure of a module over the ring $C_\infty(Q)$. Let $\widetilde{u}$ stand for the equivalence class of $u \in \mathscr{C}(Q, X)$. Given $u, v \in \mathscr{C}(Q, X)$ and $a \in C_\infty(Q)$, put

$$w(t) := u(t) + v(t) \quad (t \in \mathrm{dom}(u) \cap \mathrm{dom}(v)),$$
$$z(t) := a(t)u(t) \quad (t \in \mathrm{dom}(u) \cap \mathrm{Dom}(a)),$$

where $\mathrm{Dom}(a) := \{t \in Q : |a(t)| < +\infty\}$. Agree that $\widetilde{u} + \widetilde{v} := \widetilde{w}$ and $a \cdot \widetilde{u} := \widetilde{z}$. These definitions are clearly sound. Undoubtedly, the axioms of a module over $C_\infty(Q)$ are satisfied. Moreover, the continuous extension of the pointwise norm determines some decomposable norm on $C_\infty(Q, X)$ with range in $C_\infty(Q)$. Indeed, to each $z \in C_\infty(Q, X)$ there is a unique $x_z \in C_\infty(Q)$ such that $\|u(t)\| = x_z(t)$ for all $t \in \mathrm{dom}(u)$ for every member u of the equivalence class z. Put $\lvert z\rvert := x_z$ and note that the resulting mapping $\lvert\cdot\rvert : C_\infty(Q, X) \to C_\infty(Q)$ satisfies 1.1(a)–1.1(c). Furthermore, $\lvert ax\rvert = |a|\lvert x\rvert$ for all $a \in C_\infty(Q)$ and $x \in C_\infty(Q, X)$.

We now introduce the space

$$E(X) := \bigl\{z \in C_\infty(Q, X) : \lvert z\rvert \in E\bigr\}$$

and furnish $E(X)$ with the induced vector norm $\lvert\cdot\rvert : E(X) \to E$. The latter is decomposable, which can be shown along the lines of 2.1(1).

2.3(1). *If X is a Banach space then $E(X)$ is a BK-space whose universal completion is $C_\infty(Q, X)$.*

Proof. This is easy from 1.7(1). $\square$

2.4. Let $C_{\#}(Q,X)$ stand for the subset $C_\infty(Q,X)$ of the equivalence classes z such that $|\!|z|\!| \in C(Q)$. Thus $C_{\#}(Q,X) := E(X)$, with $E := C(Q)$. Note that $C_{\#}(Q,X)$ is also a BK-space, whereas the space $C(Q,X)$ of everywhere defined continuous vector functions from Q to X is a lattice normed space over $C(Q)$ (cp. 2.2) but fails to be d-complete in general; see [4, 2.4.8(2) and 2.4.8(3)]. In particular, $C(Q,X)$ and $C_{\#}(Q,X)$ do not coincide unless Q is finite or X is finite-dimensional; see [4, 2.4.8(5)].

2.5. We will introduce the space of weakly continuous vector functions which is analogous to $E(X)$. Assume that X is a normed space and $Z \subset X'$ is a *norming subspace* of X; i.e.,

$$\|x\|_X = \sup\{|\langle x,z\rangle| : z \in Z,\ \|z\| \le 1\} \quad (x \in X).$$

As usual X' is the dual of X and $\langle\cdot,\cdot\rangle$ is the canonical bilinear duality $X \leftrightarrow X'$.

Denote by $\mathscr{M}$ the set of $\sigma(X,Z)$-continuous vector functions $u : \operatorname{dom}(u) \to X$ such that $\operatorname{dom}(u)$ is a comeager subset of Q. Consider the factor set $C_\infty(Q,X|Z) := \mathscr{M}/{\sim}$, where $u \sim v$ means that $u(t) = v(t)$ $(t \in \operatorname{dom}(u) \cap \operatorname{dom}(v))$. Note that $C_\infty(Q,X|Z)$ admits the natural structure of a vector space: Indeed, if $\widetilde{u}$ is the equivalence class of $u \in \mathscr{M}$, then $\lambda\widetilde{u} + \mu\widetilde{v}$ is understood as the equivalence class of the pointwise linear combination $\lambda u(t) + \mu v(t)$, with $t \in \operatorname{dom}(u) \cap \operatorname{dom}(v)$. If $a \in C_\infty(Q)$ then a vector function $t \mapsto a(t)u(t)$, with $t \in \operatorname{Dom}(a) \cap \operatorname{dom}(u)$, belongs to $\mathscr{M}$ and so it defines the equivalence class $a\widetilde{w}$. Given $u \in \mathscr{M}$ and $z \in Z$, we let $\langle u,z\rangle$ stand for the continuation of $t \mapsto \langle u(t),z\rangle$ with $t \in \operatorname{dom}(u)$ on the whole of Q. If $u \sim v$ then we see that $\langle u,z\rangle = \langle v,z\rangle$. Consequently, given $w \in C_\infty(Q,X|Z)$ and $u \in w$, we may put $\langle w,z\rangle := \langle u,z\rangle$. The set $R(u) := \{\langle u,z\rangle : z \in Z, \|z\| \le 1\}$ is order bounded in $C_\infty(Q)$ as it is pointwise bounded on the comeager set $\operatorname{dom}(u)$. Granted $u \in w$, we thus can put

$$|\!|w|\!| := |\!|u|\!| := \sup\bigl\{\langle u,z\rangle : z \in Z,\ \|z\| \le 1\bigr\},$$

where the supremum is taken in $C_\infty(Q)$. Note that $\|u(\cdot)\| : t \mapsto \|u(t)\|$ $(t \in \operatorname{dom}(u))$ is the pointwise supremum of $R(u)$. Therefore, $|\!|u|\!|$ and $\|u(\cdot)\|$ coincide on a comeager subset of Q. However these functions may differ on $\operatorname{dom}(u)$.

Clearly, $|\!|\cdot|\!|$ is a decomposable norm with range in $C_\infty(Q)$. Moreover, $C_\infty(Q,X|Z)$ admits the natural structure of a faithful module over $C_\infty(Q)$ so that $|\!|au|\!| = |a||\!|u|\!|$ for $a \in C_\infty(Q)$ and $u \in C_\infty(Q,X|Z)$. Put

$$E_w(X,Z) := \bigl\{u \in C_\infty(Q,X|Z) : |\!|u|\!| \in E\bigr\}.$$

We distinguish the important particular case of $E_w(X') := E_w(X',X)$ which results in case $X := X'$ and $Z := X \subset X''$.

2.5(1). *If X is a Banach space and E is an order dense ideal of $C_\infty(Q)$; then $E_w(X,Z)$, furnished with the algebraic operations and E-valued $|\!|\cdot|\!|$ induced from $C_\infty(Q,X|Z)$, is a BK-space over E and $C_\infty(Q,X|Z)$ is the universal completion of $E_w(X,Z)$. In particular, $E_w(X')$ is a BK-space over E.*

2.6. Let (Ω,Σ,μ) be a measure space with the direct sum property, let E be an order dense ideal of $L^0(\Omega,\Sigma,\mu)$, and let X be a normed space. Denote the space of the equivalence classes of Bochner μ-measurable functions from Ω to X by $L^0(\mu,X) := L^0(\Omega,\Sigma,\mu,X)$. As usual, two vector functions on Ω are equivalent provided that they coincide almost everywhere on Ω. If $\widetilde{u} \in L^0(\mu,X)$ is the equivalence class of a measurable function $u : \Omega \to X$, then the scalar function $t \mapsto \|u(t)\|$ with $t \in \Omega$ is measurable. In this event, we denote the corresponding equivalence class by $|\!|\widetilde{u}|\!| \in L^0(\mu)$. Put

$$E(X) := \{u \in L^0(\mu,X) : |\!|u|\!| \in E\}.$$

Observe that $\bigl(E(X),E\bigr)$ is a lattice normed space with decomposable norm. Clearly, $L^p(\mu,X)$ with $1 \le p \le \infty$ coincides with $E(X)$, where $E = L^p(\mu)$.

2.6(1). *If X is a Banach space then $E(X)$ is a BK-space whose universal completion is $L^0(\mu,X)$.*

Proof. This is easy on using 1.7(1). $\square$

2.7. We will introduce a measurable version of $E_w(X)$. Let E and X be the same as in 2.6 and take some norming subspaces Z of X' (cp. 2.5). A vector function $u : \Omega \to X$ is *$\sigma(X, Z)$-measurable* or *Z-measurable* provided that the function $t \mapsto \langle u(t), z\rangle$ with $t \in \Omega$ is measurable for every $z \in Z$.

We denote the equivalence class of the latter function by $\langle u, z\rangle$ so that $\langle u, z\rangle \in L^0(\mu)$. Let $\mathscr{M}(\Omega, X|Z)$ be the set of Z-measurable vector functions $u : \Omega \to X$. Say that Z-measurable vector functions u and v are *Z-equivalent*, in symbols $u \simeq v$, whenever the measurable functions $\langle u, z\rangle$ and $\langle v, z\rangle$ coincide almost everywhere for all $z \in Z$. Consider the factor set $L^0(\mu, X|Z) := L^0(\Omega, \Sigma, \mu, X|Z) := \mathscr{M}(\Omega, X|Z)/\simeq$ with the structure of a vector space which is induced by the addition $\alpha\widetilde{u} + \beta\widetilde{v} := (\alpha u + \beta v)^\sim$. Given the equivalence class $\widetilde{u} \in L^0(\mu, X|Z)$ of $u \in \mathscr{M}(\Omega, X|Z)$, put $\langle \widetilde{u}, z\rangle := \langle u, z\rangle$.

Note that $R(\widetilde{u}) := \{\langle \widetilde{u}, z\rangle : z \in Z, \|z\| \le 1\}$ is order bounded in $L^0(\mu)$. Indeed, were this otherwise, we would find some sequence (f_n) in $R(\widetilde{u})$ which is unbounded above. This is a contradiction since $f(t) := \sup_n f_n(t)$ is measurable and $|f(t)| \le \|u(t)\|_X < \infty$ for almost all $t \in \Omega$. Given $\widetilde{u} \in \mathscr{M}/\sim$, put

$$[\![u]\!] := \sup\{\langle u, z\rangle : z \in Z,\ \|z\| \le 1\},$$

with the supremum taken in $L^0(\Omega, \Sigma, \mu)$. If $a : \Omega \to \mathbb{R}$ is a measurable function then we define $\widetilde{a} \cdot \widetilde{u}$ as the equivalence class of the vector function $t \mapsto a(t)u(t)$ with $t \in \Omega$. In this event $L^0(\mu, X|Z)$ becomes a unitary module over $L^0(\mu)$, and we have the easy equality $[\![ax]\!] = |a|[\![x]\!]$ for $a \in L^0(\mu)$ and $x \in L^0(\mu, X|Z)$. Obviously, $L^0(\mu, X|Z)$ is a decomposable lattice normed space over $L^0(\mu)$.

Introduce the set

$$E_w(X, Z) := \{u \in L^0(\mu, X|Z) : [\![u]\!] \in E\}.$$

As in 2.5, we distinguish the important particular case that X is a dual Banach space $X := X'$ and Z is the predual space $Z := X \subset X''$ of X. In this event we write $E_w(X') := E_w(X', X)$.

2.7(1). *If E is an order dense ideal of $L^0(\mu)$ then $E_w(X')$ with the operations and E-valued norm $[\![\cdot]\!]$ which are induced from $L^0(\Omega, \Sigma, \mu, X'|X)$ is a BK-space over $L^0(\Omega, \Sigma, \mu)$ and the universal completion of $E_w(X')$ is $L^0(\Omega, \Sigma, \mu, X'|X)$.*

It is clear that every measurable vector function is weakly measurable and $u \sim v$ implies $u \simeq v$ for each pair of measurable vector functions u and v. Therefore, we have the *canonical embedding* sending $\widetilde{u} \in L^0(\mu, X)$ to the equivalence class $\{v \in \mathscr{M}(\mu, X|Z) : v \simeq u\} \in L^0(\mu, X|Z)$.

2.7(2). *$L^0(\mu, X) \to L^0(\mu, X|Z)$ is a linearly isometric embedding.*

2.8. *Assume that X is a normed space and E is an order dense ideal of the Kantorovich space $C_\infty(Q)$. If $T : X \to E$ is an operator with abstract norm then there is a unique $u_T \in E_w(X')$ such that*

$$Tx = \langle x, u_T\rangle \quad (x \in X).$$

The mapping $T \mapsto u_T$ is a linear isometry between the Kantorovich spaces $L_A(X, E)$ and $E_w(X')$.

Proof. If $e := [\![T]\!]$, then the function $Tx \in C_\infty(Q)$ is finite at each point of $Q_0 := \{t \in Q : e(t) < +\infty\}$ for every $x \in X$ in view of the estimate $|Tx| \le e\|x\|$. The latter estimate shows that the functional $v(t) : x \mapsto (Tx)(t)$ with $x \in X$ is bounded and $\|v(t)\| \le e(t)$ for all $t \in Q_0$. Hence, we obtain the mapping $v : Q_0 \to X'$ which is continuous provided that X' is furnished with the weak topology $\sigma(X', X)$. Let u_T be the equivalence class of a vector function v. Then $Tx = \langle x, u_T\rangle$ for all $x \in X$. In particular, $\sup\{\langle x, u_T\rangle : \|x\| \le 1\} = e$. Consequently, $u_T \in E_w(X')$ and $[\![u_T]\!] = [\![T]\!]$. So, $T \mapsto u_T$ is an isometry from $L_A(X, E)$ to $E_w(X')$. Obviously, this mapping is linear and surjective. $\square$

2.9. Let X and Y be normed spaces. Consider $T \in L_A(X\widehat{\otimes}Y, E)$, where $X\widehat{\otimes}Y$ is the projective tensor product of X and Y. Clearly, the bilinear operator $b := T\otimes : X \times Y \to E$ has the abstract norm

$$[\![b]\!] := \sup\{|b(x, y)| : \|x\| \le 1,\ \|y\| \le 1\}$$

and $[\![b]\!] = [\![T]\!]$. Denote by $\mathscr{B}_A(X \times Y, E)$ the set of bilinear operators $b : X \times Y \to E$ with abstract norm; and by $\mathscr{B}(X \times Y)$, the set of bounded bilinear forms on $X \times Y$. Using the isometric isomorphism $(X\widehat{\otimes}Y)' \simeq \mathscr{B}(X \times Y)$ and 2.8, we come to the following fact:

2.9(1). *To each $b \in \mathscr{B}_A(X \times Y, E)$ there is a unique $u_b \in E_w(\mathscr{B}(X \times Y))$ such that*

$$b(x,y) = \langle x \otimes y, u_b \rangle \quad (x \in X,\ y \in Y).$$

In this event, $b \mapsto u_b$ is a linear isometry between $\mathscr{B}_A(X \times Y, E)$ and $E_w(\mathscr{B}(X \times Y))$.

2.10. Let G be an order dense ideal of $C_\infty(Q)$. In accord with 2.5 put

$$G_w(\mathscr{L}(X, Y')) := G_w(\mathscr{L}(X, Y'),\ X \otimes Y).$$

So, $G_w(\mathscr{L}(X, Y'))$ consists of the equivalence classes of operator functions $K : \mathrm{dom}(K) \to \mathscr{L}(X, Y')$ such that $\mathrm{dom}(K)$ is comeager in Q, the function $t \mapsto \langle y, K(t)x \rangle$ with $t \in \mathrm{dom}(K)$ is continuous for all $x \in X$ and $y \in Y$, and there exists

$$|\!|K|\!| := \sup\{|\langle y, Kx \rangle| :\ \|x\| \le 1,\ \|y\| \le 1\} \in G.$$

If $K \in G_w(\mathscr{L}(X, Y'))$ and $u \in E(X)$ then the vector function $t \mapsto K(t)u(t)$ with $t \in Q_0 := \mathrm{dom}(K) \cap \mathrm{dom}(u)$ is continuous in the weak topology $\sigma(Y', Y)$. Indeed, given $t, t_0 \in Q_0$ we have

$$\begin{gathered}\langle y, K(t)\,u(t) - K(t_0)\,u(t_0) \rangle| \\ \le |\langle y, (K(t) - K(t_0))\,u(t_0) \rangle| + |\!|K|\!|(t)\|y\|\|u(t) - u(t_0)\|.\end{gathered}$$

We may suppose that $\mathrm{dom}(K) = \{t : |\!|K|\!|(t) < +\infty\}$. Hence, $|\!|K|\!|$ is bounded in a neighborhood of t_0. Since u is strongly continuous and K is weakly continuous, we are done. The equivalence class of a weakly continuous vector function $t \mapsto K(t)u(t)$ we will denote by Ku; and the continuation of $t \mapsto \langle y, K(t)u(t) \rangle$ on the whole of Q, by $\langle y, Ku \rangle$.

2.11. Theorem. *To each bounded $T \in L_b(E(X),\ E_w(Y'))$ there is a unique $K_T \in G_w(\mathscr{L}(X, Y'))$, with $G := \mathrm{Orth}(E)$ such that $Tu = K_T u$ for all $u \in E(X)$. The mapping $T \mapsto K_T$ is a linear isometry between $L_b(E(X), E_w(Y'))$ and $G_w(\mathscr{L}(X, Y'))$.*

Proof. By 2.10 it suffices to demonstrate only the first claim. Given $x \in X$, $y \in Y$, and $e \in E$; put $S_{x,y}(e) := \langle y, T(x \otimes e) \rangle$. Clearly, $S_{x,y} \in \mathrm{Orth}(E)$. If $b(x,y) := S_{x,y}$, then $b : X \times Y \to G$ is a bilinear operator with abstract norm and $|\!|b|\!| = |\!|T|\!|$. By 2.9 there is a unique $K_T \in G_w(\mathscr{B}(X, Y))$ such that $|\!|K_T|\!| = |\!|T|\!|$ and

$$\langle y, T(x \otimes e) \rangle = \langle x \otimes y, K_T \rangle e.$$

Since we have the isometry isomorphism $\mathscr{B}(X, Y) \simeq \mathscr{L}(X, Y')$, we may assume that $K_T \in G_w(\mathscr{L}(X, Y'))$ and so

$$\langle y, T(x \otimes e) \rangle = \langle y, K_T x \rangle e = \langle y, K_T x \otimes e \rangle.$$

It suffices to observe that $X \otimes E$ is *bo*-dense in $E(X)$ and T is *bo*-continuous; see details in [4]. $\square$

3. Descents of Banach Spaces

The descent of a Banach space within a Boolean valued universe is a universally complete BK-space. Conversely, the universal completion of a lattice normed space becomes a usual Banach space after ascending into an appropriate Boolean valued universe. This opens the opportunity to transform the information on Banach spaces into facts about lattice normed spaces. Note that in the sequel we will use the modified descents and ascents of correspondences as presented in Appendix 2.

3.1. Theorem. *Let $(\mathscr{X}, \rho)$ be a Banach space within $\mathbb{V}^{(B)}$. Put $X := \mathscr{X}{\downarrow}$ and $|\!|\cdot|\!| := \rho{\downarrow}(\cdot)$. Then*
(1) *$\big(X, |\!|\cdot|\!|, \mathscr{R}{\downarrow}\big)$ is a universally complete BK-space;*
(2) *X is a faithful unitary module over $\Lambda = \mathscr{C}{\downarrow}$ such that*
   (*a*) *$(\lambda \mathbb{1})x = \lambda x$ for all $\lambda \in \mathbb{C}$ and $x \in X$;*
   (*b*) *$|\!|ax|\!| = |a|\,|\!|x|\!|$ for all $a \in \mathscr{C}{\downarrow}$ and $x \in X$;*

(c) $b \leq [\![x=0]\!] \leftrightarrow \chi(b)x = 0$ *for all* $b \in B$ *and* $x \in X$*, with* χ *the isomorphism of* B *onto* $\mathscr{P}(X)$.

PROOF. We will use the same notation $\oplus$ for the additions in $\mathscr{X}$, $\mathscr{C}$, and $\mathscr{R}$. Let $\odot$ stand for the composition $\mathscr{C} \times \mathscr{X} \to \mathscr{X}$ of the complex vector space $\mathscr{X}$ as well as the multiplications in $\mathscr{R}$ and $\mathscr{C}$. Put $+ := \oplus{\downarrow}$ and $\cdot := \odot{\downarrow}$. In more detail,

$$x + y = z \leftrightarrow [\![x \oplus y = z]\!] = \mathbb{1} \quad (x, y, z \in X);$$

$$a \cdot x = y \leftrightarrow [\![a \odot x = y]\!] = \mathbb{1} \quad (a \in \Lambda,\ x, y \in X).$$

Clearly, $(X, +)$ is an abelian group.

Given $b \in B$ and $x \in X$, put $\chi(b)x := \mathrm{mix}\{bx, b^*0\}$, where 0 is the zero element of $(X, +)$. In other words, $\chi(b)x$ is the unique element of X such that $[\![\chi(b)x = x]\!] \geq b$ and $[\![\chi(b)x = 0]\!] \geq b^*$. Hence, we have the mapping $\chi(b) : X \to X$ which is additive and idempotent. Put $\mathscr{P} := \{\chi(b) : b \in B\}$. Then $\mathscr{P}$ is a complete Boolean algebra and χ is a Boolean isomorphism. Note that $\Lambda := \mathscr{C}{\downarrow}$ is the complexification of the f-algebra $\mathscr{R}{\downarrow}$. Since $\mathscr{X}$ enjoys the axioms of vector space within $\mathbb{V}^{(B)}$, we may write

$$a \cdot (x + y) = a \odot (x + y) = a \odot x + a \odot y = a \cdot x + a \cdot y,$$

$$(a + b) \cdot x = (a + b) \odot x = a \odot x + b \odot x = a \cdot x + b \cdot x,$$

$$(ab) \cdot x = (ab) \odot x = a \odot (b \odot x) = a \cdot (b \cdot x),$$

$$\mathbb{1} \cdot x = \mathbb{1} \odot x = x \quad (a, b \in \Lambda;\ x, y \in X).$$

Since we always consider the separated universe $\mathbb{V}^{(B)}$, the operations $+$ and $\cdot$ make X a unitary Λ-module. Putting $\lambda x := (\lambda \mathbb{1}) \cdot x$ for all $\lambda \in \mathbb{C}$ and $x \in X$, we introduce the structure of a complex vector space on X. In this event, (a) is valid. Since

$$\chi(b) = 1 \ \to \ \chi(b) \odot x = x, \chi(b) = 0 \ \to \ \chi(b) \odot x = 0$$

within $\mathbb{V}^{(B)}$, it follows that for $b \leq [\![x = 0]\!]$ we have

$$b \leq [\![\chi(b) \odot x = x]\!] \wedge [\![x = 0]\!] \leq [\![\chi(b) \cdot x = 0]\!],$$

$$\neg b \leq [\![\chi(b) \odot x = 0]\!] = [\![\chi(b) \cdot x = 0]\!],$$

where $\neg b$ is the *complement* of b also denoted by b^*. Consequently, $[\![\chi(b) \cdot x = 0]\!] = \mathbb{1}$ or $\chi(b)x = 0$, which yields (c).

We now address the Banach space properties of $(\mathscr{X}, \rho)$. The subadditivity and homogeneity of the norm ρ can be written as

$$\rho \circ \oplus \leq \oplus \circ (\rho \times \rho), \quad \rho \circ \odot = \odot \circ \big(|\cdot| \times \rho\big),$$

where $\rho \times \rho : (x, y) \mapsto \big(\rho(x), \rho(y)\big)$ and $|\cdot| \times \rho : (a, x) \mapsto \big(|a|, \rho(x)\big)$. Using the Escher rules and putting $p := \lvert\!\cdot\!\rvert$, we have

$$p \circ + \leq + \circ (p \times p), \quad p \circ \cdot = \cdot \circ \big(|\cdot| \times p\big).$$

This means that $\lvert\cdot\rvert : X \to \mathrm{Re}\,\Lambda$ is a vector seminorm which satisfies (b). If $\lvert x\rvert = 0$ for some $x \in X$, then $[\![\rho(x) = \lvert x\rvert]\!] = \mathbb{1}$ implies that $[\![\rho(x) = 0]\!] = \mathbb{1}$. Thus, $[\![x = 0]\!] = \mathbb{1}$; i.e., $x = 0$. So, $\lvert\cdot\rvert$ is a vector norm whose decomposability follows from (b). Indeed, assume that $c := p(x) = c_1 + c_2$ for all $x \in X$ and $c_1, c_2 \in \Lambda^+$. There are $a_1, a_2 \in \Lambda^+$ such that $a_k c = c_k$, with $k := 1, 2$ and $a_1 + a_2 = 1$. As a_k we can take $a_k := c_k\big(c + (1 - e_c)\big)^{-1}$, where e_c is the trace of c. If $x_k := a_k \cdot x$, with $k := 1, 2$; then $x = x_1 + x_2$ and $\lvert x_k\rvert = \lvert a_k x\rvert = a_k \lvert x\rvert = c_k$.

We are left with demonstrating the *bo*-completeness of X. Let $s : \mathrm{A} \to X$ be a *bo*-Cauchy net. If $\bar{s}(\alpha, \beta) := s(\alpha) - s(\beta)$ with $\alpha, \beta \in \mathrm{A}$ then $\lim \lvert\cdot\rvert \circ \bar{s}(\alpha, \beta) = 0$. Let $\sigma : \mathrm{A}^\wedge \to \mathscr{X}$ be the modified ascent of s and $\bar{\sigma}(\alpha, \beta) := \sigma(\alpha) - \sigma(\beta)$ with $\alpha, \beta \in \mathrm{A}^\wedge$. Then $\bar{\sigma}$ is the modified ascent of $\bar{s}$ and $\rho \circ \bar{\sigma}$ is the modified ascent of $\lvert\cdot\rvert \circ s$. Hence, $[\![\lim \rho \circ \bar{\sigma} = 0]\!] = \mathbb{1}$; i.e., $\mathbb{V}^{(B)} \models$ "σ is a Cauchy net in $\mathscr{X}$." Since $\mathscr{X}$ is a Banach space within $\mathbb{V}^{(B)}$, by the maximum principle there is $x \in X$ such that $[\![\lim \rho \circ \sigma_0 = 0]\!] = \mathbb{1}$, where $\sigma_0 : \mathrm{A}^\wedge \to \mathscr{X}$ is given by the formula $\sigma_0(\alpha) := \sigma(\alpha) - x$ for all $\alpha \in \mathrm{A}^\wedge$. The modified descent of σ_0 is the net $s_0 : \alpha \mapsto s(\alpha) - x$ with $\alpha \in \mathrm{A}$. Consequently, o-$\lim \lvert\cdot\rvert \circ s_0 = 0$; i.e., o-$\lim \lvert s(\alpha) - x\rvert = 0$. □

Note that the universally complete BK-space $\mathscr{X}{\downarrow} := (\mathscr{X}, \rho){\downarrow} := (\mathscr{X}{\downarrow}, \rho{\downarrow})$ is the *descent* of a Banach space $(\mathscr{X}, \rho)$.

3.2. Theorem. *To each lattice normed space X there is a Banach space $\mathscr{X}$ within $\mathbb{V}^{(B)}$ unique up to linear isometry and such that $B \simeq \mathfrak{B}\left(|X|^{\perp\perp}\right)$. In this event $\mathscr{X}{\downarrow}$ is the universal completion of X.*

Proof. Consider a lattice normed space X whose lattice norm is $|\cdot| : X \to E$. Without loss of generality we may assume that $E = |X|^{\perp\perp} \subset mE = \mathscr{R}{\downarrow}$ and $B = \mathfrak{B}(E)$. Put

$$d(x, y) := |x - y|^{\perp\perp} \quad (x, y \in X).$$

Clearly, d is a B-metric on X; see Appendix 1. Furnish $\mathbb{C}$ with the discrete B-metric d_0. Then the addition $+ : X \times X \to X$ and the scalar multiplication $\cdot : \mathbb{C} \times X \to X$ as well as the vector norm $|\cdot|$ are contractive, which is almost evident. For instance, given $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$ we have

$$\begin{aligned} d(\alpha x, \beta y) = |\alpha x - \beta y|^{\perp\perp} &\le \left(|\alpha||x - y|\right)^{\perp\perp} \vee \left(|\alpha - \beta||y|\right)^{\perp\perp} \\ &\le d(x, y) \vee d_0(\alpha, \beta), \end{aligned}$$

which implies that scalar multiplication is contractive.

Let $\mathscr{X}_0$ be the Boolean valued realization of the B-set (X, d); see A1(2). Put $\rho_0 := \mathscr{F}^{\sim}(|\cdot|)$, $\oplus := \mathscr{F}^{\sim}(+)$, and $\odot := \mathscr{F}^{\sim}(\cdot)$, where $\mathscr{F}^{\sim}$ is as in A2(6). The mappings $\oplus$ and $\odot$ determine the structure of a vector space over $\mathbb{C}^{\wedge}$ on $\mathscr{X}_0$, while $\rho_0 : \mathscr{X}_0 \times \mathscr{X}_0 \to \mathscr{R}$ serves as a norm. By the maximum principle there are $\mathscr{X}, \rho \in \mathbb{V}^{(B)}$ such that $[\![(\mathscr{X}, \rho)$ is a complex Banach space, the completion of the normed space $(\mathscr{X}_0, \rho_0)]\!] = \mathbb{1}$. In this event we may assume that $[\![\mathscr{X}_0$ is a dense $\mathbb{C}^{\wedge}$-subspace of $\mathscr{X}]\!] = \mathbb{1}$. Let $\imath : X \to X_0 := \mathscr{X}_0{\downarrow}$ be the standard name embedding as in A2(5). Since $+$ is a contractive mapping from $X \times X$ to X; therefore, the addition in X_0, i.e., the mapping $+ := \oplus{\downarrow}$ defined uniquely, by $\imath \circ + = + \circ (\imath \times \imath)$, where $\imath \times \imath : (x, y) \mapsto (\imath x, \imath y)$ is the standard name embedding of the B-set $X \times X$. This means that $\imath$ is additive. By analogy, for $\cdot := \odot{\downarrow}$ we have $\imath \circ \cdot = \cdot \circ (\varkappa \times \imath)$, where $\varkappa \times \imath : (\lambda, x) \mapsto (\lambda^{\wedge}, \imath x)$ for all $\lambda \in \mathbb{C}$ and $x \in X$. Hence, $\imath$ is a linear operator. Arguing likewise for $|\cdot|_0 := \rho_0{\downarrow}$, we see that $\imath_E \circ |\cdot|_0 = |\cdot|_0 \circ \imath$, where $\imath_E$ is the standard name embedding of E. This means that $\imath$ is an isometry; i.e., $\imath$ preserves the vector norm.

Consider a universally complete BK-space $(Y, |\cdot|_1)$ such that $\imath X \subset Y \subset \mathscr{X}{\downarrow}$ and $|\cdot|_1$ is the restriction of $\rho{\downarrow}(y)$ to Y. Since $|\cdot|_1$ is decomposable and Y is d-complete, it follows that $X_0 \subset Y$. Indeed, $X_0 = \operatorname{mix}(\imath X)$, and by 3.1(2)(c) $x = \operatorname{mix}(b_\xi \imath x_\xi)$ for $x \in \mathscr{X}{\downarrow}$ if and only if $x = bo\text{-}\sum \chi(b_\xi)\imath x_\xi$. Since Y is decomposable and d-complete; therefore, by 1.4 and 1.6 Y is invariant under every projection $x \mapsto \chi(b)x$ $(x \in \mathscr{X}{\downarrow})$ and contains all sums of the above form. Arguing by analogy yields $Y = \operatorname{mix}(Y)$. If $\mathscr{Y} := Y{\uparrow}$ then $[\![\mathscr{X}_0 \subset \mathscr{Y} \subset \mathscr{X}]\!] = \mathbb{1}$ and $\mathscr{Y}{\downarrow} = Y$. Assume that $\sigma : \omega^{\wedge} \to \mathscr{Y}$ is a Cauchy sequence and s is the modified descent of σ. Then s is a bo-Cauchy sequence in Y. Consequently, we have some limit $y = \lim s$. Clearly, $[\![y = \lim \sigma]\!] = \mathbb{1}$, which demonstrates that $\mathscr{Y}$ is complete as well as the equivalent relations $\mathscr{X} = \mathscr{Y}$ and $X = Y$.

Assume now that $\mathscr{Z}$ is a Banach space within $\mathbb{V}^{(B)}$ and $\mathscr{Z}{\downarrow}$ is the universal completion of X. If $\imath'$ is the corresponding isometric embedding of X to $\mathscr{Z}{\downarrow}$, then $\imath' \circ \imath$ admits the unique extension to linear isometry of X_0 to some d-compete subspace Z_0 of $\mathscr{Z}{\downarrow}$. Note that $\mathscr{X}_0$ and $\mathscr{Z}_0 := Z_0{\uparrow}$ are isometric and, hence, so are their completions $\mathscr{X}$ and $\mathscr{Y} \subset \mathscr{Z}$. Since $\mathscr{Y}{\downarrow}$ is a BK-space and $\imath X \subset \mathscr{Y}{\downarrow} \subset \mathscr{Z}{\downarrow}$; therefore, $\mathscr{Y}{\downarrow} = \mathscr{Z}{\downarrow}$. Consequently, $\mathscr{Y} = \mathscr{Z}$ and so $\mathscr{X}$ and $\mathscr{Z}$ are linearly isometric. $\square$

3.3. Theorem. *Each lattice normed space (X, E) has the universal completion $(mX, |\cdot|_m, mE)$ unique up to linear isometry. Furthermore, to every $x \in mX$ and every positive real ε there are a family $(x_\xi)_{\xi \in \Xi}$ of X and a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathfrak{P}(mX)$ such that*

$$\left| x - \sum_{\xi \in \Xi} \pi_\xi \imath(x_\xi) \right|_m \le \varepsilon |x|_m.$$

Proof. Using the notation of 1.8, put $mX := \mathscr{X}{\downarrow}$ and $|\cdot|_m := \rho{\downarrow}(\cdot)$. By 3.2, $(mX, |\cdot|_m, mE, \imath)$ is the universal completion of X.

Fix $x \in mX$ and observe that there is no loss of generality in assuming that $e := |x|$ is an order unit in E. Clearly, $[\![e \in \mathscr{R}]\!] = [\![e > 0]\!] = [\![x \in \mathscr{X}]\!] = \mathbb{1}$. Since $[\![\mathscr{X}_0$ is dense in $\mathscr{X}]\!] = \mathbb{1}$; therefore, given $\varepsilon > 0$ and using the maximum principle, we can find $x_\varepsilon \in \mathbb{V}^{(B)}$ such that

$$[\![x_\varepsilon \in \mathscr{X}_0]\!] = [\![\rho(x - x_\varepsilon) \leq \varepsilon^\wedge \cdot e]\!] = \mathbb{1}.$$

Consequently, $x_\varepsilon \in X_0$ and $|x - x_\varepsilon|_m \leq \varepsilon e$. It suffices to note that $X_0 = \mathrm{mix}(\imath(X))$, and so x_ε is of the form $\sum_{\xi\in\Xi} \pi_\xi \imath(x_\xi)$, where $(x_\xi) \subset X$ and (π_ξ) is a partition of unity in $\mathscr{P}(mX)$.

Let $(Y, |\cdot|_1, mE, \imath_0)$ be the universal completion of X. By 3.1 we may assume that $Y = \mathscr{Y}{\downarrow}$, where $\mathscr{Y}$ is a Banach space within $\mathbb{V}^{(B)}$. By Theorem 3.1 $[\![$there is a linear isometry λ of $\mathscr{X}$ onto $\mathscr{Y}]\!] = \mathbb{1}$. Hence, $\lambda{\downarrow}$ is a linear isometry of $\mathscr{X}{\downarrow}$ onto Y. $\square$

3.4. Consider a lattice normed space X over E. A subspace X_0 of X is a *bo-ideal* provided that $|x| \leq |x_0|$ implies $x \in X_0$ for all $x \in X$ and $x_0 \in X_0$. A subspace X_0 is a *bo-foundation* of X provided that X_0 is a *bo*-ideal of X and X lacks nonzero elements disjoint from X_0. If X is decomposable and $|X|^{\perp\perp} = E$, then $X_0 \subset X$ is a *bo*-ideal or a *bo*-foundation if and only if $X_0 = h(L)$ for some order ideal or order foundation L of E.[2)] Indeed, take a *bo*-ideal $X_0 \subset X$ and let L stand for the *o*-ideal of E which is generated by $|X_0|$. By 1.3, $X_0 \subset h(L)$. If $x \in h(L)$ then $|x| \leq |u_1| + \dots + |u_n|$ for some $u_1, \dots, u_n \in X_0$. Since X is decomposable; $x = x_1 + \dots + x_n$, where $|x_k| \leq |u_k|$ for all $k := 1, \dots, n$. Note that $x_k \in X_0$ by the definition of *bo*-ideal, and so $x \in X_0$. Hence, $X_0 = h(L)$. Also, $x \perp X_0$ is tantamount to $|x| \perp L$ for all $x \in X$; see 1.2. Consequently, X_0 is a *bo*-foundation of X only if L is a foundation of E. Sufficiency of the claim is obvious.

3.4(1). *A lattice normed space X is linearly isometric to a bo-foundation of the universal completion of X if and only if X is a BK-space.*

Proof. Every foundation of a BK-space is clearly decomposable and *o*-complete.

Assume now that X is a BK-space. Clearly, $E_0 := |X|^{\perp\perp}$ is a Kantorovich space. So, we loose no generality in considering E_0 a foundation of $\mathscr{R}{\downarrow}$. Take $x \in mX$ and $|x|_m \in E_0$. By 3.3 there is a sequence $(x_n) \subset X$ such that

$$|x_n| \leq \left(1 + \frac{1}{n}\right)|x|_m, \quad |x - x_n|_m \leq \frac{1}{n}|x|_m \quad (n \in \omega).$$

Since every *bo*-complete decomposable space is *d*-complete and *br*-complete, it follows that $x_n \in X$ and $x \in X$. Hence,

$$X = \{x \in mX : |x|_m \in E_0\};$$

i.e., X is a *bo*-foundation of mX. $\square$

3.5. Given $U \subset Y$, put

$$rU := \Big\{y := br\text{-}\lim_{n\to\infty} y_n : (y_n)_{n\in\mathbb{N}} \subset U\Big\},$$
$$oU := \Big\{y := bo\text{-}\lim_{\alpha} y_\alpha : (y_\alpha)_{\alpha\in\mathrm{A}} \subset U\Big\},$$
$$dU := \Big\{y := bo\text{-}\sum_{\xi\in\Xi} \pi_\xi y_\xi : (y_\xi)_{\xi\in\Xi} \subset U\Big\},$$

where A is a directed set and (π_ξ) is a partition of unity of $\mathscr{P}(Y)$, where the sum and limits exist in Y.

Let X be a lattice normed space over E. The *disjoint completion* or *d-completion* of X is the *d*-complete lattice normed space Y over dE, where dE is the disjoint completion of E and dE is taken in the order completion oE of E, provided that there is a linear isometry $\imath : X \to Y$ satisfying $Y = d(\imath(X))$.

[2)] In the terminology of Russian provenance, the shorter term *foundation* is used instead of *order dense ideal*, which explains the above definition.

The *norm order completion* or *bo-completion* of X is a BK-space Y over oE with a linear isometry $\imath : X \to Y$ whenever Y equals each bo-complete decomposable subspace Z of Y such that $\imath(X) \subset Z$. If $E = mE$ then the bo-completion of X is the universal completion of X; see 1.8.

3.6. A Banach space $\mathscr{X}$ within $\mathbb{V}^{(B)}$ is a *Boolean valued realization* of X provided that $\mathscr{X}{\downarrow}$ is the universal completion of X. A few useful examples are now appropriate:

3.6(1). *Each lattice normed space X has a bo-completion and a d-completion, each of which is unique up to linear isometry.*

Proof. Let $\imath$ be the isometric linear embedding of X into mX (cp. 1.8 and 3.3). Recall that $dE \subset oE \subset mE$. Put

$$Y := \{x \in mX : \ |\!|x|\!|_m \in oE\}.$$

Then Y is the bo-completion of X, and $d(\imath(X))$ is the d-completion of X. $\square$

3.6(2). *The bo-completion $\overline{X}$ of X coincides with rdX.*

Proof. The claim follows from 3.3. $\square$

3.6(3). *A decomposable lattice normed space X is bo-complete if and only if X is d-complete and br-complete.*

Proof. Necessity was mentioned in 1.7, and sufficiency follows from 3.6(2). $\square$

3.6(4). *Let $(X, |\!|\cdot|\!|, E)$ be a BK-space, $E = |\!|X|\!|^{\perp\perp}$ and $A := \mathrm{Orth}(E)$. Then X admits the only structure of a faithful unitary A-module such that the natural representation of A in X is determined a Boolean algebra isomorphism between $\mathfrak{P}(E) \subset A$ and $\mathscr{P}(X)$. Furthermore,*

$$|\!|ax|\!| = |a||\!|x|\!| \quad (x \in X,\ a \in A).$$

Proof. By (c) of 3.1(2), $\mathscr{P}(X)$ coincides with the set of multiplication $x \mapsto \chi(b)x$ $(x \in X)$, with $b \in B$. $\square$

3.7. Theorem. *Assume that $\mathscr{X}$ and $\mathscr{Y}$ are Boolean valued realizations of BK-spaces X and Y over the same universally complete Kantorovich space E. Assume further that $\mathscr{L}^B(\mathscr{X}, \mathscr{Y})$ is the space of bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$ within $\mathbb{V}^{(B)}$, where $B := \mathfrak{B}(E)$. The embedding $T \mapsto T^\sim$ is a linear isometry between $L_b(X, Y)$ and $\mathscr{L}^B(\mathscr{X}, \mathscr{Y}){\downarrow}$.*

Proof. By Theorem 3.2 we loose no generality in supposing that $E = \mathscr{R}{\downarrow}$, $X = \mathscr{X}{\downarrow}$, and $\mathscr{Y}{\downarrow} = Y$. Consider $\mathscr{T} \in \mathscr{L}^B(\mathscr{X}, \mathscr{Y})$ within $\mathbb{V}^{(B)}$ and put $T := \mathscr{T}{\downarrow}$. Let ρ and θ be the norms on $\mathscr{X}$ and $\mathscr{Y}$. Put $p := \rho{\downarrow}$ and $q := \theta{\downarrow}$. We will use the same symbol $+$ for denoting addition on each of the spaces $\mathscr{X}$, $\mathscr{Y}$, X, and Y. Since $\mathscr{T}$ is additive and bounded; therefore,

$$\mathscr{T} \circ + = + \circ (\mathscr{T} \times \mathscr{T}), \quad \theta \circ \mathscr{T} \le k\rho,$$

with some $0 \le k \in \mathscr{R}{\downarrow}$. Using the ascending-descending machinery,[3)] we can rewrite the above in the equivalent form

$$T \circ + = + \circ (T \times T), \quad q \circ T \le kp.$$

This means that T is additive and bounded. By analogy, T is homogeneous. Let K be the set of $0 \le k \in \mathscr{R}{\downarrow}$ such that $q(Tx) \le kp(x)$ for all $x \in X$. So $K{\uparrow} = \{k \in \mathscr{R}^+ : \theta \circ \mathscr{T} \le k\rho\}$ within $\mathbb{V}^{(B)}$. Hence,

$$\mathbb{V}^{(B)} \models |\!|T|\!| = \inf K = \inf(K{\uparrow}) = \|\mathscr{T}\|.$$

Thus, $\mathscr{T} \mapsto \mathscr{T}{\downarrow}$ sends $\mathscr{L}^{(B)}(\mathscr{X}, \mathscr{Y}){\downarrow}$ to $L_b(X, Y)$ and preserves the vector norm. To demonstrate that this is a linear mapping, it suffices to check additivity. Considering $\mathscr{T}_1, \mathscr{T}_2 \in \mathscr{L}^B(\mathscr{X}, \mathscr{Y}){\downarrow}$ and $x \in X$ within $\mathbb{V}^{(B)}$, we have

$$(\mathscr{T}_1 + \mathscr{T}_2){\downarrow}(x) = (\mathscr{T}_1 + \mathscr{T}_2)(x) = \mathscr{T}_1 x + \mathscr{T}_2 x = \mathscr{T}_1{\downarrow}x + \mathscr{T}_2{\downarrow}x = (\mathscr{T}_1{\downarrow} + \mathscr{T}_2{\downarrow})x.$$

[3)] Also known as *Escher rules.*

Consequently, $(\mathscr{T}_1 + \mathscr{T}_2)\!\downarrow = \mathscr{T}_1\!\downarrow + \mathscr{T}_2\!\downarrow$. So, the descent is a linear isometry of $\mathscr{L}^B(\mathscr{X},\mathscr{Y})\!\downarrow$ onto the space of extensional bounded linear operators from X to Y. It suffices to observe that every bounded linear operator from X to Y is extensional; i.e., $[\![x = 0]\!] \le [\![Tx = 0]\!]$. Indeed, if $b := [\![x = 0]\!]$ then $\chi(b)x = 0$ by 3.1(2). Hence,

$$\chi(b)q(Tx) \le \chi(b)kp(x) = kp(\chi(b)x) = 0,$$

and so $q(\chi(b)Tx) = 0$ or $\chi(b)Tx = 0$. Using 3.1(2) once again, conclude that $b \le [\![Tx = 0]\!]$. $\square$

3.8. Theorem. *Assume that X is a normed space and $\widetilde{X}$ is the completion of X. Assume further that $\mathscr{X}$ is the completion of the $\mathscr{R}^\wedge$-normed space $X^\wedge$ within $\mathbb{V}^{(B)}$. The universally complete BK-space $\mathscr{X}\!\downarrow$ is linearly isometric to $C_\infty(Q,\widetilde{X})$, where Q is the Stone representation space of B.*

Proof. Identifying the Kantorovich spaces $\mathscr{R}\!\downarrow$ and $C_\infty(Q)$, apply Theorem 3.2 to the lattice normed space $(X, p, \mathscr{R}\!\downarrow)$, with $p(x) = \|x\|\mathbb{1}$. Using the notation of the proof of Theorem 3.2, note that $\mathscr{X}_0 = X^\wedge$. However, $\mathscr{X}\!\downarrow := (\mathscr{X}\!\downarrow,\ q,\ \mathscr{R}\!\downarrow)$ is the universal completion of $(X, p, \mathscr{R}\!\downarrow)$. Suppose for simplicity that $X \subset \mathscr{X}\!\downarrow$. Since $[\![X^\wedge$ is norm-dense in $\mathscr{X}]\!] = \mathbb{1}$, conclude that to every $u \in C_\infty(Q,\widetilde{X})$ and a positive real ε there are a family $(x_\xi) \subset X$ and a partition of unity $(Q_\xi) \subset \mathrm{Clop}(Q)$ such that $|\!|u - u_\varepsilon|\!| \le \varepsilon\mathbb{1}$ for the step function u_ε which coincides with x_ξ on Q_ξ. Put $\mathscr{T}(u_\varepsilon) := \mathrm{mix}(b_\xi x_\xi)$, where b_ξ is the member of B which corresponds to the clopen set Q_ξ. Obviously, $|\!|\mathscr{T}(u_\varepsilon)|\!| = |\!|u_\varepsilon|\!|$. Thus, $\mathscr{T}$ is an isometric linear embedding of the subspace of the vector functions of the form u_ε. If $\varepsilon \to 0$ then $|\!|u_\varepsilon - u|\!| \xrightarrow{(r)} 0$. Consequently, $(\mathscr{T}(u_{1/n}))$ is an r-Cauchy sequence. Since $\mathscr{X}\!\downarrow$ is complete, $\mathscr{X}\!\downarrow$ contains the limits $v := r\text{-}\lim \mathscr{T}(u_{1/n})$. Putting $\mathscr{T}(u) := v$, we obtain the isometric linear embedding $\mathscr{T} : C_\infty(Q,\widetilde{X}) \to \mathscr{X}\!\downarrow$. If $Z := \mathrm{im}(\mathscr{T})$ then Z is a decomposable bo-complete subspace of $\mathscr{X}\!\downarrow$ and $X \subset Z$. Using Theorem 3.2 and the definition in 1.8, we conclude that $Z = \mathscr{X}\!\downarrow$. $\square$

3.9. *Let $\mathscr{X}'$ be the dual of X within $\mathbb{V}^{(B)}$. Then $\mathscr{X}'\!\downarrow$ and $E_w(X')$, with $E = C_\infty(Q)$, are linearly isometric.*

Proof. Apply Theorem 3.3 to $Y := E$ and $X := (X, |\!|\cdot|\!|, E)$, where $|\!|x|\!| = \|x\|\mathbb{1}$. We see that $\mathscr{X}'\!\downarrow := \mathscr{L}^{(B)}(\mathscr{X},\mathscr{R})\!\downarrow$ and $L_A(X,E)$ are linearly isometric. We are done on recalling 2.8. $\square$

4. Maharam Operators

We will address some class of order continuous regular operators that mimics the behavior of functionals. In particular, so we extend the Radon–Nikodym Theorem.

4.1. Assume that a lattice normed space X is a vector lattice. The norm $|\!|\cdot|\!| : X \to E^+$ is *monotone* provided that $|x| \le |y|$ implies $|\!|x|\!| \le |\!|y|\!|$ for all $x, y \in X$. In this event, X is a *lattice normed lattice*. Furthermore, if X is a BK-space then X is a *BK-lattice*.

The norm of a lattice normed lattice $(X, |\!|\cdot|\!|)$ is *additive* in case $|\!|x + y|\!| = |\!|x|\!| + |\!|y|\!|$ for all $x, y \in X^+$; an *order continuous* in case $bo\text{-}\lim_\alpha |\!|x_\alpha|\!| = 0$ for every decreasing vanishing net $(x_\alpha) \subset X$; *order semicontinuous* in case $|\!|x|\!| = \sup_\alpha |\!|x_\alpha|\!|$; for every increasing net $(x_\alpha) \subset X$ such that $x = \sup_\alpha x_\alpha$; and *monotone complete* in case $x = \sup_\alpha x_\alpha$ for every increasing net $(x_\alpha) \subset X$ such that the net $(|\!|x_\alpha|\!|)$ is bounded in E.

4.2. To derive the Boolean valued realization of BK-lattices from 3.2, we need a few auxiliary facts:

4.2(1). *Let X be a decomposable lattice normed space over a vector lattice E. If X is a vector lattice whose vector norm is monotone, then $\mathscr{B}(X)$ is a regular subalgebra of the Boolean algebra $\mathfrak{B}(X)$. In particular, every band of X in the sense of lattice normed space theory is a band of X in the sense of vector lattice theory.*

Proof. Since the vector norm is monotone, $h(L)$ is an order ideal of X for every band $L \in \mathfrak{B}(E)$. If $0 \le x \in h(L)$ and $0 \le y \in h(L^\perp)$ then $0 \le x \wedge y \in h(L) \cap h(L^\perp) = \{0\}$ because $|\!|x \wedge y|\!| \le |\!|x|\!| \wedge |\!|y|\!|$ as the norm is monotone. So, $x \wedge y = 0$; i.e., x and y are disjoint not only in the sense of 1.2 but also in the sense of the

order on X. Let d denote the disjointness on the vector lattice X; i.e., $u\,d\,v \leftrightarrow |u|\wedge|v| = 0$. The above may be rewritten as $h(L)dh(L^{\perp})$, which implies that $h(L^{\perp}) \subset h(L)^d$, where $A^d := \{x \in X : (\forall\, a \in A)\, xda\}$. In fact $h(L^{\perp}) = h(L)^d$ for all $L \in \mathfrak{B}(E)$. Indeed, assume that $x\,d\,h(L)$ and $x \notin h(L^{\perp})$. Then $|\!|x|\!| \notin L^{\perp}$ and so there is $0 < e \in L$ such that $e \le |\!|x|\!|$. Since X is decomposable, there are $u, v \in X$ such that $x = u + v$, $|\!|u|\!| = e$, and $|\!|v|\!| = |\!|x|\!| - e$. As $u \in h(L)$; therefore, $x\,d\,u$ and so $|x| \le |v|$. But then $|\!|x|\!| \le |\!|v|\!| = |\!|x|\!| - e$ and we arrive at $0 < e \le 0$; a contradiction. Hence $x\,d\,h(L)$ implies that $x \in h(L^{\perp})$. Consequently, $h(L^{\perp}) = h(L)^d$. Replacing L with $L^{\perp}$, we see that $h(L) = h(L^{\perp})^d$. It follows that $h(L) \in \mathfrak{B}(X)$; i.e., $\mathscr{B}(X) \subset \mathfrak{B}(X)$. Using 1.3, we find that $h(L)^{\perp} = h(L^{\perp}) = h(L)^d$. Therefore, the Boolean complement in $\mathscr{B}(X)$ is induced from $\mathfrak{B}(X)$. Since the meets in both algebras coincide with the set theoretic intersection, we conclude that $\mathscr{B}(X)$ is a regular subalgebra of $\mathfrak{B}(X)$. $\square$

4.2(2). *Let X be the same as in 4.2(1), and E is a vector lattice with the projection property. Then $\mathscr{P}(X)$ is a regular subalgebra of $\mathfrak{P}(X)$. In particular, every band of X in the sense of lattice normed space theory is a band of X in the sense of vector lattice theory.*

Proof. Follows from 4.2(1) and 1.4. $\square$

Say that a lattice normed space X over E *admits the coherent module structure* over $A := \mathfrak{P}(E)$ whenever X can be made into a faithful unitary A-module so that $|\!|ax|\!| = |a||\!|x|\!|$ for all $a \in A$ and $x \in X$, while the natural representation of A in X is the same isomorphism between $\mathfrak{P}(E)$ and $\mathscr{P}(X)$ as in 1.4. If X is a vector lattice then we also assume that $\mathscr{B}(X)$ is a regular subalgebra of $\mathfrak{B}(X)$.

4.2(3). *Let X be a d-decomposable sequentially bo-complete lattice normed space or lattice over a sequentially complete vector lattice E, where $E = |\!|X|\!|^{\perp\perp}$. Then X admits the coherent module structure over* $\mathrm{Orth}(E)$.

Proof. Let $a \in \mathrm{Orth}(E)$ be a *finite valued element*; i.e., $a = \sum_{k=1}^{n} \lambda_k \pi_k$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\{\pi_1, \dots, \pi_n\}$ is a finite partition of unity in $\mathfrak{P}(E)$. Put $ax := \sum_{k=1}^{n} \lambda_k \pi_k x$. Identifying $\mathfrak{P}(E)$ with $\mathscr{P}(X)$ and recalling 1.4, we can write

$$|\!|ax|\!| = \left|\!\left|\sum_{k=1}^{n} \lambda_k \pi_k x\right|\!\right| = \sum_{k=1}^{n} |\lambda_k| \pi_k |\!|x|\!| = a|\!|x|\!|.$$

Note that each $a \in \mathrm{Orth}(E)$ is the limit of an increasing sequence of finite valued elements $(a_n) \subset \mathrm{Orth}(E)$. Note that $(a_n x) \subset X$ is bo-Cauchy as $|\!|a_n x - a_m x|\!| = |a_n - a_m||\!|x|\!| \xrightarrow{(o)} 0$. Putting $ax := bo\text{-}\lim a_n x$, we have

$$|\!|ax|\!| = |\!|bo\text{-}\lim a_n x|\!| = o\text{-}\lim |a_n||\!|x|\!| = a|\!|x|\!|.$$

The remaining claims are obvious. In case X is a lattice normed lattice, we are done on applying 4.2(2). $\square$

4.3. Theorem. *Let $(X, |\!|\cdot|\!|)$ be a BK-space and let $(\mathscr{X}, \|\cdot\|) \in \mathbb{V}^{(B)}$ be the Boolean valued realization of $(X, |\!|\cdot|\!|)$. Then*

(1) *X is a BK-lattice if and only if $\mathbb{V}^{(B)} \models$ "$\mathscr{X}$ is a Banach lattice";*

(2) *X is an order complete BK-lattice if and only if $\mathbb{V}^{(B)} \models$ "$\mathscr{X}$ is an order complete Banach lattice";*

(3) *$|\!|\cdot|\!|$ is order continuous, or order semicontinuous, or monotone complete, or additive if and only if so is $\|\cdot\|$ within $\mathbb{V}^{(B)}$.*

Proof. Claims (1) and (2) follow from 3.1 and 3.2. Claim (3) results from the Escher rules. $\square$

4.4. Assume that E is a vector lattice, F is a Kantorovich space, and T is a positive operator from E to F. Say that T possesses the *Maharam property* whenever for all $x \in E^+$ and $0 \le f \le Tx \in F^+$ there is $0 \le e \le x$ such that $f = Te$. Clearly, $T \in L^+(E, F)$ possesses the Maharam property if and only if T sends the order interval $[0, x]$ onto the order interval $[0, Tx]$ for all $x \in E^+$; i.e., $T([0, x]) = [0, Tx]$ for all $x \in X$. A *Maharam operator* is an order continuous positive operators with the Maharam property. In other words, a Maharam operator is an interval preserving order continuous operator.

4.4(1). *Let T be an essentially positive operator from E to F which possesses the Maharam property. Put* $\lvert e\rvert := T(|e|)$ $(e \in E)$. *Then* $(E, \lvert\cdot\rvert)$ *is a d-decomposable lattice normed space over F.*

Proof. Clearly, $\lvert\cdot\rvert$ is a monotone vector norm on E. By definition, we see that conditions 1.1(a)–1.1(c) are satisfied. If $\lvert e\rvert = f_1 + f_2$ for some $e \in E^+$, $f_1, f_2 \in F^+$, and $f_1 \perp f_2$; then by the Maharam property there are $e_1, e_2 \in [0, |e|]$ such that $Te_k = f_k$ for $k := 1, 2$. Since $T(e_1 \wedge e_2) \leq f_1 \wedge f_2 = 0$; therefore, $e_1 \perp e_2$ as T is essentially positive; see A3(11). It follows that $e_1 + e_2 \leq |e|$. Since $T(|e| - e_1 - e_2) = 0$ and T is essentially positive, we conclude that $|e| = e_1 + e_2$. The modulus of every element of a vector lattice is obviously a decomposable norm. So $e = u_1 + u_2$, $|u_1| = e_1$, and $|u_2| = e_2$ for some $u_1, u_2 \in E$. $\square$

Put $F_T := \{T(|x|) : x \in E\}^{\perp\perp}$ and let $\mathscr{D}_m(T)$ denote the greatest foundation of the universal completion mE of E on which T can be extended by o-continuity. In other words, $z \in \mathscr{D}_m(T)$ if and only if $z \in mE$ and $\{T(x) : x \in E,\ 0 \leq x \leq |z|\}$ is bounded in F. In this event the minimal extension of T to $\mathscr{D}_m(T)$ exists and presents an order continuous positive operator; cp. A3(8)–A3(10).

4.4(2). *Let F be a Kantorovich space, and let $T : E \to F$ be an essentially positive Maharam operator. Then there are a foundation of $X \subset mE$ which includes E and an essentially positive Maharam operator $\Phi : X \to mF$ such that $X = \mathscr{D}_m(\Phi)$ and $\Phi(e) = Te$ $(e \in E)$.*

Proof. Consider the bo-completion $\overline{E}$ of the lattice normed space $(E, \lvert\cdot\rvert)$ in 4.4(1). By 3.6(2) $\overline{E} = rd(E)$, where $d(E)$ is taken with respect to the regular subalgebra $\mathscr{P}(X)$ of $\mathscr{D}_m(T)$); cp. 4.2(2). Thus, $\overline{E}$ is a foundation of mE. Let $\mathscr{X}$ be the Boolean valued realization of the BK-space $(\overline{E}, \lvert\cdot\rvert)$ and $X := \mathscr{X}{\downarrow}$. We will assume that $mE = \mathscr{R}{\downarrow}$. Since $\overline{E}$ is a Kantorovich space whose vector norm is additive and order continuous, by 4.3 X is a Kantorovich space with the additive o-continuous norm. By 3.4 $\overline{E}$ is a bo-foundation of X, while $\overline{E}$ is an order dense ideal of X because the norm is monotone; see 4.1. So, X is a foundation of mE. Put $\Phi(x) = \lvert x^+\rvert - \lvert x^-\rvert$ for all $x \in X$. Clearly, $\Phi : X \to mF$ is an order continuous positive linear operator. Moreover, X admits a coherent module structure over $\mathrm{Orth}(mF)$ (cp. 4.2(3)), which yields the Maharam property of Φ. Indeed, if $0 \leq y \leq \Phi(x)$ and $0 \leq x \in X$ then $y = \alpha\Phi(x)$ for some $\alpha \in \mathrm{Orth}(mF)^+$. So $y = \alpha\Phi(x) = \Phi(\alpha x)$ and, furthermore, $0 \leq \alpha x \leq x$. It is also clear that Φ extends T. $\square$

4.5. Theorem. *Let E and F be Kantorovich spaces and let $T : E \to F$ be an essentially positive Maharam operator. Put $X := \mathscr{D}_m(T)$ and* $\lvert x\rvert := \Phi(|x|)$ $(x \in X)$, *where Φ is the extension of T to X by o-continuity. Then* $(X, \lvert\cdot\rvert)$ *is a BK-lattice with order continuous additive norm.*

Proof. Appreciating 4.4 and Theorem 1.7(1), it suffices to demonstrate the d-completion and br-completion of X. Take a *metrically disjoint* family $(x_\xi)_{\xi\in\Xi}$ in X with $\lvert x_\xi\rvert \leq f \in F$ for all $\xi \in \Xi$. Note that $(x_\xi)_{\xi\in\Xi}$ is bo-summable if and only if so is $(|x_\xi|)_{\xi\in\Xi}$. Thus, we may assume that $x_\xi \geq 0$ for all $\xi \in \Xi$. Put $\Theta := \mathscr{P}_{\mathrm{fin}}(\Xi)$ and $x_\theta := \sum_{\xi\in\theta} x_\xi$ for $\theta \in \Theta$. Clearly, $(x_\theta)_{\theta\in\Theta}$ is an increasing family, while $(Tx_\theta)_{\theta\in\Theta}$ is bounded above as $Tx_\theta = \lvert x_\theta\rvert \leq f$. By hypothesis there is x such that $x := \sup_\theta x_\theta$ and $Tx = \sup_\theta Tx_\theta$. It follows that x is the bo-sum of (x_ξ) because

$$\lvert x - x_\theta\rvert = T(x - x_\theta) \xrightarrow{(o)} 0.$$

Consider a br-Cauchy sequence (x_n) in X. Dropping to a subsequence, if need be, we may assume that $\lvert x_n - x_{n-1}\rvert \leq (1/n^3)f$ for all $n \in \mathbb{N}$ and some $f \in F^+$. Put $f_m := \sum_{n=2}^m n\lvert x_n - x_{n-1}\rvert$ and $e_m := \sum_{n=2}^m n|x_n - x_{n-1}|$. Note that (e_m) is an increasing sequence and (f_m) is a bounded above increasing sequence, since the series $\sum_{n=2}^\infty n\lvert x_n - x_{n-1}\rvert$ is o-convergent as dominated by the o-convergent series $\sum_{n=2}^\infty (1/n^2)f$. Since $Te_m = f_m$, there exist $e := \sup_m e_m$ by hypothesis. Thus

$$n|x_{n+k} - x_n| = n\left|\sum_{l=n+1}^{n+k}(x_l - x_{l-1})\right| \leq \sum_{l=n+1}^{n+k} l|x_l - x_{l-1}| \leq e_{n+k} - e_n \leq e,$$

implying that $|x_{n+k} - x_n| \leq (1/n)e$. So (x_n) is r-Cauchy. Therefore, there exists $x := r\text{-}\lim_n x_n$ in X. Hence, we may pass to the r-limit as $k \to \infty$ in the above inequality and conclude that $|x_n - x| \leq (1/n)e$.

Consequently,

$$|\!|x_n - x|\!| \leq (1/n)|\!|e|\!|;$$

i.e., $br\text{-}\lim_n x_n = x$. Thus, X is br-complete. $\square$

Recall that a linear operator $S : E \to F$ is *absolutely continuous with respect to* T, in symbols $S \ll T$, provided that $|S(x)| \in \{T(|x|)\}^{\perp\perp}$ for all $x \in E$.

4.6. Theorem. *Let F be a Kantorovich space and let $T : X \to Y$ is a Maharam operator. Then there is a lattice homomorphism h from the universally complete Kantorovich space $\mathrm{Orth}^\infty(F_T)$ onto a regular subspace of $\mathrm{Orth}^\infty(E_T)$ such that*

(1) *$h(\mathfrak{P}(F_T))$ is a regular subalgebra of the complete Boolean algebra $\mathfrak{P}(E_T)$;*

(2) *$h(\mathscr{Z}(F_T))$is a sublattice and subring of $\mathscr{Z}(E_T)$;*

(3) *if $S : E \to F$ is a linear operator and $S \ll T$ then $\pi \circ S(x) = S \circ h(\pi)\,(x)$ for all $\pi \in \mathfrak{P}(F_T)$;*

(4) *if $S : E \to F$ is an order continuous positive linear operator and $S \ll T$ then $\pi \circ S(x) = S \circ h(\pi)\,(x)$ for all $\pi \in \mathrm{Orth}^\infty(F_T)$ and $x \in \mathscr{D}(h(\pi))$; in particular, S is a Maharam operator.*

Proof. Without loss of generality we may assume that $E = E_T$ and $F = F_T$. Given $e \in E$, put $|\!|e|\!| := T(|e|)$. Using 4.4, we see that $(X, |\!|\cdot|\!|)$ is a disjointly decomposable lattice normed lattice. By 4.2(3) E admits the coherent module structure over $A := \mathrm{Orth}(F)$. Let $h : A \to \mathrm{Orth}(E)$ be the natural representation of A in $\mathrm{Orth}(E)$. Then (1) and (2) are straightforward from 4.2(3). The Boolean isomorphism h has the unique extension to a lattice isomorphism of $\mathrm{Orth}^\infty(F)$ onto the order closed sublattice of $\mathrm{Orth}^\infty(E)$ which consists of those members of $\mathrm{Orth}^\infty(E)$ whose spectral functions range in $\mathscr{B} = h(\mathfrak{P}(F))$. We will keep the notation h for this extension.

To prove (3), let $S : E \to F$ be such that $S \ll T$. The definition of h (cp. 1.3 and 1.4) implies that

$$S \circ h(\pi)x \in \{T \circ h(\pi)x\}^{\perp\perp} \subset \pi\,(F)$$

for all $\pi \in \mathfrak{P}(F)$ and $x \in E$. Consequently, $\pi^\perp \circ S \circ h(\pi) = 0$. Replacing π with $\pi^\perp$ yields $\pi \circ S \circ h(\pi^\perp) = 0$. The former equality implies that $S \circ h(\pi) = \pi \circ S \circ h(\pi)$ and the latter, $\pi \circ S = \pi \circ S \circ h(\pi)$. So we arrive at the required; i.e., $\pi \circ S = S \circ h(\pi)$.

We turn now to proving (4). If $\alpha := \sum_{l=1}^n \lambda_l \pi_l$, with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\{\pi_1, \dots, \pi_n\}$ a partition of unity in $\mathfrak{P}(F)$; then it is clear that

$$\pi_l \circ \alpha \circ S = \pi_l \circ S(\lambda_l h(\pi_l)) = \pi_l \circ S \circ h(\alpha)$$

for all l. Summation over l yields $\alpha \circ S = S \circ h(\alpha)$. Finally, if $\alpha \in \mathrm{Orth}^\infty(F)^+$ then $\alpha = \sup(\alpha_\xi)$ for some updirected net (α_ξ) in $\mathscr{Z}(F)$; whereas the members of $\mathscr{Z}(F)$ are r-limits of orthomorphisms of the form $\sum_{l=1}^n \lambda_l \pi_l$. Therefore, (4) ensues from the o-continuity of S. $\square$

4.7. Theorem. *Let X be a Kantorovich space and let E be a foundation of the universally complete Kantorovich space $\mathscr{R}{\downarrow}$. Assume that $\Phi : X \to E$ is a Maharam operator, while $X = X_\Phi$ and $E = E_\Phi$. Then there are $\mathscr{X}$ and ϕ within $\mathbb{V}^{(B)}$ such that*

(1) *$[\![\mathscr{X}$ is a Kantorovich space and $\phi : \mathscr{X} \to \mathscr{R}$ is an o-continuous positive functional such that $\mathscr{X} = \mathscr{X}_\phi = \mathscr{D}_m(\phi)]\!] = \mathbb{1}$;*

(2) *if $X' := \mathscr{X}{\downarrow}$ and $\Phi' = \phi{\downarrow}$; then X' is a Kantorovich space, $\Phi' : X' \to E$ is a Maharam operator, and $X' = \mathscr{D}_m(\Phi') = X'_{\Phi'}$;*

(3) *there is a linear and lattice isomorphism h from X to some foundation of X' such that $\Phi = \Phi' \circ h$.*

Proof. Consider the lattice normed lattice $X := \mathscr{D}_m(T)$ with the norm $|\!|x|\!| := \Phi(|x|)$ for all $x \in X$. By 4.5, X is a BK-space. By 3.2 we may assume that $X = \mathscr{X}{\downarrow}$ for some Banach space $\mathscr{X}$ within $\mathbb{V}^{(B)}$. Furthermore, by 4.3(2, 3) $\mathbb{V}^{(\mathbb{B})} \models$"$\mathscr{X}$ is a Kantorovich space whose norm is order continuous, monotone complete, and additive." Using the maximum principle yields $\phi \in \mathbb{V}^{(\mathbb{B})}$ such that

$$[\![\phi : \mathscr{X} \to \mathscr{R}]\!] = [\![(\forall\, x \in \mathscr{X})\, \phi(x) = \|x^+\| - \|x^-\|]\!] = \mathbb{1}.$$

Since the norm is order continuous and additive; therefore, $[\![\phi$ is an essentially positive order continuous functional$]\!] = \mathbb{1}$. The monotone completeness yields $[\![\mathscr{X} = \mathscr{D}_m(\phi)]\!] = \mathbb{1}$. Put $\Phi' := \phi{\downarrow}$ and note that we can write on using descent that

$$[\![\phi{\downarrow}(x) = \phi(x) = \|x^+\| - \|x^-\| = |x^+| - |x^-| = \Phi(x)]\!] = \mathbb{1}$$

for all $x \in X$. The remaining details follow from 4.3. $\square$

4.8. Corollary. *Let $(X, |\cdot|)$ be as in Theorem 4.5. Then there exists an AL-space $\mathscr{X}$ within $\mathbb{V}^{(B)}$ such that X is isometrically lattice isomorphic to some foundation of the BK-space $\mathscr{X}{\downarrow}$.*

Proof. It suffices to put $\|x\| := \phi(|x|)$ for all $x \in \mathscr{X}$ and observe that $(X, \|\cdot\|)$ is an AL-space within $\mathbb{V}^{(B)}$ by transfer; see [2, Section 1.4].

4.9. Theorem 4.7 allows us to state that each statement about o-continuous positive linear functionals on Kantorovich spaces has an analog for Maharam operators. We will exhibit a few relevant examples on assuming that X, Φ, $\mathscr{X}$, and ϕ are the same as in 4.7.

4.9(1). *A linear operator S is absolutely continuous with respect to Φ if and only if there is $\sigma \in \mathbb{V}^{(B)}$ such that $\mathbb{V}^{(B)} \models$ "$\sigma : \mathscr{X} \to \mathscr{R}$ is a linear functional" and $S = (\sigma{\downarrow}) \circ h$.*

Proof. Note that $S \ll \Phi$ if and only if S is extensional. Indeed, necessity follows from 4.6(3), and sufficiency is obvious. So, the ascent $\sigma := S{\uparrow}$ of S exists and sends $\mathscr{X}$ to $\mathscr{R}$. In this event, S coincides with the descent of σ in view of ascending-descending machinery. $\square$

Let $L_\Phi(X, \mathscr{R}{\downarrow})$ stand for the space of linear operators absolutely continuous with respect to Φ. Clearly, $L_\Phi(X, \mathscr{R}{\downarrow})$ is a unitary faithful $\mathscr{R}{\downarrow}$-module. Let $\mathscr{X}^{\#}$ be such that $\mathbb{V}^{(B)} \models$ "$\mathscr{X}^{\#} := L(\mathscr{X}, \mathscr{R})$ is the dual space of $\mathscr{X}$." Then $\mathscr{X}^{\#}{\downarrow}$ is a unitary faithful module over $\mathscr{R}{\downarrow}$.

4.9(2). *The mapping $\sigma \mapsto \sigma{\downarrow}$ as well as $S \mapsto S{\uparrow}$ is a module isomorphism between $\mathscr{X}^{\#}{\downarrow}$ and $L_\Phi(X, \mathscr{R}{\downarrow})$.*

Proof. Note that $\sigma \mapsto \sigma{\downarrow}$ is a bijection by 4.9(1) and the arrow cancellation rule. The additivity and homogeneity of the mapping follows from A3(11)–A3(13). $\square$

Let $\mathscr{X}_n^{\sim}$ be the space of order continuous regular functionals on $\mathscr{X}$. In other words, if $[\![\sigma \in \mathscr{X}_n^{\sim}]\!] = \mathbb{1}$ then $[\![\sigma : \mathscr{X} \to \mathscr{R}$ is an order continuous regular functional$]\!] = \mathbb{1}$.

4.9(3). *The mapping $\sigma \mapsto \sigma{\downarrow}$, as well as $S \mapsto S{\uparrow}$, is an order and algebraic isomorphism between the lattice ordered modules $\mathscr{X}_n^{\sim}{\downarrow}$ and $\{\Phi\}^{\perp\perp}$.*

Proof. In view of 4.9(1), it suffices to note that S is positive or regular if and only if $[\![\sigma$ is positive or regular$]\!] = \mathbb{1}$. The claim of positivity results for instance as follows:

$$\sigma(\mathscr{X}^+) \subset \mathscr{R} \leftrightarrow \sigma(\mathscr{X}^+){\downarrow} \subset \mathscr{R}{\downarrow} \leftrightarrow \sigma{\downarrow}(\mathscr{X}^+{\downarrow}) \subset \mathscr{R}{\downarrow} \leftrightarrow S(X^+) \subset \mathscr{R}{\downarrow}. \quad \square$$

4.10. A few corollaries are now relevant:

4.10(1). *Two operators in $\{\Phi\}^{\perp\perp}$ are disjoint if and only if so are their supports.*

Proof. It suffices to apply A3(11) and 4.9(3) to functionals in $\{\phi\}^{\perp\perp}$ within $\mathbb{V}^{(\mathbb{B})}$. $\square$

4.10(2). *If $\Phi : E \to F$ is an essentially positive Maharam operator, then there is a Boolean algebra isomorphism $\iota : \mathfrak{E}(\Phi) \to \mathfrak{P}(E)$ such that $S = \Phi \circ \iota(S)$ for all $S \in \mathfrak{E}(\Phi)$.*

Proof. Take the projection to the support of S as $\iota(S)$. If S is a fragment of Φ,[4)] then $\neg S := I_E - S$ is the fragment of Φ disjoint from S. By 4.10(1) the supports of S and $\neg S$ are disjoint; hence, so are $\iota(S)$ and $\iota(\neg S^\perp)$. Since $S + \neg S^\perp = \Phi$, the essential positivity of Φ implies that $\iota(S) + \iota(\neg S) = I_E$. Finally, $S = \Phi \circ \iota(S)$. $\square$

[4)] An element v in a vector lattice X is a *component* or *fragment* of $u \in X$ if $|v| \wedge |u - v| = 0$.

4.10(3). Hahn Decomposition Theorem. *Let E and F be Kantorovich spaces and let $T : E \to F$ be an order bounded operator whose modulus is a Maharam operator. Then there is $\pi \in \mathfrak{P}(E)$ such that $T^+ = T \circ \pi$ and $T^- = T \circ \pi^\perp$.*

Proof. It suffices to put $\Phi := |T|$ and $\pi := \iota(T^+)$ in 4.10(2). $\square$

4.10(4). Luxemburg–Schep Theorem. *Assume that E and F are Kantorovich spaces. Assume further that S and T are order continuous positive operators from E to F and T enjoys the Maharam property. Then the following are equivalent:*

(*a*) *$S \in \{T\}^{\perp\perp}$;*
(*b*) *$S \ll T$;*
(*c*) *there is $0 \le \rho \in \mathrm{Orth}^\infty(E)$ such that $Sx = T(\rho x)$ for all x in the domain of ρ;*
(*d*) *there is a sequence (ρ_n) in $\mathrm{Orth}(E)$ such that $Sx = \sup_n T(\rho_n x)$ for all $x \in E^+$.*

Proof. All claims are straightforward from 4.9 and 4.10. $\square$

5. Mixed Normed Spaces

The section addresses the mixed normed spaces and their basic properties.

5.1. Recall that a normed lattice E is defined as a vector lattice endowed with a *monotone norm* $\|\cdot\|$; i.e., $|x| \leqslant |y|$ implies that $\|x\| \leqslant \|y\|$ for all $x, y \in E$. If a normed lattice E is norm-complete then E is a *Banach lattice*.

A Banach lattice E is an *abstract L-space* or an *AL-space* whenever

$$\|x + y\| = \|x\| + \|y\| \quad (x, y \in E^+).$$

A Banach lattice E is an *abstract M-space* or an *AM-space* whenever

$$\|x \vee y\| = \|x\| \vee \|y\| \quad (x, y \in E^+).$$

If the unit ball of an AM-space E has the top element u, then u is a strong unit and the unit ball of E is the symmetric order interval $[-u, u]$. In this event E is a *unital AM-space.*

Recall further than if E is a vector lattice and $I(u) := \bigcup_{n=1}^\infty [-nu, nu]$ is an order ideal generated by $0 \le u \in E$, then we introduce the seminorm

$$\|x\|_\infty := \|x\|_u := \inf\{\lambda \in \mathbb{R} :\ |x| \le \lambda u\} \quad (x \in I(u)).$$

If $I(u) = E$ then u is a *strong unit* and $I(u)$ is a *vector lattice of bounded elements.* The seminorm $\|\cdot\|_u$ is a norm if and only if $I(u)$ is an Archimedean vector lattice.

The simplest example of a unital AM-space is an r-complete Archimedean vector lattice of bounded elements E with strong unit u which is furnished with the norm $\|\cdot\|_\infty := \|\cdot\|_u$. Indeed, the required properties of $\|\cdot\|_u$ are straightforward, and the norm completeness of E amounts to the r-completeness of E with the regulator u.

If (X, E) is a lattice normed space with E a normed lattice then $\lvert x\rvert \in E$ for all $x \in X$ by definition. Hence, we may furnish X with the *mixed norm* $|\!|\!|\cdot|\!|\!|$ as follows:

$$|\!|\!|x|\!|\!| := \big\|\lvert x\rvert\big\| \quad (x \in X).$$

In this event $(X, |\!|\!|\cdot|\!|\!|)$ is a *mixed normed space.*

All concepts of Sections 1 and 2 are clearly sensible for mixed normed spaces. In particular, we will use decomposability, *bo*-ideals, *br*-completeness, *d*-completeness, *bo*-completeness, etc. The two simple properties are of relevance:

5.1(1). *The vector norm $\lvert\cdot\rvert$ is a norm continuous operator from $(X, |\!|\!|\cdot|\!|\!|)$ to E.*

Proof. Indeed, since $\big|\lvert x\rvert - \lvert y\rvert\big| \leqslant \lvert x - y\rvert$ and the norm on E is monotone, we have

$$\big\|\lvert x\rvert - \lvert y\rvert\big\| \leqslant |\!|\!|x - y|\!|\!| \quad (x, y \in X),$$

which demonstrates the claim. $\square$

5.1(2). *Let (X, E) be a disjointly decomposable mixed normed space and let E be a lattice with the projection property. Then every projection $\pi \in \mathscr{P}(X)$ is bounded and has norm 1.*

Proof. Indeed, by 1.4

$$|\!|\!|\pi x|\!|\!| = \|\,|\pi x|\,\| = \|\pi |x|\| \leq |\!|\!|x|\!|\!|,$$

and we are done. $\square$

5.2. A *Banach mixed normed space* is a pair (X, E) such that E is a Banach lattice while X is a br-complete lattice normed space with E-valued norm. The next fact justified the definition:

5.2(1). *Assume that E is a Banach lattice. Then $(X, |\!|\!|\cdot|\!|\!|)$ is a Banach space if and only if (X, E) is r-complete.*

Proof. $\leftarrow$: Take a Cauchy sequence $(x_n) \subset X$. Without loss of generality, we may assume that $|\!|\!|x_{n+1} - x_n|\!|\!| \leqslant 1/n^3$ for all $n \in \mathbb{N}$. Put

$$e_n := |x_1| + \sum_{k=1}^{n} k|x_{k+1} - x_k| \quad (n \in \mathbb{N}).$$

Then

$$\|e_{n+l} - e_n\| = \left\| \sum_{k=n+1}^{n+l} k|x_{k+1} - x_k| \right\| \leqslant \sum_{k=n+1}^{n+l} k|\!|\!|x_{k+1} - x_k|\!|\!| \leqslant \sum_{k=n+1}^{n+l} \frac{1}{k^2} \underset{n,l\to\infty}{\longrightarrow} 0.$$

Hence, (e_n) is a Cauchy sequence and so we have the limit $e := \lim_{n\to\infty} e_n$. We can pass to the limit in inequalities in normed lattices, and if a monotone increasing sequence has a norm limit, then the limit is the supremum of the sequence. Since $e_{n+k} \geqslant e_n$ for all $n, k \in \mathbb{N}$; therefore, $e = \sup e_n$.

If $n \geqslant m$ then

$$m|x_{n+l} - x_n| \leqslant \sum_{k=n+1}^{n+l} k|x_{k+1} - x_k| \leqslant e_{n+l} - e_n \leqslant e.$$

Consequently, $|x_{n+l} - x_n| \leqslant (1/m)e$, which means that (x_n) is br-Cauchy. Since X is br-complete, we have the limit $x := br\text{-}\lim_{n\to\infty} x_n$. Moreover, $\lim_{n\to\infty} |\!|\!|x - x_n|\!|\!| = 0$.

$\rightarrow$: Assume now that $(x_n) \subset X$ is br-Cauchy; i.e., $|x_n - x_m| \leqslant \lambda_k e$ for all $m, n, k \in \mathbb{N}$ and $m, n \geqslant k$, where $0 \leqslant e \in E$ and $\lim_{k\to\infty} \lambda_k = 0$. Then $|\!|\!|x_n - x_m|\!|\!| \leqslant \lambda_k \|e\|$ and $\lambda_k\|e\| \to 0$ as $k \to \infty$. Hence, we have the norm limit $x := \lim_{n\to\infty} x_n$. Since the vector norm is continuous, we can pass to the limit in the inequality $|x_n - x_m| \leqslant \lambda_k e$ as $m \to \infty$. In this event, we conclude that $|x - x_n| \leqslant \lambda_k e$ for all $n \geq k$, and so $x = br\text{-}\lim x_n$. $\square$

5.3. Let X be a lattice normed space over E, and let F be an ideal of E. The space $Y := \{x \in X : |x| \in F\}$ with the F-valued norm $|y|_Y := |y|_X$ is the F-*restriction* of X. If X is a BK-space then so is Y. If X is br-complete and F is a Banach lattice, then Y is a Banach mixed normed space.

Take a Banach space $\mathscr{X}$ within $\mathbb{V}^{(B)}$, and let F be a foundation of the Kantorovich space $\mathscr{R}{\downarrow}$. The F-restriction of $\mathscr{X}{\downarrow}$ is the F-*descent* of $\mathscr{X}$ which is denoted by $\mathscr{X}{\downarrow}^F$. In more detail, the F-descent of $\mathscr{X}$ is the triple $(\mathscr{X}{\downarrow}^F, |\cdot|, F)$, where

$$\mathscr{X}{\downarrow}^F := \{x \in \mathscr{X}{\downarrow} : |x| \in F\}, \quad |\cdot| := (\|\cdot\|){\downarrow}|_{\mathscr{X}{\downarrow}^F}.$$

5.3(1). *If a Banach lattice E is an ideal of $\mathscr{R}{\downarrow}$ then $\mathscr{X}{\downarrow}^E$ is a Banach mixed normed space.*

In case $E := \Lambda$ is a Kantorovich space of bounded elements, the order ideal $\mathscr{R}{\downarrow}$ is generated by $\mathbb{1} := 1^{\wedge} \in \mathscr{C}{\downarrow}$ and furnished with the norm $\|e\|_\infty := \inf\{\lambda > 0 : |e| \leq \lambda\mathbb{1}\}$; cp. 5.1. In this event the Λ-descent of $\mathscr{X}$ is called the *restricted descent* of $\mathscr{X}$ which is denoted by $\mathscr{X}{\Downarrow}$. Thus,

$$\mathscr{X}{\Downarrow} := \mathscr{X}{\downarrow}^{\Lambda} := \{x \in \mathscr{X}{\downarrow} : |x| \in \Lambda\}, \quad |\!|\!|x|\!|\!| := \|\,|x|\,\|_\infty \quad (x \in \mathscr{X}{\Downarrow}).$$

Take another Banach space $\mathscr{Y}$ and a bounded linear operator $\mathscr{T} : \mathscr{X} \to \mathscr{Y}$ within $\mathbb{V}^{(B)}$. By Theorem 3.7, $\mathscr{T}$ is a bounded linear operator from $\mathscr{X}{\downarrow}$ to $\mathscr{Y}{\downarrow}$. If $|\mathscr{T}{\downarrow}| \in \Lambda$, then $\mathscr{T}{\Downarrow}$ is the restriction of $\mathscr{T}{\downarrow}$ to $\mathscr{X}{\Downarrow}$ by definition.

5.3(2). *The restricted descent $\mathscr{T}\Downarrow$ of $\mathscr{T}$ is a bounded linear operator from $\mathscr{X}\Downarrow$ to $\mathscr{Y}\Downarrow$.*

Proof. If $T := \mathscr{T}\Downarrow$ then

$$\|Tx\| = \|\lvert \mathscr{T}\downarrow x \rvert\|_\infty \le \|\lvert \mathscr{T}\downarrow\rvert \cdot \lvert x\rvert\|_\infty \le \|\lvert \mathscr{T}\downarrow\rvert\|_\infty \|x\|$$

for all $x \in \mathscr{X}\Downarrow$. □

5.4. We arrive at the natural question: Which Banach spaces are linearly isometric to E-descents, in particular, the restricted descents of Banach spaces within $\mathbb{V}^{(B)}$? Clearly, the answer depends essentially on the geometry of the Banach space under consideration. We will confine exposition only to the case of restricted descent.

Let X be a normed space. Assume that $\mathscr{L}(X)$ includes some complete Boolean algebra $\mathscr{B}$ of norm one projection which is isomorphic to B; see 1.1(c). In this event we identify $\mathscr{B}$ and B, and so we will write $B \subset \mathscr{L}(X)$. Say that X is a *normed B-space* provided that $B \subset \mathscr{L}(X)$ and for every partition of unity in $(b_\xi)_{\xi\in\Xi}$ in B the following hold:

(a) if $b_\xi x = 0$ for all $\xi \in \Xi$ for some $x \in X$ then $x = 0$;

(b) if $b_\xi x = b_\xi x_\xi$ for all $\xi \in \Xi$, some $x \in X$, and $(x_\xi)_{\xi\in\Xi} \subset X$; then $\|x\| \le \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}$.

Conditions (a) and (b) may be rewritten equivalently as follows:

(a′) for every $x \in X$ there is a maximal projection $b \in B$ such that $bx = 0$;

(b′) if x, (x_ξ), and (b_ξ) are the same as in (b), then $\|x\| = \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}$.

In particular, (b′) implies that

$$\left\| \sum_{k=1}^{n} b_k x \right\| = \max_{k=1,\dots,n} \|b_k x\|$$

for $x \in X$ and pairwise disjoint projections $b_1, \dots, b_n \in B$.

Say that $x \in X$ is a *mixing* of $(x_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$, provided that $b_\xi x = b_\xi x_\xi$ for all $\xi \in \Xi$. Clearly, (a) means that zero is the sole mixing of any family of zeros, and so (a) is equivalent to the uniqueness of mixing. Also, (b) can be reformulated as follows: The unit ball U_X of X contains all existent mixings of its elements; i.e., U_X is mix-*closed*.

5.5. Theorem. *Let X be a Banach space. Then the following are equivalent:*

(1) *X is a decomposable mixed normed space whose norming lattice is an order complete unital AM-space;*

(2) *X is a Banach B-space.*

Proof. (1) → (2): The claim is straightforward by definition on using 1.4.

(2) → (1): Assume that X is a Banach B-space, and $J : B \to \mathscr{B}$ is the corresponding isomorphism of B onto the complete Boolean algebra of projections $\mathscr{B}$ in X. Let E stand for the ideal of the universally complete Kantorovich space of all B-valued resolutions of unity which is generated by $\bar{\mathbb{1}} \in E := \mathfrak{K}(B)$; see 1.14. Take the finite valued element $\alpha := \sum_{i=1}^{n} \lambda_i \bar{b}_i \in E$ such that $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\{b_1, \dots, b_n\}$ is a finite partition of unity in B, while $\bar{b}$ and $\lambda\bar{b}$ are the spectral functions; see 1.12. Put $J(\alpha) := \sum_{i=1}^{n} \lambda_i J(b_i)$ and observe that $J(\alpha)$ is a bounded linear operator in X. The norm of $J(\alpha)$ is as follows:

$$\|J(\alpha)\| = \sup_{\|x\|\le 1} \|J(\alpha)x\| = \sup_{\|x\|\le 1} \sup_{l:=1,\dots,n} \{\|\pi_l x\| |\lambda_l|\}$$
$$= \sup_{l:=1,\dots,n} \sup\{\|\pi_l x\|\ |\lambda_l| : \|x\| \le 1\} = \max\{|\lambda_1|, \dots, |\lambda_n|\}.$$

Also, the norm $\|\alpha\|_\infty$ of α in an AM-space E coincides with $\max\{|\lambda_1|, \dots, |\lambda_n|\}$. Consequently, J is a linear isometry from the subspace E_0 of finite valued elements of E to $\mathscr{L}(X)$. Furthermore, $J(\alpha\beta) = J(\alpha) \circ J(\beta)$ for all α, $\beta \in E_0$. Since E_0 is norm dense in E and $\mathscr{L}(X)$ is a Banach algebra, J admits

a continuous extension to an isometric isomorphism of E onto a closed subalgebra of $\mathscr{L}(X)$. Putting $x\alpha := \alpha x := J(\alpha)x$ for all $x \in X$ and $\alpha \in E$, furnish X with the structure of a unitary E-module so that

$$\|\alpha x\| \le \|x\|\,\|\alpha\|_\infty \quad (\alpha \in E,\ x \in X).$$

Moreover, $\alpha U_X + \beta U_X \subset U_X$ if $|\alpha| + |\beta| \le \mathbb{1}$. Define $p : X \to E^+$ by the formula

$$p(x) := \inf\{\alpha \in E^+ :\ x \in \alpha U_X\} \quad (x \in X),$$

where the infimum is taken in the Kantorovich space E. If $p(x) = \bar{\mathbb{0}}$, then to every $\varepsilon > 0$ there are a partition of unity $(\pi_\xi) \subset B$ and a family $(\alpha_\xi) \subset E^+$ such that $\pi_\xi \alpha_\xi \le \varepsilon \bar{\mathbb{1}}$ and $x \in \alpha_\xi U_X$ for all ξ. In this event $\pi_\xi x \in \pi_\xi \alpha_\xi U_X \subset \varepsilon U_X$. Since U_X is closed under mixing by hypothesis; therefore, $x = \operatorname{mix}(\pi_\xi x_\xi) \in \varepsilon U_X$. As $\varepsilon > 0$ is arbitrary, we see that $x = \bar{\mathbb{0}}$. If $x \in \alpha U_X$ and $y \in \beta U_X$ for some $\alpha, \beta \in E^+$; then, letting $\gamma := \alpha + \beta + \varepsilon \bar{\mathbb{1}}$, we can write

$$x + y = \gamma\,(\gamma^{-1}x + \gamma^{-1}y) \in \gamma\,(\gamma^{-1}\alpha U_X + \gamma^{-1}\beta U_X) \subset \gamma U_X.$$

Consequently, $p(x+y) \le \alpha+\beta+\varepsilon\bar{\mathbb{1}}$, and taking the infimum over α, β, and ε yields $p(x+y) \le p(x)+p(y)$. Moreover, if $\pi \in B$ and $x \in X$ then

$$\pi p(x) = \inf\{\pi\alpha :\ \bar{\mathbb{0}} \le \alpha \in E,\ x \in \alpha U_X\} = \inf\{\alpha \in E^+ :\ \pi x \in \alpha U_X\} = p(\pi x).$$

Hence, if $\alpha = \sum_{k=1}^n \lambda_k \pi_k$, with $\{\pi_1, \dots, \pi_n\}$ a partition of unity in B; then

$$p(\alpha x) = \sum_{k=1}^n \pi_k p(\lambda_k x) = \sum_{k=1}^n \pi_k |\lambda_k| p(x) = |\alpha| p(x).$$

Thus, $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in E$, and so (X, p, E) is a decomposable lattice normed space.

Show that X has a mixed norm; i.e., $\|x\| = \|p(x)\|_\infty$ for all $x \in X$. Take $0 \ne x \in X$ and put $y := x/\|x\|$. So $y \in U_X$ and $p(y) \le \bar{\mathbb{1}}$. Consequently, $p(x) \le \|x\|\bar{\mathbb{1}}$ implying that $\|p(x)\|_\infty \le \|x\|\|\bar{\mathbb{1}}\|_\infty = \|x\|$. Conversely, to every $\varepsilon > 0$ we can find a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathfrak{P}(E)$ and a family $(\alpha_\xi)_{\xi\in\Xi} \subset E^+$ such that $\pi_\xi \alpha_\xi \le p(x) + \varepsilon\bar{\mathbb{1}} \le (\|p(x)\|_\infty + \varepsilon)\bar{\mathbb{1}}$ and $x \in \alpha_\xi U_X$ $(\xi \in \Xi)$. Hence,

$$\pi_\xi x_\xi \in \pi_\xi \alpha_\xi U_X \subset (\|p(x)\|_\infty + \varepsilon)\pi_\xi \bar{\mathbb{1}} U_X \subset (\|p(x)\|_\infty + \varepsilon) U_X.$$

Consequently, $\|\pi_\xi x_\xi\| \le \|p(x)\|_\infty + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, using 5.4(b) yields $\|x\| \le \|p(x)\|_\infty$. □

5.6. A normed space X is *B-cyclic* provided that X is a B-space and every norm bounded family has a mixing by an arbitrary partition of unity in B. By 5.4 we see that X is B-cyclic if and only if for every partition of unity $(b_\xi) \subset B$ and every family $(x_\xi) \subset U_X$ there is a unique $x \in U_X$ such that $b_\xi x = b_\xi x_\xi$ for all ξ.

5.6(1). *A Banach B-space X is B-cyclic if and only if X is a d-complete lattice normed space.*

Proof. Straightforward by definition. □

A *B-isometry* of normed B-spaces is a linear isometry commuting with every projection in B. Say that Y is a *B-cyclic completion* of a B-space X provided that Y is a B-cyclic space and there is a B-isometry $\imath : X \to Y$ such that every B-cyclic subspace of Y, including $\imath(X)$, coincides with Y.

5.6(2). *A normed B-space is a B-cyclic Banach space if and only if the relevant lattice normed space is bo-complete.*

Proof. The claim follows from 1.7(1) and 5.6(1) on recalling that the norm completeness amounts to r-completeness; cp. 5.2. □

5.6(3). *Each Banach B-space has the B-cyclic completion unique up to B-isometry.*

Proof. The claim follows from 3.6(1) and 3.6(2). $\square$

We are now ready to answer the question that is posed in 5.4.

5.7. Theorem. *A Banach space X is linearly isometric to a bo-complete mixed normed space whose norming lattice is a unital order complete AM-space if and only if X is $\mathscr{B}$-cyclic with respect to some complete Boolean algebra of contractive projections of $\mathscr{B}$.*

Proof. In view of 3.2 it suffices to note that a Banach B-space is B-cyclic if and only if the space is disjointly complete as a lattice normed space. $\square$

5.8. Theorem. *A Banach space X is linearly isometric to the restricted descent of some Banach space within $\mathbb{V}^{(B)}$ if and only if X is B-cyclic.*

Proof. Use 3.1, 3.2, 5.2, and 5.7. $\square$

5.9. Consider a normed B-space X and the norm completion $\widetilde{X}$ of X which is clearly a Banach B-space since every projection $b \in B$ can be extended with the same norm by continuity to the whole space $\widetilde{X}$. By 5.6(1) $\widetilde{X}$ has the cyclic B-completion which will be denoted by $\overline{X}$. Using 5.6(2), take some Banach space $\mathscr{X}$ within $\mathbb{V}^{(B)}$ whose restricted descent is B-isometric to $\overline{X}$. Note that $\mathscr{X} \in \mathbb{V}^{(B)}$ is the *Boolean valued realization* of X.

Assume that X and Y are normed spaces, while $B \subset \mathscr{L}(X)$ and $B \subset \mathscr{L}(Y)$. A linear operator $T : X \to Y$ is *B-linear* provided that T commutes with every projection in B; i.e., $b \circ T = T \circ b$ for all $b \in B$.

Let $\mathscr{L}_B(X, Y)$ stand for the set of all bounded B-linear operators from X to Y. Clearly, $W := \mathscr{L}_B(X, Y)$ is a Banach space and $B \subset W$. If Y is a B-cyclic space then so is W. The projection $b \in B$ acts on W by the rule $T \mapsto b \circ T$ for all $T \in W$.

The space $X^{\#} := \mathscr{L}_B(X, \Lambda)$ is the *B-dual* of X. If $X^{\#}$ and Y are B-isometric then we will refer to Y also as the *B-dual* of X and call X a *B-predual* of Y. In symbols, $Y = X^{\#}$ and $X = Y_{\#}$.

5.10. Theorem. *Assume that X is a normed B-space and Y is a Banach B-cyclic space. Let $\mathscr{X}$ and $\mathscr{Y}$ be some Boolean valued realizations of X and Y. Then $\mathscr{L}_B(X, Y)$ is B-isometric to the restricted descent of $\mathscr{L}^B(\mathscr{X}, \mathscr{Y})$ within $\mathbb{V}^{(B)}$. In this event to $T \in \mathscr{L}_B(X, Y)$ there corresponds the element $\mathscr{T} := T{\uparrow}$ of $\mathbb{V}^{(B)}$ such that $[\![\mathscr{T} : \mathscr{X} \to \mathscr{Y}]\!] = \mathbb{1}$ and $[\![\mathscr{T}\imath x = \imath T x]\!] = \mathbb{1}$ for all $x \in X$, where $\imath$ stands for both embeddings of X to $\mathscr{X}{\downarrow}$ and Y to $\mathscr{Y}{\downarrow}$.*

Proof. Without loss of generality we will assume that X and Y are the restricted descents of some Banach spaces $\mathscr{X}$ and $\mathscr{Y}$; cp. 5.6(3) and 5.8. Put $X_0 := \mathscr{X}{\downarrow}$ and $Y_0 := \mathscr{Y}{\downarrow}$. By 3.6 $\mathscr{L}^B(\mathscr{X}, \mathscr{Y}){\downarrow}$ and $\mathscr{L}_{\mathscr{R}{\downarrow}}(X_0, Y_0)$ are linearly isometric. Moreover, the restriction of $\mathscr{L}_{\mathscr{R}{\downarrow}}(X_0, Y_0)$ to the ideal $\mathscr{R}{\downarrow}$ generated by $\mathbb{1}$, is the restricted descent of $\mathscr{L}^B(\mathscr{X}, \mathscr{Y})$. It suffices to note that each $T \in \mathscr{L}_B(X, Y)$ has the unique extension to $\widetilde{T} \in \mathscr{L}_{\mathscr{R}{\downarrow}}(X_0, Y_0)$, and in this event $\|T\| = \| |\widetilde{T}| \|_\infty$. $\square$

5.11. Let $\mathscr{X}'$ be the dual of $\mathscr{X}$ within $\mathbb{V}^{(B)}$. Denote the relations of isometric isomorphy and isometric B-isomorphy between Banach spaces and Banach B-spaces by $\simeq$ and $\simeq_B$. Assume also that X, Y, $\mathscr{X}$, and $\mathscr{Y}$ are the same as in 5.10.

5.11(1). $X^{\#} \simeq_B Y \leftrightarrow [\![\mathscr{X}' \simeq \mathscr{Y}]\!] = \mathbb{1}$.

5.11(2). *If $\overline{X}$ is the cyclic completion of X, then $X^{\#} = \overline{X}^{\#}$.*

6. Boolean Valued Duality of Banach Spaces

This section deals with decompositions of Banach space by means of projections satisfying certain norm conditions, namely, L-projections and M-projections, whose presence indicates that the Banach space has some features of an AL-space or AM-space. It is demonstrated that a Banach space admits

a Boolean valued representation which is a Banach space without nontrivial L-projections, while the dual space is represented as the dual Banach space without nontrivial M-projections. We will understand the Boolean algebra of projections in a real vector spaces in accordance with 1.1(c).

6.1. A projection π in a Banach space X is an *L-projection* provided that $\|x\| = \|\pi x\| + \|x - \pi x\|$ for all $x \in X$, while π is an *M-projection* provided that $\|x\| = \|\pi x\| \vee \|x - \pi x\|\}$ for all $x \in X$. The L-projections and M-projections different from the zero and the identity are referred to as *nontrivial*. The sets of all L-projections and M-projections on X will be denoted by $\mathbb{P}_L(X)$ and $\mathbb{P}_M(X)$, respectively.

Observe the obvious duality between L- and M-projections: π is an L-projection or an M-projection in X if and only if so is the adjoint π' of π in the dual space X'.

The following result is due to Cunningham [9, Theorem 2.5] and [10, Theorem 5].

6.2. Theorem. *Let X be a Banach space. Then*
(1) *$\mathbb{P}_L(X)$ is a complete Boolean algebra.*
(2) *$\mathbb{P}_M(X)$ is a (generally not complete) Boolean algebra.*
(3) *If X' is the dual of X then $\mathbb{P}_M(X')$ is isomorphic to $\mathbb{P}_L(X)$.*

Proof. Demonstration can be found in [11, Theorem 1.10]. $\square$

It follows that the Boolean algebra $\mathbb{P}_M(X')$ is complete. There are other examples of complete Boolean algebras of M-projections which are delivered by injective Banach lattices: The Boolean algebra $\mathbb{P}_M(X)$ is complete for an arbitrary injective Banach lattice X; moreover the unit ball of X is $\mathbb{P}_M(X)$-complete; see Theorem 7.7 and Corollary 7.11(2). Simple induction shows that for all $x \in X$ every finite collection of pairwise disjoint projections $\{\pi_1, \dots, \pi_n\}$ in $\mathbb{P}_L(X)$ or, respectively, $\mathbb{P}_M(X)$ with $\pi_0 = \pi_1 + \dots + \pi_n$ we have

$$\|\pi_0 x\| = \|\pi_1 x\| + \dots + \|\pi_n x\| \quad (\text{respectively, } \|\pi_0 x\| = \{\|\pi_1 x\| \vee \dots \vee \|\pi_n x\|). \tag{1}$$

Cunningham [9, Lemma 2.4] proved also that $\mathbb{P}_L(X)$ is *Badé complete*: If (π_α) is an increasing net of L-projections and $\pi := \sup_\alpha \pi_\alpha$ then $(\pi_\alpha x)$ is norm convergent to πx for all $x \in X$. It follows that for every $0 \neq x \in X$ the function $\mu_x : \mathbb{P}_L(X) \to \mathbb{R}$ defined as $\mu_x : \pi \mapsto \|\pi x\|$ is a nonzero order continuous measure. Hence, $\mathbb{P}_L(X)$ has a separating set of order continuous measures (normal measures) or, equivalently, the representation space of $\mathbb{P}_L(X)$ is *hyperstonean*.

6.3. Lemma. *Let X be a Banach space. Assume that $a \in X$ and $(x_\xi)_{\xi\in\Xi}$ in X are such that $\|\pi x_\xi\| \le \|\pi a\|$ for all $\xi \in \Xi$ and $\pi \in \mathbb{P}_L(X)$. Then for every partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{P}_L(X)$ there is a unique $x \in X$ satisfying $\pi_\xi x_\xi = \pi_\xi x$ for all $\xi \in \Xi$. Moreover, $x = \sum_{\xi\in\Xi} \pi_\xi x_\xi$ and $\|\pi x\| \le \|\pi a\|$ for all $\pi \in \mathbb{P}_L(X)$.*

Proof. Suppose that (x_ξ) and $a \in X$ meet the hypotheses of the lemma. Let Θ be the set of all finite subsets of Ξ. Given $\theta \in \Theta$, put

$$y_\theta := \sum_{\xi\in\theta} \pi_\xi x_\xi, \quad \sigma_\theta := \bigvee_{\xi\in\theta} \pi_\xi, \quad \sigma := \bigvee_{\xi\in\Xi} \pi_\xi, \quad \rho_\theta := \sigma - \sigma_\theta.$$

Take $\theta, \theta_1, \theta_2 \in \Theta$ with $\theta \subset \theta_1 \cap \theta_2$ and denote by θ' and $\theta_1 \triangle \theta_2$ the complement of θ and the symmetric difference of θ_1 and θ_2, respectively. Using (1) implies that

$$\|y_{\theta_1} - y_{\theta_2}\| = \Bigg\| \sum_{\xi\in\theta_1\triangle\theta_2} \pi_\xi x_\xi \Bigg\| = \sum_{\xi\in\theta_1\triangle\theta_2} \|\pi_\xi x_\xi\| \le \sum_{\xi\in\theta} \|\pi_\xi a\| = \|\rho_\theta a\|.$$

By definition $(\rho_\theta a)_{\theta\in\Theta}$ decreases to zero and (ρ_θ) is norm convergent to zero by Badé completeness of $\mathbb{P}_L(X')$, so that $(y_\theta)_{\theta\in\Theta}$ is Cauchy and there exists

$$x := \lim_{\theta\in\Theta} y_\theta = \sum_{\xi\in\Xi} \pi_\xi x_\xi$$

in X. If $\xi \in \theta$ then evidently $\pi_\xi y_\theta = \pi_\xi x_\xi$, and passage to the limit yields $\pi_\xi x = \pi_\xi x_\xi$. Moreover, $\|\pi\pi_\xi y_\theta\| \le \|\pi a\|$ by hypothesis and so $\|\pi\sigma_\theta x\| \le \|\pi a\|$ for all $\theta \in \Theta$; again, passage to the limit in the inequality reveals that $\|\pi x\| \le \|\pi a\|$. $\square$

Consider a Banach space $(\mathscr{X}, \rho)$ in $\mathbb{V}^{(B)}$ and an order dense ideal E in $\mathscr{R}{\downarrow}$. Recall that the *E-descent* of $\mathscr{X}$ is the subspace $\mathscr{X}{\downarrow}^E := \{x \in \mathscr{X}{\downarrow} : |x| \in E\}$ of $\mathscr{X}{\downarrow}$ with the induced E-valued norm; see 5.3.

6.4. Theorem. *If $\mathscr{X}$ is a Banach space within $\mathbb{V}^{(B)}$, then the E-descent $X := \mathscr{X}{\downarrow}^E$ is a BK-space over E and X can be endowed with the structure of a faithful unitary module over the f-algebra $\mathrm{Orth}(E)$ so that $b \le [\![x = 0]\!]$ if and only if $\chi(b)x = 0$ for all $b \in B$ and $x \in X$. Conversely, if X is a BK-space over E with $E = |X|^{\perp\perp}$ and $B := \mathfrak{P}(E)$, then there exists a Banach space $\mathscr{X}$ unique up to linear isometry within $\mathbb{V}^{(B)}$ whose E-descent is linearly isometric to X (in the sense of E-valued norms).*

Proof. This is immediate from Theorems 3.1 and 3.2. $\square$

It is worth to note the two particular cases of Theorem 6.4 which concern AL-spaces and AM-spaces. Let M be the *bounded part* of the universally complete vector lattice $\mathscr{R}{\downarrow}$; i.e., M consists of all $x \in \mathscr{R}{\downarrow}$ with $|x| \le C\mathbb{1}$ for some $C \in \mathbb{R}$. Endow M with the norm $\|m\|_\infty := \inf\{0 < \lambda \in \mathbb{R} : |m| \le \lambda\mathbb{1}\}$. Then M is an order complete AM-space with unit and $B = \mathfrak{P}(M)$. Putting $E := M$ in Theorem 6.4, we conclude that the M-descent $X := \mathscr{X}{\downarrow}^M$ endowed with the mixed norm $\|x\| := \big\||x|\big\|_\infty$ is a Banach space; see Theorem 5.8. The M-descent X is also called the *restricted descent* of $\mathscr{X}{\downarrow}^M$ and still denoted by $\mathscr{X}{\Downarrow}$; see 5.3(1).

6.5. Consider a lattice normed space $(X, |\cdot|)$ over an order complete Banach lattice E and put $B := \mathfrak{P}(E)$. Endow X with the mixed norm defined as $\|x\| := \|x\|_X := \big\||x|\big\|_E$ for all $x \in X$. A subset $S \subset X$ is *B-bounded* if there exists $a \in X$ such that $\|\pi x\| \le \|\pi a\|$ for all $x \in S$ and $\pi \in B$. Say that X is *B-complete* whenever, given a partition of unity (b_ξ) in B and a B-bounded family (x_ξ) in X, there exists a unique $x \in X$ such that $b_\xi x = b_\xi x_\xi$ for all $\xi \in \Xi$.

6.5(1). Lemma. *A decomposable lattice normed space $(X, |\cdot|)$ over a Banach lattice E presents a BK-space if and only if the associated mixed normed space $(X, |\cdot|)$ is a B-complete Banach space.*

Proof. Observe first that the condition $\|\pi x\| \le \|\pi a\|$ for all $\pi \in B$ is equivalent to the inequality $|x| \le |a|$. Indeed, $|x| \le |a|$ implies obviously that $\|x\| = \big\||a|\big\|_E \le \big\||x|\big\|_E = \|a\|$ by the monotonicity of $\|\cdot\|_E$. If $|x| \le |a|$ is false then, using Proposition 2.3, we can choose some $\pi_0 \in B$ and a real $\varepsilon > 0$ so that $|\pi_0 x| \ge (1+\varepsilon)|\pi_0 a|$. It follows that $\|\pi_0 x\| \ge (1+\varepsilon)\|\pi_0 a\| > \|\pi_0 a\|$; a contradiction. Now it is clear that $A \subset X$ is B-bounded if and only if $|A|$ is order bounded in E. It remains to recall that a decomposable lattice normed space over E is bo-complete if and only if it is d-complete and br-complete; see 1.7(1). $\square$

6.5(2). Lemma. *Let X be as above and let E be an order complete AM-spaces with unit. Then X is B-complete if and only if X is B-cyclic.*

Proof. A subset of an AM-space with unit E is norm bounded if and only if it is order bounded. It follows that a subset A of X is B-bounded if and only if $|A|$ is order bounded in E. $\square$

6.6. Recall that the B-dual $(X^\#, |\cdot|)$ of a lattice normed space X over E is defined as the lattice normed space over $\mathrm{Orth}(E)$, where $X^\#$ consists of all linear operators $x^\# : X \to E$ such that there exists a positive orthomorphism $S \in \mathrm{Orth}(E)$ with $|(x, x^\#)| := |x^\#(x)| \le S(|x|)$ for all $x \in X$; the least S satisfying the above is denoted by $|x^\#|$.

6.6(1). Lemma. *A linear operator $T : X \to E$ belongs to $X^\#$ if and only if T is norm bounded and B-linear; moreover, $\|T\| = \|\,|T|\,\|_\infty$, where $\|T\| = \inf\{\lambda > 0 : |T| \le \lambda I_E\}$.*

Proof. Note first that $\mathrm{Orth}(E) = \mathscr{Z}(E)$ for every normed space E (see [12, Theorem 3.1.11]) and $\mathscr{Z}(E)$ can be identified with M. Consider $T \in L(X, E)$. If $T \in X^\#$, $\pi \in \mathbb{P}(E)$, and $x \in X$; then $|T(\pi x)| \le \pi|T||x|$ by the above definition, and so $\pi^* T(\pi x) = 0$. It follows that $\pi T x = \pi T(\pi x)$. Similarly,

$\pi T(\pi^* x) = 0$ and $T(\pi x) = \pi T(\pi x)$. Thus, $T \circ \pi = \pi \circ T$; i.e., T is B-linear. Moreover, $|Tx| \leq \| \,|T|\, \|_\infty |x|$ so that

$$\|Tx\| = \||Tx|\| \leq \| \,|T|\, \|_\infty \| \,|x|\, \| = \| \,|T|\, \|_\infty \|x\|,$$

whence $\|T\| \leq \| \,|T|\, \|_\infty$. Conversely, if T is norm bounded and B-linear then

$$\|\pi Tx\|_E = \|T(\pi x)\|_E \leq \|T\| \, |\pi x| \, \|_\infty = \big\|\pi(\|T\| |x|)\big\|_E$$

for all $\pi \in B$ and we get $|Tx| \leq \|T\| |x|$ for all $x \in X$. It follows that $T \in X^{\#}$ and $\|T\| = \| \,|T|\, \|_\infty$. $\square$

6.7. Lemma. *If $\mathscr{X}$ is a Banach space within $\mathfrak{V}^{(B)}$ and an order ideal $E \subset \mathscr{R}{\downarrow}$ is a Banach lattice then $\mathscr{X}'{\downarrow}^{\mathscr{X}(E)}$ is B-isometric to $X^{\#}$. In particular, $X^{\#}$ is a BK-space over M (with the obvious identification of M and $\mathscr{X}(E)$).*

Proof. This is straightforward from Theorem 5.10, 5.11(1), 5.11(2), and Lemma 6.6. $\square$

Assume now that the representation space of $\mathfrak{P}(\mathscr{R}{\downarrow})$ is hyperstonean. Then there is an order dense ideal $L \subset \mathscr{R}{\downarrow}$ which is an AL-space. Moreover, there is a strictly positive order continuous functional $\phi : L \to \mathbb{R}$ such that $\|u\|_L = \phi(|u|)$ for all $u \in L$. Put $L^1(B, \phi) := (L, \| \cdot \|_L)$.

6.8. Lemma. *Let $(X, |\cdot|)$ be a BK-space over $L^1(B, \phi)$ and $\| \cdot \| := \phi \circ |\cdot|$. Then $h : x^{\#} \mapsto \phi \circ x^{\#}$ is a B-isometry from $(X, |\cdot|)^{\#}$ onto $(X, |\cdot|)'$. In particular, X' is a BK-space over M with the obvious identification of M and $L^1(B, \phi)'$.*

Proof. Observe first that X' is a BK-space over M and $\langle x, x' \rangle \leq \phi(|x||x'|)$ for all $x \in X$ and $x' \in X'$; see [4, Theorem 7.1.4]. It is immediate from Lemma 6.6(1) that h is a linear operator from $X^{\#}$ into X'. If $\phi \circ x^{\#} = 0$ for some $x^{\#} \in X^{\#}$, then $\phi(\pi(x, x^{\#})) = 0$ for all $x \in X$ and $\pi \in B$. This implies that $x^{\#} = 0$ as ϕ is essentially positive so that h is injective. Denote by $D^{\#}$ and D' the unit balls of $X^{\#}$ and X'. Then $D^{\#} = \{x^{\#} \in X^{\#} : |x^{\#}| \leq \mathbb{1}\}$ and $D' = \{x' \in X' : (\forall x \in X) \langle x, x' \rangle \leq \phi(|x|)\}$. If $x^{\#} \in D^{\#}$ then

$$|\langle x, \phi \circ x^{\#} \rangle| = |\phi(\langle x, x^{\#} \rangle)| \leq \phi(|x||x^{\#}|) \leq \phi(|x|) = \|x\|$$

and this implies that $h(D^{\#}) \subset D'$. To prove the converse inclusion take $x' \in X'$; i.e., $|\langle x, x' \rangle| \leq |x'| \phi(|x|)$ for all $x \in X$. Since ϕ is positive and order continuous, there exists a linear operator $T : X \to L$ such that $x' = \phi \circ T$ and $Tx \leq \|x'\| |x|$ for all $x \in X$; see, for instance, [13, Theorem 4.5.2]. So $T \in X^{\#}$ and $h(D') \subset D^{\#}$. Thus, h is a B-isometry of $X^{\#}$ onto X'. $\square$

6.9. Lemma. *For every positive order continuous measure μ on $B = \mathbb{P}_L(X)$ there exists a unique $N(\mu) \in L^1(B, \phi)$ such that $\mu(b) = \phi(bN(\mu))$ for all $b \in B$.*

Proof. Identify B with the Boolean algebra of the components of $\mathbb{1} \in M$, so that $B \subset M$. Each order continuous measure $\mu : B \to \mathbb{R}$ admits the unique extension to some order continuous functional f_μ on M. Since the order continuous dual M'_n is lattice isometric to $L^1(B, \phi)$, there exists exactly one $N(\mu) \in L^1(B, \phi)$ such that $f_\mu(u) = \phi(uN(\mu))$ for all $u \in M$; see [4, Theorem 4.10(4)]. $\square$

6.10. Theorem. *Let X be a Banach space with the dual X' and the duality pairing $\langle \cdot, \cdot \rangle$. If $B := \mathbb{P}_L(X)$ then there exists a Banach space $\mathscr{X}$ unique up to linear isometry within $\mathbb{V}^{(B)}$ such that*

(1) $\mathscr{X}$ has no nontrivial L-projections and $\mathscr{X}'$ has no nontrivial M-projections.

(2) X is linearly B-isometric to $\mathscr{X}{\downarrow}^L$ and X' is linearly B-isometric to $\mathscr{X}'{\downarrow}^M$.

(3) There exists a bilinear operator $\langle\!\langle \cdot, \cdot \rangle\!\rangle : X \times X' \to L^1(B, \phi)$ satisfying

$$\langle \pi x, x' \rangle = \phi(\pi \langle\!\langle x, x' \rangle\!\rangle) \quad (x \in X, \; x' \in X', \; \pi \in B).$$

Proof. Let X be a Banach space and $B := \mathbb{P}_L(X)$. As was mentioned after Theorem 6.2, B is Badé complete, and so the function $\mu_x : B \to \mathbb{R}$ defined as $\mu_x(b) := \|bx\|$ $(b \in B)$ is an order continuous measure for every $x \in X$. By Lemma 6.9 there exists a unique $N(\mu_x)$ in $L^1(B, \phi)_+$ such that $|bx| = \phi(bN(\mu_x))$ for all $b \in B$. Put $|x| := N(\mu_x)$ and observe that $|\cdot| : X \to L^1(B, \phi)$ is a decomposable norm. Indeed,

the d-decomposability of $\lvert\cdot\rvert$ is trivial and $(X, \lvert\cdot\rvert)$ is $L^1(B,\phi)$-uniformly complete by 5.2(1). Lemma 6.3 provides the B-completeness of X, so that X is a BK-space Lemma 6.5(1) and $\lvert x\rvert = \| |x| \|_L$ for all $x \in X$ by definition. So Theorem 6.4 is applicable and guarantees the existence within $\mathbb{V}^{(B)}$ of a Banach space $\mathscr{X}$ unique up to linear isometry whose $L^1(B,\phi)$-descent $\mathscr{X}{\downarrow}^L$ is B-linearly isometric to X. By Lemmas 6.7 and 6.8 X' is linearly B-isometric to $\mathscr{X}'{\downarrow}^M$ so that 6.10(2) holds.

To prove 6.10(1) consider an L-projection $\rho \in \mathbb{P}_L(\mathscr{X})$ within $\mathbb{V}^{(B)}$ and denote by P the restriction of the descent $\rho{\downarrow}$ to X. Using the fact that the descent of the composition of mappings within $\mathbb{V}^{(B)}$ is the composition of their descents, we see that P is a projection and $\lvert x\rvert = \lvert Px\rvert + \lvert(I_X - P)x\rvert$ for all $x \in X$. Using ϕ, we get $\|x\| = \|Px\| + \|(I_X - P)x\|$ for all $x \in X$ so that $P \in \mathbb{P}_L(X)$. It follows that there is $b \in B$ such that P is multiplication by $\chi(b)$; see Theorem 6.4. At the same time, $\chi(b) \in \{0,1\} \subset \mathscr{R}$; see [1, 4.2.2]. It follows that ρ is multiplication by $\chi(b)$ and so ρ is trivial. Thus, $\mathbb{P}_L(\mathscr{X}) = \{0, I_{\mathscr{X}}\}$ and, by Theorem 6.2, we also have $\mathbb{P}_M(\mathscr{X}') = \{0, I_{\mathscr{X}}\}$.

Finally, consider the duality $(\mathscr{X}, \mathscr{X}')$ and denote by δ the natural pairing $\delta(x,x') := x'(x)$. Then δ is a bilinear form on $\mathscr{X} \times \mathscr{X}'$ within $\mathbb{V}^{(B)}$ and its descent $\delta{\downarrow}$ is a bilinear operator from $\mathscr{X}{\downarrow} \times \mathscr{X}'{\downarrow}$ to $\mathscr{R}{\downarrow}$. Using Lemmas 6.6(1) and 6.7, define the bilinear operator $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ from $X \times X'$ to $L^1(B,\phi)$ by letting $\langle\!\langle x, x'\rangle\!\rangle := \delta{\downarrow}(x, x^{\#})$ where $x \in X$, $x^{\#} \in X^{\#}$, and $x' = h(x^{\#})$. Using ϕ, we deduce that

$$\phi(\langle\!\langle x, x'\rangle\!\rangle) = \phi(\langle\!\langle x, x^{\#}\rangle\!\rangle) = \langle x, \phi \circ x^{\#}\rangle = \langle x, x'\rangle. \quad \square$$

6.11. Corollary. *Let X be a Banach space with the dual X' and the duality pairing $\langle\cdot,\cdot\rangle$ and let $B := \mathbb{P}_L(X)$. Then*

(1) *X is a BK-space with mixed norm over L and $\mathscr{P}(X) = \mathbb{P}_L(X)$.*

(2) *X' is a BK-space with mixed norm over M and $\mathscr{P}(X') = \mathbb{P}_M(X')$.*

(3) *There exists a bilinear operator $\langle\!\langle\cdot,\cdot\rangle\!\rangle : X \times X' \to L^1(B,\phi)$ satisfying*

$$\langle \pi x, x'\rangle = \phi(\pi\langle\!\langle x, x'\rangle\!\rangle) \text{ and } |\langle\!\langle x, x'\rangle\!\rangle| \leq \lvert x\rvert \lvert x'\rvert$$

for all $x \in X$, $x' \in X'$, and $\pi \in B$.

Proof. This is immediate from Theorem 6.4 and 6.10. $\square$

7. Injective Banach Lattices

In this section we briefly discuss the Boolean valued interpretation of Banach lattice theory. We confine exposition only to the basic. As regards the needed information on Banach lattices, we refer to [8] and [13].

7.1. Consider a Banach lattice X. An *M-band projections* of X is a band projection that is also an M-projection. Denote by $M(X)$ the set of all M-band projections of X; i.e., $M(X) := \mathfrak{B}(X) \cap \mathbb{P}_M(X)$. Note that $M(X)$ is a subalgebra of the Boolean algebra of band projections $\mathfrak{P}(X)$ in X. A Banach lattice X is *B-cyclic* whenever X is a B-cyclic Banach space for some regular subalgebra B of $M(X)$.

If X has the Fatou and Levi properties, then $M(X)$ itself is an order closed subalgebra of the complete Boolean algebra $\mathfrak{P}(X)$. Recall that a Banach lattice X has the *Levi property* or X is *monotone complete*) provided that $0 \leq x_\alpha\uparrow$ and $\|x_\alpha\| \leq 1$ imply that $\sup_\alpha x_\alpha$ exists in X. Say that X has the *Fatou property* or X is *order semicontinuous* provided that $0 \leq x_\alpha \uparrow x$ implies $\|x_\alpha\| \uparrow \|x\|$; (see [12, Definition 2.4.18]).

7.2. Theorem. *The restricted descent of a nonzero Banach lattice within $\mathbb{V}^{(B)}$ is a B-cyclic Banach lattice. Conversely, if X is a B-cyclic Banach lattice, then there exists a Banach lattice $\mathscr{X}$ within $\mathbb{V}^{(B)}$ unique up to the isometric lattice isomorphism whose restricted descent is lattice B-isometric to X. Moreover, $\pi \mapsto \pi{\Downarrow} := \pi{\downarrow}|_X$ is an isomorphism of the Boolean algebras $M(\mathscr{X}){\downarrow}$ and $M(X)$; in symbols, $M(\mathscr{X}){\downarrow} \simeq M(\mathscr{X}{\Downarrow})$.*

Proof. Assuming that X is a B-cyclic Banach lattice with the norm $\|\cdot\|_X$, we will demonstrate that X is B-isometric to the restricted descent $\mathscr{X}{\Downarrow}$ of some Banach lattice $\mathscr{X} \in \mathbb{V}^{(B)}$. The Banach part

of the claim follows from Theorems 5.7 and 5.8. In particular, we may suppose that X coincides with the B-cyclic Banach space $\mathscr{X}{\Downarrow}$ for some Banach space $\mathscr{X}$ within $\mathbb{V}^{(B)}$.

Put $\mathscr{X}_+ := X_+{\uparrow}$. If f is an extensional mapping then $f(A){\uparrow} = f{\uparrow}(A{\uparrow})$ where $A \subset \operatorname{dom}(f)$; see [1, Theorem 3.3.11(2)]. Applying this to the addition $f : (x, y) \mapsto x + y$ with $x, y \in X$ and $A := X_+ \times X_+$ and to the Λ-multiplication $f : (\lambda, x) \mapsto \lambda x$ with $\lambda \in \Lambda$, $x \in X$, and $A := \Lambda_+ \times X_+$; we find $[\![\mathscr{X}_+ + \mathscr{X}_+ = \mathscr{X}_+]\!] = \mathbb{1}$ and $[\![\mathscr{R}_+ \cdot \mathscr{X}_+ = \mathscr{X}_+]\!] = \mathbb{1}$; i.e., $[\![\mathscr{X}_+ \text{ is a convex cone}]\!] = \mathbb{1}$. Moreover, $[\![\mathscr{X}_+ \text{ is pointed}]\!] = \mathbb{1}$, since $[\![\pm x \in \mathscr{X}_+ \text{ and } \|x\| \le 1]\!] = \mathbb{1}$ implies that $\pm x \in \mathscr{X}_+{\downarrow} \cap X \subset X_+$ and so $x = 0$. Now, define the order on $\mathscr{X}$ by $[\![(\forall x, y \in \mathscr{X})(x \le y \leftrightarrow y - x \in \mathscr{X}_+)]\!] = \mathbb{1}$. By transfer $(\mathscr{X}, \mathscr{X}_+)$ is an ordered Banach space within $\mathbb{V}^{(B)}$. Moreover, $[\![x \le y]\!] = \mathbb{1} \Longleftrightarrow x \le y$ for all $x, y \in X$.

Consider the sentence $\sigma \equiv (\forall\, a \in \{0,1\})(\forall x, y \in \mathscr{X})\,(ax \le ay \leftrightarrow (a \ne 1 \vee x \le y))$ which is a very simple ZFC-Theorem. By transfer $[\![\sigma]\!] = \mathbb{1}$. Calculating the Boolean truth values for quantifiers, we rephrased as follows: $[\![ax \le ay]\!] = [\![a = 1]\!]^* \vee [\![x \le y]\!]$ for all $a \in \{0,1\}{\downarrow}$ and $x, y \in \mathscr{X}{\downarrow}$. Using the Boolean isomorphism $\chi : B \to \{0,1\}{\downarrow}$ (see [1, 4.2.2]), we may replace $a \in \{0,1\}{\downarrow}$ by $\chi(b)$ for some $b \in B$ and write $b^* \vee [\![x \le y]\!] = [\![\chi(b)x \le \chi(b)y]\!]$. Now it is easy to see (cp. the Gordon Theorem in [1, 5.2.2]) that

$$b \le [\![x \le y]\!] \Longleftrightarrow \chi(b)x \le \chi(b)y \quad (b \in B;\ x, y \in \mathscr{X}{\downarrow}).$$

The last relation allows us to treat the interplay between X and $\mathscr{X}$. By way of example we prove that $\mathscr{X}$ is a vector lattice; i.e., the sentence $(\forall x \in \mathscr{X})(\exists y \in \mathscr{X}) y = x \vee (-x)$ is true within $\mathbb{V}^{(B)}$. Using the rules for calculating Boolean truth values (see [1, 3.3.2]) and the maximum principle it suffices to prove that for every $x \in X$ there exists $y \in X$ for which $[\![y = x \vee (-x)]\!] = \mathbb{1}$. By hypothesis there exists $y = |x|$ in X, and by the definition of the order on $\mathscr{X}$ we have $[\![\pm x \le y]\!] = \mathbb{1}$. Thus, it remains to check that $[\![(\forall u \in \mathscr{X})(\pm x \le u \to y \le u)]\!] = \mathbb{1}$. Again by [1, 2.1.7 and 3.3.2] the latter is equivalent to the relation $[\![\pm x \le u]\!] \le [\![y \le u]\!]$ for all $u \in X$. Now, if $b = [\![\pm x \le u]\!]$ then $\pm\chi(b)x \le \chi(b)u$ and $\chi(b)y \le \chi(b)u$ by the Gordon Theorem. It follows that $b \le [\![y \le u]\!]$.

The Λ-valued norm $\lvert\!\cdot\!\rvert$ of X is the descent of the norm $\|\cdot\|_{\mathscr{X}}$ of $\mathscr{X}$ and $\|x\|_X = \|\lvert x\rvert\|_\infty$ for all $x \in X$. Therefore, $\|\cdot\|_{\mathscr{X}}$ is a lattice norm if and only if $|x| \le |y|$ implies $\lvert x\rvert \le \lvert y\rvert$ for all $x, y \in X$. By hypothesis $\|\cdot\|_X$ is a lattice norm. Assume that $|x| \le |y|$ for some $x, y \in X$. If $\lvert x\rvert \le \lvert y\rvert$ were false, there would be $\pi \in B$ and $0 < \varepsilon \in \mathbb{R}$ with $\pi\lvert x\rvert\pi(\lvert y\rvert + \varepsilon\mathbb{1})$. Hence, $\|\pi x\|_X = \|\pi\lvert x\rvert\|_\infty \ge \|\pi\lvert y\rvert\|_\infty + \varepsilon > \|\pi y\|_X$ which would contradict the hypothesis, since $\pi x \le \pi y$. Thus, $(\mathscr{X}, \mathscr{X}_+)$ is a Banach lattice within $\mathbb{V}^{(B)}$.

Assume that π is an M-projection in $\mathscr{X}$ and Π is the restriction of $\pi{\downarrow}$ to X. Then $[\![\pi \circ \pi = \pi]\!] = \mathbb{1}$, $[\![0 \le \pi x \le x \text{ for all } x \in \mathscr{X}_+]\!] = \mathbb{1}$, and $[\![\|x\| = \|\pi x\| \vee \|\pi^\perp x\|\} \text{ for all } x \in \mathscr{X}]\!] = \mathbb{1}$ by definition. By [1, Theorems 3.2.12 and 3.2.13] $\pi{\downarrow} = (\pi \circ \pi){\downarrow} = \pi{\downarrow} \circ \pi{\downarrow}$ and so $\Pi = \Pi \circ \Pi$. Since $[\![\pi x = \Pi x]\!] = \mathbb{1}$ for all $x \in X$, we have $0 \le \Pi x \le x$ for all $x \in X_+$. Finally, the relations $[\![\|x\| = \|\pi x\| \vee \|\pi^\perp x\| \text{ for all } x \in \mathscr{X}]\!] = \mathbb{1}$ and $\lvert x\rvert = \lvert \Pi x\rvert \vee \left\lvert \Pi^\perp x\right\rvert$ for all $x \in X$ are equivalent, whence we deduce $\|x\|_X = \|\,\lvert\Pi x\rvert \vee \left\lvert\Pi^\perp x\right\rvert\,\|_\infty = \|\Pi x\|_X \vee \|\Pi^\perp x\|_X$. Thus, Π is an M-projection in X; i.e., $\Pi \in M(X)$. A similar reasoning in the converse direction shows that if Π is an M-projection in X, then $\pi := \Pi{\uparrow}$ is also an M-projection in $\mathscr{X}$; i.e., $\pi \in M(\mathscr{X}){\downarrow}$. Clearly, the descent and ascent implement bijections between $M(\mathscr{X}){\downarrow}$ and $M(X)$. Each of them is a Boolean isomorphism.

We leave to the reader the straightforward verification of the fact that the restricted descent of a nonzero Banach lattice is a B-cyclic Banach lattice. $\square$

The element $\mathscr{X} \in \mathbb{V}^{(B)}$ from Theorem 3.1 is the *Boolean valued realization of X*.

7.3. Corollary. *If $\mathscr{X}$ is a Boolean valued representation of a B-cyclic Banach lattice X, then the conditions $B = M(X)$ and $[\![M(\mathscr{X}) = \{0, I_{\mathscr{X}}\}]\!] = \mathbb{1}$ are equivalent.*

Proof. This is immediate from Theorem 3.1, since B is isomorphic to the descent of the two-element Boolean algebra $\{0, I_{\mathscr{X}}\}$ (see [1, 4.2.2]). $\square$

7.4. Corollary. *A Banach lattice X is lattice isometric to the restricted descent of some Banach lattice $\mathscr{X}$ within $\mathbb{V}^{(B)}$ if and only if X is B-cyclic relative to a complete Boolean algebra of M-band projections in X.*

Say that a downward directed set A in a B-cyclic Banach lattice X is *B-vanishing* if for every $0 < \varepsilon \in \mathbb{R}$ there exists a partition of unity $(\pi_a)_{a \in A}$ in B such that $\|\pi_a a\| \le \varepsilon$ for all $a \in A$. The norm in X is *order B-continuous* if every downward directed set $A \subset X$ with $\inf A = 0$ is B-vanishing; cp. with the notion of m-convergence by Takeuti [6].

7.5. Theorem. *Suppose that X is a B-cyclic Banach lattice and $\mathscr{X} \in \mathbb{V}^{(B)}$ is the Boolean valued representation of X. Then*

(1) *X is order complete $\iff [\![\mathscr{X}$ is order complete$]\!] = \mathbb{1}$.*

(2) *X has a Fatou (Levi) norm $\iff [\![\mathscr{X}$ has a Fatou (Levi) norm$]\!] = \mathbb{1}$.*

(3) *X has an order B-continuous norm $\iff [\![\mathscr{X}$ has an order continuous norm$]\!] = \mathbb{1}$.*

(4) *X has an order B-continuous Levi norm $\iff [\![\, \mathscr{X}$ is a BK-space$]\!] = \mathbb{1}$.*

(5) *$S \in X^{\#}$ is order continuous $\iff$ $[\![\, \sigma := S{\uparrow} \in \mathscr{X}^*$ is order continuous$]\!] = \mathbb{1}$.*

Proof. (1): Just as in [1, Theorem 4.4.10] we can demonstrate that, given an order bounded set $A \subset X_+$, there exists $a = \sup(A)$ if and only if $[\![$there exists $\sup(A{\uparrow})]\!] = \mathbb{1}$ and in this case $[\![a = \sup(A{\uparrow})]\!] = \mathbb{1}$. Thus, the order completeness of $\mathscr{X}$ within $\mathbb{V}^{(B)}$ implies that X is order complete. Conversely, suppose that X is order complete and take a nonempty set $\mathscr{A} \subset \mathscr{X}_+$ bounded above by $u \in \mathscr{X}$. There is no loss of generality in assuming that $[\![\|u\| \le 1]\!] = \mathbb{1}$. Then $A := \mathscr{A}{\downarrow}$ lies in X and, using the cancelation rule $\mathscr{A}{\downarrow}{\uparrow} = \mathscr{A}$ (cp. [1, 3.3.3]), we see that there exists $a = \sup(\mathscr{A}{\downarrow})$ if and only if $[\![$there exists $\sup(\mathscr{A})]\!] = \mathbb{1}$ and in this case $[\![a = \sup(\mathscr{A})]\!] = \mathbb{1}$.

(2): We may assume without loss of generality that the upward directed sets in the definitions of Fatou norm and Levi norm belong to the unit balls $U(X)$ and $U(\mathscr{X})$. Moreover, if $A \subset X$ is upward directed then $[\![A{\uparrow}$ is upward directed$]\!] = \mathbb{1}$ and $[\![\mathscr{A} \subset \mathscr{X}$ is upward directed$]\!] = \mathbb{1}$ imply that $\mathscr{A}{\downarrow}$ is upward directed. Finally, observe that $U(\mathscr{X}){\downarrow} = \{x \in \mathscr{X}{\downarrow} : \lvert x \rvert \le \mathbb{1}\} = U(X)$. Now, let $\mathscr{X}$ have a Levi norm and take an upward directed set $A \subset U(X)$. It follows that $\{\lvert a \rvert : a \in A\} \subset [-\mathbb{1}, \mathbb{1}]$ and thus $[\![\{\|a\| : a \in A{\uparrow}\} \subset [-1, 1]\,]\!] = \mathbb{1}$; i.e., $[\![A{\uparrow} \subset U(\mathscr{X})]\!] = \mathbb{1}$. By hypothesis $a = \sup(A{\uparrow})$ exists in $\mathscr{X}$, whence $a = \sup(A)$. The argument for the converse is similar. To ensure the claim on the Fatou norm it suffices to observe that $\lvert a \rvert = \sup\{\lvert a' \rvert : a' \in A\}$ in Λ if and only if $\|a\| = \|\lvert a \rvert\|_\infty = \sup\{\|\lvert a' \rvert\|_\infty : a' \in A\}$, since the AM-space Λ has a Levi norm.

(3): Using the remarks in (2) it is easy to see that $[\![\, \mathscr{X}$ has an order continuous norm$\,]\!] = \mathbb{1}$ if and only if for every downward directed set $A \subset X_+$ with $\inf(A) = 0$ we have $\inf\{\lvert a \rvert : a \in A\} = 0$ in Λ. By [1, 5.2.7(2)] the latter can be rephrased as follows: For every $\varepsilon > 0$ there exists a partition of unity $(\pi_a)_{a \in A}$ in B such that $\lvert \pi_a a \rvert = \pi_a \lvert a \rvert < \varepsilon \mathbb{1}$ for all $a \in A$. Thus, we are done, since the relations $\lvert \pi_a a \rvert \le \varepsilon \mathbb{1}$ and $\|\pi_a a\| \le \varepsilon$ are equivalent.

(4): This is immediate from (2) and (3).

(5): By Theorem 5.11(1) $S \in X^{\#}$ if and only if $[\![\, \sigma := S{\uparrow} \in \mathscr{X}^*]\!] = \mathbb{1}$. Moreover, S and σ are positive or not simultaneously. Thus, we can inspect only the case of S positive. Observe also that if $[\![\mathscr{A} \subset \mathscr{X}_+ \cap U(\mathscr{X})]\!] = \mathbb{1}$ and $A = \mathscr{A}{\downarrow}$ then $S(A) = \sigma(\mathscr{A}){\downarrow}$ by [1, Theorem 3.2.13] and if $A \subset X_+$ and $\mathscr{A} = A{\uparrow}$ then $[\![\sigma(\mathscr{A}) = S(A){\uparrow}]\!] = \mathbb{1}$ by [1, Theorem 3.3.11]. Now, using the same argument as in (1), but with the infimum replaced by the supremum, we see that if S is order continuous then $[\![\inf(A) = 0]\!] = \mathbb{1}$ implies $[\![\inf \sigma(\mathscr{A}) = 0]\!] = \mathbb{1}$ and if $[\![\sigma$ is order continuous$]\!] = \mathbb{1}$ then $\inf(A) = 0$ implies $\inf S(A) = 0$. $\square$

7.6. A real Banach lattice X is *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \to X$ there exists a positive linear extension $T : Y \to X$ with $\|T_0\| = \|T\|$.[5)] Equivalently, X is an injective Banach lattice if, whenever X is lattice isometrically imbedded into a Banach lattice Y, there exists a contractive positive projection from Y onto X.

A Banach lattice X has the *Cartwright property* if, given $x_1, x_2, y \in X_+$ and $\lambda_1, \lambda_2 \in \mathbb{R}_+$ with $\|x_1\| \le \lambda_1$, $\|x_2\| \le \lambda_2$, and $\|x_1 + x_2 + y\| \le \lambda_1 + \lambda_2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$,

[5)] In case $\|T\| \le \lambda \|T_0\|$ with $1 \le \lambda \in \mathbb{R}$ replaces $\|T_0\| = \|T\|$ in the definition of injective Banach lattice, the terms like *λ-injectivity* are applied. In this article we confine exposition to the isometric theory; i.e., we consider only 1-injective Banach lattices.

$\|x_1+y_1\| \leq \lambda_1$, and $\|x_2+y_2\| \leq \lambda_2$. A Banach lattice X has the *splitting property* if X has the Cartwright property with $\lambda_1 = \lambda_2 = 1$.

A Banach lattice X has the *finite order intersection property* if, given $z \in X_+$ and finite collections $x_1, \dots, x_n \in X_+$, $y_1, \dots, y_m \in X_+$ and strictly positive reals $r_1, \dots, r_n \in \mathbb{R}_+$, $s_1, \dots, s_m \in \mathbb{R}_+$ such that $\|x_i\| \leq r_i$, $\|y_j\| \leq s_j$, and $\|x_i + y_j + z\| \leq r_i + s_j$ for all $i := 1, \dots, n$ and $j := 1, \dots, m$, there exist $u, v \in X_+$ with $z = u + v$, $\|x_i + u\| \leq r_i$, and $\|y_j + v\| \leq s_j$ for all $i := 1, \dots, n$ and $j := 1, \dots, m$.

7.7. Theorem. *An injective Banach lattice is order complete and has the Fatou and Levi properties.*

Proof. For every Banach lattice X the natural embedding $\varkappa : X \to X''$ is a lattice isometry and $\varkappa(X)$ is a closed sublattice in X''. If X is injective, then there exists a positive contractive projection from X'' onto $\varkappa(X)$ by definition. Given an order or norm bounded set U in X, there exists $y := \sup \varkappa(U)$ in X'', since X'' is complete and has the Levi property. Moreover, the identities $x := \varkappa^{-1}(Py) = \sup(U)$ and $\|x\| = \sup_{u \in U} \|u\|$ evidently are true in X because X'' has also the Fatou property. $\square$

The following geometric characterization was discovered by Cartwright:

7.8. Theorem. *For a Banach lattices the Cartwright property, the splitting property, and the finite order intersection property are equivalent. A Banach lattice X has one of these equivalent properties if and only if the second dual X'' is an injective Banach lattice.*

Proof. See [14, Theorems 2.9 and 3.6]. $\square$

We will also need the Haydon results in [15] which can be summarized as follows:

7.9. Theorem. *A Banach lattice X is injective if and only if X has Cartwright, Fatou, and Levi properties. An injective Banach lattice X is an AL-space if and only if there is no band M-projection in X other than 0 and I_X.*

Proof. Demonstration can be found in [15, Theorem 3F(ii) and Corollary 5D]; see also [2, Theorem 5.11.9]. $\square$

The above allow us to obtain some representation results for injective Banach lattices.

7.10. Theorem. *Let X be a B-cyclic Banach lattice and let $\mathscr{X}$ be the Boolean valued representation within $\mathbb{V}^{(B)}$. Then*

(1) $\mathbb{V}^{(B)} \models$ *"$\mathscr{X}$ is injective" if and only if X is injective;*

(2) $\mathbb{V}^{(B)} \models$ *"$\mathscr{X}$ is an AM-space" if and only if X is an AM-space;*

(3) $\mathbb{V}^{(B)} \models$ *"$\mathscr{X}$ is an AL-space" if and only if X is injective and $B = M(X)$.*

Proof. (1): Theorems 7.8 and 7.9 are valid within $\mathbb{V}^{(B)}$ by transfer. By Theorem 7.5 we have to show only that $[\![\mathscr{X}$ has the splitting property$]\!] = \mathbb{1}$ if and only if X has the splitting property. It is easy to see that $[\![\mathscr{X}$ has the splitting property$]\!] = \mathbb{1}$ is equivalent to the following: For all $x, y, z \in X_+$ with $|x| \leq \mathbb{1}$, $|y| \leq \mathbb{1}$, and $|x + y + z| \leq 2\mathbb{1}$, there exist $u, v \in X_+$ such that $z = u + v$, $|x + u| \leq \mathbb{1}$, and $|y + v| \leq \mathbb{1}$. But the latter amounts to the splitting property in X, since the relations $|x| \leq C\mathbb{1}$ and $\|x\|_X = \||x|\|_\infty \leq C$ are equivalent.

(2): Since the Λ-valued norm $|\cdot|$ in X is the restricted descent of the norm $\|\cdot\|_{\mathscr{X}}$ and the lattice operation $(x, y) \mapsto x \vee y$ in X is the restriction to X^2 of the descent of the similar operation in $\mathscr{X}$, it follows that $[\![\, \|\cdot\|_{\mathscr{X}}$ is an M-norm$]\!] = \mathbb{1}$ if and only if $|x \vee y| = |x| \vee |y|$ for all $x, y \in X^+$. Since $(\Lambda, \|\cdot\|_\infty)$ is an AM-space, we deduce that $\|x \vee y\|_X = \||x \vee y|\|_\infty = \||x|\|_\infty \vee \||y|\|_\infty = \|x\|_X \vee \|y\|_X$. Conversely, if X is an AM-space but $|x \vee y| > |x| \vee |y|$ for some $x, y \in X$, then we can choose a strictly positive $\varepsilon \in \mathbb{R}$ and a nonzero band projection $\pi \in \mathbb{P}(\Lambda)$ with $|\pi x \vee \pi y| - \epsilon\pi\mathbb{1} = \pi(|x \vee y| - \epsilon\mathbb{1}) \geq \pi(|x| \vee |y|) = |\pi x| \vee |\pi y|$, which leads to a contradiction as

$$\|\pi x \vee \pi y\| - \varepsilon = \||\pi x \vee \pi y| - \varepsilon\pi\mathbb{1}\|_\infty \geq \||\pi x| \vee |\pi y|\|_\infty = \|\pi x\| \vee \|\pi y\| = \|\pi x \vee \pi y\|.$$

(3): By transfer and Theorem 7.9 we can claim that $[\![\mathscr{X}$ is an AL-space if and only if $\mathscr{X}$ is injective and $M(\mathscr{X}) = \{0, I_X\}]\!] = \mathbb{1}$. Therefore, the claim is immediate from (1) and Theorem 7.2. $\square$

7.11(1). Corollary. *A nonzero Banach lattice X with the Fatou and Levi properties is injective if and only if the Boolean valued representation $\mathscr{X} \in \mathbb{V}^{(B)}$ of X within $\mathbb{V}^{(B)}$, where B is isomorphic to the complete Boolean algebra $M(X)$, is an AL-space.*[6)]

7.11(2). Corollary. *$M(X) = \mathbb{P}_M(X)$ for every injective Banach lattice X.*

Proof. Follows from Theorem 7.7 and Corollary 7.11(1). □

7.12. Consider some vector lattices X and Y. A positive operator $T : X \to Y$ has the *Levi property* if $Y = T(X)^{\perp\perp}$ and $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$ provided that the net (Tx_α) is order bounded in Y.

Assume that Y is order complete, and $\Phi \in L_+(X, Y)$ is strictly positive (i.e., $x > 0$ implies $\Phi(x) > 0$) and put $\lvert x \rvert := \Phi(|x|)$ for $x \in X$. Then $(X, \lvert\cdot\rvert)$ is a lattice normed space with the additive lattice norm. The *bo*-completion of X, denoted by $L^1(\Phi)$, is a BK-lattice presenting a *bo*-completion of $(X, \lvert\cdot\rvert)$; see [2, Theorems 2.2.8 and 2.2.11]. It is easy to see that $L^1(\Phi) = X$ if and only if Φ is a strictly positive Maharam operator with the Levi property (see [2, 3.5.1]).

Putting together Theorems 7.9, 7.10(3), and Corollary 7.11(1), we arrive at our main representation theorem for injective Banach lattices:

7.13. Theorem. *Let X be a Banach lattice, let B be a complete Boolean algebra, and let Λ be a unital order complete AM-space such that $\mathfrak{P}(\Lambda)$ is isomorphic to B. Then the following are equivalent:*

(1) *X is injective and the Boolean algebra $M(X)$ is isomorphic to B.*

(2) *X is lattice isometric to the restricted descent of some nonzero AL-space within $\mathbb{V}^{(B)}$.*

(3) *There exists an essentially positive Maharam operator $\Phi : X \to \Lambda$ with the Levi property such that $\|x\| = \|\Phi(|x|)\|_\infty$ for all $x \in X$.*

(4) *There is a Λ-valued additive norm on X such that $(X, \lvert\cdot\rvert)$ is a BK-lattice, $\lvert X\rvert^{\perp\perp} = \Lambda$, and $\|x\| = \|\lvert x\rvert\|_\infty$ for all $x \in X$.*

Proof. (1) $\iff$ (2) follows from Corollary 7.11(1) and Theorem 7.10(3).

(2) $\implies$ (3): Assume that the Boolean valued representation $\mathscr{X}$ of X is an AL-space within $\mathbb{V}^{(B)}$. Working within $\mathbb{V}^{(B)}$ and using the transfer principle, we can find some strictly positive order continuous functional $\phi : \mathscr{X} \to \mathscr{R}$ with the Levi property such that $\|x\|_{\mathscr{X}} = \phi(|x|)$ for all $x \in \mathscr{X}$. The descent $\Phi' := \phi\!\downarrow$ as well as its restriction $\Phi := \Phi'|_X : X \to \Lambda$ is a strictly positive Maharam operator with the Levi property; see Theorem 4.7. Since $\lvert\cdot\rvert = (\|\cdot\|_{\mathscr{X}})\!\downarrow$, we have $\lvert x\rvert = \Phi(|x|)$ for all $x \in X$. By the definition of restricted descent $\|x\|_X = \|\lvert x\rvert\|_\infty = \|\Phi(|x|)\|_\infty$.

(3) $\implies$ (4): If (3) is true then a Λ-valued additive norm on X is defined as $\lvert x\rvert := \Phi(|x|)$ for all $x \in X$. The fact that $(X, \lvert\cdot\rvert)$ is a BK-space follows from Theorem 4.5.

(4) $\implies$ (2): This is immediate from Theorems 7.2, 7.9, and 7.10(3). □

7.14. Corollary. *If Φ is a strictly positive Maharam operator with the Levi property taking values in a unital order complete AM-space Λ and $|\!|\!|x|\!|\!| = \|\Phi(|x|)\|_\infty$ with $x \in L^1(\Phi)$, then $(L^1(\Phi), |\!|\!|\cdot|\!|\!|)$ is an injective Banach lattice and there is a Boolean isomorphism φ from $B := \mathbb{P}(\Lambda)$ onto $M(L^1(\Phi))$ with $\pi \circ \Phi = \Phi \circ \varphi(\pi)$ for all $\pi \in B$. Conversely, each injective Banach lattice X is lattice B-isometric to $(L^1(\Phi), |\!|\!|\cdot|\!|\!|)$ for some strictly positive Maharam operator Φ with the Levi property taking values in a unital order complete AM-space Λ, where $B = \mathbb{P}(\Lambda) \simeq M(X)$.*

8. Comments

1. Lattice normed spaces and mixed norms. The concept of lattice normed space was introduced for the first time by Kantorovich in 1936. Also he propounded the unusual axiom of decomposability 1.1(d). The decomposability axiom implies the existence of a Boolean algebra of linear projections in

[6)] The zero Banach lattice $X = \{0\}$ is injective formally, but X fails to satisfy the claims of Corollary 7.11(1); since the Boolean algebra $B = M(X) = \{0\}$ and the unverse $\mathbb{V}^{(B)}$ degenerate in this case. By default we assume X nonzero in all similar situations.

a lattice normed space, so leading to some Boolean valued representation as a normed space. Presenting the basic facts about lattice normed spaces and their Boolean valued representation, we follow the book [4] in which some additional related material is also discussed. The classical Lebesgue space with mixed norm $L_{p,q}$ is a mixed norm space in the sense of Section 5 since $\|f\|_{L_{p,q}} = \| \, |f|_q \, \|_{L_p}$, where the vector norm $|\cdot|_q : L_{p,q} \to L_p$ is defined as $|f|_q : t \to \|f(t,\cdot)\|_{L_q}$.

The machinery of Boolean valued representations presented here can be useful in the various topics of the geometry of Banach spaces and Banach lattices; see [16–18]. Consider by way of illustration a counterpart of the well-known Ando Theorem stating that a Banach lattice X with $\dim(X) \geq 3$ admits a contractive positive Boolean projection onto each closed sublattice if and only if X is lattice isometric to either $L^p(\mu)$ for some $1 \leq p \in \mathbb{R}$ and measure μ or to $c_0(\Gamma)$ for some nonempty set Γ.

Let F be a Kantorovich space with order unit $\mathbb{1}$, and let $\Phi : E \to F$ be an essentially positive Maharam operator. By 4.4(2), Φ admits the unique extension to an essentially positive Maharam operator from the foundation $\mathscr{D}_m(\Phi) \subset E^u$ to F, for which we keep the same symbol Φ. Let Λ be the order ideal in F^u generated by $\mathbb{1}$ and $L^1(\Phi) := \mathscr{D}_m(\Phi)$; see 7.12. Take $\mathbb{1} \leq p \in \Lambda^u$ and put

$$L^p(\Phi) := \Big\{ x \in E^u : \ |x|^p \in L^1(\Phi), \ |x|_p := \big(\Phi(|x|^p)\big)^{\frac{1}{p}} \Big\},$$
$$L^p_\Lambda(\Phi) := \big\{ x \in E^u : \ |x|^p \in L^1(\Phi), \ |x|_p \in \Lambda \big\}, \quad \|x\|_p := \big\| \, |x|_p \, \big\|_\infty.$$

The next result was established by Kusraev and Kutateladze in [16].

Theorem. *Let B be a complete Boolean algebra and let Q be the Stone representation space of B. For a B-cyclic Banach lattice X with $B - \dim \geq 3$ the following are equivalent:*

(1) *There is a contractive positive projection onto each B-complete closed sublattice in X which commutes with the projections from B.*

(2) *There exists a partition of unity $(\pi_\gamma)_{\Gamma \cup \{0\}}$ in B with Γ a nonempty set of cardinals such that $\pi_0 X \simeq_{\pi_0 B} L^p(\Phi)$ for some $\mathbb{1} \leq p \in \Lambda^u$ and an injective Banach lattice $L := L^1(\Phi)$ with $M(L) \simeq \pi_0 B$ and $\pi_\gamma X \simeq_{\pi_\gamma B} C_\#(Q_\gamma, c_0(\gamma))$ for all $\gamma \in \Gamma$, where Q_γ is a clopen subset of Q corresponding to π_γ.*

2. Continuous and measurable Banach bundles. The representation by continuous or measurable Banach bundles is among the most interesting and important chapters of the theory of lattice normed spaces. The invention of Banach bundles is customarily connected with von Neumann who proposed in 1937 some ideas about varying Banach spaces. The corresponding formal descriptions appeared circa 1950 in the papers of Godement, Kaplansky, and Gelfand and Naimark. Presently, the theory of continuous Banach bundles is a rather wide area of research. Various useful connections of this theory, including some aspects of the theory of Banach lattices, are extensively reflected in [19].

Significant contribution to the theory of Banach bundles and lattice normed spaces was made by Gutman. In [20] he developed the theory of *ample continuous Banach bundles* and proved that every BK-space X over an order dense ideal $E \subset C_\infty(Q)$ is linearly isometric to $E(\mathscr{X})$ for some ample continuous Banach bundle $\mathscr{X}$ over Q. Moreover, such bundle $\mathscr{X}$ is unique to within linear isometry. Ample Banach bundles have other advantages. For instance, having continuous Banach bundles $\mathscr{X}$ and $\mathscr{Y}$ with $\mathscr{X}$ ample, enables one to form the operator bundle $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ whose q-fibres are the spaces of bounded operators $L(\mathscr{X}(q), \mathscr{Y}(q))$ between the corresponding fibers $\mathscr{X}(q)$ and $\mathscr{Y}(q)$; moreover, $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ is ample if and only if so is $\mathscr{Y}$; see [20, Part I].

The measurable part of the theory builds on the idea of *measurability structure* which had been proposed by Dinculeanu as early as in 1966, but has not been much studied since then. Gutman introduces the measurable structure $\mathscr{C}$ axiomatically as some distinguished set of sections, and then defines the measurable section as the limit almost everywhere (on sets of finite measure) of a convergent sequence of sections of the form $\sum_{k=1}^n \chi_{A_k} c_k$ with $c_1, \dots, c_n \in \mathscr{C}$ and measurable $A_1, \dots, A_n$. Then he introduced the notion of lifting in the space of measurable sections and created the theory of *liftable measurable Banach bundles* which play, in a sense, the same role in the class of all measurable Banach bundles as ample continuous Banach bundles in the class of all continuous Banach bundles. The measurable Banach

bundles of operator spaces are studied in the same spirit; see [20, Part II]. We also note the important contribution by Ganiev which is reflected in the survey [21] and contains his results on the measurable bundles of Boolean algebras, metrizable lattices, Banach lattices, and Banach–Kantorovich lattices as well as some applications.

It is also worth noting the important contribution of Ganiev (reflected in the survey [21]) containing his results on measurable bundles of Boolean algebras, metrizable lattices, Banach lattices, Banach-Kantorovich lattices, as well as some applications of these results.

3. Maharam and Luxemburg. In the articles published from 1949 to 1955, Maharam propounded an original approach to studying positive operators in function lattices (for the main results and relevant references; see the survey by Maharam [22]). The concept of Maharam operator and the main ideas of Section 4 from these papers. The concept of interval preserving operator was introduced by Maharam under the name *full-valued F-integral.* It turned out that "full-valuedness" is exactly what is needed for transferring the results of the classical integration theory to operators in function lattices. Following this idea, Maharam established, in particular, the seminal version of the Radon–Nikodym Theorem for full-valued F-integrals in [23].

Luxemburg was the first to appreciate Maharam's contribution. In his joint articles with Schep [24] and de Pagter [25] some portion of Maharam theory was extended to positive operators in Kantorovich spaces (i.e., Dedekind complete vector lattices). Luxemburg was also a pioneer and promoter of blending model theory and functional analysis. He pointed out that Maharam operators may play a fundamental role not only in the theory of positive operators but also in Boolean valued analysis. His article [26] in the Maharam anniversary volume states:

Finally we like to mention that the F-measure algebras introduced by Maharam in [27] *have also recently appeared in the literature in the form of Boolean valued models of standard numerical measure algebras. It may be of interest to explore further the properties of such abstract F-measure algebras from the point of view of the theory developed by D. Maharam. For further details we have to refer the reader to* [6].

The development of Maharam's ideas within Boolean valued analysis is due to other authors. The articles [24–26] made these ideas fruitfully implanted into operator theory, convex analysis, and elsewhere; see [1, 2, 4, 13].

4. Maharam traces. The concept of Maharam operator has also infiltrated into the noncommutative area. Let M be a von Neumann algebra and let F be a complex Dedekind complete vector lattice. An *F-valued trace* on M is a linear operator $\Phi : M \to F$ with $\Phi(x^*x) = \Phi(xx^*)$ for all $x \in M$. A trace Φ is *faithful* if the equality $\Phi(x^*x) = 0$ implies $x = 0$, and Φ is *normal* if $\Phi(x_\alpha) \uparrow \Phi(x)$ for all $x_\alpha, x \in M_h$ with $x_\alpha \uparrow x$. The trace $\Phi : M \to F$ possesses the Maharam property if for all $x \in M_+$ and $f \in F$ with $0 \le f \le \Phi(x)$ there exists a positive $y \le x$ such that $\Phi(y) = f$. A faithful normal F-valued trace Φ with the Maharam property is a *Maharam trace* (cf. Definition 4.4). Denote the algebra of all measurable operators affiliated with M by $S(M)$. In much the same way as it was done in 8.1, we can define the new class of BK-spaces and accompanying Banach spaces with mixed norm, the noncommutative $L^p(\Phi)$ and $L^p_\Lambda(\Phi)$ spaces, with $\mathbb{1} \le p \in \Lambda^u$, which are associated with a Maharam trace Φ. In [28] Chilin and Zakirov introduced such spaces $L^p(\Phi)$ for $1 \le p \in \mathbb{R}$ and described their duals. The geometric study of these spaces in the spirit of [18, 29–32] seems promising; cp. 8.10.

5. L^p-Structures in Banach spaces. The concept of L^p-projection, with $p \ge 1$, introduced by Cunningham [33] in 1953, is a natural generalization of M- and L-projections. The main characterization results for L^p-projections were obtained later; see Behrends et al. [34].

A linear projection π on a Banach space X is an *L^p-projection* provided that $\|x\|^p = \|\pi x\|^p + \|(I - \pi)x\|^p$ for all $x \in X$. An L^1-projection is referred to as *L-projection.* For $p \neq 2$ every two L^p-projections commute and the collection of all L^p-projections forms some complete Boolean algebra denoted by $\mathbb{P}_p(X)$. As in the case $p = 1$, this algebra is Badé complete. If $1 \le p, q \le \infty$ and $p \neq q$; then either $\mathbb{P}_p(X)$ or $\mathbb{P}_q(X)$ is reduced to the trivial algebra $\{0, I_X\}$. Moreover, there is a complete duality between L^p-

and L^q-projections: P is an L^p-projection on a Banach space X if and only if the conjugate P' is an L^q-projection on the dual X' of X. Detailed presentation of this concept is in [33–35].

The L^p-version ($p \neq 2$) of Theorem 6.10 can be stated as follows: *Let $1 < p < \infty$, $q := p/(p-1)$, and let X be a Banach space with the dual X' and the duality pairing $\langle \cdot, \cdot \rangle$. If $B := \mathbb{P}_p(X)$ then there exists some Banach space $\mathscr{X}$ unique up to linear isometry within $\mathbb{V}^{(B)}$ such that*

(1) *$\mathscr{X}$ has no nontrivial $L^{\hat{p}}$-projections and $\mathscr{X}'$ has no nontrivial $L^{\hat{q}}$-projections.*

(2) *X is linearly B-isometric to $\mathscr{X}{\downarrow}^{L^p}$ and X' is linearly B-isometric to $\mathscr{X}'{\downarrow}^{L^q}$.*

(3) *There exists a bilinear operator $\langle\!\langle \cdot, \cdot \rangle\!\rangle : X \times X' \to L^1(B, \phi)$ satisfying*

$$\langle \pi x, x' \rangle = \phi(\pi \langle\!\langle x, x' \rangle\!\rangle) \quad (x \in X,\ x' \in X',\ \pi \in B).$$

Here $\widehat{p} \in \mathbb{V}^{(B)}$ denotes the standard name of $p \in \mathbb{R}$. In this regard, it seems natural to ask: *Is there a meaningful theory of the L^p-structure in Banach spaces with mixed norm for a variable exponent $p \in \mathscr{R}{\Downarrow}$?*

6. Injective Banach lattices. The Boolean valued approach to the injective Banach lattices presented in Section 7 is due to Kusraev [16]. Theorem 7.13 states that each injective Banach lattice embeds into an appropriate Boolean valued model as an AL-space. According to this fact and the principles of Boolean valued models, each theorem about an AL-space within Zermelo–Fraenkel set theory has an analog in the original injective Banach lattice interpreted as a Boolean valued AL-space. This transfer principle is a new powerful tool in studying injective Banach lattices; see [2].

The concept of injective Banach lattice was introduced by Lotz in [36]. He proved therein that unital Dedekind complete AM-spaces and AL-spaces are injective Banach lattices (see [36, Propositions 2.1 and 3.2]). The statement about AM-spaces was previously stated by Abramovich [37] without using the term "injective." By definition, injective Banach lattices are injective objects in the category of Banach lattices and positive contractions. This fact shows that there is an essential difference between injective Banach lattices and injective Banach spaces, since Dedekind complete AM-spaces are the only injective objects (up to isomorphism) in the category of Banach spaces and linear contractions (the Nachbin–Goodner–Kelley Theorem).

Important contribution to the study of injective Banach lattices belongs to Cartwright [14] who proposed the *order intersection property* (see 7.7) and proved Theorem 7.8. Another important contribution is due to Haydon who discovered that each injective Banach lattice has a mixed AM-AL-structure and proved three representation theorems for injectives. One of his results [15, Theorem 6H] tells us that an injective Banach lattice can be represented as $L^1(\mathbf{m})$ for some Λ-valued modular Maharam measure $\mathbf{m}$. This is immediate from Theorem 7.13, since for such measure $\mathbf{m}$ the mapping $\Phi : L^1(\mathbf{m}) \to \Lambda$ defined by $\Phi : f \mapsto \int f\, d\mathbf{m}$ is a Maharam operator with the Levi property and $L^1(\mathbf{m}) = L^1(\Phi)$; see [4, Theorem 6.1.10]. Two other Haydon's representation results [15, Theorems 5C and 7B] can be also deduced from Theorem 7.13 by interpreting the fact that $L^1(\phi)$ is linearly isometric to the vector lattice of order continuous functionals on $L^\infty(\phi)$ (see [16, Corollary 4.10 and Remark 4.12]) on using the bundle representation of BK-spaces [4, § 2.4].

7. Preduals of injective Banach lattices. One of the intriguing problems that stemmed from the works of Grothendieck [38] and Lindenstrauss [39] is to describe the Banach spaces whose duals are isometric (isomorphic) to an AL-space. The injective version of this problem was posed in [40, Problem 5.16]: *Classify and characterize the Banach spaces whose duals are injective Banach lattices.* The results of Section 6 enables us to apply the Boolean valued transfer principle of Section 7 to the problem.

Consider a B-complete Banach space and $\pi \in B$. The *π-diameter* $\delta_\pi(A)$ and the *Chebyshev π-radius* $r_\pi(A)$ of A are defined as $\delta_\pi(A) = \sup\{\|\pi(a-b)\| : a, b \in A\}$ and $r_\pi(A) = \inf_{x \in X} r_\pi(A, x)$ where $r_\pi(A, x) = \sup_{a \in A} \|\pi(x-a)\|$ for all $x \in X$. It is clear that $\delta_\pi(A) \leq 2r_\pi(A)$. A subset $A \subset X$ of a B-complete Banach space X is *B-centerable* if $2r\pi(\mathrm{mix}(A)) = \delta\pi(\mathrm{mix}(A))$ for all $\pi \in B$, where $\mathrm{mix}(A)$ denoted the set of all mixings of all families in A by all partitions of unity in B.

Theorem. *For a real Banach space X with $B := \mathbb{P}_L(X)$ the following are equivalent:*
(1) *X' is an injective Banach lattice;*
(2) *every four-point subset of X is B-centerable;*
(3) *every finite subset of X is B-centerable;*
(4) *every B-bounded* mix-*compact subset of X is B-centerable.*

If $B = \{0, I_X\}$ then we get the geometric characterization L^1-predual space due to Duan and Lin in [41].

8. Uniqueness of IBL-preduals. The Grothendieck Uniqueness Theorem states that a Banach space is an AL-space provided that its dual is isometrically isomorphic to a Banach space of bounded continuous functions on a locally compact space (see [42, Theorem 27.4.1]). Interpreting this result in an appropriate Boolean valued model leads to the uniqueness of the predual injective Banach lattice; see [16, Theorem 5.11]. The norm on a Banach lattice X has the *Nakano property* if, for every updirected order bounded subset $A \subset X_+$, we have $\inf\{\|b\| : b \in B\} = \sup\{\|a\| : a \in A\}$, where B is the set of all upper bounds for A.

Theorem. *Let Y be simultaneously a B-cyclic Banach lattice and an AM-space whose norm has the Nakano property. Suppose that X is a B-cyclic Banach space and there is a linear B-isometry Φ from Y onto the B-dual $X^\#$. Then $X_+ := \{x \in X : (\forall\, y \in Y_+)\, \langle x, \Phi(y)\rangle \geq 0\}$ is a convex cone and X, equipped with the induced order, is an injective Banach lattice with $B = M(X)$.*

A more subtle result in terms of the dual space X' (rather than the B-dual space $X^\#$; see 5.9) was obtained in [18, Theorems 7 and 8].

9. Ball intersection properties. Many interesting geometric characterizations of Banach spaces are due to various ball intersection properties; see [11, 39]. The injective objects in the category of Banach spaces can be also characterized geometrically in terms of the Nachbin *binary intersection property*: A Banach space is injective if and only if each collection of pairwise intersecting closed balls $\{x \in X : \|x - x_i\| \leq r_i\}$ has nonempty intersection. Alfsen and Effros in [43] initiated the study of M-ideals; one of the main results is the characterization of M-ideals by means of the 3-balls intersection property; see [11, Theorem 2.2]. Recall that, by definition, an M-ideal of a Banach space X is a closed subspace whose polar is an L^1-summand in the dual space X'.

Lindenstrauss proved in [39, Theorem 6.1] that the dual X' of a real Banach space X is isometric to an L^1-space if and only if X has the $(4,2)$-intersection property; i.e., every collection of four mutually intersecting closed balls has nonempty intersection. To formulate the Boolean version of this result, we define a *B-cell* as a set of the form $B(a,r) := \{x \in X : \|\pi(x-a)\| \leq \|\pi r\|_L$ for all $\pi \in B\}$, where $B := \mathbb{P}_L(X)$, $a \in X$, and $r \in L^1(B, \phi)$. The above technique allows us to state that the dual X' of a real Banach space X is an injective Banach lattice with $M(X')$ isomorphic to $B := \mathbb{P}_L(X)$ if and only if every collection of four mutually intersecting B-cells in X has nonempty intersection; see [18, Remark 5.9].

10. Geometric description of state spaces. The Choquet Integral Representation Theorem led quite naturally to exploring the related geometric structure of state spaces. The complete solutions of this problem for JB-algebras (Alfsen and Shultz), C^*-algebras (Alfsen, Hanche-Olsen, and Shultz), JBW-algebras and W-algebras (Iohum and Shultz) are presented in [44]. One of the recent interesting developments is some program of classifying and characterizing the preduals of von Neumann algebras and JBW^*-triples that was initiated in the series of papers by Friedman and Rousso which started with [45]. In particular, they proved that *a neutral strongly facially symmetric Banach space over the reals is isometrically isomorphic to an $L_1(\mu)$-space, provided that every maximal geometric tripotent from the dual is unitary.* Furthermore, they pointed out that the complex analog of this claim fails. In fact, the technology of Boolean valued analysis is fully applicable to this study. In particular, the geometric description by Ibragimov and Kudaybergenov in [29, 30] of an L^1-space as a strongly facially symmetric space admits a counterpart for injective Banach lattices. Another illustration is the important result by Ayupov and Yadgorov in [46]: *The normal state spaces of modular Jordan algebras are characterized in*

the class of compact convex sets as precisely those that are modular, spectral, and symmetric. We also note some related results by Yadgorov, Ibragimov, and Kudaybergenov [47] and Zakirov and Chilin [48].

Appendix 1. Boolean Sets and Correspondences

The descent of every set within $\mathbb{V}^{(B)}$ has some extra structure induced by B. We will shortly present the relevant details.

A1(1). Let X be a set. A mapping $d : X \times X \to B$ is a *B-semimetric* whenever that, given x, y, and $z \in X$, we have

(a) $d(x,x) = \mathbb{0}$;

(b) $d(x,y) = d(y,x)$;

(c) $d(x,y) \le d(x,z) \vee d(z,y)$.

Furthermore, if $d(x,y) = \mathbb{0}$ implies that $x = y$; then d is *B-metric* or *Boolean metric* on X. Every pair (X,d) is a *B-set* or a *Boolean set*.

If X is a subset of $\mathbb{V}^{(B)}$ then X has the *canonical B-metric*

$$d(x,y) := [\![x \ne y]\!] = \neg d(x,y) \quad (x,y \in X).$$

Recall that we always work in the separated universe $\mathbb{V}^{(B)}$ and furnish subsets of $\mathbb{V}^{(B)}$ with the canonical B-metric.

Given an arbitrary Boolean algebra D, we may take as a D-metric the *symmetric difference*; i.e., $x \vartriangle y := (x \wedge \neg y) \vee (y \wedge \neg x)$. We proceed now with a relevant construction. Let ψ be an ultrafilter on a Boolean algebra D. Consider a Boolean set (X, d_X) with some D-metric d_X. Introduce the binary relation $\sim_\psi$ on X by the rule

$$(x,y) \in \sim_\psi \leftrightarrow \neg\, d_X(x,y) \in \psi.$$

From the definition of Boolean metric it follows that $\sim_\psi$ is an equivalence. Let $X/\sim_\psi$ stand for the factor-set of X by $\sim_\psi$ and let $\pi_X : X \to X/\sim_\psi$ be the canonical factor-mapping. If we start with the Boolean set $(D, \vartriangle)$ then we see that $D/{\sim_\psi}$ is the binary Boolean algebra $\mathbb{2}$. In this event there is a unique mapping $\overline{d} : X/{\sim_\psi} \to \mathbb{2}$ such that

$$\overline{d}\big(\pi_X(x), \pi_X(y)\big) = \pi_D\big(d(x,y)\big)$$

for all $x, y \in X$. Moreover, $\overline{d}$ is the discrete Boolean metric on $X/{\sim_\psi}$.

A1(2). Let $\psi := \psi_\imath$, with $\imath$ the identity homomorphism on B. The standard name embedding guaranties that $\mathbb{V}^{(B)} \models$"ψ is an ultrafilter on $B^\wedge$ and $A^\wedge \subset \psi$ implies that $\bigwedge(A)^\wedge \in \psi$" for every $A \subset B$.

Take a B-set (X,d). Clearly, $(X^\wedge, d^\wedge)$ is a B-set within $\mathbb{V}^{(B)}$. Using the maximum principle, we see that there are $X^\sim$, $\sim := \sim_\psi$, and $\pi_X \in \mathbb{V}^{(B)}$ such that

(a) $\mathbb{V}^{(B)} \models$ "$\sim$ is an equivalence on $X^\wedge$";

(b) $\mathbb{V}^{(B)} \models \widetilde{X} := X^\wedge/\sim$;

(c) $\mathbb{V}^{(B)} \models$ "$\pi_X : X \to X^\sim$ is a factor-mapping";

(d) $[\![(x^\wedge, y^\wedge)^B \in \sim]\!] \neg d(x,y)$ for all $x, y \in X$.

Note that $\sim = I_{X^\wedge}$ and $X^\sim = X^\wedge$ for a discrete B-set (X,d), the *Boolean valued realization* of (X,d).

A1(3). Let (b_ξ) be a partition of unity in B and let (x_ξ) be a family in a B-set X. A *mixing* of (x_ξ) by (b_ξ) is $x \in X$ such that $b_\xi \wedge d(x, x_\xi) = \mathbb{0}$ for all ξ; in symbols, $x = \operatorname{mix}(b_\xi x_\xi)$. If such mixing exists then it is unique. Note that mixing can fail to exist in an arbitrary B-set in contrast to mixing within $\mathbb{V}^{(B)}$.

Consider a B-set (X,d). Given $A \subset X$, let $\operatorname{mix}(A)$ stand for all mixings of families in A. If $\operatorname{mix}(A) = A$ then A is a *cyclic subset* of X. Denote the intersection of cyclic supersets of A by $\operatorname{cyc}(A)$. Call X *universally complete* in case X contains all $\operatorname{mix}(b_\xi x_\xi)$ of all $(x_\xi) \subset X$ by all partitions of unity $(b_\xi) \subset B$. In case the mixings exist only for finite families, and X is *decomposable*. Clearly, if X is universally complete then $\operatorname{mix}(A) = \operatorname{cyc}(A)$ for all $A \subset X$. A cyclic subset of a B-set can fail to be universally complete, whereas a cyclic subset of $\mathbb{V}^{(B)}$ furnished with the canonical B-metric is universally complete.

A1(4). *Let* A *be a set and let some B-set (X_α, d_α) be for every $\alpha \in \mathrm{A}$. Put $X := \Pi_{\alpha\in\mathrm{A}} X_\alpha$ and define the mapping $d : X \times X \to B$ as follows:*

$$d(x, y) := \bigvee \{d_\alpha(x(\alpha), y(\alpha)) : \alpha \in \mathrm{A}\}.$$

Then d is a Boolean metric on X Moreover, (X, d) is universally complete if and only if so is X_α for every $\alpha \in \mathrm{A}$.

We always furnish the product of B-set as a B-set with the just defined metric d.

A1(5). *Let A be a subset of a universally complete B-set (X, d). Then the Boolean distance from x to A for $x \in X$, given as*

$$\operatorname{dist}(x, A) := \bigwedge \{d(x, a) : a \in A\},$$

is attained at some $a \in \operatorname{mix}(A)$. In other words, to each $x \in X$ there is $a \in \operatorname{mix}(A)$ such that $\operatorname{dist}(x, A) = d(x, a)$.

The Boolean distance from $x \in X$ to a subset A of a universally complete B-set X vanishes if and only if $x \in \operatorname{mix}(A)$.

A1(6). The *Boolean distance* between $A_1 \subset X$ and $A_2 \subset X$ is defined as

$$\bar{d}(A_1, A_2) := \bigvee_{\alpha\in A_1} \operatorname{dist}(a, A_2) \vee \bigvee_{\alpha\in A_2} \operatorname{dist}(A_1, a).$$

Clearly, $\bar{d}$ is a Boolean semimetric on $\mathscr{P}(X)$ which fails to be a metric in general. Call $\bar{d}$ the *B-Hausdorff semimetric* associated with d.

If X is universally complete then $\bar{d}(A_1, A_2) = \mathbb{0}$ whenever $\operatorname{mix}(A_1) = \operatorname{mix}(A_2)$. Let $\mathscr{P}_{\mathrm{cyc}}(X)$ consist of all cyclic subsets of a B-set (X, d). Then (X, d) is universally complete if and only if so is $(\mathscr{P}_{\mathrm{cyc}}(X), \bar{d})$.

A1(7). Consider B-sets (X, d_X) and (Y, d_Y). A correspondence Φ from X to Y is *contractive* provided that

$$\bar{d}_Y(\Phi(x), \Phi(y)) \leq d_X(x, y) \quad (x, y \in \operatorname{dom}(\Phi)),$$

where $\bar{d}_Y$ is the Hausdorff B-semimetric associated with d_Y.

A1(8). *A correspondence Φ is contractive if and only if one* (and, hence, both) *of the following holds:*

(*a*) *if $d_X(x_1, x_2) \leq b$ for some $b \in B$ and $x_1, x_2 \in \operatorname{dom}(\Phi)$ then*

$$b \vee \operatorname{dist}(y, \Phi(x_1)) = b \vee \operatorname{dist}(y, \Phi(x_2))$$

for all $y \in Y$;

(*b*) $\operatorname{dist}(y_1, \Phi(x_2)) \leq d_X(x_1, x_2)$ *for all $x_1, x_2 \in \operatorname{dom}(\Phi)$ and $y_1 \in \Phi(x_1)$.*

If X and Y are subsets of $\mathbb{V}^{(B)}$ then we speak about contractive and extensional correspondences. To avoid confusion recall extensionality relates to the truth value of the equality $[\![\cdot = \cdot]\!]$, whereas contractibility to the *B-metric* under study.

A correspondence Φ is *totally contractive* provided that Φ is contractive and

$$\Phi(x) = \operatorname{mix}(\Phi(x)) \quad (x \in \operatorname{dom}(\Phi)).$$

A correspondence Φ is *totally extensional* provided that Φ is extensional and

$$\Phi(x) = \operatorname{mix}(\Phi(x)) \quad (x \in \operatorname{dom}(\Phi)).$$

A1(9). *The descent of a correspondence is totally contractive and totally extensional.*

A mapping $f : X \to Y$ is contractive whenever

$$d_Y(f(x), f(x')) \le d_X(x, x') \quad (x, x' \in X).$$

If the equality holds then f is a *B-isometry*. A bijective B-isometry is an *isomorphism* of B-sets.

A1(10). Each $X \in \mathbb{V}$ transforms into a B-set provided that X is furnished with the *discrete B-metric* on letting $d(x, y) = \mathbb{1}$ in case $x \ne y$ and $d(x, y) = \mathbb{0}_B$ in case $x = y$ for all $x, y \in X$. In this event (X, d) is a *discrete B-set*. A discrete B-set lacks $\operatorname{mix}(b_\xi x_\xi)$, if (x_ξ) is not a singleton or the partition of unity is not $\{\mathbb{0}_B, \mathbb{1}_B\}$.

Discrete and universally complete B-sets are in a sense extreme examples. Boolean valued analysis uses the B-sets of another provenance.

A1(11). *Let π be a complete monomorphism of B to a Boolean algebra C. Put*

$$d_\pi(x, y) := \bigwedge\{b^* : \pi(b) \wedge x = \pi(b) \wedge y\} \quad (x, y \in C).$$

Then d_π is a B-metric on C and the Boolean operations on C are contractive.

Appendix 2. Modified Descents and Ascents

A2(1). We need some more constructions concerning B-sets.

A few set-theoretic operations on X and $X/{\sim_\psi}$ are simply interrelated. If (X_α) is a family of subsets of X then

$$\Big(\bigcup X_\alpha\Big)/{\sim_\psi} = \bigcup (X_\alpha/{\sim_\psi}).$$

Also there is a natural bijection between $X^n/{\sim_\psi}$ and $(X/{\sim_\psi})^n$, with $n \in \mathbb{N}$ such that

$$\pi_{X^n}(x_1, \dots, x_n) \mapsto (\pi_X x_1, \dots, \pi_X x_n) \quad (x_1, \dots, x_n \in X).$$

Note that if $A \subset X$ then $A/{\sim_\psi} = \pi_X(A)$ and $\pi_A = \pi_X \restriction A$.

Take another B-set (Y, d_Y) and assume that $F \subset X \times Y$. Then

$$\operatorname{dom}(F/{\sim_\psi}) = \operatorname{dom}(F)/{\sim_\psi}, \quad \operatorname{im}(F/{\sim_\psi}) = \operatorname{im}(F)/{\sim_\psi}.$$

A2(2). Say that subsets A and C of some B-set (X, d) are in *general position* provided that

$$d(a, c) \ge \bigwedge\{d(a, b) \vee d(b, c) : b \in A \cap C\}$$

for all $a \in A$ and $c \in C$. In fact the equality holds as $d(a, c) \le d(a, b) \vee d(b, c)$. Note that A and C are in general position if and only if

$$\mathbb{V}^{(B)} \models (A \cap C)^\sim = A^\sim \cap C^\sim.$$

A2(3). *Let (X, d_X) and (Y, d_Y) be B-sets and let Φ be a contractive correspondence from X to Y. Then there is a unique correspondence $\Phi^\sim$ from $X^\sim$ to $Y^\sim$ within $\mathbb{V}^{(B)}$ such that*

$$\operatorname{dom}(\Phi^\sim) = (\operatorname{dom}(\Phi))^\sim,$$

$$[\![\Phi^\sim(\pi_X x^\wedge) = \pi_Y(\Phi(x)^\wedge)]\!] = \mathbb{1} \quad (x \in \operatorname{dom}(\Phi)).$$

Furthermore,

(a) if $A \subset X$ and $\operatorname{dom}(\Phi)$ are in general position then

$$\mathbb{V}^{(B)} \models \Phi(A)^\sim = \Phi^\sim(A^\sim);$$

(b) if correspondences Φ and Ψ are contractive then so is $\Psi \circ \Phi$; if $\operatorname{dom}(\Psi \circ \Phi) = \operatorname{dom}(\Phi)$ while $\operatorname{dom}(\Psi)$ and $\Phi(x)$ are in general position for all $x \in \operatorname{dom}(\Phi)$ then

$$\mathbb{V}^{(B)} \models (\Psi \circ \Phi)^\sim = \Psi^\sim \circ \Phi^\sim;$$

(c) $\mathbb{V}^{(B)} \models (I_X)^\sim = I_{X^\sim}$.

A2(4). *If $f : X \to Y$ is a contractive mapping then there is a unique $f^\sim \in \mathbb{V}^{(B)}$ such that*

$$[\![f^\sim : X^\sim \to Y^\sim]\!] = [\![f^\sim \circ \pi_X = \pi_Y \circ f^\wedge]\!] = \mathbb{1}.$$

In this event

(a) $\mathbb{V}^{(B)} \models f(A)^\sim = f^\sim(A^\sim)$ *for all* $A \subset X$*;*

(b) if $g : Y \to Z$ *is a contractive mapping then so is* $g \circ f$ *and* $\mathbb{V}^{(B)} \models (g \circ f)^\sim = g^\sim \circ f^\sim$*;*

(c) $\mathbb{V}^{(B)} \models$ *"$f^\sim$ is injective" if and only if f is a B-isometry;*

(d) $\mathbb{V}^{(B)} \models$ *"$f^\sim$ is surjective" if and only if* $\bigvee\{d(f(x), y) : x \in X\} = \mathbb{1}$ *for all* $y \in Y$*.*

A2(5). *Let (X, d_X) be a B-set and $X' := X^\sim\!\downarrow$. Then*

(a) there is an injection $\imath_X : X \to X'$ such that

$$d_X(x_1, x_2) = [\![\imath_X x_1 \neq \imath_X x_2]\!] \quad (x_1, x_2 \in X);$$

(b) for every $x' \in X'$ there are a partition of unity (b_ξ) and a family $(x_\xi) \subset X$ such that

$$x' = \mathrm{mix}(b_\xi \imath(x_\xi));$$

(c) if Φ is a contractive correspondence from X to a B-set Y, while $Y' := Y^\sim\!\downarrow$ and $\Phi' := \Phi^\sim\!\downarrow$; then Φ' is the unique totally extensional correspondence from X' to Y' such that $\mathrm{dom}(\Phi') = \mathrm{mix}(\imath_X(\mathrm{dom}(\Phi)))$ *and*

$$\Phi'(\imath_X x) = \mathrm{mix}(\imath_X(\Phi(x))) \quad (x \in \mathrm{dom}(\Phi)).$$

A2(6). Assume that X is a nonempty B-set and $Y \in \mathbb{V}^{(B)}$ is such that $[\![Y \neq \varnothing]\!] = \mathbb{1}$. Consider $\Phi \in \mathbb{V}^{(B)}$ satisfying $\mathbb{V}^{(B)} \models$ "$\Phi = (F, X^\sim, Y)$ is a correspondence from $X^\sim$ to Y." Clearly, $\Phi\!\downarrow$ is a correspondence from $X' := X^\sim\!\downarrow$ to $Y\!\downarrow$. Put $\Phi\!\Downarrow := \Phi\!\downarrow \circ\, \imath_X$ and call $\Phi\!\Downarrow$ the *modified descent* of Φ. Obviously, $\Phi\!\Downarrow$ is the unique totally contractive correspondence from X to $Y\!\downarrow$ such that

$$y \in \Phi\!\Downarrow(x) \leftrightarrow [\![y \in \Phi(\imath_X x)]\!] = \mathbb{1} \quad (x \in X).$$

Note also that $\Phi\!\Downarrow = (F\!\downarrow^-, X, Y\!\downarrow)$, where

$$F\!\downarrow^- := \big\{(x, y) \in X \times Y\!\downarrow :\ (\imath_X x, y)^B \in F\big\}.$$

A2(7). Assume further that $\Psi := (F, X, Y\!\downarrow)$ is a contractive correspondence. It is impossible to ascend Ψ without circumlocutions, whereas $\Psi \circ \imath_X$. So, we put $\Psi\!\Uparrow := (\Psi \circ \imath_X^{-1})\!\uparrow$ and call $\Psi\!\Uparrow$ the *modified ascent* of Ψ. Clearly, $\Psi\!\Uparrow$ is the unique correspondence from $X^\sim$ to Y within $\mathbb{V}^{(B)}$ such that

$$[\![\mathrm{dom}(\Psi)\!\Uparrow = (\mathrm{dom}(\Psi))^\sim]\!] = \mathbb{1},$$
$$[\![\Psi\!\Uparrow(\imath_X x) = \Psi(x)\!\uparrow]\!] = \mathbb{1} \quad (x \in \mathrm{dom}(\Psi)).$$

Observe that $\Psi\!\Uparrow = (F_-\!\uparrow, X^\sim, Y)$, where

$$F_- := \big\{(\imath_X x, y)^B :\ (x, y) \in F\big\}.$$

A2(8). Assume that X is a discrete B-set. Then $\Phi\!\Downarrow$ is the unique correspondence from X to $Y\!\downarrow$ which is determined from the relation

$$y \in \Phi\!\Downarrow(x) \leftrightarrow [\![y \in \Phi(x^\wedge)]\!] = \mathbb{1} \quad (x \in X).$$

In this event every correspondence Ψ from X to $Y\!\downarrow$ is contractive and so there is a unique correspondence $\Psi\!\Uparrow$ from $X^\wedge$ to Y such that

$$[\![\Psi\!\Uparrow(x^\wedge) = \Psi(x)\!\uparrow]\!] = \mathbb{1} \quad (x \in X).$$

A2(9). *Let $\mathscr{A}$ be the set of $\Phi \in \mathbb{V}^{(B)}$ such that $[\![\Phi$ is a correspondence from $X^\sim$ to $Y]\!] = \mathbb{1}$, and let $\mathscr{B}$ be the set of totally contractive correspondences from X to $Y\!\downarrow$. The modified descent and modified ascent are mutually inverse bijections between $\mathscr{A}$ and $\mathscr{B}$.*

Appendix 3. Order Bounded Operators

We will recall the auxiliaries that are needed for Boolean valued analysis.

A3(1). Let E and F be vector lattices over the same ordered field $\mathbb{F}$. A linear operator $T : E \to F$ is
positive in case $T(E^+) \subset F^+$;
regular in case T is a difference of positive operators;
order bounded or *o-bounded* if $T(U)$ is order bounded for every order bounded subset U of E.

Recall that $S \in L(E, F)$ *dominates* $T \in L(E, F)$ provided that $|Tx| \le S(|x|)$ for all $x \in E$. Say that T is *dominated* whenever T has a positive operator that dominates T. A positive $T : E \to F$ dominates itself; i.e., $|Tx| \le T(|x|)$ for all $x \in E$.

A3(2). *A linear operator T between vector lattices is dominated if and only if T is regular.*

Note that $L^r(E, F)$, the space of regular operators, and $L^\sim(E, F)$, the space of order continuous operators, are subspaces of $L(E, F)$. A regular operator from E to F lattices is order bounded; whereas the converse, failing in general, holds if F is a Kantorovich space.

A3(3). Riesz–Kantorovich Theorem. *Let E be a vector lattice and let F be a Kantorovich space. Then $L^\sim(E, F)$, ordered by $L^+(E, F)$, is a Kantorovich space.*

Note that the Riesz–Kantorovich Theorem provides the natural formulas for the lattice operations on $L^\sim(E, F)$, the so-called *order operator calculus*.

A3(4). *Let E be a vector lattice, let F be a real vector space, and let S be a positive homogeneous additive mapping from E^+ to F. Then S has the unique linear extension T on the whole of E. Furthermore, if F is a vector lattice and $S(E^+) \subset F^+$ then T is positive.*

A3(5). An operator $T : E \to F$ is *order continuous* whenever the net (Tx_α) is order convergent to Tx for every net $(x_\alpha)_{\alpha\in A}$ order convergent to x; in symbols, $T \in L_n^\sim(E, F)$. If order convergence holds for sequences, then T is *sequentially order continuous* or *order σ-continuous*; in symbols, $T \in L_{n\sigma}^\sim(E, F)$. If $F = \mathbb{R}$ then we write $E_n^\sim$ instead of $L_n^\sim(E, \mathbb{R})$. A positive $T \in L^\sim(E, F)$ is order continuous or order σ-continuous if and only if $Tx_\alpha \xrightarrow{(o)} 0$ for every decreasing net or sequence (x_α) in E such that $\inf_\alpha x_\alpha = 0$.

Note that $L_n^\sim(E, F)$ and $L_{n\sigma}^\sim(E, F)$ are bands of $L^\sim(E, F)$.

A3(6). A linear operator $T : E \to F$ is a *lattice homomorphism* provided that T enjoys one and hence all of the properties

$$T(x \vee y) = Tx \vee Ty \quad (x, y \in E),$$
$$T(x \wedge y) = Tx \wedge Ty \quad (x, y \in E),$$
$$x \wedge y = 0 \to Tx \wedge Ty = 0 \quad (x, y \in E),$$
$$T(x^+) = (Tx)^+ \quad (x \in E),$$
$$T(|x|) = |Tx| \quad (x \in E).$$

An injective lattice homomorphism T is a *lattice monomorphism* or an *order monomorphism* as well as an *isomorphic embedding* or a *lattice isomorphism* from E to F. If T is a bijection then E and F are *lattice isomorphic* or *order isomorphic*, etc. Let $\mathrm{Hom}(E, F)$ stand for the set of all lattice homomorphisms from E to F.

A linear operator $T : E \to F$ is *disjointness preserving* provided that $Tx \perp Ty$ for all $x \perp y$. Clearly, every lattice homomorphism is disjointness preserving, while a positive operator is disjointness preserving if and only if it is a lattice homomorphism.

Consider a vector lattice E and a sublattice D of E. A linear operator $T : D \to E$ is *band preseving* or *contractive* provided that T enjoys one and hence all of the properties

$$Te \in \{e\}^{\perp\perp} \quad (e \in D),$$
$$e \perp f \to Te \perp f \quad (e \in D,\ f \in E),$$
$$T(K \cap D) \subset K \quad (K \in \mathfrak{B}(E)),$$

where the disjoint complements are taken in E. Note that a contractive operator can fail to be order bounded. If π is an order projection in E then the restriction of π to a foundation D of E is an order projection in D, and we will keep for the latter the previous notation π.

A3(7). *Assume that E is a vector lattice enjoying the principal projection property; i.e., each principal band of E admits the band projection. Assume further that T is a linear operator from a foundation D of E to E. Then the following are equivalent:*
(a) *T is contractive;*
(b) *$T\pi = \pi T\pi$ for all $\pi \in \mathfrak{P}(E)$;*
(c) *$\pi T = \pi T\pi$ for all $\pi \in \mathfrak{P}(E)$;*
(d) *$\pi T = T\pi$ for all $\pi \in \mathfrak{P}(E)$.*

A3(8). Let E be a vector lattice and let F be a Kantorovich space. Consider a positive operator $S : G \to F$ on an ideal G of E. Let $\mathscr{D}_m(T)$ stand for all $e \in E$ such that $S([0, |e|] \cap G)$, with $[0, |e|] := \{x \in E : 0 \le x \le |e|\}$, is order bounded in F. Clearly, $\mathscr{D}_m(T)$ is an ideal of E; and, given $e \in E$, we may put

$$\widetilde{S}e := \sup\{Sg : \ g \in G,\ 0 \le g \le e\} := \sup\{S(g \wedge e) : \ g \in G\} \quad (e \in \mathscr{D}_m(T)^+).$$

The operator $\widetilde{S} : E^+ \to F$ is additive and positive homogeneous. And so we can extend $\widetilde{S}$ as in A3(4). The resulting operator is the *minimal extension* of S to $\mathscr{D}_m(T)$ which we will keep the notation $\widetilde{S}$ for.

A3(9). *The minimal extension $\widetilde{S}$ of S is a positive operator coinciding with S on G and vanishing on $G^\perp \subset \mathscr{D}_m(T)$.*

A3(10). *If S is order continuous or order σ-continuous then so is $\widetilde{S}$.*

A3(11). Let $\mathscr{N}_T := \mathscr{N}(T) := \{x \in E : |T|(|x|) = 0\}$ be the *null ideal* of $T \in L^\sim(E, F)$. The disjoint complement of $\mathscr{N}(T)^\perp$ is the *support* of T which is denoted by $\mathscr{C}_T$. The null ideal of T is a foundation of $\mathscr{C}_T^\perp$. Consequently, if T is order continuous then T vanishes on $\mathscr{N}_T$; i.e., $\mathscr{N}_T$ is a band. In this event the support of T is called the *essential positivity band* of T. An operator T is *essentially positive* provided that $\mathscr{N}(T)^\perp = E$. If the supports of two order bounded operators are disjoint then so are the operators. The converse fails in general but holds for order continuous operators dominated by the same Maharam operator.

A3(12). Let G be a universally complete Kantorovich space $\mathscr{R}{\downarrow}$. Recall that G is also a faithful f-ring with unity $\mathbb{1} := 1^\wedge$.

Let $\mathrm{End}_N(G)$ be the collection of all contractive linear operators in G. Clearly, $\mathrm{End}_N(G)$ is a vector space. Moreover, $\mathrm{End}_N(G)$ turns into a unital faithful module over G on defining the operator gT by the rule $gT : x \mapsto g \cdot Tx$ for all $x \in G$. Indeed, multiplication by an element of G is a contractive operator and the composition of contractive operators is contractive. Denote by $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R})$ the member of $\mathbb{V}^{(B)}$ which represents the space of $\mathbb{R}^\wedge$-linear mappings from $\mathscr{R}$ to $\mathscr{R}$. Then $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R})$ is a vector space over $\mathscr{R}$ within $\mathbb{V}^{(B)}$ and $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R}){\downarrow}$ is a unital faithful module over G.

A3(13). *A linear operator $T : G \to G$ is contractive if and only if T is extensional.*

A3(14). *The modules $\mathrm{End}_N(G)$ and $\mathrm{End}_{\mathbb{R}^\wedge}(\mathscr{R}){\downarrow}$ are isomorphic by sending each contractive operator to its ascent.*

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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