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ORIENTED ROTATABILITY EXPONENTS OF SOLUTIONS TO HOMOGENEOUS AUTONOMOUS LINEAR DIFFERENTIAL SYSTEMS

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Abstract—We fully study the oriented rotatability exponents of solutions to homogeneous autonomous linear differential systems and establish that the strong and weak oriented rotatability exponents coincide for each solution to an autonomous system of differential equations. We also show that the spectrum of this exponent (i.e., the set of values of nonzero solutions) is naturally determined by the number-theoretic properties of the set of imaginary parts of the eigenvalues of the matrix of a system. This set (in contrast to the oscillation and wandering exponents) can contain other than zero values and the imaginary parts of the eigenvalues of the system matrix; moreover, the power of this spectrum can be exponentially large in comparison with the dimension of the space. In demonstration we use the basics of ergodic theory, in particular, Weyl's Theorem. We prove that the spectra of all oriented rotatability exponents of autonomous systems with a symmetrical matrix consist of a single zero value. We also establish relationships between the main values of the exponents on a set of autonomous systems. The obtained results allow us to conclude that the exponents of oriented rotatability, despite their simple and natural definitions, are not analogs of the Lyapunov exponent in oscillation theory.

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1. Introduction

The Lyapunov characteristics of oscillation of solutions to differential equations and systems were first introduced by Sergeev in [1–4]. The motivation to their consideration is as follows. As is known, the Lyapunov and Perron exponents of a linear differential system coincide in the autonomous case with the real parts of the eigenvalues of the coefficient matrix; and so they can be considered as analogs of the real parts of the eigenvalues (the Perron exponents after regularization in the Millionshchikov sense) for systems with variable coefficients. But the analogs of the imaginary parts of eigenvalues for linear differential equations with variable coefficients are Sergeev's frequencies (which were called earlier the characteristic frequencies) after regularization in the Millionshchikov sense [2–5] and, for linear differential systems, the oscillation exponents (they were earlier called full and vector frequencies) [6–10]. Hence, the definition (in addition to the Lyapunov and Perron exponents) of these characteristics allow us to achieve some natural necessary completeness of considerations of linear differential equations and systems from the viewpoint of asymptotic behavior at infinity of their solutions to the extent that this behavior is defined by the roots of their characteristic polynomials.

The collection of adopted characteristics increases constantly. Oriented rotatability exponents were defined in the report by Sergeev [11] in 2013. Their spectra for autonomous systems contain the collection of the moduli of the imaginary parts of eigenvalues of the system matrix but they do not agree with this set in the general case [12]. The question of typical values of an oriented rotatability exponent and its spectrum was pointed out in this report. Burlakov managed to prove that a wide class of systems with constant coefficients (which have real eigenvalues or two complex ones with incommensurate imaginary parts) has zero as a typical value of this exponent and to define its spectrum for the systems with simple purely imaginary eigenvalues [13]. Thereby, the problem of determining the spectra of oriented rotatability exponents is partially solved. In the present article, we solve this problem on using the Burlakov ideas.

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Given $n \in \mathbb{N}$, denote by $\mathscr{M}^n$ the set of the linear systems

$$\dot{x} = A(t)\,x, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}_+ \equiv [0, +\infty),$$

with *bounded continuous* operator-functions $A : \mathbb{R}_+ \operatorname{End}\mathbb{R}^n$ (each of which is identified with the corresponding system). A subset of $\mathscr{M}^n$ comprising *autonomous systems* is denoted by $\mathscr{C}^n$. Denote the solution space of the system $A \in \mathscr{M}^n$ by $\mathscr{S}(A)$; and the subset of the nonzero solutions of A, by $\mathscr{S}_*(A)$. Below, the star subindex designates a vector space without zero. Let

$$\mathscr{S}_*^n = \bigcup_{A \in \mathscr{M}^n} \mathscr{S}_*(A).$$

2. Oriented Rotatability Exponents

Definition 1 [11]. Given $x \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ and a finite time $t > 0$, define the functional $\Theta(x,t)$ as a continuous branch of the oriented angle between the vectors $x(t)$ and $x(0)$ such that $\Theta(x,0) = 0$. If there exists a time $\tau \in [0,t]$ for which $x(\tau) = 0$, then by definition (despite the loss of continuity) we take $\Theta(x,t) = +\infty$.

Definition 2 [11]. The lower (upper) strong and weak oriented rotatability exponents of a vector function $x \in \mathscr{S}_*^n$ are given by the formulas

$$\check{\theta}^\bullet(x) = \inf_{L \in \operatorname{End}_2 \mathbb{R}^n} \varliminf_{t \to +\infty} \frac{1}{t}|\Theta(Lx,t)| \ \left(\hat{\theta}^\bullet(x) = \inf_{L \in \operatorname{End}_2 \mathbb{R}^n} \varlimsup_{t \to +\infty} \frac{1}{t}|\Theta(Lx,t)|\right),$$
$$\check{\theta}^\circ(x) = \varliminf_{t \to +\infty} \inf_{L \in \operatorname{End}_2 \mathbb{R}^n} \frac{1}{t}|\Theta(Lx,t)| \ \left(\hat{\theta}^\circ(x) = \varlimsup_{t \to +\infty} \inf_{L \in \operatorname{End}_2 \mathbb{R}^n} \frac{1}{t}|\Theta(Lx,t)|\right),$$

where $\operatorname{End}_2 \mathbb{R}^n$ is a subset of the collection $\operatorname{End}\mathbb{R}^n$ of linear operators of rank 2.

If the upper and lower values of one of the above exponents $\hat{\varkappa}(x) = \check{\varkappa}(x)$ coincide then we say that the exponent $\varkappa(x)$ is *exact* and if so do the weak and strong exponents $\varkappa^\circ(x) = \varkappa^\bullet(x)$ then we say that $\varkappa(x)$ ia absolute.

Remark. Note that rotatability exponents for $n = 1$ are no senseless. Hence, we will assume that $n \geq 2$.

Definition 3 [11]. For all

$$\omega = \check{\theta}^\bullet, \hat{\theta}^\bullet, \check{\theta}^\circ, \hat{\theta}^\circ \tag{1}$$

the ith upper and lower principal (or regularized in the Millionshchikov sense) values $\omega_{\bar{i}}(A)$ and $\omega_{\underline{i}}(A)$ of the corresponding oriented rotatability exponents of a system $A \in \mathscr{M}^n$ are defined as

$$\omega_{\bar{i}}(A) \equiv \inf_{V \in G^i(A)} \sup_{x \in V_*} \omega(x), \qquad \omega_{\underline{i}}(A) \equiv \sup_{V \in G^{n-i+1}(A)} \inf_{x \in V} \omega(x),$$

where $i = 1, 2, \ldots, n$. Let $G^i(A)$ stand for the set of i-dimensional subspaces of $\mathscr{S}(A)$.

The proof of Theorem VI [3] can be applied to (1). Thereby, for every system $A \in \mathscr{M}^n$, we have

$$0 \leq \omega_{\bar{1}}(A) \leq \cdots \leq \omega_{\bar{n}}(A), \quad 0 \leq \omega_{\underline{1}}(A) \leq \cdots \leq \omega_{\underline{n}}(A), \tag{2}$$

$$\omega_{\underline{i}}(A) \leq \omega_{\bar{i}}(A), \quad i = 1, \ldots, n, \tag{3}$$

$$\omega_{\underline{1}}(A) = \omega_{\bar{1}}(A) = \inf_{x \in \mathscr{S}_*(A)} \omega(x), \quad \omega_{\underline{n}}(A) = \omega_{\bar{n}}(A) = \sup_{x \in \mathscr{S}_*(A)} \omega(x).$$

Denote the last quantities by $\omega_1(A)$ and $\omega_n(A)$, respectively.

3. The Basic Results

DEFINITION 4 [13]. Let $\gcd^*$ be the function on the set of nonnegative numbers $Q = \{q_1, \ldots, q_r\}$ such that $\gcd^*(\{q_i\}) = q_i$ for all $i = 1, 2, \ldots, r$. If q_i, with $i = 1, \ldots, r$, are pairwise rationally incommensurable then $\gcd^*(Q) = 0$. Otherwise, we consider the largest α such that $S = \left\{\frac{q_1}{\alpha}, \ldots, \frac{q_r}{\alpha}\right\}$ consists of integers. If S contains at least one even integer then $\gcd^*(Q) = 0$; otherwise $\gcd^*(Q) = \alpha$.

Theorem 1. *For an arbitrary $A \in \mathscr{C}^n$, the exponent θ is exact and absolute, while the spectrum of A is of the form*

$$\operatorname{Spec}_\theta(A) = \left\{ \gcd^*(S) \,|\, S \neq \varnothing,\ S \subset \{|\operatorname{Im}\lambda_1(A)|, \ldots, |\operatorname{Im}\lambda_n(A)|\} \right\}, \tag{4}$$

where $\lambda_i(A)$, with $i = 1, 2, \ldots, n$, are the roots of the characteristic polynomial of the system ordered by moduli of their imaginary parts.

Corollary 1. *For an arbitrary system $A \in \mathscr{C}^n$, the spectrum of the exponent θ is discrete and its power is at most $2^{\frac{n}{2}} - 1$.*

As is known from algebra, all eigenvalues of a symmetric matrix are real. Hence, the following holds:

Corollary 2. *Spectra of all oriented rotatability exponents of autonomous systems with a symmetric matrix consist only of zero.*

Theorem 2. *For all $A \in \mathscr{C}^n$ and $\omega = \check{\theta}^\bullet, \hat{\theta}^\bullet, \check{\theta}^\circ, \hat{\theta}^\circ$, we have*

$$\omega_1(A) = \omega_{\underline{2}}(A) = \omega_{\overline{2}}(A) = \cdots = \omega_{\underline{n-1}}(A) = \omega_{\overline{n-1}}(A) \leq \omega_n(A) = |\operatorname{Im}\lambda_n(A)|. \tag{5}$$

4. Auxiliary Definitions and Facts

DEFINITION 5 [8, 9]. The upper (lower) strong and weak oscillation exponents of hyperroots of a function $x \in \mathscr{S}^n_*$ are defined by the formulas

$$\hat{\nu}^*_\bullet(x) \equiv \inf_{m\in\mathbb{R}^n} \varlimsup_{t\to+\infty} \frac{\pi}{t}\nu^*(x, m, t) \quad \left(\check{\nu}^*_\bullet(x) \equiv \inf_{m\in\mathbb{R}^n} \varliminf_{t\to+\infty} \frac{\pi}{t}\nu^*(x, m, t)\right),$$

$$\hat{\nu}^*_\circ(x) \equiv \varlimsup_{t\to+\infty} \inf_{m\in\mathbb{R}^n} \frac{\pi}{t}\nu^*(x, m, t) \quad \left(\check{\nu}^*_\circ(x) \equiv \varliminf_{t\to+\infty} \inf_{m\in\mathbb{R}^n} \frac{\pi}{t}\nu^*(x, m, t)\right),$$

where $\nu^*(x, m, t)$ is the number of hyperroots of the inner product $\langle x, m\rangle$ on $(0, t]$: every not multiple roots is taken once while a multiple root infinitely many times.

Lemma 1 [10]. *For all $x \in \mathscr{S}^n_*$ we have*

$$\hat{\theta}^\bullet(x) \leq \hat{\nu}^*_\bullet(x), \quad \check{\theta}^\bullet(x) \leq \check{\nu}^*_\bullet(x), \quad \hat{\theta}^\circ(x) \leq \hat{\nu}^*_\circ(x), \quad \check{\theta}^\circ(x) \leq \check{\nu}^*_\circ(x).$$

DEFINITION 6 [14]. The following objects are called similar:
(a) matrices $A, B \in \mathscr{C}^n$ if $B = CAC^{-1}$ for some nondegenerate matrix $C \in \mathscr{C}^n$;
(b) vector-functions $x, y \in \mathscr{S}^n_*$ if $y = Cx$ for some nondegenerate matrix $C \in \mathscr{C}^n$.

Lemma 2. *Similar vectors $x, y \in \mathscr{S}^n_*$ satisfy the equality*

$$\varkappa(x) = \varkappa(y), \tag{6}$$

where $\varkappa$ is one of the oriented rotatability exponents $\check{\theta}^\bullet$, $\hat{\theta}^\bullet$, $\check{\theta}^\circ$, and $\hat{\theta}^\circ$.

PROOF. Let $x, y \in \mathscr{S}^n_*$ be similar. There exists a nondegenerate matric $C \in \mathscr{C}^n$ such that $y = Cx$. The lower oriented rotatability exponent enjoys the relation

$$\check{\theta}^\circ(y) = \varliminf_{t\to+\infty} \inf_{L\in\operatorname{End}_2 \mathbb{R}^n} \frac{1}{t}|\Theta(Ly, t)| = \varliminf_{t\to+\infty} \inf_{L\in\operatorname{End}_2 \mathbb{R}^n} \frac{1}{t}|\Theta(LCx, t)| = \check{\theta}^\circ(x).$$

Equality (6) for the remaining oriented rotatability exponents is proven by analogy. $\square$

The candidate thesis by Burlakov contains the proofs of the three lemmas:

Lemma 3. *Let the vector-function*

$$x(\tau)=\begin{pmatrix} g_1(\tau)\\ g_2(\tau)\end{pmatrix}\in C^1(\mathbb{R}_+,\mathbb{R}^2)$$

meets the conditions $|x(t)|\neq 0$, $g_1(0)=0$, and all zeros of g_1 are simple. Denote by t_k the zeros of g_1 beginning with $t_0=0$. Then

$$|\Theta(x,t_k)|=\left|\frac{\pi}{2}\sum_{i=1}^{k}(-1)^i\Big(\operatorname{sgn} g_2(t_i)-\operatorname{sgn} g_2(t_{i-1})\Big)\right|,\quad k>0.$$

Lemma 4. *In the notations of Lemma 3,*

$$\left||\Theta(x,t)|-\left|\pi\sum_{i=1}^{k}(-1)^i\operatorname{sgn} g_2(t_i)\right|\right|\le 2\pi$$

for every $t\in\mathbb{R}_+$, where k is the minimal index such that $t_k\le t$.

Lemma 5. *For an arbitrary $x\in C^1(\mathbb{R}_+,\mathbb{R}^n_*)$, a real $\alpha>0$, and the function y with $y(t)=x(\alpha t)$, we have*

$$\varkappa(y)=\alpha\varkappa(x),$$

where $\varkappa$ is one of the oriented rotatability exponents $\check{\theta}^\bullet$, $\hat{\theta}^\bullet$, $\check{\theta}^\circ$, and $\hat{\theta}^\circ$.

Definition 7 [15]. By *a real analog of a Jordan matrix*, we mean a block-diagonal matrix with blocks of the form

$$J_{\lambda,s}=\begin{pmatrix}\lambda & 1 & 0 & \dots & 0\\ 0 & \lambda & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \dots & \lambda & 1\\ 0 & 0 & \dots & 0 & \lambda\end{pmatrix}\in\mathscr{C}^s,$$

$$J_{\alpha,\beta,p}=\begin{pmatrix}\alpha & \beta & 1 & 0 & \dots & \dots & \dots & \dots & 0\\ -\beta & \alpha & 0 & 1 & 0 & \dots & \dots & \dots & 0\\ 0 & 0 & \alpha & \beta & 1 & 0 & \dots & \dots & 0\\ 0 & 0 & -\beta & \alpha & 0 & 1 & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \dots & \dots & 0 & \alpha & \beta & 1 & 0\\ 0 & 0 & \dots & \dots & 0 & -\beta & \alpha & 0 & 1\\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & \alpha & \beta\\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & -\beta & \alpha\end{pmatrix}\in\mathscr{C}^p,$$

where $\alpha,\beta\in\mathbb{R}$, $s,j\in\mathbb{N}$, and $p=2j$; in this case the former blocks are called *single* and the latter *paired.*

Denote by $\mathscr{G}^2$ the collection of all two-dimensional subspaces G of $\mathbb{R}^n$ endowed with the finite Lebesgue measure and standard topology and by P_G the orthogonal projection onto a subspace $G\subset\mathbb{R}^n$.

Lemma 6 [11]. *If $L=P_G$ in (6) of Definition 2 and we take the greatest lower bound over $G\in\mathscr{G}^2$ rather than the greatest lower bound over $L\in\operatorname{End}_2\mathbb{R}^n$ then the quantities defined by these equalities coincide.*

5. Proofs of the Main Results

PROOF OF THEOREM 1. 1. As is known [15], a matrix $A \in \mathscr{C}^n$ is similar to a real analog of a Jordan matrix, passing to which replaces every solution to the system A with a similar vector-function while the values of the oriented rotatability exponents in question do not change (Lemma 2). Hence, A can be assumed to be a real analog of a Jordan matrix.

2. Order the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A by nondecreasing moduli of their imaginary parts and construct the real fundamental system of solutions

$$x^1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \quad x^2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \quad \dots, \quad x^n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} \tag{7}$$

to $A \in \mathscr{C}^n$ from the subsystems of solutions corresponding to the blocks of A:

(a) To every single block $J_{\lambda_k,s}$ of order s relating to $\lambda_k \in \mathbb{R}$, there corresponds a subsystem of s solutions

$$x^k(t) = e^{\lambda_k t} h_k, \dots, x^{k+s-1}(t) = \frac{t^{s-1}}{(s-1)!} e^{\lambda_k t} h_k + \dots + t e^{\lambda_k t} h_{k+s-2} + e^{\lambda_k t} h_{k+s-1},$$

where jth component of every vector h_j in this subsystem is equal to 1 and the remaining components vanish.

(b) To every paired block $J_{\alpha,\beta,p}$ of order $p = 2s$ relating to $\lambda_k = \alpha + i\beta$, there corresponds a subsystem $x^k, x^{k+1}, \dots, x^{k+2s-1}$ of the solutions

$$\begin{pmatrix} x_{1k} \\ \vdots \\ x_{k-1k} \\ x_{kk} \\ x_{k+1k} \\ x_{k+2k} \\ \vdots \\ x_{nk} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{\alpha t}\cos\beta t \\ -e^{\alpha t}\sin\beta t \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_{1k+1} \\ \vdots \\ x_{k-1k+1} \\ x_{kk+1} \\ x_{k+1k+1} \\ x_{k+2k+1} \\ \vdots \\ x_{nk+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{\alpha t}\sin\beta t \\ e^{\alpha t}\cos\beta t \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} x_{1k+2} \\ \vdots \\ x_{kk+2} \\ x_{k+1k+2} \\ x_{k+2k+2} \\ x_{k+3k+2} \\ \vdots \\ x_{nk+2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ te^{\alpha t}\cos\beta t \\ -te^{\alpha t}\sin\beta t \\ e^{\alpha t}\cos\beta t \\ -e^{\alpha t}\sin\beta t \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_{1k+3} \\ \vdots \\ x_{kk+3} \\ x_{k+1k+3} \\ x_{k+2k+3} \\ x_{k+3k+3} \\ \vdots \\ x_{nk+3} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ te^{\alpha t}\sin\beta t \\ te^{\alpha t}\cos\beta t \\ e^{\alpha t}\sin\beta t \\ e^{\alpha t}\cos\beta t \\ \vdots \\ 0 \end{pmatrix}, \dots,$$

$$\begin{pmatrix} x_{1k+2s-2} \\ \vdots \\ x_{kk+2s-2} \\ x_{k+1k+2s-2} \\ \vdots \\ x_{k+2s-4k+2s-2} \\ x_{k+2s-3k+2s-2} \\ x_{k+2s-2k+2s-2} \\ x_{k+2s-1k+2s-2} \\ \vdots \\ x_{nk+2s-2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \frac{t^{s-1}}{(s-1)!}e^{\alpha t}\cos\beta t \\ -\frac{t^{s-1}}{(s-1)!}e^{\alpha t}\sin\beta t \\ \vdots \\ te^{\alpha t}\cos\beta t \\ -te^{\alpha t}\sin\beta t \\ e^{\alpha t}\cos\beta t \\ -e^{\alpha t}\sin\beta t \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_{1k+2s-1} \\ \vdots \\ x_{kk+2s-1} \\ x_{k+1k+2s-1} \\ \vdots \\ x_{k+2s-4k+2s-1} \\ x_{k+2s-3k+2s-1} \\ x_{k+2s-2k+2s-1} \\ x_{k+2s-1k+2s-1} \\ \vdots \\ x_{nk+2s-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \frac{t^{s-1}}{(s-1)!}e^{\alpha t}\sin\beta t \\ \frac{t^{s-1}}{(s-1)!}e^{\alpha t}\cos\beta t \\ \vdots \\ te^{\alpha t}\sin\beta t \\ te^{\alpha t}\cos\beta t \\ e^{\alpha t}\sin\beta t \\ e^{\alpha t}\cos\beta t \\ \vdots \\ 0 \end{pmatrix}.$$

(c) All above solutions form the ordered fundamental system of solutions (7).

Hence, the general solution to the system A takes the form

$$x = c_1 x^1(t) + c_2 x^2(t) + \dots + c_n x^n(t), \tag{8}$$

where c_1, $c_2, \dots, c_n$ are arbitrary constants.

It is proven in [10] that

$$\check{\theta}^\circ(x^j) = \hat{\theta}^\circ(x^j) = \check{\theta}^\bullet(x^j) = \hat{\theta}^\bullet(x^j) = |\operatorname{Im}\lambda_j(A)|, \quad j = 1, 2, \dots, n. \tag{9}$$

Take an arbitrary solution

$$x = c_q x^q(t) + c_{q+1}x^{q+1}(t) + \dots + c_r x^r(t), \quad c_q \neq 0,\ 1 \le q \le r \le n, \tag{10}$$

to $A \in \mathscr{C}^n$.

3. Let $\lambda_q \in \mathbb{R}$. The results of [16] allow us to justify the chain of the equalities

$$\hat{\nu}^*_\bullet(x) = \check{\nu}^*_\bullet(x) = \hat{\nu}^*_\circ(x) = \check{\nu}^*_\circ(x) = 0$$

and Lemma 1 yields

$$\check{\theta}^\circ(x) = \hat{\theta}^\circ(x) = \check{\theta}^\bullet(x) = \hat{\theta}^\bullet(x) = 0. \tag{11}$$

4. Let $\beta \equiv \operatorname{Im}\lambda_q(A) = \dots = \operatorname{Im}\ \lambda_r(A) \neq 0$. In view of Lemma 6, the vector-function Lx is of the form

$$Lx = \begin{pmatrix} e^{\gamma_1 t}\sqrt{P_1^2(t) + P_2^2(t)}\sin(\beta t + \varphi(t)) \\ e^{\gamma_2 t}\sqrt{P_3^2(t) + P_4^2(t)}\cos(\beta t + \phi(t)) \end{pmatrix},$$

where $\varphi(t)$ and $\phi(t)$ are auxiliary arguments for $t \ge 0$ and $P_1(t)$, $P_2(t)$, $P_3(t)$, are $P_4(t)$ polynomials of degree at most $n/2$. Therefore, we have

$$|\Theta(Lx, t) - \Theta(z, t)| \le 2\pi, \quad t \ge 0,$$

where

$$z = \begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix},$$

and so

$$\check{\theta}^\circ(x) = \hat{\theta}^\circ(x) = \check{\theta}^\bullet(x) = \hat{\theta}^\bullet(x) = \lim_{t\to+\infty}\frac{1}{t}|\Theta(z, t)| = \beta.$$

5. Assume that $\lambda_q \notin \mathbb{R}$ and the imaginary parts Im $\lambda_q(A), \ldots,$ Im $\lambda_r(A)$ of the eigenvalues of A those in (10) include a rationally incommensurable pair. Denote it by β_1, β_2. There exists a transformation L_1 such that the vector-function L_1x is representable as

$$L_1x = \begin{pmatrix} e^{\alpha_1 t}\sin\beta_1 t \\ e^{\alpha_2 t}\cos\beta_2 t \end{pmatrix}.$$

The sequence $t_k = \frac{\pi k}{\beta_1}$ of moments enjoys the inequality

$$|\Theta(L_1x, t) - \Theta(L_1x, t_k)| \le \pi.$$

Lemma 4 implies that

$$\left| |\Theta(L_1x,t)| - \left| \pi \sum_{i=1}^{k} (-1)^i \operatorname{sgn} e^{\alpha_2 t_i} \cos\beta_2 t_i \right| \right|$$
$$= \left| |\Theta(L_1x,t)| - \left| \pi \sum_{i=1}^{k} \operatorname{sgn}\cos\left(\frac{\beta_2 \pi i}{\beta_1} + \pi i\right) \right| \right| \le 2\pi,$$

where k is the maximal index such that $t_k \le t$. Two last inequalities and passing to the limit over t validate the chain of equalities

$$\overline{\lim_{t\to+\infty}} \frac{1}{t}|\Theta(L_1x,t)| = \overline{\lim_{k\to+\infty}} \frac{1}{t_k}|\Theta(L_1x,t_k)| = \overline{\lim_{k\to+\infty}} \frac{\pi}{t_k}\left|\sum_{i=1}^{k}\operatorname{sgn}\cos\omega i\right|, \tag{12}$$

where $\omega = \pi\bigl(1 + \frac{\beta_2}{\beta_1}\bigr)$.

Since β_1 and β_2 are rationally incommensurable, so are ω and π. Take the Riemann integrable function $f(s) = \beta_1 \operatorname{sgn}\cos s$ and the map $T(\alpha) = \alpha + \omega \bmod 2\pi$ presenting a rotation of a circle by an irrational angle. The equality

$$\frac{1}{k}\sum_{i=1}^{k} f(T^i 0) = \frac{\beta_1}{k}\sum_{i=1}^{k}\operatorname{sgn}\cos\omega i = \frac{\pi}{t_k}\sum_{i=1}^{k}\operatorname{sgn}\cos\omega i, \quad k \ge 0,$$

and the Weyl Theorem [17] applied to the sequence $f(T^k 0)$ imply

$$\lim_{k\to+\infty}\frac{1}{k}\sum_{i=1}^{k} f(T^i 0) = \frac{1}{2\pi}\int_0^{2\pi} f(s)\,ds = \frac{\beta_1}{2\pi}\int_0^{2\pi}\operatorname{sgn}\cos s\,ds = 0,$$

yielding the required relations

$$0 \le \hat{\theta}^{\bullet}(x) \le \overline{\lim_{t\to+\infty}}\frac{1}{t}|\Theta(L_1x,t)| = \overline{\lim_{k\to+\infty}}\frac{\pi}{t_k}\left|\sum_{i=1}^{k}\operatorname{sgn}\cos\omega i\right| = \lim_{k\to+\infty}\frac{1}{k}\left|\sum_{i=1}^{k} f(T^i 0)\right| = 0.$$

The above-constructed solution x satisfies (11).

6. Suppose that $\lambda_q \notin \mathbb{R}$ and among the imaginary parts Im $\lambda_q(A), \ldots, \operatorname{Im}\lambda_r(A)$ of eigenvalues of A in (10), there is a pair β_3, β_4 such that $\frac{\beta_4}{\beta_3} = \frac{2p}{l}$, where $\frac{p}{l}$ is an irreducible fraction and q is odd. In this case there exists a homomorphism L_2 such that the vector-function L_2x is representable as

$$L_2x = \begin{pmatrix} e^{\alpha_3 t}\sin\beta_3 t \\ e^{\alpha_4 t}\cos\beta_4 t \end{pmatrix}.$$

Demonstrate that as before this case leads to (11).

Note that the functions $\sin\beta_3 t$ and $\cos\beta_4 t$ are periodic with period $T = \frac{2\pi l}{\beta_3} = \frac{4\pi p}{\beta_4}$. Take the sequence of times $t_k = \frac{\pi k}{\beta_3}$. Lemma 3 yields

$$|\Theta(L_2 x, kT)| = \left| \frac{\pi}{2} \sum_{i=1}^{\frac{\beta_3 T k}{\pi}} (-1)^i \Big(\operatorname{sgn}\left(e^{\alpha_4 t_i} \cos\beta_4 t_i\right) - \operatorname{sgn}\left(e^{\alpha_4 t_{i-1}} \cos\beta_4 t_{i-1}\right)\Big) \right|$$

$$= \left| \frac{\pi}{2} \sum_{i=1}^{\frac{\beta_3 T k}{\pi}} (-1)^i \Big(\operatorname{sgn}\cos\beta_4 t_i - \operatorname{sgn}\cos\beta_4 t_{i-1} \Big) \right| = \left| \pi \sum_{i=1}^{2lk} (-1)^i \operatorname{sgn}\cos\beta_4 t_i \right|$$

$$= \left| \pi \sum_{i=1}^{2lk} \operatorname{sgn}\cos\left(\frac{2p\pi i}{l} + \pi i\right) \right| = \left| \pi \sum_{i=1}^{2lk} \operatorname{sgn}\cos\frac{2p+l}{l}\pi i \right|$$

$$= \left| \pi \sum_{i=1}^{lk} \operatorname{sgn}\cos\frac{2p+l}{l}\pi i + \pi \sum_{i=lk+1}^{2lk} \operatorname{sgn}\cos\frac{2p+l}{l}\pi i \right|$$

$$= \left| \pi \sum_{i=1}^{lk} \operatorname{sgn}\cos\frac{2p+l}{l}\pi i + \pi \sum_{i=1}^{lk} \operatorname{sgn}\cos\left(\frac{2p+l}{l}\pi i + (2p+l)\pi\right) \right|$$

$$= \left| \pi \sum_{i=1}^{lk} \operatorname{sgn}\cos\frac{2p+l}{l}\pi i - \pi \sum_{i=1}^{lk} \operatorname{sgn}\cos\frac{2p+l}{l}\pi i \right| = 0.$$

Thus we established the required inequality

$$0 \le \hat{\theta}^{\bullet}(x) \le \varlimsup_{k\to+\infty} \frac{1}{kT} |\Theta(L_2 x, kT)| = 0.$$

7. Suppose that $\lambda_r \notin \mathbb{R}$ and every pair from the modulus of the imaginary parts $\beta_q, \ldots, \beta_r$ of eigenvalues of A in (10) is rationally incommensurable. Next, choose the largest number α such that the set $\left\{\frac{\beta_q}{\alpha}, \ldots, \frac{\beta_r}{\alpha}\right\}$ consists of odd naturals whose greatest common divisor is equal to 1. Denote two numbers from this set by ω_1 and ω_2. Lemma 5 ensures that the function

$$y(t) = x\left(\frac{t}{\alpha}\right) = c_q x^q \left(\frac{t}{\alpha}\right) + c_{q+1} x^{q+1} \left(\frac{t}{\alpha}\right) + \cdots + c_r x^r \left(\frac{t}{\alpha}\right), \tag{13}$$

where $c_q \cdot c_r \neq 0$, $1 \le q \le r \le n$, satisfies the equalities

$$\varkappa(x) = \alpha \varkappa(y), \quad \varkappa \in \left\{\check{\theta}^{\circ}, \hat{\theta}^{\circ}, \check{\theta}^{\bullet}, \hat{\theta}^{\bullet}\right\}.$$

(d) Find a linear transform L_3 such that the vector-function $L_3 y$ takes the form

$$L_3 y = \begin{pmatrix} e^{\gamma_1 t} \sin\omega_1 t \\ e^{\gamma_2 t} \cos\omega_2 t \end{pmatrix}, \quad \omega_1 < \omega_2,$$

and prove the equality $\Theta(L_3 y, t_{\omega_1 k}) = \pi k$ for the sequence $t_k = \frac{\pi k}{\omega_1}$.

Indeed, Lemma 3 and the periodicity of $\sin\omega_1 t$ and $\cos\omega_2 t$, where k is a natural, yield

$$
\begin{aligned}
|\Theta(L_3y, t_{\omega_1 k})| &= \left|\frac{\pi}{2}\sum_{i=1}^{\omega_1 k}(-1)^i\left(\operatorname{sgn}\left(e^{\gamma_2 t_i}\cos\omega_2 t_i\right) - \operatorname{sgn}\left(e^{\gamma_2 t_{i-1}}\cos\omega_2 t_{i-1}\right)\right)\right| \\
&= \left|\frac{\pi}{2}\sum_{i=1}^{\omega_1 k}(-1)^i\left(\operatorname{sgn}\cos\left(\frac{\omega_2\pi i}{\omega_1}\right) - \operatorname{sgn}\cos\left(\frac{\omega_2\pi(i-1)}{\omega_1}\right)\right)\right| \\
&= \left|\pi\sum_{i=1}^{\omega_1 k}(-1)^i\operatorname{sgn}\cos\left(\frac{\omega_2\pi i}{\omega_1}\right)\right| = \left|\pi\sum_{i=1}^{\omega_1 k}(-1)^i\operatorname{sgn}\cos\left(\frac{(\omega_1+2q)\pi i}{\omega_1}\right)\right| \\
&= \left|\pi\sum_{i=1}^{\omega_1 k}(-1)^i\operatorname{sgn}\cos\left(\pi i + \frac{2q\pi i}{\omega_1}\right)\right| = \left|\pi\sum_{i=1}^{\omega_1 k}\operatorname{sgn}\cos\left(\frac{2q\pi i}{\omega_1}\right)\right| = \pi k.
\end{aligned}
$$

The validity of the last equality follows from the fact that the values $\frac{2q\pi i}{\omega_1}$ lying on the unit circle centered at the origin for $i = 1, 2, \ldots, \omega_1$ are vertices of the regular ω_1-gon with the fixed vertex $(1, 0)$, and, for $i = \omega_1 + 1, \ldots, 2\omega_1$, the previous values $\frac{2q\pi i}{\omega_1}$ are repeated sequentially. The situation is similar for the remaining values of i. As a result, the indices of the vertices of the polygon which lie on the different sides of the oy-axis differ by one.

(e) In view of Lemma 6, for every admissible value of L, the vector-function Ly is of the form

$$
Ly = \begin{pmatrix} e^{\gamma_3 t}\sqrt{P_5^2(t) + P_6^2(t)}\,\sin(\omega_3 t + \psi_1(t)) \\ e^{\gamma_4 t}\sqrt{P_7^2(t) + P_8^2(t)}\,\cos(\omega_4 t + \psi_2(t)) \end{pmatrix},
$$

where $\psi_1(t)$ and $\psi_2(t)$ are auxiliary angles, $t \geq 0$, and $P_5(t)$, $P_6(t)$, $P_7(t)$, and $P_8(t)$ are polynomials of degree at most $n/2$. Hence, we infer

$$
|\Theta(Ly, t) - \Theta(u, t)| \leq 2\pi, \quad t \geq 0,
$$

where

$$
u = \begin{pmatrix} \sin\omega_3 t \\ \cos\omega_4 t \end{pmatrix}.
$$

Note that the values ω_3 and ω_4 can coincide for some L. In this case the infimum in the definition of the weak exponents is not realized.

(f) The last two subsections imply that

$$
\begin{aligned}
\theta^\circ(y) &= \lim_{t\to+\infty}\inf_{L\in\mathrm{End}_2\,\mathbb{R}^n}\frac{1}{t}|\Theta(Ly, t)| = \lim_{k\to+\infty}\inf_{L\in\mathrm{End}_2\,\mathbb{R}^n}\frac{1}{t_{\omega_1 k}}|\Theta(Ly, t_{\omega_1 k})| \\
&= \lim_{k\to+\infty}\frac{1}{t_{\omega_1 k}}|\Theta(L_3y, t_{\omega_1 k})| = \lim_{k\to+\infty}\frac{1}{\pi k}|\Theta(L_3y, \pi k)| = \lim_{k\to+\infty}\frac{1}{\pi k}\cdot\pi k = 1.
\end{aligned}
$$

Considering the definitions of oriented rotatability exponents, we find that

$$
1 = \theta^\circ(y) \leq \theta^\bullet(y) \leq \lim_{k\to+\infty}\frac{1}{t_{\omega_1 k}}|\Theta(L_3y, t_{\omega_1 k})| = 1.
$$

Thus, (10) satisfies the equalities

$$
\check{\theta}^\circ(x) = \hat{\theta}^\circ(x) = \check{\theta}^\bullet(x) = \hat{\theta}^\bullet(x) = \alpha. \quad \square
$$

Proof of Theorem 2. Fix a matrix $A \in \mathscr{C}^n$.

1. The last equality in (5) results from (4) automatically.

2. Let the moduli of eigenvalues of A coincide. In this case items 3 and 4 of the proof of Theorem 1 demonstrate the equalities of the principal values of the oriented rotatability exponents.

3. If at least one of the eigenvalues of A is real then (11) hold for every $x \in \operatorname{span}\{x^1, x^2, \dots, x^{n-1}\}$ and thereby

$$\omega_{\overline{n-1}}(A) = 0.$$

Hence, (2) and (3) imply that

$$\omega_1(A) = \omega_{\underline{2}}(A) = \omega_{\overline{2}}(A) = \cdots = \omega_{\underline{n-1}}(A) = \omega_{\overline{n-1}}(A) = 0. \tag{14}$$

4. The conditions of items 5 and 6 of the proof of Theorem 1 for all eigenvalues of A also lead to (14). Indeed, in every case there exists an $n-1$-dimensional subspace of $\mathscr{S}(A)$ whose solutions enjoy (11).

5. Suppose that, among the moduli of the imaginary parts $\beta_1, \dots, \beta_r$ (enumerated in increasing order) of eigenvalues of A, every pair is rationally incommensurable and $\left\{\frac{\beta_1}{\alpha}, \dots, \frac{\beta_r}{\alpha}\right\}$ consists of odd naturals whose greatest common divisor is equals to 1. The value α is largest in this case. The definition of gcd* implies that

$$\gcd{}^* \{\beta_1, \dots, \beta_{k-1}, \beta_k, \beta_{k+1}, \dots, \beta_r\} = \alpha,$$

$$\beta_1 \geq \alpha \leq \gcd{}^* \{\beta_1, \dots, \beta_{k-1}, \beta_{k+1}, \dots, \beta_r\}.$$

Hence, on the one hand, $\omega_1(A) = \alpha$ and, on the other hand,

$$\omega_{\overline{n-1}}(A) = \alpha.$$

Indeed, the last equality is realized on the subspace

$$V = \operatorname{span}\{x^1, x^2, \dots, x^{n-1}\},$$

since

$$\check{\theta}^\circ(x) = \hat{\theta}^\circ(x) = \check{\theta}^\bullet(x) = \hat{\theta}^\bullet(x) = \alpha$$

for $x \in V$. Thus, in view of (2) and (3), we derive that

$$\begin{aligned} \alpha = \omega_1(A) = \omega_{\underline{2}}(A) = \omega_{\overline{2}}(A) = \cdots = \omega_{\underline{n-1}}(A) \\ = \omega_{\overline{n-1}}(A) < \omega_n(A) = \beta_r. \quad \square \end{aligned}$$

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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