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ON A NEW FAMILY OF COMPLETE G_2-HOLONOMY RIEMANNIAN METRICS ON $S^3 \times \mathbb{R}^4$

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Abstract: Studying a system of first-order nonlinear ordinary differential equations for the functions determining a deformation of the standard conic metric over $S^3 \times S^3$, we prove the existence of a one-parameter family of complete G_2-holonomy Riemannian metrics on $S^3 \times \mathbb{R}^4$.

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1. Introduction

This article is a sequel to [1] which studied a special class of Riemannian manifolds with holonomy group G_2. The main idea is to consider the conic metric over a Riemannian manifold with special geometry and deform it to resolve the singularity at the cone apex. The deformation of the metric is determined by two functions $A_i(t)$ and $B_i(t)$ of a variable changing along the cone element. Thus, when the base is the space $M = S^3 \times S^3$, we can express the deformed metric as

$$d\bar{s}^2 = dt^2 + \sum_{i=1}^{3} A_i(t)^2 (\eta_i + \tilde{\eta}_i)^2 + \sum_{i=1}^{3} B_i(t)^2 (\eta_i - \tilde{\eta}_i)^2, \tag{$*$}$$

where η_i and $\tilde{\eta}_i$ constitute the basis for left-invariant 1-forms described in Section 2, while the functions $A_i(t)$ and $B_i(t)$ determine the deformation of the conic metric. The system of differential equations of [2] ensures that the holonomy group of $d\bar{s}^2$ lies in G_2.

In this article we continue studying this class of metrics, putting $A_2 = A_3$ and $B_2 = B_3$ and considering a different boundary condition from that of [1]; namely, we require that A_i for $i = 1, 2, 3$ vanish at the cone apex. In result, the Riemannian metric $d\bar{s}^2$ becomes defined on $S^3 \times \mathbb{R}^4$. Observe that the authors of [2] (see also [3–5]) imposed the same boundary condition and found one particular solution to this system which determines a G_2-holonomy metric on $S^3 \times \mathbb{R}^4$, but its asymptotic behavior is different.

Let us state the main result of this article.

Theorem. *For each parameter $p < 0$ there exists a complete G_2-holonomy Riemannian metric on $S^3 \times \mathbb{R}^4$ of the form $(*)$ such that $p = \frac{12}{B_1^2(0)(A_1'''(0) - A_2'''(0))}$.*

The metrics in this family are approximated however closely as $t \to \infty$ by the direct product $S^1 \times C(S^2 \times S^3)$, where $C(S^2 \times S^3)$ is the cone over the product of spheres.

In Section 3 we give a precise definition of approximation in this class of metrics. For $p = -\frac{1}{5}$ the metric $(*)$ coincides with the metric found in [2]. For $A_1(0) = A_2(0)$ the metric $(*)$ coincides with the metric found in [6] and is asymptotically approximated by the cone $C(S^3 \times S^3)$. We may assume that this case corresponds to the parameter values $p = \pm\infty$.

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2. A G_2-Structure on the Cone over $S^3 \times S^3$

Our notation and the main stages of our construction of a G_2-structure follow [1].

Consider the Lie group $G = SU(2)$ with the standard bi-invariant metric $\langle X, Y\rangle = -\operatorname{tr}(XY)$, where $X, Y \in \mathbf{su}(2)$. Consider on G the three Killing vector fields

$$\xi^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \xi^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

On $M = G \times G$ the six Killing fields ξ^i and $\tilde{\xi}^i$ for $i = 1, 2, 3$ arise, tangent respectively to the first and second factors, as well as six dual 1-forms η_i and $\tilde{\eta}_i$. Equip the cone $\overline{M} = \mathbb{R}_+ \times M$ with the metric

$$d\bar{s}^2 = dt^2 + \sum_{i=1}^{3} A_i(t)^2(\eta_i + \tilde{\eta}_i)^2 + \sum_{i=1}^{3} B_i(t)^2(\eta_i - \tilde{\eta}_i)^2,$$

where $A_i(t)$ and $B_i(t)$ are positive functions defining a deformation of the standard conic metric.

Introducing the orthonormal coframe in the metric $d\bar{s}^2$ as

$$e^1 = A_1(\eta_1 + \tilde{\eta}_1), \quad e^2 = A_2(\eta_2 + \tilde{\eta}_2), \quad e^3 = A_3(\eta_3 + \tilde{\eta}_3),$$
$$e^4 = B_1(\eta_1 - \tilde{\eta}_1), \quad e^5 = B_2(\eta_2 - \tilde{\eta}_2), \quad e^6 = B_3(\eta_3 - \tilde{\eta}_3), \quad e^7 = dt,$$

define the 3-form

$$\Psi = e^{564} + e^{527} + e^{513} + e^{621} + e^{637} + e^{432} + e^{417},$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. The form Ψ determines a G_2-structure on $\overline{M}$ which is parallel provided that

$$d\Psi = 0, \quad d * \Psi = 0. \tag{1}$$

In this article we consider the particular case $A_3 = A_2$ and $B_3 = B_2$. The following lemma is straightforward from calculations.

Lemma 1. *Equations* (1) *are equivalent to the system of ordinary differential equations*

$$\begin{aligned} \frac{dA_1}{dt} &= \frac{1}{2}\left(\frac{A_1^2}{A_2^2} - \frac{A_1^2}{B_2^2}\right), & \frac{dA_2}{dt} &= \frac{1}{2}\left(\frac{B_2^2 - A_2^2 + B_1^2}{B_1 B_2} - \frac{A_1}{A_2}\right), \\ \frac{dB_1}{dt} &= \frac{A_2^2 + B_2^2 - B_1^2}{A_2 B_2}, & \frac{dB_2}{dt} &= \frac{1}{2}\left(\frac{A_2^2 - B_2^2 + B_1^2}{A_2 B_1} + \frac{A_1}{B_2}\right). \end{aligned} \tag{2}$$

Proof. Using the relations

$$d\eta_i = -2\eta_{i+1} \wedge \eta_{i+2}, \quad d\tilde{\eta}_i = -2\tilde{\eta}_{i+1} \wedge \tilde{\eta}_{i+2}$$

in [7], where the indices $i = 1, 2, 3$ are reduced modulo 3, we can calculate $d\Psi$:

$$\begin{aligned} d\Psi = \Big(&-4B_3A_3 + B_2A_3\frac{dA_1}{dt} + B_2A_1\frac{dA_3}{dt} + A_1A_3\frac{dB_2}{dt} \\ &+B_2B_3\frac{dB_1}{dt} + B_1B_3\frac{dB_2}{dt} + B_1B_2\frac{dB_3}{dt} - B_3A_2\frac{dA_1}{dt} - B_3A_1\frac{dA_2}{dt} \\ &-A_2A_1\frac{dB_3}{dt} + B_1A_3\frac{dA_2}{dt} + B_1A_2\frac{dA_3}{dt} \\ &+A_3A_2\frac{dB_1}{dt}\Big)\eta_1 \wedge \eta_2 \wedge \tilde{\eta}_3 \wedge dt + \dots + \Big(-B_2A_3\frac{dA_1}{dt} - B_2A_1\frac{dA_3}{dt} - A_1A_3\frac{dB_2}{dt} \\ &+B_2B_3\frac{dB_1}{dt} + B_1B_3\frac{dB_2}{dt} + B_1B_2\frac{dB_3}{dt} - B_3A_2\frac{dA_1}{dt} - B_3A_1\frac{dA_2}{dt} - A_2A_1\frac{dB_3}{dt} \\ &-B_1A_3\frac{dA_2}{dt} - B_1A_2\frac{dA_3}{dt} - A_3A_2\frac{dB_1}{dt}\Big) dt \wedge \tilde{\eta}_2 \wedge \tilde{\eta}_1 \wedge \tilde{\eta}_3. \end{aligned}$$

(We include only a part of the full expression for $d\Psi$ which is bulky.) Then the equation $d\Psi = 0$ leads to four independent first-order ordinary differential equations on $A_i(t)$ and $B_i(t)$ for $i = 1, 2, 3$. Putting here $A_3 = A_2$ and $B_3 = B_2$, we obtain three differential equations on $A_i(t)$ and $B_i(t)$ for $i = 1, 2$:

$$-4B_2A_2 + (B_2)^2\frac{dB_1}{dt} + 2B_1B_2\frac{dB_2}{dt} + 2B_1A_2\frac{dA_2}{dt} + (A_2)^2\frac{dB_1}{dt} = 0,$$

$$4B_1A_1 - 2(B_2)^2\frac{dB_1}{dt} - 4B_1B_2\frac{dB_2}{dt} + 4B_1A_2\frac{dA_2}{dt} + 2(A_2)^2\frac{dB_1}{dt} = 0,$$

$$4B_1A_1 - 4B_2A_2\frac{dA_1}{dt} - 4B_2A_1\frac{dA_2}{dt} - 4A_1A_2\frac{dB_2}{dt} = 0.$$

Considering the form $*\Psi$ and its exterior derivative similarly, we obtain two differential equations on $A_i(t)$ and $B_i(t)$ for $i = 1, 2$:

$$-4A_1A_2^2 - 8A_2B_1B_2 + 4A_1B_2^2 + 8A_2B_2^2\frac{dA_2}{dt} + 8A_2^2B_2\frac{dB_2}{dt} = 0,$$

$$\begin{gathered}4A_1A_2^2 - 4A_2B_1B_2\frac{dA_1}{dt} - 4A_1B_1B_2\frac{dA_2}{dt} - 4A_1A_2B_2\frac{dB_1}{dt} \\ -4A_1A_2B_1\frac{dB_2}{dt} + 4A_1B_2^2 = 0.\end{gathered}$$

Solving this system of five linear equations for the derivatives of the unknown functions $A_i(t)$ and $B_i(t)$ for $i = 1, 2$, we obtain

$$\frac{dA_1}{dt} = \frac{1}{2}\left(\frac{A_1^2}{A_2^2} - \frac{A_1^2}{B_2^2}\right), \quad \frac{dA_2}{dt} = \frac{1}{2}\left(\frac{B_2^2 - A_2^2 + B_1^2}{B_1B_2} - \frac{A_1}{A_2}\right),$$

$$\frac{dB_1}{dt} = \frac{A_2^2 + B_2^2 - B_1^2}{A_2B_2}, \quad \frac{dB_2}{dt} = \frac{1}{2}\left(\frac{A_2^2 - B_2^2 + B_1^2}{A_2B_1} + \frac{A_1}{B_2}\right).$$

The proof of the lemma is complete.

We can resolve the conic singularity of $\overline{M}$ for $t = 0$ by choosing initial values for A_i and B_i as follows.

TYPE 1. $B_1(0) = 0$, $B_2(0) \neq 0$, and $A_i(0) \neq 0$. We studied this case in detail in [1].

TYPE 2. $A_i(0) = 0$ and $B_i(0) \neq 0$. In this case the integral three-dimensional spheres generated by the vector fields $\xi^i + \tilde{\xi}^i$ collapse. These spheres are the orbits of the free action of $SU(2) = S^3$ on M defined as

$$h = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2) : (U, V) \mapsto \left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} U, \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} V\right), \ |a|^2 + |b|^2 = 1.$$

The diffeomorphism

$$\phi : M \to M : (U, V) \mapsto (V^{-1}U, V)$$

transforms the above action of $SU(2)$ into the action

$$h \in SU(2) : (U, V) \mapsto \left(U, \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} V\right).$$

Taking the quotient by the action of $SU(2)$ on the second factor, we obtain $S^3 \times \{*\}$. Upon contracting the orbits of this action on the ambient space $[0, \infty) \times G$ to a point for $t = 0$, we obtain the product $S^3 \times \mathbb{R}^4$, where the punctured $\mathbb{R}^4 \setminus \{0\}$ is a bundle over the open ray $(0, \infty)$ with fibers the concentric spheres S^3. Thus, the metric $d\bar{s}^2$ on $\overline{M}$ extends to a space homeomorphic to $S^3 \times \mathbb{R}^4$, which we denote by $\mathscr{M}$.

Lemma 2. *The following conditions are necessary and sufficient for the metric $d\bar{s}^2$ to extend to a smooth metric on $\mathscr{M}$:*

(1) $A_1(0) = A_2(0) = 0$ and $|A_1'(0)| = |A_2'(0)| = \frac{1}{2}$;

(2) $B_1(0) = B_2(0) \neq 0$ and $B_1'(0) = B_2'(0) = 0$;

(3) A_i and B_i keep sign on $(0, \infty)$.

The proof is almost identical to that of Lemma 4 of [7]. Indeed, [7] deals with the cone over a 3-Sasakian manifold M having the structure of a bundle over a 4-manifold (orbifold) with fiber S^3. Smoothing out the cone amounts to contracting the fiber of this bundle at $t = 0$ to a point. We have the cone over $S^3 \times S^3$ fibered over S^3 with fiber S^3 and the same type of smoothing out the singularity. The dimension and particulars of the base of the bundle play no role in the proof. Thus, the function B of Lemma 4 of [7] controls the diameter of the base as t varies, while here the pair of functions B_1 and B_2 play this role; our conditions (2) and (3) correspond to (2) and (3) of Lemma 4 of [7]. Furthermore, the condition of collapse of a sphere to a point in Lemma 4 of [7] is $A_i(0) = 0$ for $i = 1, 2, 3$, which fully agrees with our conditions $A_1(0) = A_2(0) = 0$. It remains to relate the conditions on the derivatives. In [7] the fiber S^3 is the sphere of radius 1, and the conditions on the derivative of A_i at $t = 0$ amount to the conditions of the smoothness of the metric expressed in the spherical coordinate system on $\mathbb{R}^4$. In our situation the fields ξ^i have norms equal to $\sqrt{2}$. In addition, the spherical fiber in $S^3 \times S^3$ lies diagonally, that is, we have to multiply the size of the sphere S^3 by $\sqrt{2}$. In result, the unit frame on the spherical fiber in question looks as $\frac{\eta_i + \tilde{\eta}_i}{2}$; hence, the functions $2A_i$ for $i = 1, 2$ correspond to the sphere of unit size, and their derivatives at zero must be equal to 1, which explains why (1) is equivalent to condition (1) of Lemma 4 of [7]. □

The exact solution for (2) found in [2] is of the form (the remaining solutions of the family found in [2] are homothetic to this one)

$$\begin{gathered} A_1(r) = \sqrt{\frac{(r - 9/4)\,(r + 9/4)}{(r - 3/4)\,(r + 3/4)}}, \quad A_2(r) = \frac{1}{\sqrt{3}}\sqrt{(r + 3/4)\,(r - 9/4)}, \\ B_1(r) = 2r/3, \quad B_2(r) = \frac{1}{\sqrt{3}}\sqrt{(r - 3/4)\,(r + 9/4)}, \end{gathered} \tag{3}$$

where $r \geq 9/4$, and r is related to t by the change of variables

$$dt = \frac{dr}{A_1(r)}, \quad t|_{r=\frac{9}{4}} = 0.$$

The metric (3) is a complete G_2-holonomy metric on $S^3 \times \mathbb{R}^4$. In the case $A_1 = A_2 = A_3 = A$ and $B_1 = B_2 = B_3 = B$ we can integrate (2) in elementary functions and obtain another complete G_2-holonomy metric on $S^3 \times \mathbb{R}^4$:

$$d\bar{s}^2 = \frac{dr^2}{1 - \frac{1}{r^3}} + \frac{r^2}{9}\left(1 - \frac{1}{r^3}\right)\sum_{i=1}^{3} (\eta_i + \tilde{\eta}_i)^2 + \frac{r^2}{3}\sum_{i=1}^{3} (\eta_i - \tilde{\eta}_i)^2 . \tag{4}$$

This metric was constructed for the first time in [6] (also see [8]). The metrics (3) and (4) exhaust the list of available explicit solutions to (2) corresponding to complete G_2-holonomy Riemannian metrics.

Making the formal substitution $r \to -r$ into (4), we obtain the following solution to (2):

$$d\bar{s}^2 = \frac{dr^2}{1 + \frac{1}{r^3}} + \frac{r^2}{9}\left(1 + \frac{1}{r^3}\right)\sum_{i=1}^{3} (\eta_i + \tilde{\eta}_i)^2 + \frac{r^2}{3}\sum_{i=1}^{3} (\eta_i - \tilde{\eta}_i)^2 . \tag{5}$$

This solution is defined for $0 < r < \infty$, but fails to determine a complete smooth Riemannian metric since it has a singularity at $r = 0$.

3. A Family of New Solutions

By analogy with [7], consider the standard space $\mathbb{R}^4$ and denote by $R(t) \in \mathbb{R}^4$ the vector consisting of $A_1(t)$, $A_2(t)$, $B_1(t)$, and $B_2(t)$. Look at the function $V : \mathbb{R}^4 \to \mathbb{R}^4$ of R defined by the right-hand side of (1) (certainly, it is defined only on the domain with $A_i, B_i \neq 0$). Thus, (1) becomes

$$\frac{dR}{dt} = V(R).$$

Using the invariance of V under the homothety of $\mathbb{R}^4$, insert $R(t) = f(t)S(t)$ with $f(t) = |R(t)|$ and $|S(t)| = 1$, where $S(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t))$. Our system splits into the radial and tangential parts:

$$\frac{dS}{du} = V(S) - \langle V(S), \quad S\rangle S = W(S), \tag{6}$$

$$\frac{1}{f}\frac{df}{du} = \langle V(S), S\rangle, \quad dt = f\,du. \tag{7}$$

Consequently, we have to solve firstly the autonomous system (6) on the three-dimensional sphere $S^3 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mid \sum_{i=1}^4 \alpha_i^2 = 1\}$, and then find the solutions to (2) by the usual integration of (7).

Lemma 3. *The systems* (2) *and* (6) *admit the symmetries*

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto (-\alpha_1, \alpha_4, \alpha_3, \alpha_2),$$
$$((\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)) \mapsto (-\alpha_1(-u), \alpha_2(-u), \alpha_3(-u), -\alpha_4(-u)),$$
$$((\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)) \mapsto (-\alpha_1(-u), -\alpha_2(-u), \alpha_3(-u), \alpha_4(-u)),$$
$$((\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)) \mapsto (\alpha_1(u), \alpha_2(u), -\alpha_3(u), -\alpha_4(u)),$$
$$((\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)) \mapsto (\alpha_1(u), -\alpha_2(u), -\alpha_3(u), \alpha_4(u)).$$

By Lemma 2, to construct a regular metric on $\mathscr{M}$, we need a trajectory of (6) emanating from the point $S_0 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The remaining solutions result from the case we considered once we apply the symmetries of (6).

To construct a smooth metric on $\mathscr{M}$, blow up the sphere S^3 at $S_0 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The blowup operation looks as follows. Choose local coordinates $\left(\alpha_1, \alpha_2, \alpha_3 - \frac{1}{\sqrt{2}}\right)$ on a neighborhood of S_0 and take the ball $B = \left\{\left(\alpha_1, \alpha_2, \alpha_3 - \frac{1}{\sqrt{2}}\right) \mid \alpha_1^2 + \alpha_2^2 + \left(\alpha_3 - \frac{1}{\sqrt{2}}\right)^2 \leq \varepsilon^2\right\}$ of radius ε. Its intersection with the plane $\alpha_3 = \frac{1}{\sqrt{2}}$ is the disk $U = \left\{(\alpha_1, \alpha_2) \mid \sum_{i=1}^2 \alpha_i^2 \leq \varepsilon^2\right\}$ of radius ε.

Introduce the geodesic coordinate system on U; that is, consider the radial coordinate $-\varepsilon < r < \varepsilon$ and the tangential coordinate $s \in S^1$, where $S^1 = \left\{(\alpha_1, \alpha_2) \mid \sum_{i=1}^2 \alpha_i^2 = 1\right\}$. Thus, $(\alpha_1, \alpha_2) = rs$. Consider the product $(-\varepsilon, \varepsilon) \times S^1$ with the action $(r, s) \mapsto (-r, -s)$ of the group $\mathbb{Z}_2$. The action is clearly free, and we obtain the quotient space $\widetilde{U} = (-\varepsilon, \varepsilon) \times S^1/\mathbb{Z}_2$ amounting to the Möbius band. The assignment $\pm(r, s) \mapsto rs$ determines a smooth mapping $\widetilde{U} \to U$, which is obviously a diffeomorphism $\widetilde{U}\backslash P \to U\backslash S_0$, where $P = \{(r, s) \mid r = 0\}$ is a projective line embedded into $\widetilde{U}$.

Remove the point S_0 from the neighborhood U and glue in $\widetilde{U}$ using the diffeomorphism just constructed. The resulting manifold is said to be the *blowup of* S^3 *at* S_0.

Denote by $\widetilde{S}$ the sphere S^3 blown up at S_0 (we can represent $\widetilde{S}$ as the connected sum of the sphere S^3 and the real projective space $\mathbb{R}P^3$). We need local coordinates on a neighborhood of P. Consider $U_i = \{\pm(r, s) \mid \alpha_i \neq 0\}$, $i = 1, 2$. On U_i put

$$\alpha_i^i = \alpha_i, \quad \alpha_j^i = \frac{\alpha_j}{\alpha_i} \text{ for } i \neq j.$$

This defines local coordinates α_1^i, α_2^i on $\widetilde{U}$ in the neighborhood U_i, $i = 1, 2$. Extend $\widetilde{U}$ to a three-dimensional neighborhood of the point S_0 by putting $\alpha_3^i = \alpha_3$ for $i = 1, 2$.

Lemma 4. *There exists a one-parameter family of trajectories of* (6) *emanating from the point* $S_0 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ *and entering into the domain* $\alpha_2 \geq \alpha_1 > 0$.

PROOF. Carry the system (6) over to $\widetilde{S}$, and then the projections of the trajectories onto S^3 yield the required solutions. By the previous arguments and Lemma 2, we must study the trajectories of (6) on $\widetilde{S}$ emanating from the point $\alpha_1^2 = 1$, $\alpha_2^2 = 0$, $\alpha_3^2 = \frac{1}{\sqrt{2}}$. Express the field W on U_2 in the new coordinates. For simplicity, put $x = \alpha_1^2$, $y = \alpha_2^2$, and $z = \alpha_3^2$. Then (6) is equivalent to

$$\begin{aligned}
\frac{dx}{dv} &= W_1(xy, y, z) - xW_2(xy, y, z) = \widetilde{W_1}(x, y, z),\\
\frac{dy}{dv} &= yW_2(xy, y, z) = \widetilde{W_2}(x, y, z),\\
\frac{dz}{dv} &= yW_3(xy, y, z) = \widetilde{W_3}(x, y, z),
\end{aligned} \tag{8}$$

where $du = y\,dv$.

We can verify directly that the vector field $\widetilde{W}$ vanishes at $p = \left(1, 0, \frac{1}{\sqrt{2}}\right)$. Consider the linearization of (8) in a neighborhood of this point:

$$\frac{dx}{dv} = x, \quad \frac{dy}{dv} = \frac{1}{2}y, \quad \frac{dz}{dv} = -3z.$$

Thus, in a neighborhood of the point $p = \left(1, 0, \frac{1}{\sqrt{2}}\right)$ we have the surface swept by the trajectories of (8) emanating from p exponentially in v. Furthermore, this surface is tangent at $p = \left(1, 0, \frac{1}{\sqrt{2}}\right)$ to the two-dimensional plane spanned by the first two eigenvectors $e_1 = \{1, 0, 0\}$ and $e_2 = \{0, 1, 0\}$. Namely, in the phase plane with coordinates $\tilde{x} = x - 1$ and $\tilde{y} = y$ our trajectories amount to the parabolas $\tilde{y}^2 = 2p\tilde{x}$ emanating parallel to the chosen direction e_2. Each of these parabolas is the parametrized curve $\gamma(v) = (\alpha e^v, \beta e^{\frac{v}{2}})$ or $\gamma(u) = \left(\frac{\alpha u^2}{4\beta^2}, \frac{u}{2}\right)$, and so $\frac{d\gamma}{du} = \left(\frac{\alpha u}{2\beta^2}, \frac{1}{2}\right)$ is its velocity vector and $\frac{d^2\gamma}{du^2} = \left(\frac{\alpha}{2\beta^2}, 0\right)$ is its acceleration vector. Hence, $\frac{\alpha}{2\beta^2} = \frac{d^2x}{du^2}$.

It is not difficult to calculate that

$$\frac{d^2x}{du^2} = f\frac{d}{dt}\left(f\frac{d}{dt}\left(\frac{A_1}{A_2}\right)\right)\bigg|_{t=0} = \frac{1}{8}b_0^2(a_1 - a_2),$$

where $b_0 = B_1(0) = B_2(0)$, $a_1 = \frac{A_1'''(0)}{6}$, and $a_2 = \frac{A_2'''(0)}{6}$. Then we find the focal parameter of the parabola p from the condition $2p = \frac{\beta^2}{\alpha}$, which yields

$$p = \frac{12}{B_1^2(0)(A_1'''(0) - A_2'''(0))}.$$

Observe that it determines our trajectory uniquely (up to homothety). Furthermore, the trajectory enters into the domain $\alpha_1 < \alpha_2$ when $p < 0$, and into the domain $\alpha_1 > \alpha_2$ when $p > 0$. It is worth noting that the previously available particular solutions (3) and (4) to (1) at the initial time are also tangent to the vector e_2; moreover, (4) amounts to the straight line $\tilde{x} = 0$ (we may assume that this solution corresponds to the limit values $p = \pm\infty$), while (3) lies in the family of parabolas $\tilde{y}^2 = 2p\tilde{x}$ and corresponds to $p = -1/5$.

REMARK. Our arguments show that there exists a family of solutions for $p > 0$, but in this article we do not address the global behavior of the corresponding trajectories.

Thus, in a neighborhood of the point $S_0 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ we have a one-parameter family of trajectories of (6) emanating from S_0 in finite time with respect to the variable u; furthermore, this family is tangent at S_0 to the plane parallel to the coordinate plane $O\alpha_1\alpha_2$, and the tangent vector at the initial time is of the form $\{\alpha_1, \alpha_2, 0, 0\}$ with $\alpha_1 < \alpha_2$. $\square$

The next lemma is established in [1].

Lemma 5. *The stationary solutions to* (6) *on* S^3 *are exhausted by the following list of zeros of the vector field* W *up to the symmetries of Lemma* 3*:*

$$\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right), \quad \left(0, \frac{\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{2}}{\sqrt{5}}, \frac{\sqrt{3}}{\sqrt{10}}\right).$$

Definition. Refer to a point $S \in S^3$ at which the field W is undefined as *conditionally stationary* whenever there exists a real-analytic curve $\gamma(u)$ on S^3 for $u \in (-\varepsilon, \varepsilon)$ with $\gamma(0) = S$, such that W is defined at all points $\gamma(u)$ for $u \in (-\varepsilon, \varepsilon)$ with $u \neq 0$, extends continuously to the whole curve $\gamma(u)$, and $\lim_{u\to 0} W(\gamma(u)) = 0$.

The next lemma is also established in [7].

Lemma 6. *The system* (6) *lacks conditionally stationary solutions on* S^3.

Definition. A metric $d\bar{s}^2$ is called *asymptotically locally conic* whenever there exist functions $\widetilde{A}_i(t)$ and $\widetilde{B}_i(t)$ linear in t up to translations such that

$$\left|1 - \frac{A_i}{\widetilde{A}_i}\right| \to 0, \ \left|1 - \frac{B_i}{\widetilde{B}_i}\right| \to 0 \quad \text{as} \quad t \to \infty.$$

The metric defined by $\widetilde{A}_i(t)$ and $\widetilde{B}_i(t)$ is called *locally conic.*

The next lemma is established in [7].

Lemma 7. *Associated to the stationary solutions to* (6) *there are locally conic metrics on* $\overline{M}$, *while associated to the trajectories of* (6) *tending asymptotically to the stationary solutions there are asymptotically locally conic metrics on* $\overline{M}$.

The next lemma follows from a straightforward inspection of (2) and (6).

Lemma 8. *If* $S = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *is a solution to* (6) *then*

$$\frac{d}{dt}\big(2A_1A_2B_2 - B_1\big(B_2^2 - A_2^2\big)\big) = 0, \tag{9}$$

$$\frac{d}{du}\left(\frac{\alpha_1\alpha_2\alpha_4}{2\alpha_1\alpha_2\alpha_4 - \alpha_3\left((\alpha_4)^2 - (\alpha_2)^2\right)}\right) = \frac{\alpha_1\alpha_3}{2\alpha_1\alpha_2\alpha_4 - \alpha_3\left((\alpha_4)^2 - (\alpha_2)^2\right)}, \tag{10}$$

$$\frac{d}{du}\left(\log\frac{\alpha_3((\alpha_4)^2 - (\alpha_2)^2)}{\alpha_1\alpha_2\alpha_4}\right) = \frac{2\alpha_1\alpha_2\alpha_4 - \alpha_3((\alpha_4)^2 - (\alpha_2)^2)}{2\alpha_2\alpha_4((\alpha_4)^2 - (\alpha_2)^2)}, \tag{11}$$

$$\frac{d}{du}\log\frac{\alpha_2}{\alpha_4} = -\frac{\alpha_1}{(\alpha_4)^2} \quad \textit{for} \ \ \alpha_2 = \alpha_4, \tag{12}$$

$$\frac{d}{du}\left(\frac{\alpha_3}{\alpha_4}\right) = \frac{3}{2\alpha_4}\left(\frac{2}{\sqrt{3}} + \frac{\alpha_3}{\alpha_4}\right)\left(\frac{2}{\sqrt{3}} - \frac{\alpha_3}{\alpha_4}\right) \quad \textit{for} \ \ \alpha_1 = 0, \ \alpha_2 = \alpha_4. \tag{13}$$

Proof. Let us verify (10) and (12) for instance. We have

$$\frac{d}{du}\left(\frac{\alpha_1\alpha_2\alpha_4}{2\alpha_1\alpha_2\alpha_4 - \alpha_3\left((\alpha_4)^2 - (\alpha_2)^2\right)}\right) = \frac{d}{du}\left(\frac{A_1A_2B_2}{2A_1A_2B_2 - B_1\left((B_2)^2 - (A_2)^2\right)}\right)$$
$$= f\frac{d}{dt}\left(\frac{A_1A_2B_2}{2A_1A_2B_2 - B_1\left((B_2)^2 - (A_2)^2\right)}\right) = f\frac{A_1B_1}{2A_1A_2B_2 - B_1\left((B_2)^2 - (A_2)^2\right)}$$
$$= \frac{\alpha_1\alpha_3}{2\alpha_1\alpha_2\alpha_4 - \alpha_3\left((\alpha_4)^2 - (\alpha_2)^2\right)},$$

$$\frac{d}{du}\log\left(\frac{\alpha_2}{\alpha_4}\right) = \frac{d}{du}\log\left(\frac{A_2}{B_2}\right) = f\frac{d}{dt}\log\left(\frac{A_2}{B_2}\right)$$
$$= f\left(\frac{1}{2}\frac{2A_2B_2((B_2)^2 - (A_2)^2) - A_1B_1((A_2)^2 + (B_2)^2)}{(A_2)^2B_1(B_2)^2}\right)\bigg|_{A_2=B_2} = f\left(-\frac{A_1}{(B_2)^2}\right) = -\frac{\alpha_1}{(\alpha_4)^2}.$$

Remark. The function $F(t) = 2A_1A_2B_2 - B_1\big(B_2^2 - A_2^2\big)$ is an integral of (2).

Lemma 9. *The trajectory of* (6) *determined by the initial point* $S_0 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ *tends as* $u \to \infty$ *to the stationary point* $S_\infty = \left(0, \frac{\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{2}}{\sqrt{5}}, \frac{\sqrt{3}}{\sqrt{10}}\right)$.

Proof. Demonstration is similar to that in [7], but for the sake of rigor of our exposition we present it in full. Label the following points of S^3:

$$O = (0,0,1,0), \quad A = (0,0,0,1), \quad B = (1,0,0,0), \quad C = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

Consider the domain $\Pi \subset S^3$ defined by the inequalities

$$\Pi : \alpha_4 \geq \alpha_2 \geq 0,\ \alpha_1 \geq 0,\ \alpha_3 \geq 0.$$

It is not difficult to verify that Π is a spherical tetrahedron $(OABC)$. The boundary of the domain is the union of the sets

$$\Pi_1 = (OAB) = \{\alpha_4 \geq 0, \alpha_2 = 0, \alpha_1 \geq 0, \alpha_3 \geq 0\},$$

$$\Pi_2 = (OBC) = \{\alpha_2 = \alpha_4, \alpha_2 \geq 0, \alpha_1 \geq 0, \alpha_3 \geq 0\},$$

$$\Pi_3 = (OAC) = \{\alpha_4 - \alpha_2 \geq 0, \alpha_2 \geq 0, \alpha_1 = 0, \alpha_3 \geq 0\},$$

$$\Pi_4 = (ABC) = \{\alpha_4 - \alpha_2 \geq 0, \alpha_2 \geq 0, \alpha_1 \geq 0, \alpha_3 = 0\}.$$

The initial point $S_0 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ belongs to (OA). For small u the trajectory of (6) determined by the initial point S_0 lies in Π.

Consider firstly the possibility that the trajectory reaches the boundary of Π in finite time. Consider also Π_1 and define the function F_1 on S^3 as

$$F_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{\alpha_1 \alpha_2 \alpha_4}{F(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}.$$

At the initial moment $F_1(S_0) = 0$. Since $F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = f(t)^{-3} F(S_0) < 0$, we infer from (10) that the derivative of F_1 is negative, and so the function strictly decreases along the trajectories of (2) inside Π. On $\Pi_1 \backslash ((AB) \cup (OB))$ we have $F_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$; consequently, the trajectory cannot return and intersect this wall, with the possible exception of the arcs $(AB) = \{\alpha_3 = 0\}$ and $(OB) = \{\alpha_4 = 0\}$. Furthermore, on Π_2 we have

$$\frac{d(\alpha_4 - \alpha_2)}{du} = \frac{\alpha_1}{\alpha_2} > 0$$

for $\alpha_1 \neq 0$, that is, the trajectory cannot intersect some neighborhood of Π_2 in finite time nor even approach sufficiently close, with the exception of the arc $(OC) = \{\alpha_1 = 0\}$. The same reasons also exclude a neighborhood of the arc (OB). Finally, on Π_4 the derivative of $\alpha_3(u)$ is strictly positive and separated from zero:

$$\begin{aligned}\frac{d\alpha_3}{du} = \frac{d}{du}\left(\frac{B_1}{f}\right) = f \frac{d}{dt}\left(\frac{B_1}{f}\right)\bigg|_{B_1=0} &= \frac{2A_1^2\left(A_2^2 + B_2^2\right) + 3A_2^4 + 2A_2^2B_2^2 + 3B_2^4}{2A_2B_2\left(A_1^2 + A_2^2 + B_2^2\right)} \\ &= \frac{2\alpha_1^2\left(\alpha_2^2 + \alpha_4^2\right) + 3\alpha_2^4 + 2\alpha_2^2\alpha_4^2 + 3\alpha_4^4}{2\alpha_2\alpha_4\left(\alpha_1^2 + \alpha_2^2 + \alpha_4^2\right)};\end{aligned}$$

thus, the trajectory is disjoint from Π_4 and some neighborhood of the latter (this also excludes the remaining possibility of approach to the arc (AB)). Since Π_3 is an invariant subset for (6), the trajectory cannot intersect Π_3 in finite time (including the arc (OC)).

Suppose that C is the limit set of the trajectory under consideration. Then it may contain the following points.

(1) The stationary or conditionally stationary points of (2) (by Lemmas 5 and 6, the only two possibilities are open: the points S_∞ and $S_1 = \left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right)$).

(2) The points lying on a critical level of the function F_1.

(3) Finally, suppose that $p \in C$ is not of type 1 and 2. If $p \in \operatorname{Int}\Pi$ then the trajectory has nonzero velocity at p, and since F_1 decreases, it can never again return to some neighborhood of p; this contradicts the assumption that p is a limit point. Thus, $p \in \partial\Pi$. Similar reasoning shows that p lies on the minimal level of F_1.

In the same fashion, consider the function

$$F_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \log \frac{\alpha_3\left(\alpha_4^2 - \alpha_2^2\right)}{\alpha_1\alpha_2\alpha_4}.$$

It follows from (11) that F_2 decreases along the trajectories, and so the set $C \cap \partial\Pi$ lies on the minimal level of F_2 in Π. Observe that the minimal level of F_2 in Π is the set $\Pi_2 \cup \Pi_4$. We ruled out above the approach to a neighborhood of Π_4; consequently, only the case $C \cap \partial\Pi \subset \Pi_2$ is possible.

Furthermore, by (12) the function $F_3 = \log\frac{\alpha_2}{\alpha_4}$ decreases along the trajectories (for sufficiently large u) to the minimal values of Π_2, attained at $\alpha_1 = 0$. Thus, our trajectory tends as $u \to \infty$ to the one-dimensional invariant set $\Pi_2 \cap \Pi_3 = (OC)$. By (13), in a neighborhood of (OC) the function $F_4 = \frac{\alpha_3}{\alpha_4}$ increases when $F_4 \le \frac{2}{\sqrt{3}}$ and decreases when $F_4 \ge \frac{2}{\sqrt{3}}$; consequently, $C \cap \partial\Pi$ can contain only the point S_∞ determined by the condition $F_4 = \frac{2}{\sqrt{3}}$.

Thus, we conclude that the trajectory under consideration tends to either S_1 or S_∞. To complete the proof of the lemma, it remains to verify the lack of convergence to S_1.

Consider the linearization of (6) in a neighborhood of the stationary point S_1 in the local coordinates $(\alpha_1, \alpha_2, \alpha_3)$. Straightforward calculations show that the linearized system has the three eigenvalues of multiplicity 1:

$$\lambda_1 = -2\sqrt{2}, \quad \lambda_2 = -\frac{7}{3}\sqrt{2} - \frac{1}{3}\sqrt{290}, \quad \lambda_3 = -\frac{7}{3}\sqrt{2} + \frac{1}{3}\sqrt{290}.$$

Thus, in a neighborhood of S_1 there exists a (locally defined) surface swept by the trajectories entering S_1; furthermore, this surface is tangent at S_1 to the two-dimensional plane spanned by the first two eigenvectors e_1 and e_2. The remaining trajectories in a neighborhood of S_1 emanate from S_1. Furthermore, the first eigenvector e_1 has in $\mathbb{R}^4$ the coordinates $(-\sqrt{3}, -\sqrt{3}, 1, 1)$ and is tangent to the trajectory defined as $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$. It is not difficult to see that the eigenvalue λ_1 corresponds precisely to the solutions (4) and (5) (the trajectories enter S_1 from the opposite sides; (4) corresponds to $F < 0$, and (5) to $F > 0$). Since $|\lambda_2| > |\lambda_1|$, the remaining trajectories entering S_1, except one, are tangent at S_1 to the trajectory (4) or (5). The unique trajectory not tangent to either (4) or (5) corresponds to the eigenvalue λ_2, and we can verify directly that it lies on the invariant surface $F = 0$, and consequently, cannot coincide with our trajectory.

Consider the two functions $G_1 = \alpha_2\alpha_4 - \alpha_1\alpha_3$ and $G_2 = \alpha_1\alpha_4 - \alpha_2\alpha_3$. The initial point S_0 and the stationary point S_1 lie in the domain $\{G_1 = 0, G_2 = 0\}$. Straightforward calculations show that the vector e_2 is directed inside the domain $\{G_1 > 0, G_2 > 0\}$ or $\{G_1 < 0, G_2 < 0\}$ depending on the choice of the direction of e_2:

$$e_2 = \pm\left\{-\frac{1}{2} + \frac{\sqrt{145}}{10}, 1, -\frac{11\sqrt{3}}{6} - \frac{\sqrt{3}\sqrt{145}}{6}, \frac{5}{\sqrt{3}} + \frac{2\sqrt{145}}{5\sqrt{3}}\right\}.$$

It is easy to verify that $\frac{d}{du}G_1 = -\frac{2}{\alpha_2}G_2$ at those points where $G_1 = 0$, and $\frac{d}{du}G_2 = -\frac{2}{\alpha_2}G_1$ at those points where $G_2 = 0$. Hence, the trajectory can reach S_1 only from the domain $\{G_1 > 0, G_2 > 0\}$. If it crosses into one of the domains $\{G_1 > 0, G_2 < 0\}$ or $\{G_1 < 0, G_2 > 0\}$ then it will be unable to leave them (observe that S_∞ lies in $\{G_1 > 0, G_2 < 0\}$). This reason determines the direction of e_2: it is directed inside the domain $\{G_1 > 0, G_2 > 0\}$.

On the other hand, at the initial time the tangent vector to our trajectory is of the form $\{\alpha_1, \alpha_2, 0, 0\}$ with $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$; thus, it is directed into one of the domains $\{G_1 > 0, G_2 < 0\}$ or $\{G_1 < 0, G_2 > 0\}$ depending on the sign of $(\alpha_2 - \alpha_1)$. It only remains to recall that $\alpha_2 > \alpha_1$ (see the proof of Lemma 4), and so the trajectory belongs to the domain $\{G_1 > 0, G_2 < 0\}$, where the unique limit point is S_∞. $\square$

The main theorem follows from Lemmas 4 and 9. The initial point of the trajectory determines the topological structure of the space carrying our metric, whose holonomy group obviously coincides with the whole of G_2. The limit point S_∞ means that at infinity the function A_1 is approximated by a constant, while the remaining functions defining the metric, by nonconstant linear functions.

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