

# ON NEW EXPLICIT RIEMANNIAN $SU(2(n + 1))$ -HOLONOMY METRICS

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**Abstract:** Generalizing the Calabi metrics in dimension  $4(n + 1)$ , we construct a family of complete noncompact Ricci-flat metrics with holonomy  $SU(2(n + 1))$  in explicit algebraic form.

**Keywords:** holonomy, Calabi metric

## 1. Introduction

This article, naturally continuing [1–4], studies Ricci-flat Riemannian metrics with exceptional holonomy. In the systematic study [4] of the Riemannian metrics with holonomy  $\text{Spin}(7)$  on the cones over 7-dimensional 3-Sasakian manifolds, some continuous family was explicitly found of 8-dimensional complete noncompact metrics  $\bar{g}_\alpha$  depending on a real parameter  $0 \leq \alpha \leq 1$ . The metric  $\bar{g}_0$  coincides with the Calabi metric with holonomy  $SU(4)$ , and the metric  $\bar{g}_1$ , with the hyper-Kähler Calabi metric with holonomy  $Sp(2) \subset SU(4)$ . Every metric  $\bar{g}_\alpha$ ,  $0 < \alpha < 1$ , has the holonomy  $SU(4)$  and is automatically Ricci-flat. Therefore, this family “connects” Calabi metrics.

Both Calabi metrics (originally constructed in [5]) are however defined not only in dimension 8, but also in all dimensions divisible by four. Thus, the natural question arises of the possibility of generalizing the families of metrics of [4] to higher dimensions. In this article we answer the question in the affirmative: we explicitly construct a continuous family of metrics  $\bar{G}_\alpha$  “connecting” higher-dimensional Calabi metrics in every real dimension  $4(n + 1)$ .

**Theorem.** *The following family consists of  $4(n + 1)$ -dimensional complete Ricci-flat Riemannian metrics:*

$$\begin{aligned} \bar{G}_\alpha = & \frac{r^4(r^4 - \alpha^4)^n}{(r^4 - \alpha^4)^{n+1} - (1 - \alpha^4)^{n+1}} dr^2 + \frac{(r^4 - \alpha^4)^{n+1} - (1 - \alpha^4)^{n+1}}{r^2(r^4 - \alpha^4)^n} \eta_1^2 \\ & + r^2(\eta_2^2 + \eta_3^2) + (r^2 + \alpha^2) \sum_{\beta=1}^n (\eta_{4\beta}^2 + \eta_{5\beta}^2) + (r^2 - \alpha^2) \sum_{\beta=1}^n (\eta_{6\beta}^2 + \eta_{7\beta}^2), \end{aligned}$$

where  $0 \leq \alpha \leq 1$  and  $r \geq 1$ . The metrics  $\bar{G}_0$  and  $\bar{G}_1$  have holonomy  $SU(2(n + 1))$  and  $Sp(n + 1)$  respectively and coincide with the higher-dimensional Calabi metrics of [5]. The metrics  $\bar{G}_\alpha$  for  $0 < \alpha < 1$  have holonomy  $SU(2(n + 1))$ , and for  $n = 1$  coincide with the family constructed in [4]. For  $0 \leq \alpha < 1$  the metrics  $\bar{G}_\alpha$  are defined on the  $(n + 1)$ th tensor power of a complex line bundle over the space of complex flags in  $\mathbb{C}^{2n+1}$ , and the metric  $\bar{G}_1$  is defined on  $T^*\mathbb{C}P^{n+1}$ .

In the next section we explain our construction in detail of  $\bar{G}_\alpha$  and prove the theorem.

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## 2. The Proof

The existence of 8-dimensional metrics of the form

$$dt^2 + A_1(t)^2\eta_1^2 + A_2(t)^2\eta_2^2 + A_3(t)^2\eta_3^2 + B(t)^2(\eta_4^2 + \eta_5^2) + C(t)^2(\eta_6^2 + \eta_7^2)$$

with holonomy  $\text{Spin}(7)$  on the cone over a 7-dimensional 3-Sasakian manifold  $M$  whose 4-dimensional quaternionic Kähler orbifold  $\mathcal{O}$  possesses a Kähler structure was studied in [4]. The form  $dt$  corresponds to a generator of the cone,  $\eta_i$  for  $i = 1, 2, 3$  are the characteristic forms of the 3-Sasakian manifold,  $\eta_i$  for  $i = 4, \dots, 7$  are 1-forms on the orbifold. The condition that the holonomy group is included into  $\text{Spin}(7)$  reduces to a system of differential equations on the functions  $A_i$ ,  $B$ , and  $C$ . Detailed study of this system yields the family of solutions

$$\begin{aligned} \bar{g}_\alpha = & \frac{r^4(r^2 - \alpha^2)(r^2 + \alpha^2)}{r^8 - 2\alpha^4(r^4 - 1) - 1} dr^2 + \frac{r^8 - 2\alpha^4(r^4 - 1) - 1}{r^2(r^2 - \alpha^2)(r^2 + \alpha^2)} \eta_1^2 + r^2(\eta_2^2 + \eta_3^2) \\ & + (r^2 + \alpha^2)(\eta_4^2 + \eta_5^2) + (r^2 - \alpha^2)(\eta_6^2 + \eta_7^2), \end{aligned} \quad (*)$$

where  $0 \leq \alpha \leq 1$  and  $r \geq 1$ . The metrics  $(*)$  are defined on a smooth manifold only in the case that  $M$  is the Aloff–Wallach space  $N_{1,1} = SU(3)/S^1$ .

Calabi constructed his metrics  $\bar{G}_0$  and  $\bar{G}_1$  on complex bundles over complex Kähler–Einstein manifolds [5]. In particular, he constructed metrics with holonomy  $SU(n)$  on line bundles over compact Kähler–Einstein manifolds, and hyper-Kähler metrics on  $T^*\mathbb{C}P^n$ . Nevertheless, the metrics constructed were not written out explicitly in [5].

The expression for  $\bar{G}_0$  was obtained in [6]:

$$[1 - (1/\rho)^{2m+2}]^{-1} d\rho^2 + [1 - (1/\rho)^{2m+2}] \rho^2 (d\tau - 2A)^2 + \rho^2 ds^2, \quad (1)$$

where  $ds^2$  is the metric on the  $m$ -dimensional Hodge Kähler–Einstein manifold  $F$ , while  $dA$  is the Kähler form on  $F$ . For  $m = 3$  the metric (1) becomes  $\bar{g}_0$  for  $F = SU(3)/T^2$ . The metric (1) is defined on the  $(m+1)$ th power of the canonical line bundle over  $F$ .

The metrics on  $T^*\mathbb{C}P^{n+1}$  of cohomogeneity 1 were studied in [7]. It is not difficult to see that the spherical subbundle in  $T^*\mathbb{C}P^{n+1}$  fibers over  $SU(n+2)/(U(n) \times U(1))$ . On the Lie algebra  $\mathfrak{su}(n+2)$  we can choose a basis of left-invariant 1-forms  $L_A^B$  the exterior algebra generated by which satisfies  $dL_A^B = iL_A^C \wedge L_C^B$ . Here  $A$  runs over the values  $(1, 2, \beta)$ , while  $\beta$  takes values from 1 through  $n$ ; henceforth we never fix  $\beta$ , thus avoiding confusion. It is obvious that  $\mathfrak{u}(n) \oplus \mathfrak{u}(1)$  is the Lie subalgebra of  $\mathfrak{su}(n+2)$ , but it fails to be an exterior subalgebra. The forms  $L_1^\beta = \sigma_\beta$ ,  $L_2^\beta = \Sigma_\beta$ , and  $L_1^2 = \nu$  constitute a basis on the quotient space  $\mathfrak{su}(n+2)/(\mathfrak{u}(n) \oplus \mathfrak{u}(1))$ . Then we define the real forms  $\sigma_{1\beta} + i\sigma_{2\beta} = \sigma_\beta$ , and so forth. The form  $\lambda = L_1^1 - L_2^2$  is real by definition. The metrics

$$dt^2 + a(t)^2|\sigma_\beta|^2 + b(t)^2|\Sigma_\beta|^2 + c(t)^2|\nu|^2 + f(t)^2\lambda^2, \quad (2)$$

with summation over  $\beta$  from 1 to  $n$ , were considered in [7]. The expression for the hyper-Kähler metric  $\bar{g}_1$  on  $T^*\mathbb{C}P^{n+1}$  was found:

$$\frac{dr^2}{1 - r^{-4}} + \frac{1 - r^{-4}}{4} r^2 \lambda^2 + r^2(\nu_1^2 + \nu_2^2) + \frac{r^2 + 1}{2} (\Sigma_{1\beta}^2 + \Sigma_{2\beta}^2) + \frac{r^2 - 1}{2} (\sigma_{1\beta}^2 + \sigma_{2\beta}^2). \quad (3)$$

In the case  $n = 1$ , in order to match the notation of [4] and [7], we should put

$$\lambda = 2\eta_1, \quad \nu_1 = \eta_3, \quad \nu_2 = \eta_2, \quad \Sigma_1 = \sqrt{2}\eta_4, \quad \Sigma_2 = \sqrt{2}\eta_5, \quad \sigma_1 = \sqrt{2}\eta_6, \quad \sigma_2 = \sqrt{2}\eta_7.$$

The Ricci tensor is also written out in [7]. It obviously has five terms:

$$\text{Ric} = R_0 dt^2 + R_a |\sigma_\beta|^2 + R_b |\Sigma_\beta|^2 + R_c |\nu|^2 + R_f \lambda^2,$$

and depends on four functions, which we refrain from recalling here.

Observe that for every dimension  $n$  the coefficients of (3) are of the same form, while the coefficients of (1) depend on  $n$  explicitly. Consequently, the required family of metrics must also depend on  $n$  explicitly and become (3) for  $\alpha = 1$ . We seek the metrics

$$\frac{dr^2}{W^2} + \frac{W^2 r^2}{4} \lambda^2 + r^2 (\nu_1^2 + \nu_2^2) + \frac{(r^2 - \alpha^2)}{2} (\sigma_{1\beta}^2 + \sigma_{2\beta}^2) + \frac{(r^2 + \alpha^2)}{2} (\Sigma_{1\beta}^2 + \Sigma_{2\beta}^2),$$

where  $W = W(r, \alpha, n)$  is an unknown function. When we insert the corresponding functions into the expression for the Ricci tensor, the terms  $R_a$ ,  $R_b$ , and  $R_c$  become

$$R_a = -\frac{2Q}{(r^2 - \alpha^2)^2(r^2 + \alpha^2)}, \quad R_b = \frac{2Q}{(r^2 + \alpha^2)^2(r^2 - \alpha^2)}, \quad R_c = -\frac{2Q}{r^2(r^4 - \alpha^4)},$$

where  $Q = \frac{dW}{dr}(r^5 - r\alpha^4) + 4W^2\alpha^4 + 4(n+1)(r^4 - \alpha^4 - r^4W^2)$ , and we substitute  $\frac{dr}{dt} = W$ . This equation can be integrated without effort:

$$W^2 = \frac{(r^4 - \alpha^4)^{n+1} + C}{r^4(r^4 - \alpha^4)^n},$$

where  $C$  is a constant of integration, which we can fix by a translation with respect to  $r$  and should put equal to  $-(1 - \alpha^4)^{n+1}$ ; then  $r \geq 1$  and  $W(1) = 0$ . When we use this function, the terms  $R_0$  and  $R_f$ , which are of the second order in  $W$ , automatically vanish.

Furthermore, consider the 2-form

$$\Omega = r dr \wedge \lambda + 2r^2 \nu_1 \wedge \nu_2 - (r^2 + \alpha^2) \Sigma_{1\beta} \wedge \Sigma_{2\beta} + (r^2 - \alpha^2) \sigma_{1\beta} \wedge \sigma_{2\beta}.$$

Using the exterior algebra relations of [7], we can easily verify that this form is closed, and up to the multiplication by  $\frac{1}{2}$  it is the Kähler form of the metric (2). Since the Ricci tensor vanishes and the Kähler form  $\Omega$  is closed, the claim of the theorem about holonomy follows.

Let us study the topology of the spaces on which our metrics are defined, carrying over the construction of [4] to a higher-dimensional case. Consider the complex space  $\mathbb{C}^{n+2}$  with the diagonal action of the circle  $S^1$ . This action determines equivalence classes, which we denote by brackets, for instance  $[u]$  and  $[V]$ , where  $u$  and  $V$  is a vector and a subspace.

Consider the space

$$\tilde{H} = \{(u_1, u_2, V) \mid |u_1| = 1, u_1 \perp_{\mathbb{C}} u_2 \perp_{\mathbb{C}} V\} \subset S^{2n+3} \times \mathbb{C}^{n+2} \times G_n(\mathbb{C}^{n+2})$$

and the projection  $\tilde{\pi} : (u_1, u_2, V) \rightarrow (u_1, V)$  of  $\tilde{H}$  onto the space  $\tilde{F} = \{(u_1, V) \mid |u_1| = 1, u_1 \perp_{\mathbb{C}} V\}$ . We can identify the total space of the spherical subbundle  $\tilde{H}^1 = \{(u_1, u_2, V) \mid |u_i| = 1, u_1 \perp_{\mathbb{C}} u_2 \perp_{\mathbb{C}} V\}$  with  $SU(n+2)/SU(n)$ . Via the diagonal action of  $S^1$  the bundle  $\tilde{\pi}$  generates the complex line bundle

$$\pi : H = \tilde{H}/S^1 \rightarrow F = \tilde{F}/S^1.$$

The spherical subbundle  $\pi$  coincides with the mapping

$$\pi^1 : H^1 = SU(n+2)/S[U(n) \times U(1)] \rightarrow F = SU(n+2)/T,$$

where

$$T = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & \bar{z} \det \bar{A} & 0 \\ 0 & 0 & A \end{pmatrix} \mid z \in U(1), A \in U(n) \right\}.$$

Observe that  $H^1$  is a 3-Sasakian manifold coinciding with the Aloff–Wallach space for  $n = 1$ , while  $\pi^1$  makes it a bundle over the corresponding twistor space  $\mathcal{Z} = F$ , which is the space of complex flags

$$\{([u], V) \mid u \in S^{2n+3}, V \in G_n(\mathbb{C}^{n+2}), u \perp_{\mathbb{C}} V\}$$

in  $\mathbb{C}^{n+2}$ . It is not difficult to verify that in  $\bar{G}_\alpha$  the length of the characteristic vector field corresponding to the form  $\eta_1$  at the initial moment of time  $r = 1$  is equal to  $2(n+1)$ . In order to define  $\bar{G}_\alpha$  correctly we must mod out the circle generated by the corresponding characteristic vector field by the discrete subgroup  $\mathbb{Z}_{n+1}$  since in the 3-Sasakian bundle  $\pi^1$  we already “made” the corresponding factorization modulo  $\mathbb{Z}_2$  (see [1]). Consequently, the metrics  $\bar{G}_\alpha$  for  $0 \leq \alpha < 1$  are defined on the tensor power  $\pi^{n+1}$ .

The proof of the theorem is complete.

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