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EQUIVALENCE OF THE CATEGORY OF PRECUBICAL SETS AND THE CATEGORY OF TRANSITIONAL CHU SPACES WITH PRESERVATION OF THE OPENNESS PROPERTY OF MORPHISMS

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We use directed algebraic topology for establishing categorical relationships between two geometrical models of concurrency, precubical sets and transitional Chu spaces. We study the universal dicovering functor from the category of precubical sets to the category of simply diconnected counterparts of precubical sets. We prove that the category of transitional Chu spaces is equivalent to the category of simply diconnected precubical sets and show that the openness of morphisms is preserved. Bibliography: 19 titles. Illustrations: 7 figures.

Introduction

Directed algebraic topology (ditopology) [1] deals with directed toplogical spaces, i.e., spaces with a distinguished direction (order) and continuous mappings preserving the direction. Directed paths (dipaths), unlike usual paths, cannot be reversible. Many concepts of algebraic theory have been extended to directed algebraic theory (cf., for example, [1, 2]).

Precubical sets (counterparts of semisymplectic sets in algebraic topology) form a family of sets of cubes of different dimension that are glued together along common faces. In concurrency theory, precubical sets are usually referred to as *higher dimensional automata* (cf. [3, 4]). The model of higher dimensional automata is the most expressive (cf. [5]) and, at the same time, the least studied structural model of concurrency. This model seems to be promised since it is represented via precubical sets and, consequently, tools of ditopology can be used for studying concurrent computations (cf. [6]). For example, two processes are concurrent if and only if the dipaths representing these processes are dihomotopic. A homological approach was used in [7]-[10], where precubical sets were presented as algebraic complexes.

Chu spaces [11, 12] are topological spaces equipped with a set of points, a set of open sets, and the membership relation. In concurrency theory, Chu spaces are formalized up to an isomorphism [13] by the so-called configuration structures.

In this paper, we use methods of ditopology for establishing categorical relationships between

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two geometric models of concurrent processes: precubical sets and transitional Chu spaces. We clarify relationships between the model of higher dimensional automata and configuration structures and thereby develop concurrency semantics of higher dimensional automata in terms of configuration structures. In particular, we construct categories of models under consideration and subcategories of dipaths. We define and study the universal dicovering functor from the category of precubical sets to the category of their simply diconnected counterparts. This functor is the right conjugate of the inclusion functor. We also establish that the category of transitional Chu spaces is equivalent to the category of simply diconnected precubical sets and show that the openness property of morphisms is preserved.

1 Precubical Sets and Cubical Paths

In this section, we introduce the category **PS** of precubical sets and the subcategory **cP** of cubical paths.

Definition 1. A (nondegenerate) *precubical set* M is a collection of sets of cubes $(M_n)_{n\in\mathbb{N}}$ such that $M_n \cap M_{n'} = \varnothing$ $(n \neq n')$ and, together with boundary function $M_n \overset{d_i^0}{\underset{d_j^1}{\rightrightarrows}} M_{n-1}$ $(n \in \mathbb{N}$, $i, j = 1 \dots n)$ such that $|\{d_i^m(x) \mid i = 1 \dots n\}| = n$ for any $m = 0, 1$ and $x \in M_n$, satisfy the commutativity of the following diagram:

$$\begin{array}{ccc} M_n & \xrightarrow{d_j^m} & M_{n-1} \\ {\scriptstyle d_i^k}\downarrow & & \downarrow{\scriptstyle d_i^k} \\ M_{n-1} & \xrightarrow[d_{j-1}^m]{} & M_{n-2} \end{array}$$

for all $i < j$ and $k, m = 0, 1$.

Definition 2. A (*labeled*) *precubical set* (*with a labeled point*) is a triple $\mathrm{M} = (M, i_0, l_L)$ where M is a precubical set, $i_0 \in M_0$ is a labeled point, and $l_L : M_1 \to L$ is a *labeling function* from the set of 1-cubes to the set L of actions such that $l_L(d_i^0(x)) = l_L(d_i^1(x))$ $(i = 1, 2)$ for all $x \in M_2$[1].

As was already mentioned, precubical sets are called higher dimensional automata [3, 4] and present the most powerful structural model [5] of concurrency theory. Using higher dimensional automata, it is possible to simulate concurrency in a natural way: the concurrent execution of n actions is represented by an n-dimensional cube, whereas the sequential execution of the same actions is represented by the edges of this cube. An example of higher dimensional automata is represented in Figure 1. The concurrent execution of two actions a and b which is simulated by the 2-cube x (the grey square) whose boundaries are the 1-cubes (the segments) x_1, y_2, y_1, x_2 is shown at the right in Figure 1. The square boundaries are presented by functions of two types: the source functions d_1^0, d_2^0 and the target functions d_1^1, d_2^1 (in a certain sense, $x_2 = d_1^1(x)$ and $y_2 = d_2^1(x)$ are copies of $x_1 = d_1^0(x)$ and $y_1 = d_2^0(x)$ respectively). Then the boundary functions

[1] The extension of l_L to any $x \in M_n$ is defined by $l_L(x) = \varnothing$ for $n = 0$ and $l_L(x) := (l_L(y_1), \dots, l_L(y_n))$ for $n > 1$, where $y_i = d_1^0 \circ \dots d_{i-1}^0 \circ d_{i+1}^0 \circ \dots \circ d_n^0(x)$ for all $1 \leqslant i \leqslant n$.

determine a direction. For example, the 1-cube x_1 starts at the 0-cube (the point) $d_1^0(x_1) = i_0$ and ends up with the 0-cube $d_1^1(x_1) = s$. The sequential execution of two actions a and b which is simulated by the higher dimensional automaton constructed from the 1-cubes x_1, y_2 and y_1, x_2 is shown at the left in Figure 1. The processes at the left and right start at a fixed point i_0.

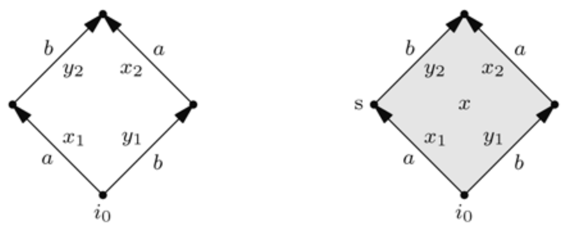


Figure 1. An example of the sequential (at the left) and concurrent (at the right) executions of actions a and b in a higher dimensional automaton.

We introduce the relation $\sim_\square \in (M_1)^2$ as the minimal equivalence such that the existence of $y \in M_2$ with $x_1 = d_i^k(y)$ and $x_2 = d_i^{1-k}(y)$ for some $i = 1, 2$ and $k = 0, 1$ implies the equivalence of x_1 and x_2. Let $\ll x \gg$ be the equivalence class $\sim_\square$. Informally speaking, all $x_1 \in \ll x \gg$ simulate the same event of a concurrent process.

We introduce the mappings $D^0, D^1 : M_n \to M_0$, $n \geqslant 0$, by $D^0(x) = d_1^0 \circ \ldots \circ d_n^0(x)$ and $D^1(x) = d_1^1 \circ \ldots \circ d_n^1(x)$ respectively for all $x \in M_n$.

In what follows, we consider only precubical sets with $x_1, \ldots x_n \in M_1$ and $y \in M_n$ satisfying the following axioms.

A0 If $d_1^0(x_1) = d_1^0(x_2)$ and $x_1 \sim_\square x_2$, then $x_1 = x_2$.

A1 If $d_1^1(x_1) = d_1^0(x_2)$, $d_1^1(x_2) = d_1^0(x_3)$, …, $d_1^1(x_{n-1}) = d_1^0(x_n)$ and $x_i \sim_\square d_1^{m_1^i} \circ \ldots \circ d_{\sigma(i)-1}^{m^i_{\sigma(i)-1}} \circ d_{\sigma(i)+1}^{m^i_{\sigma(i)+1}} \circ \ldots \circ d_n^{m_n^i}(y)$ for all $1 \leqslant i \leqslant n$, then there is a unique cube $x \in M_n$ such that $x_i = d_1^{m_1^i} \circ \ldots \circ d_{\sigma(i)-1}^{m^i_{\sigma(i)-1}} \circ d_{\sigma(i)+1}^{m^i_{\sigma(i)+1}} \circ \ldots \circ d_n^{m_n^i}(x)$ for all $1 \leqslant i \leqslant n$, where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation of order n. Here, $m_j^i = 1$ if $j = \sigma(k)$ and $k < i$; otherwise, $m_j^i = 0$ for all $1 \leqslant i, j \leqslant n$.

Definition 3. Let $\mathrm{M}^1 = (M^1, i_0^1, l_{L^1}^1)$ and $\mathrm{M}^2 = (M^2, i_0^2, l_{L^2}^2)$ be precubical sets. A mapping $\mathrm{f} = \langle f, \alpha \rangle$, where $f = \cup f_n$, $f_n : (M^1)_n \to (M^2)_n$, and $\alpha : L^1 \to L^2$, is called a *morphism* from M^1 to M^2 if the following conditions are satisfied:

(1) $f_0(i_0^1) = i_0^2$, (2) $l_{L^2}^2 \circ f = \alpha \circ l_{L^1}^1$, (3) $f_n \circ d_i^m = d_i^m \circ f_{n+1}$.

Precubical sets, together with morphisms between precubical sets, form the category **PS**.

A *cubical path* in a precubical set M is a sequence $P = p_0 p_1 \ldots p_k$ of cubes such that $p_{s-1} = d_i^0(p_s)$ or $p_s = d_j^1(p_{s-1})$ for all $s = 1 \ldots k$ and, in addition, $p_0 = i_0$. We denote by $\mathscr{CP}(\mathrm{M})$ ($\mathscr{CP}_u(\mathrm{M})$) the set of all cubical paths (ending with a cube $u \in M$) in a precubical set M. Let $\mathscr{CP}_1(\mathrm{M})$ be the set of *one-dimensional cubical paths*, i.e., cubical paths $P = p_0 p_1 \ldots p_k$ in M such that p_i is either a 0-cube or a 1-cube for any $0 \leqslant i \leqslant k$. A cubical path $Q = q_0 \ldots q_n$ is an *extension* of a cubical path $P = p_0 \ldots p_k$ (denoted by $P \to Q$) if $n \geqslant k$ and $p_0 \ldots p_k = q_0 \ldots q_k$. In particular, we write $P \xrightarrow{d_i^m} Q$ if $n = k + 1$ and $q_k = d_i^0(q_{k+1})$ for $m = 0$ or $q_{k+1} = d_i^1(q_k)$ for $m = 1$. A cubical path P is *acyclic* if it does not contain the same cubes and precubes.

On the analogy of the notion of a path homotopy in algebraic topology, the notion of a dihotomy of cubical paths is introduced with taking into account that cubical sets are always directed (i.e., there is an order). A dihomotopy (denoted by $\sim$) is the least equivalence on cubical paths in a precubical set M such that if cubical paths P and P' are *s-adjacent* (denoted by $P \overset{s}{\leftrightarrow} P'$), i.e., P' can be obtained from P by replacing (for some $i < j$ and $m = 0, 1$) either $\xrightarrow{d_i^0} p_s \xrightarrow{d_j^m}$ with $\xrightarrow{d_{j-1}^m} p'_s \xrightarrow{d_i^0}$ (or vice versa) or $\xrightarrow{d_j^m} p_s \xrightarrow{d_i^1}$ with $\xrightarrow{d_i^1} p'_s \xrightarrow{d_{j-1}^m}$ (or vice versa), then P and P' are equivalent. Furthermore, cubical paths P and P' are said to be (s, u, v)-*adjacent* (denoted by $P \overset{(s,u,v)}{\longleftrightarrow} P'$) if P' can be obtained from $P = \widehat{p}_0 \dots \widehat{p}_k$ by an s-adjacency replacement of its segment $\xrightarrow{d_u^n} \widehat{p}_s \xrightarrow{d_v^l}$. For every cubical path P we denote by $[P]$ its dihomotopy class.

We consider a precubical set M in Figure 2. The set M consists of the 3-cube, the 2-cube convoluted to a cylinder, and all subcubes obtained under the action of the boundary functions. For an example of a cubical path we can consider the sequence $P = i_0p_1p_2p_3p_4p_5p_6p_7p_8p_7$. The cubical paths P and $Q = i_0p_1p_2q_1q_2p_5p_6p_7p_8p_7$ are dihomotopic since $P \overset{4}{\leftrightarrow} (i_0p_1p_2q_1p_4p_5p_6p_7p_8p_7) \overset{5}{\leftrightarrow} Q$. The cubical path $i_0p_1p_2p_3p_4p_5$ is an example of an acyclic cubical path.

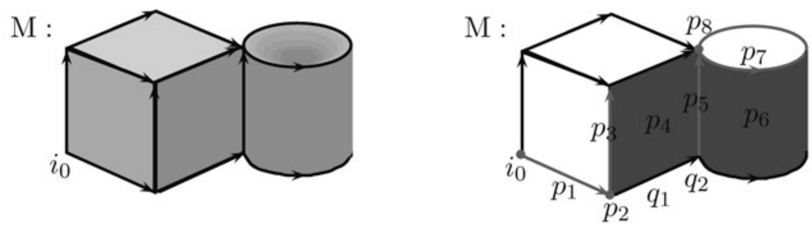


Figure 2. An example of a cubical path in a precubical set M.

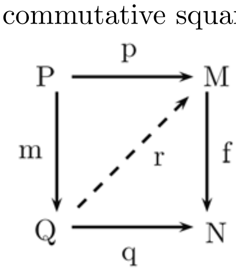
A *cubical path object* is a precubical set having shape of an acyclic cubical path. Let **cP** denote the full subcategory of the category **PS** consisting of cubical path objects.

We endow the category **PS** with a fibred structure. We denote by $\mathbf{PS}_L$ the subcategory whose objects are precubical sets with the same set L of actions and morphisms have the identity second component.

2 Open Morphisms

The notion of an open morphism is important in category theory [14] and, as shown in [15], can be used for defining bisimulation of models in different categories.

Definition 4. A morphism $\mathrm{f} : \mathrm{M} \to \mathrm{N}$ of a category **M** is **P**-*open* if for any morphism $\mathrm{m} : \mathrm{P} \to \mathrm{Q}$ of a subcategory **P** and a commutative square



there is a morphism $\mathrm{r} : \mathrm{Q} \to \mathrm{M}$ dividing the diagram into two commutative triangles.

We note that the objects of **M** and **P**-open morphisms form a subcategory of **M** since the identity morphism and compositions of **P**-open morphisms are **P**-open morphisms.

Theorem 1. *Let* M *be a precubical set. A morphism* $\mathrm{f} = \langle f, 1_L\rangle : \mathrm{M} \to \mathrm{M}'$ *of the category* $\mathbf{PS}_L$ *is* $\mathbf{cP}_L$*-open if and only if for any cubical path* P *in* M *the following conditions are satisfied*:

(1) *if* $f(P) \longrightarrow Q'$ *in* M', *then* $P \longrightarrow P'$ *and* $f(P') = Q'$ *for some cubical path* P' *in* M,

(2) *if* $f(P) \sim Q'$ *in* M', *then* $P \sim P'$ *and* $f(P') = Q'$ *for some cubical path* P' *in* M.

Proof. *Necessity.* Let $\mathrm{f} = \langle f, 1_L\rangle : \mathrm{M} \to \mathrm{M}'$ be a $\mathbf{cP}_L$-open morphism. We prove only assertion (1) since (2) is proved in a similar way. Let P be a cubical path in M. Without loss of generality we can assume that $f(P) \xrightarrow{d_i^l} Q'$ in M'. We consider a precubical subset P (Q') of the shape of a cubical path P (Q') in M (M'). It is clear that there is a cubical path object $\widehat{\mathrm{P}}$ ($\widehat{\mathrm{Q}}$) and the corresponding maximal[2] cubical path $\widehat{P}$ ($\widehat{Q}$) such that the mapping p (q) acting by the rule $p(\widehat{P}) = P$ ($q(\widehat{Q}) = Q'$) can be extended to a morphism $\mathrm{p} = \langle p, 1_L\rangle : \widehat{\mathrm{P}} \to \mathrm{M}$ ($\mathrm{q} = \langle q, 1_L\rangle : \widehat{\mathrm{Q}} \to \mathrm{M}'$) in $\mathbf{PS}_L$. We note that the morphism has the form $\mathrm{p} : \widehat{\mathrm{P}} \to \mathrm{P} \hookrightarrow \mathrm{M}$ ($\mathrm{q} : \widehat{\mathrm{Q}} \to \mathrm{Q}' \hookrightarrow \mathrm{M}'$).

Assume that $\widehat{P} = p_0 \dots p_k$ and $\widehat{Q} = q_0 \dots q_k q_{k+1}$. Then we set $m(p_i) = q_i$ and $m(d_{j_1}^{\varepsilon_1} \circ \dots \circ d_{j_s}^{\varepsilon_s}(p_i)) = d_{j_1}^{\varepsilon_1} \circ \dots \circ d_{j_s}^{\varepsilon_s}(q_i)$ for all $\varepsilon_r = 0, 1$, $r = 1 \dots s$, $1 \leqslant j_1 < \dots < j_s \leqslant \dim p_i$, $1 \leqslant s \leqslant \dim p_i$, and $0 \leqslant i \leqslant k$. It is clear that $\mathrm{m} = \langle m, 1_L\rangle : \widehat{\mathrm{P}} \to \widehat{\mathrm{Q}}$ is a morphism in $\mathbf{cP}_L$. By the definition of m, we find $\mathrm{f} \circ \mathrm{p} = \mathrm{q} \circ \mathrm{m}$.

Since f is a $\mathbf{cP}_L$-open morphism, there is a morphism $\mathrm{r} : \widehat{\mathrm{Q}} \to \mathrm{M}$ such that $\mathrm{p} = \mathrm{r} \circ \mathrm{m}$ and $\mathrm{q} = \mathrm{f} \circ \mathrm{r}$. Consequently, there exists a cubical path $r(\widehat{Q})$ in M. It is easy to see that $m(\widehat{P}) = (q_0 \dots q_k) \xrightarrow{d_i^l} (q_0 \dots q_{k+1}) = \widehat{Q}$ in view of the definition of $\xrightarrow{d_i^l}$. Hence we can write $r(m(\widehat{P})) \xrightarrow{d_i^l} r(\widehat{Q})$ because r is a morphism in $\mathbf{PS}_L$. Since $\mathrm{p} = \mathrm{r} \circ \mathrm{m}$ and $\mathrm{q} = \mathrm{f} \circ \mathrm{r}$, we have $p(\widehat{P}) = P \xrightarrow{d_i^l} r(\widehat{Q})$ and $f(r(\widehat{Q})) = q(\widehat{Q}) = Q'$.

Sufficiency. Let $\mathrm{f} = \langle f, 1_L\rangle : \mathrm{M} \to \mathrm{M}'$ be a morphism in $\mathbf{PS}_L$, and let the assumptions of the theorem be satisfied. We show that f is $\mathbf{cP}_L$-open.

We introduce additional notions. Let O_1 and O_2 be cubical path objects. A morphism $\iota_{\mathrm{l(w)}} = \langle \iota_{l(w)}, 1_L\rangle : \mathrm{O}_1 \to \mathrm{O}_2$ is called the *l-step* (the *w-step*) if there are maximal cubical paths O_1 and O_2 in O_1 and O_2 respectively such that $\iota_l(O_1) \xrightarrow{d_i^m} \xrightarrow{d_i^m} O_2$ ($\iota_w(O_1) \xleftrightarrow{(s,u,v)} O_2$).

It is easy to see that any morphism $\langle m, 1_L\rangle$ of the category $\mathbf{cP}_L$ is a finite (say, of length n) composition of an isomorphism, l-steps, and w-steps. By induction on n, it suffices to prove that f is a $\mathbf{cP}_L$-open morphism relative to the isomorphism, l-steps or w-steps. Let, for example, $\mathrm{m} = \iota_{\mathrm{w}} : \mathrm{P} \to \mathrm{Q}$ be a w-step (the remaining cases are treated in a similar and even simpler way). Then there are maximal cubical paths P and Q of the cubical path objects P and Q such that $\iota_w(P) \xleftrightarrow{(s,u,v)} Q$. We consider arbitrary morphisms $\mathrm{p} : \mathrm{P} \to \mathrm{M}$ and $\mathrm{q} : \mathrm{Q} \to \mathrm{M}'$ in $\mathbf{PS}_L$ such that $\mathrm{f} \circ \mathrm{p} = \mathrm{q} \circ \iota_{\mathrm{w}}$. Since q is a morphism, we have $q(\iota_w(P)) \xleftrightarrow{(s,u,v)} q(Q)$ in M'. Since $f(p(P)) = q(\iota_w(P))$, from the assumptions of the theorem it follows that there is a cubical path

[2] By the *maximal cubical path* of a cubical path object we mean any acyclic cubical path such that this cubical path object has the same shape.

P' in M such that $p(P) \overset{(s,u,v)}{\longleftrightarrow} P'$ and $f(P') = q(Q)$. We introduce a mapping r by $r(Q) = P'$ and extend r in such a way that the pair of mappings $\langle r, 1_L \rangle$ satisfies the morphism conditions in $\mathbf{PS}_L$. It is obvious that $\mathrm{p} = \mathrm{r} \circ \iota_{\mathrm{w}}$ and $\mathrm{q} = \mathrm{f} \circ \mathrm{r}$. Consequently, f is a $\mathbf{cP}_L$-open morphism. □

In [15], the notion of a "span" of **P**-open morphisms was introduced for defining a **P**-bisimulation on objects of a category **M**.

Definition 5. Two objects M′ and M″ of a category **M** are **P**-*bisimular* if there exists a construction of **P**-open morphisms $\mathrm{M}' \overset{\mathrm{f}'}{\longleftarrow} \mathrm{M} \overset{\mathrm{f}''}{\longrightarrow} \mathrm{M}''$.

The notion of a bisimulation plays an important role in concurrency theory. A categorical (in terms of "spans" of open morphisms) characterization of bisimulation can be found in [15]–[17].

3 Universal Dicover of Precubical Sets

Definition 6. A precubical set M is *simply diconnected* if for any $u \in M$ (1) there is a cubical path $P \in \mathscr{CP}_u(\mathrm{M})$ and (2) $P \sim Q$ for any cubical paths $P,Q \in \mathscr{CP}_u(\mathrm{M})$.

Let **oPS** denote the full subcategory of **PS** with simply diconnected precubical sets for objects. It is clear that **cP** is a subcategory of **oPS**.

For a cubical path $P \in \mathscr{CP}_{p_k}(\mathrm{M})$ with $\dim p_k > 0$ the *i-start* $d_i^0(P)$ is a cubical path in $\mathscr{CP}_{d_i^0(p_k)}(\mathrm{M})$ such that (i) $P = d_i^0(P)p_k$ or (ii) $P \overset{m+1}{\longleftrightarrow} P_1 \overset{m+2}{\longleftrightarrow} \ldots \overset{k-2}{\longleftrightarrow} P_{k-m-2} \overset{k-1}{\longleftrightarrow} d_i^0(P)p_k$ for some $0 \leqslant m \leqslant (k-2)$ and the *i-end* $d_i^1(P)$ is a cubical path from $\mathscr{CP}_{d_i^1(p_k)}(\mathrm{M})$ such that $d_i^1(P) = P d_i^1(p_k)$.

The following assertion is obtained from [5, p. 280–281].

Lemma 1. *For any cubical path* $P \in \mathscr{CP}_{p_k}(\mathrm{M})$ *with* $\dim p_k > 0$ *there exists a unique cubical path* $d_i^l(P) \in \mathscr{CP}_{d_i^l(p_k)}(\mathrm{M})$ *for any* $l = 0, 1$.

Definition 7. Let $\mathrm{M} = (M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$ be a precubical set. A *universal dicover* of M is a mapping $\rho_{\mathrm{M}} = \langle \rho_M, 1_L \rangle : \mathscr{U}(\mathrm{M}) \to \mathrm{M}$ such that $\rho_M([p_0 \ldots p_k]) = p_k$ for all $[p_0 \ldots p_k] \in A$, where the dicovering $\mathscr{U}(\mathrm{M}) = (A, i_0, l_L)$ of M is defined by

$$A_n = \{[P = p_0 \ldots p_k] \mid P \in \mathscr{CP}(\mathrm{M}),\ p_k \in M_n\}, \quad n \geqslant 0, \quad \text{and} \quad \widetilde{d_i^m}([P]) = [d_i^m(P)],$$

$$i_0 = [i_0^{\mathrm{M}}], \quad \text{and} \quad l_L([p_0 \ldots p_k]) = l_L^{\mathrm{M}}(p_k) \quad \text{for} \quad [p_0 \ldots p_k] \in A_1.$$

An example of a universal dicover $\rho_{\mathrm{M}} : \mathscr{U}(\mathrm{M}) \to \mathrm{M}$ is shown in Figure 3.

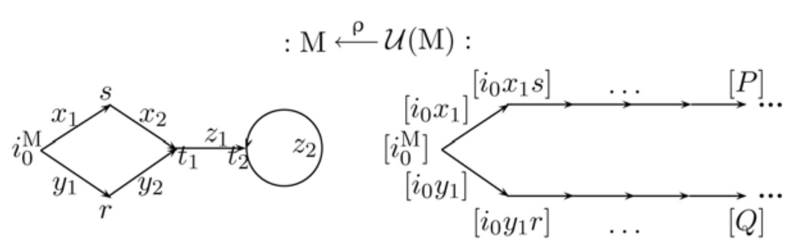


Figure 3. An example of a universal dicover.

The cubical paths $P = i_0x_1sx_2t_1z_1t_2z_2t_2z_2t_2$ and $Q = i_0y_1ry_2t_1z_1t_2z_2t_2z_2t_2$ are not dihomotopic in M. Hence the 0-cubes $[P]$ and $[Q]$ do not coincide in $\mathscr{U}(\mathrm{M})$, although $\rho_M([P]) = t_2 = \rho_M([Q])$.

Theorem 2. *Let* M *be a precubical set labeled by a set* L*, and let* $\mathscr{U}(\mathrm{M})$ *be the universal dicovering of* M*. Then the following assertions hold.*

1. $\mathscr{U}(\mathrm{M})$ *is an object of the category* **PS***, and* $\rho_{\mathrm{M}} = \langle \rho_M, 1_L\rangle : \mathscr{U}(\mathrm{M}) \to \mathrm{M}$ *is a* $\mathbf{cP}_L$*-open morphism of the category* $\mathbf{PS}_L$*, i.e.,* M *and* $\mathscr{U}(\mathrm{M})$ *are* $\mathbf{cP}_L$*-bisimular.*

2. $\mathscr{U}(\mathrm{M})$ *is a simply diconnected precubical set.*

3. $\mathscr{U}$ *is the right conjugate of the inclusion functor* $\iota : \mathbf{oPS} \hookrightarrow \mathbf{PS}$*, and, consequently,* **oPS** *is a coreflective subcategory of* **PS***.*

Proof. 1. We show the commutativity of thediagram in the definition of a precubical set. The condition concerning an n-cube starting and ending with n different faces is obvious for $\mathscr{U}(\mathrm{M})$. We have $\widetilde{d}_i^m(\widetilde{d}_j^k([P])) = [d_i^m(d_j^k(P))] = [d_{j-1}^k(d_i^m(P))] = \widetilde{d}_{j-1}^k(\widetilde{d}_i^m([P]))$ for $i < j$. We verify Axiom A0 (the case of Axiom A1 is similar). Assume that $[p_0 \dots p_k], [q_0 \dots q_m] \in A_1$, $\widetilde{d}_1^0([p_0 \dots p_k]) = \widetilde{d}_1^0([q_0 \dots q_m])$, and $[p_0 \dots p_k] \sim_\Box [q_0 \dots q_m]$. Then $d_1^0(p_k) = d_1^0(q_m)$ since the boundary functions are equal and $p_k \sim_\Box q_m$ by the definition of $\sim_\Box$. Since Axiom A0 is true for M, we have $p_k = q_m$. Thus, $[p_0 \dots p_k] = [d_1^0(p_0 \dots p_k)p_k] = [d_1^0(q_0 \dots q_m)p_k] = [d_1^0(q_0 \dots q_m)q_m] = [q_0 \dots q_m]$.

We show that ρ_{M} is a $\mathbf{cP}_L$-open morphism. Since it is a morphism, we can use Theorem 1. It suffices to verify only the condition (2), since the condition (1) is verified in a similar way. We consider an arbitrary cubical path $O = o_0 \dots o_k$ in $\mathscr{U}(\mathrm{M})$. Let $\rho_M(O) = p_0 \dots p_k$. By induction on k, it is easy to show that $o_i = [p_0 \dots p_i]$ for all $0 \leqslant i \leqslant k$. Let $\rho_M(O) \overset{(s,u,v)}{\longleftrightarrow} P'$ for some cubical path $P' = p_0 \dots p'_s \dots p_k$ in M. Then we set $o'_s = [p_0 \dots p_{s-1}p'_s]$. By the construction of $\mathscr{U}(\mathrm{M})$, we see that $O' = o_0 \dots o'_s \dots o_k$ is a cubical path in $\mathscr{U}(\mathrm{M})$ and $O \overset{(s,u,v)}{\longleftrightarrow} O'$. It is clear that $\rho_M(O') = P'$.

2. The first condition in the definition of simple diconectedness is obviously satisfied, whereas the second condition follows from Theorem 1 applied to ρ_{M}.

3. Suppose that M, N are precubical sets and $\mathscr{U}(\mathrm{M})$, $\mathscr{U}(\mathrm{N})$ are their universal dicoverings respectively. For a morphism $\mathrm{f} = \langle f, \alpha\rangle : \mathrm{M} \to \mathrm{N}$ we define $\mathscr{U}(\mathrm{f}) = \langle \mathscr{U}(f), \alpha\rangle : \mathscr{U}(\mathrm{M}) \to \mathscr{U}(\mathrm{N})$ by $\mathscr{U}(f)([P]) = [f(P)]$, where $P \in \mathscr{CP}(\mathrm{M})$. It is clear that $\mathscr{U}(\mathrm{f})$ is a morphism of the category **oPS**. Hence $\mathscr{U}$ is a functor.

We assume that M is an object of the category **PS** and O is an object of the category **oPS**. We need to find a bijection between morphisms $\mathrm{f} : \mathrm{O} \to \mathrm{M}$ of the category **PS** and morphisms $\mathrm{g} : \mathrm{O} \to \mathscr{U}(\mathrm{M})$ of the category **oPS** and show that this bijection is natural with respect to O and M as well. For a morphism $\mathrm{f} = \langle f, \alpha\rangle : \mathrm{O} \to \mathrm{M}$ we define the morphism $\varphi_{\mathrm{O,M}}(\mathrm{f}) = \langle \varphi_{\mathrm{O,M}}(f), \alpha\rangle : \mathrm{O} \to \mathscr{U}(\mathrm{M})$ by $\varphi_{\mathrm{O,M}}(f)(v) = [f(P_v)]$ for all $v \in O$, where $P_v \in \mathscr{CP}_v(\mathrm{O})$. For a morphism $\mathrm{g} = \langle g, \alpha\rangle : \mathrm{O} \to \mathscr{U}(\mathrm{M})$ we define the morphism $\psi_{\mathrm{O,M}}(\mathrm{g}) = \langle \psi_{\mathrm{O,M}}(g), \alpha\rangle : \mathrm{O} \to \mathrm{M}$ by $\psi_{\mathrm{O,M}}(\mathrm{g}) = \rho_{\mathrm{M}} \circ \mathrm{g}$. It is easy to show that $\varphi_{\mathrm{O,M}}$ and $\psi_{\mathrm{O,M}}$ are well defined.

Let us prove that $\varphi_{\mathrm{O,M}}$ is a bijection. We assume that $\mathrm{f}, \mathrm{f}' : \mathrm{O} \to \mathrm{M}$ are morphisms such that $\varphi_{\mathrm{O,M}}(\mathrm{f}) = \varphi_{\mathrm{O,M}}(\mathrm{f}')$. Then $[f(P_v)] = \varphi_{\mathrm{O,M}}(f)(v) = \varphi_{\mathrm{O,M}}(f')(v) = [f'(P_v)]$ for any $v \in O$ and $P_v \in \mathscr{CP}_v(\mathrm{O})$. Hence $f(v) = f'(v)$ for any v. Further, we consider a morphism $\mathrm{g} = \langle g, \alpha\rangle : \mathrm{O} \to \mathscr{U}(\mathrm{M})$ of the category **oPS**. We need to show that there exists a morphism $\mathrm{f} = \langle f, \alpha\rangle : \mathrm{O} \to \mathrm{M}$ of the category **PS** such that $\varphi_{\mathrm{O,M}}(\mathrm{f}) = \mathrm{g}$. We set $\mathrm{f} = \langle f, \alpha\rangle = \psi_{\mathrm{O,M}}(\mathrm{g})$. Indeed, if $P_v = p_0 \dots (p_k = v)$, then $\varphi_{\mathrm{O,M}}(f)(v) = [f(P_v)] = [\rho_M(g(P_v))] = [\rho_M(g(p_0)) \dots \rho_M(g(p_k))] = g(\rho_O([P_v])) = g(v)$. Consequently, $\varphi_{\mathrm{O,M}}$ is a bijection such that $\varphi^{-1}_{\mathrm{O,M}} = \psi_{\mathrm{O,M}}$.

We show that $\varphi_{\mathrm{O,M}}$ is the natural bijection with respect to O (the case of M is similar). For this purpose we consider the diagram

$$\begin{array}{ccc}
\mathbf{PS}(\mathrm{O},\mathrm{M}) & \xleftarrow{\ \varphi^{-1}_{\mathrm{O,M}}\ } & \mathbf{oPS}(\mathrm{O},\mathcal{U}(\mathrm{M})) \\
\downarrow{\scriptstyle (r)^*} & & \downarrow{\scriptstyle r^*} \\
\mathbf{PS}(\mathrm{O}',\mathrm{M}) & \xleftarrow[\ \varphi^{-1}_{\mathrm{O}',\mathrm{M}}\ ]{} & \mathbf{oPS}(\mathrm{O}',\mathcal{U}(\mathrm{M}))
\end{array}$$

for arbitrary objects O, O′ and an arbitrary morphism $\mathrm{r} : \mathrm{O}' \to \mathrm{O}$ of the category **oPS**.

The mappings r^* and $(r)^*$ are defined by $r^*(\mathrm{g}) = \mathrm{g} \circ \mathrm{r}$ for any morphism $\mathrm{g} : \mathrm{O} \to \mathscr{U}(\mathrm{M})$ and $(r)^*(\mathrm{f}) = \mathrm{f} \circ \mathrm{r}$ for any morphism $\mathrm{f} : \mathrm{O} \to \mathrm{M}$. The diagram is commutative. Indeed, $(r)^*(\varphi^{-1}_{\mathrm{O,M}}(\mathrm{g})) = (r)^*(\rho_{\mathrm{M}} \circ \mathrm{g}) = \rho_{\mathrm{M}} \circ \mathrm{g} \circ \mathrm{r}$. On the other hand, $\varphi^{-1}_{\mathrm{O}',\mathrm{M}}(r^*(\mathrm{g})) = \varphi^{-1}_{\mathrm{O}',\mathrm{M}}(\mathrm{g} \circ \mathrm{r}) = \rho_{\mathrm{M}} \circ \mathrm{g} \circ \mathrm{r}$. Thus, we conclude that $\mathscr{U} : \mathbf{PS} \to \mathbf{oPS}$ is the right conjugate of the inclusion functor $\mathbf{oPS} \hookrightarrow \mathbf{PS}$ (cf. [18]). □

4 Transitional Chu Spaces and Paths

In this section, we define the category **SCS** of transitional Chu spaces and the subcategory **P** of paths in Chu spaces. Denote by $\mathscr{L}(\mathscr{P}_{fin}(E))$ the set of all linear orders on finite subsets of a set E.

Definition 8. A (*labeled*) *transitional Chu space* is a triple $\mathrm{E} = (E, \diamond, l_L)$, where

- E is a set of *events*,
- $\diamond = \bigcup\limits_{n\geqslant 0} \diamond^n = \bigcup\limits_{n\geqslant 0, <\in\mathscr{L}(\mathscr{P}_{fin}(E))} \diamond^n_< \subseteq \mathscr{P}_{fin}(E) \times \mathscr{P}_{fin}(E)$ is the *transition relation*, i.e., if $F \diamond^n_< G$, then $F \subseteq G$, a linear order $<$ acts on the n-element set $G\backslash F$, and $F \diamond^m_{<|_{H\backslash F}} H \diamond^{n-m}_{<|_{G\backslash H}} G$ for all $F \subseteq H \subseteq G$, and
- $l_L : E \to L$ is a *labeling* mapping from the set of events to the set of actions.

In concurrency theory, such spaces are referred to as structures of events of higher dimension [19] and form a subclass of configuration structures [13].

We consider Chu spaces satisfying the following axioms.

B0 If $F \diamond$[3] $F \sqcup$[4]$\{e\}$ and $G \diamond G \sqcup \{e\}$, then

$$F \cap G \diamond (F \cap G) \sqcup \{e\} \diamond^{k_1}_{<_1} \begin{cases} \cdots \diamond^{k_m}_{<_m} F \sqcup \{e\}, \\ \cdots \diamond^{k^l}_{<_l} G \sqcup \{e\}. \end{cases}$$

B1 If $F \diamond F \sqcup \{e_1\} \diamond F \sqcup \{e_1, e_2\} \diamond \cdots \diamond F \sqcup \{e_1, \dots, e_n\}$ and $G \diamond^n_< G \sqcup \{e_1, \dots, e_n\}$, then $F \diamond^n_< F \sqcup \{e_1, \dots, e_n\}$.

[3] We omit the super(sub)script in the transitional relation $\diamond$ if no confusion arises.
[4] Hereinafter, $\sqcup$ denotes the union of disjoint sets.

A *path* in a transitional Chu space E is a sequence $F = (\varnothing \diamond^{k_1}_{<_1} F_1 \diamond^{k_2}_{<_2} \cdots \diamond^{k_n}_{<_n} F_n)$ in E. Let $\Pi(\mathrm{E})$ ($\Pi_{F_n}(\mathrm{E})$) denote the set of all paths (ending with a common subset F_n) in a transitional Chu space E, and let $\Pi_1(\mathrm{E})$ be the set of one-element paths in E, i.e., paths with $k_1 = \cdots = k_n = 1$. A *dihomotopy* on paths in E is a minimal equivalence such that if a path F is obtained from a path G by eliminating a single set that is neither starting nor ending for the path, then F and G are equivalent. For every path F we denote by $[F]$ its dihomotopy class.

In the general case, not all events and not all paris of points connected by the transition relation can be reached from the set $\{\varnothing\}$ by the relation $\diamond$. To avoid such a situation, we introduce a special subclass of transitional Chu spaces. We say that a transitional Chu space $\mathrm{E} = (E, \diamond, l_L)$ is *reachable* if for every event $e \in E$ there is a pair $(F, F) \in \diamond$ such that $e \in F$, for each pair $(F, F) \in \diamond$ there is a path ending at a subset of F, and all paths ending at the same subset are dihomotopic. In what follows, we consider only reachable transitional Chu spaces and call them transitional Chu spaces.

An example of a transitional Chu space E is presented in Figure 4. It consists of the set $E = \{e_1, \dots, e_6\}$, the transition relation

$$\begin{aligned}\diamond = (\diamond^3 = \{(\ \varnothing\}, \{e_1, e_2, e_3\})\}) \cup (\diamond^2 = \{(\{e_2, e_3\}, \{e_i \mid i = 1\dots 4\}), (\{e_1\}, \{e_1, e_2, e_6\}),\\ (\{e_1\}, \{e_1, e_3, e_6\}), (\{e_1\}, \{e_1, e_2, e_3\}), (\{e_1, e_6\}, \{e_i \mid i = 1\dots 4\}),\\ (\{e_1, e_2\}, \{e_i \mid i = 1\dots 4\}), (\{e_1, e_3\}, \{e_i \mid i = 1\dots 4\})\} \cup A)\\ \cup (\diamond^1 = \{(\{e_i \mid i = 1\dots 4\}, \{e_i \mid i = 1\dots 5\})\} \cup B),\end{aligned}$$

where the sets A and B consist of transition subrelations of $\diamond^3$ and $\diamond^2 \cup \diamond^3$ respectively, and the labeling mapping l_L acting by the rule $l_L(e_i) = a_i$ for all $1 \leqslant i \leqslant 6$. For an example of a path we can consider the sequence $F = (\varnothing \diamond^2 \{e_2, e_3\} \diamond^2 \{e_i \mid i = 1\dots 4\} \diamond^1 \{e_i \mid i = 1\dots 5\})$ in Figure 4. The paths F and $G = (\varnothing \diamond^3 \{e_1, e_2, e_3\} \diamond^1 \{e_i \mid i = 1\dots 4\} \diamond^1 \{e_i \mid i = 1\dots 5\})$ are dihomotopic via the path $\varnothing \diamond^2 \{e_2, e_3\} \diamond^1 \{e_i \mid i = 1\dots 3\} \diamond^1 \{e_i \mid i = 1\dots 4\} \diamond^1 \{e_i \mid i = 1\dots 5\}$.

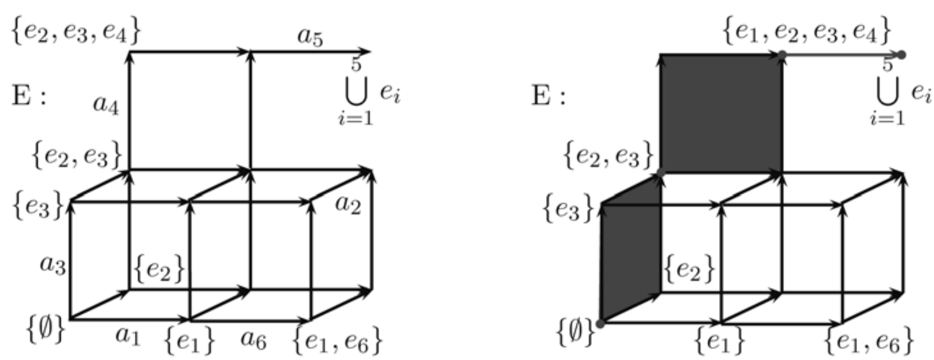


Figure 4. An example of a path in a transitional Chu space E.

Let $\mathrm{E}^1 = (E^1, \diamond^1, l^1_{L^1})$ and $\mathrm{E}^2 = (E^2, \diamond^2, l^2_{L^2})$ be transitional Chu spaces. A *morphism* from E^1 to E^2 is a pair of functions $f : E^1 \to E^2$ and $\alpha : L^1 \to L^2$ such that $\alpha \circ l^1_{L^1} = l^2_{L^2} \circ f$, $F(\diamond^1)^n_< G \Rightarrow f(F)(\diamond^2)^n_\prec f(G)$, $e_i < e_j \Rightarrow f(e_i) \prec f(e_j)$ for all $1 \leqslant i, j \leqslant n$ and $F(\diamond^1)^n_<(F \sqcup \{e_1, \dots, e_n\})$.

Let **SCS** denote the category of transitional Chu spaces with the above morphisms.

Since it is not essential what path leads to a prescribed subset of events, it is convenient to write the equivalence class $[F = (\varnothing \diamond^{k_1}_{<_1} F_1 \diamond^{k_2}_{<_2} \cdots \diamond^{k_n}_{<_n} F_n)]$ as $[F_n]$. Furthermore, for $[F_n]$ in a transitional Chu space E one can construct a *computation* $\mathrm{F_n}$, i.e., a transitional Chu subspace

E equipped with the set of events $F_n \subseteq E$, the transition relation, and a labeling mapping of E bounded on events in F_n.

Let **P** be the full subcategory of the category **SCS** with computations for objects. We can consider $\mathbf{P}_L$-open morphisms of the category $\mathbf{SCS}_L$ and $\mathbf{P}_L$-bisimulation on objects of $\mathbf{SCS}_L$.

5 Equivalence of the Categories oPS and SCS with Preserving the Openness of Morphisms

We begin by constructing mappings $\mathscr{F} : \mathbf{SCS} \to \mathbf{oPS}$ and $\mathscr{G} : \mathbf{oPS} \to \mathbf{SCS}$ and then prove that these mappings are functors. Using the functor $\mathscr{G}$, we show that $\mathscr{F}$ is a faithful, full, and dense functor, i.e., the categories **oPS** and **SCS** are equivalent (cf., for example, [18]).

Proposition 1. *Suppose that a mapping* $\mathscr{F} : \mathbf{SCS} \to \mathbf{oPS}$ *sends an object* $\mathrm{E} = (E, \diamond, l_L)$ *of the category* **SCS** *to an object* $\mathscr{F}(\mathrm{E}) = (M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$ *of the category* **oPS***, where*

- $M_n = \{(F,G)_< \mid F, G \in \mathscr{P}_{fin}(E), <\in \mathscr{L}(\mathscr{P}_{fin}(E)), F \diamond_<^n G\}$, $\widehat{d}_i^0(F,G)_< = (F, G \setminus pr_i(G \setminus F))_{<|_H}$ *and* $\widehat{d}_i^1(F,G)_< = (F \sqcup pr_i(G \setminus F), G)_{<|_H}$, *where* $pr_i(e_1 < \cdots < e_n) = e_i$ *and* $H = G \setminus (F \sqcup pr_i(G \setminus F))$,
- $i_{0_{\mathrm{M}}} = (\varnothing, \varnothing)$, *and* $l_L^{\mathrm{M}}(F, F \sqcup e) = l_L(e)$ *for all* $(F, F \sqcup e) \in \diamond^1$;

moreover, $\mathscr{F}$ *associates a morphism* $\mathrm{f} = \langle f, \alpha \rangle : \mathrm{E}^1 \to \mathrm{E}^2$ *of the category* **SCS** *with a morphism* $\mathscr{F}(\mathrm{f}) = \langle \widehat{f}, \alpha \rangle : \mathscr{F}(\mathrm{E}^1) \to \mathscr{F}(\mathrm{E}^2)$ *of the category* **oPS***, where* $\widehat{f}(F,G)_< = (f(F), f(G))_\prec$ *for all* $(F,G)_< \in (M^1)_n$*. Then* $\mathscr{F}$ *is a functor.*

Proof. We prove that $\mathscr{F}(\mathrm{E})$ is a precubical set. We show that the diagram in the definition of a precubical set is commutative for $k = m = 0$ (the remaining cases are considered in a similar way). Assume that $(F, G) \in M_n$ and $G \setminus F = e_1 < \cdots < e_n$. For $i < j$ we have

$$\begin{aligned}
\widehat{d}_i^0(\widehat{d}_j^0(F,G)) &= (F, (G \setminus pr_j(G \setminus F)) \setminus pr_i((G \setminus pr_j(G \setminus F)) \setminus F)) \\
&= (F, G \setminus (pr_j(e_1 < \cdots < e_n) \cup pr_i(e_1 < \cdots < e_{j-1} < e_{j+1} < \cdots < e_n))) \\
&= (F, G \setminus (e_j \cup e_i)) = (F, G \setminus (pr_i(e_1 < \cdots < e_n) \\
&\quad \cup pr_{j-1}(e_1 < \cdots < e_{i-1} < e_{i+1} < \cdots < e_n))) \\
&= (F, (G \setminus pr_i(G \setminus F)) \setminus pr_{j-1}((G \setminus pr_i(G \setminus F)) \setminus F)) = \widehat{d}_{j-1}^0(\widehat{d}_i^0(F,G)).
\end{aligned}$$

We verify Axiom A0 for $\mathscr{F}(\mathrm{E})$ (Axiom A1 is verified in a similar way). Assume that $(F, F \sqcup e_1), (G, G \sqcup e_2) \in M_1$, $\widehat{d}_1^0(F, F \sqcup e_1) = \widehat{d}_1^0(G, G \sqcup e_2)$, and $(F, F \sqcup e_1) \sim_\Box (G, G \sqcup e_2)$. Since the boundary functions are equal, we have $F = G$. The relation $\sim_\Box$ between 1-cubes implies $e_1 = e_2$. Thus, $(F, F \sqcup e_1) = (G, G \sqcup e_2)$.

We show that the second condition of Definition 6 is satisfied by $\mathscr{F}(\mathrm{E})$ (the condition (1) is verified in a similar way). Suppose that $\widehat{P}$, $\widehat{Q} \in \mathscr{CP}_{(F,G)_<}(\mathscr{F}(\mathrm{E}))$, where $(F,G)_< \in M_n$, $n \geqslant 0$. Without loss of generality we can assume that

$$\widehat{P} = P\widehat{d}_2^0 \circ \cdots \circ \widehat{d}_n^0(F,G)_< \ldots \widehat{d}_n^0(F,G)_<(F,G)_< ,$$

$$\widehat{Q} = Q\widehat{d}_2^0 \circ \cdots \circ \widehat{d}_n^0(F,G)_< \ldots \widehat{d}_n^0(F,G)_<(F,G)_< ,$$

where $P, Q \in \mathscr{C}\mathscr{P}_{1,(F,F)}(\mathscr{F}(\mathrm{E}))$. Suppose that

$$P = (\varnothing, \varnothing) \ldots (F_i, F_i)(F_i, F_{i+1})(F_{i+1}, F_{i+1}) \ldots (F, F),$$
$$Q = (\varnothing, \varnothing) \ldots (G_j, G_j)(G_j, G_{j+1})(G_{j+1}, G_{j+1}) \ldots (F, F).$$

By the definition of $\mathscr{F}$, there exist the preimages of P and Q:

$$P_1 = (\varnothing \diamond \cdots \diamond F_i \diamond \cdots \diamond F),$$
$$Q_1 = (\varnothing \diamond \cdots \diamond G_j \diamond \cdots \diamond F)$$

in $\Pi_{1,F}(\mathrm{E})$. Since the transitional Chu space E is reachable, P_1 and Q_1 are dihomotopic. It is easy to verify that, under the action of $\mathscr{F}$, dihomotopic paths in $\Pi_1(\mathrm{E})$ become dihomotopic cubical paths in $\mathscr{C}\mathscr{P}_1(\mathscr{F}(\mathrm{E}))$. Hence P and Q are also dihomotopic. Consequently, $\widehat{P}$ and $\widehat{Q}$. are dihomotopic.

The mapping $\mathscr{F}(\mathrm{f}) = \langle \widehat{f}, \alpha \rangle$ is a morphism of the category **oPS** since f is a morphism of the category **SCS**. Now, It is easy to see that $\mathscr{F}$ is a functor. □

Lemma 2. *Suppose that* M *is a simply diconnected precubical set and* $E = \{\ll x \gg \mid x \in M_1\}$. *Let a mapping* $\pi : M_0 \to \mathscr{P}_{fin}(E)$ *be defined as follows: for any point* $u \in M_0$ *we set* $\pi(u) = \{\ll x_0 \gg, \ldots, \ll x_k \gg\} \subseteq E$ *for some cubical path* $P = (p_0 x_0 p_1 \ldots p_k x_k u) \in \mathscr{C}\mathscr{P}_1(\mathrm{M})$. *Then the mapping* π *is injective.*

Proof. Since M is simply diconnected, it is obvious that π is a mapping.

Let $\pi(u) = \pi(v) = \{\ll x_0 \gg, \ldots, \ll x_k \gg\}$. We show that $u = v$ by induction on k. The case $k = -1$, i.e., $\pi(u) = \pi(v) = \varnothing$, is obvious. The case $\pi(u) = \pi(v) = \ll x \gg$ follows from Axiom A0 for M. Suppose that π is an injection for k. We show that the same is true for $k+1$. We consider the cubical paths $P = (p_0 x_0 p_1 \ldots p_k x_k u)$, $Q = (q_0 y_0 q_1 \ldots q_k y_k v) \in \mathscr{C}\mathscr{P}_1(\mathrm{M})$ and assume that $\pi(u) = \pi(v)$. It is clear that $\ll x_k \gg = \ll y_{\sigma(k)} \gg$, where $\sigma : \{0, \ldots, k\} \to \{0, \ldots, k\}$ is a permutation of order $k+1$. Without loss of generality we can assume that there exists a chain of 2-cubes $w_1, \ldots, w_l$ such that $x_k = d^{1-\varepsilon}_{i_1}(w_1)$, $d^{\varepsilon}_{i_1}(w_1) = d^{1-\varepsilon}_{i_2}(w_2), \ldots, d^{\varepsilon}_{i_{s_1-1}}(w_{s_1-1}) = d^{\varepsilon}_{i_{s_1}}(w_{s_1}), \ldots, d^{1-\varepsilon}_{i_{l-1}}(w_{l-1}) = d^{\varepsilon}_{i_l}(w_l), d^{1-\varepsilon}_{i_l}(w_l) = y_{\sigma(k)}$ for $\varepsilon = 0, 1$. We assume that $\varepsilon = 0$, i.e., there exists a unique $s_1 \in \{1, \ldots, l\}$ such that $d^0_1(w_{s_1-1}) = d^0_1(w_{s_1})$. Since M is a simply diconnected precubical set, there is a cubical path $T = (t_0 z_0 t_1 \ldots t_m z_m t_{m+1}) \in \mathscr{C}\mathscr{P}_1(\mathrm{M})$ and $t_{m+1} = d^0_1(d^0_2(w_{s_1}))$.

Since $[T d^0_{3-i_{s_1-1}}(w_{s_1-1}) \ldots d^0_{3-i_1}(w_1) p_k] = [p_0 x_0 p_1 \ldots p_k][T d^0_{3-i_{s_1}}(w_{s_1}) \ldots d^0_{3-i_l}(w_l) q_{\sigma(k)}] = [q_0 y_0 q_1 \ldots q_{\sigma(k)}]$ and $\pi(u) = \pi(v)$, we conclude that for any j ($\sigma(k) + 1 \leqslant j \leqslant k$) there is a number r_j ($1 \leqslant r_j \leqslant (s_1 - 1)$) such that $\ll y_j \gg = \ll d^0_{3-i_{r_j}}(w_{r_j}) \gg$. Using this argument and Axiom A1 for M, it is easy to prove by induction on $n = k - \sigma(k)$ that there is a cubical path $Q' = (q_0 y_0 q_1 \ldots q_{\sigma(k)} y'_{\sigma(k)} q'_{\sigma(k)+1} \ldots q'_k y'_k v)$ such that $Q \sim Q'$ and $y_{\sigma(k)} \sim_\square y'_k$. The cubical paths $\overline{P} = (p_0 x_0 p_1 \ldots p_k)$ and $\overline{Q} = (q_0 y_0 q_1 \ldots q_{\sigma(k)} y'_{\sigma(k)} q'_{\sigma(k)+1} \ldots q'_k)$ satisfy the induction assumption. Thus, $p_k = q'_k$. Furthermore, $x_k \sim_\square y_{\sigma(k)} \sim_\square y'_k$. By Axiom A0 for M, we find $x_k = y'_k$, i.e., $u = v$. Thus, π is injective. □

Proposition 2. *Suppose that a mapping* $\mathscr{G} : \mathbf{oPS} \to \mathbf{SCS}$ *sends an object* $\mathrm{M} = (M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$ *of the category* **oPS** *to an object* $\mathscr{G}(\mathrm{M}) = (E, \diamond, l_L)$ *of the category* **SCS**, *where*

- $E = \{\ll x \gg \mid x \in M_1\}$ *is a set of events,*
- $\diamond \subseteq \mathscr{P}_{fin}(E) \times \mathscr{P}_{fin}(E)$ *is defined as follows:* $F \diamond^n_{\prec} G \overset{def}{\Leftrightarrow}$ *there is a cube* $x_{(F,G,\prec)} \in M_n$ *such that* $F = \pi(D^0(x_{(F,G,\prec)}))$, $G = \pi(D^1(x_{(F,G,\prec)}))$, *and* $\prec = <_{x_{(F,G,<)}}$, *where* $(e, \epsilon) \in <_{x_{(F,G,<)}}$ *in* $G \setminus F \overset{def}{\Leftrightarrow} e = \ll d^0_1 \circ \cdots \circ d^0_{i-1} \circ d^0_{i+1} \circ \cdots \circ d^0_n(x_{(F,G,<)}) \gg$, $\epsilon = \ll d^0_1 \circ \cdots \circ d^0_{j-1} \circ d^0_{j+1} \circ \cdots \circ d^0_n(x_{(F,G,<)}) \gg$ *and* $i < j$,
- $l_L(\ll x \gg) = l^{\mathrm{M}}_L(x)$ *for all* $x \in M_1$;

moreover, $\mathscr{G}$ *associates a morphism* $\mathrm{g} = \langle g, \alpha \rangle : \mathrm{M}^1 \to \mathrm{M}^2$ *of the category* **oPS** *with a morphism* $\mathscr{G}(\mathrm{g}) = \langle \widehat{g}, \alpha \rangle : \mathscr{G}(\mathrm{M}^1) \to \mathscr{G}(\mathrm{M}^2)$ *of the category* **SCS**, *where* $\widehat{g}(\ll x \gg) = \ll g(x) \gg$ *for all* $\ll x \gg \in E^1$. *Then* $\mathscr{G}$ *is a functor.*

Proof. Let M be a simply diconnected precubical set. We show that $\mathscr{G}(\mathrm{M})$ is a transitional Chu space. By definition, there exists $x_{(F,G,<)} \in M_n$ such that $F = \pi(D^0(x_{(F,G,<)}))$ and $G = \pi(D^1(x_{(F,G,<)}))$, i.e., $F \subseteq G$. Let $F \subseteq H \subseteq G$, and let $G \setminus F = (e_1 < \ldots < e_n)$, where $e_i = \ll d^0_1 \circ \cdots \circ d^0_{i-1} \circ d^0_{i+1} \circ \cdots \circ d^0_n(x_{(F,G,<)}) \gg$ for all $1 \leqslant i \leqslant n$, and $H \setminus F = (e_{i_1} < \ldots < e_{i_m})$. Then the m-cube $d^0_{j_1} \circ \ldots \circ d^0_{j_{n-m}}(x_{(F,G,<)})$ corresponds to $F \diamond^m_{<|_{H \setminus F}} H$ and the $(n-m)$-cube $d^1_{i_1} \circ \ldots \circ d^1_{i_m}(x_{(F,G,<)})$ corresponds to $H \diamond^{(n-m)}_{<|_{G \setminus H}} G$, where the ordered sequences $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_{n-m})$ form the ordered sequence $(1, \ldots, n)$. We note that $x_{(F,G,<)}$ is a unique cube corresponding to the transition relation $F \diamond^n_< G$ in view of Axioms A0 and A1 for M.

Let us show that Axiom B0 is satisfied by $\mathscr{G}(\mathrm{M})$. Suppose that $F \diamond F \sqcup \{e\}$ and $G \diamond G \sqcup \{e\}$. Without loss of generality we can assume that there exist different $z^1_1, \ldots z^1_{k^1}, z^2_1, \ldots z^2_{k^2} \in M_2$ such that $x_{(F,F \sqcup e)} = d^{1-m}_{i_{k^1}}(z^1_{k^1})$, $d^m_{i_{k^1}}(z^1_{k^1}) = d^{1-m}_{i_{k^1-1}}(z^1_{k^1-1}), \ldots, d^m_{i_1}(z^1_1) = d^m_{j_1}(z^2_1), \ldots, d^{1-m}_{j_{k^2-1}}(z^2_{k^2-1}) = d^m_{j_{k^2}}(z^2_{k^2}), d^{1-m}_{j_{k^2}}(z^2_{k^2}) = x_{(G,G \sqcup e)}$ for some $m = 0, 1$. Arguing in the same way as in the proof of Lemma 2, we verify Axiom B0 by induction on the number of elements of the set $(F \cap G) \setminus S$ if $m = 0$ or the set $S \setminus (F \cap G)$ if $m = 1$, where $S = \pi(d^0_1(d^m_{i_1}(z^1_1)))$.

Axiom B1 for $\mathscr{G}(\mathrm{M})$ follows from Lemma 2 and Axiom A1 for M.

The proof of the reachability property of the transitional Chu space $\mathscr{G}(\mathrm{M})$ is reduced to the proof of the dihomotopy of two paths ending at the same subset of events. We consider arbitrary paths $P_1 = (\varnothing \diamond^{k_1}_{<_1} F_1 \diamond^{k_2}_{<_2} \cdots \diamond^{k_n}_{<_n} (F_n = F))$ and $Q_1 = (\varnothing \diamond^{m_1}_{\prec_1} G_1 \diamond^{m_2}_{\prec_2} \cdots \diamond^{m_l}_{\prec_l} (G_m = F))$ in $\Pi_F(\mathscr{G}(\mathrm{M}))$, where $F \in \mathscr{P}_{fin}(E)$. Without loss of generality, we can assume that these paths consist of a single element. By the definition of $\mathscr{G}$, for P_1 and Q_1 there exist their preimages $P = x_{(\varnothing,\varnothing)}\ x_{(\varnothing,F_1)} x_{(F_1,F_1)} \ldots x_{(F,F)}$ and $Q = x_{(\varnothing,\varnothing)}\ x_{(\varnothing,G_1)} x_{(G_1,G_1)} \ldots x_{(F,F)}$ in $\mathscr{C}\mathscr{P}_{1,x_{(F,F)}}(\mathrm{M})$. Since M is a simply diconnected precubical set, P and Q are dihomotopic. It is easy to verify that, under the action of $\mathscr{G}$, dihomotopic cubical paths in $\mathscr{C}\mathscr{P}_1(\mathrm{M})$ go to dihomotopic paths in $\Pi_1(\mathscr{G}(\mathrm{M}))$. Hence the paths P_1 and Q_1 in $\mathscr{G}(\mathrm{M})$ are also dihomotopic.

Let $\mathrm{g} = \langle g, \alpha \rangle : \mathrm{M}^1 \to \mathrm{M}^2$ be a morphism of the category **oPS**. We show that $\mathscr{G}(\mathrm{g}) = \langle \widehat{g}, \alpha \rangle$ is a morphism of the category **SCS**. Since M^1 and M^2 are simply diconnected precubical sets and g is a morphism of **oPS**, it is easy to show that if an n-cube $x_{(F,G,<)}$ corresponds to the transition relation $F \diamond^n_< G$ in $\mathscr{G}(\mathrm{M}^1)$, then the n-cube $g(x_{(F,G,<)})$ corresponds to the transition relation $\widehat{g}(F) \diamond^n_{\prec} \widehat{g}(G)$ in $\mathscr{G}(\mathrm{M}^2)$. Now, it is obvious how to verify that $\mathscr{G}(\mathrm{g})$ is a morphism of **SCS**. Thus, it is obvious that $\mathscr{G}$ is a functor. □

Theorem 3. *The categories* ***SCS*** *and* ***oPS*** *are equivalent.*

This assertion follows from Lemmas 3–5 below.

Lemma 3. *$\mathscr{F}$ is a faithful functor.*

Proof. Let E^1, E^2 be objects and $\mathrm{f}_1, \mathrm{f}_2 : \mathrm{E}^1 \to \mathrm{E}^2$ morphisms of the category **SCS**. We need to show that $\mathscr{F}(\mathrm{f}_1) = \mathscr{F}(\mathrm{f}_2)$ implies $\mathrm{f}_1 = \mathrm{f}_2$. By the definition of $\mathscr{F}$, the equality $\mathscr{F}(\mathrm{f}_1) = \mathscr{F}(\mathrm{f}_2)$ implies the equality of the second components of f_1 and f_2. Let f_i and $\widehat{f_i}$ be the first components of $\mathrm{f_i}$ and $\mathscr{F}(\mathrm{f_i})$ respectively ($i = 1, 2$). Then $(f_1(F), f_1(G))_{\prec_1} = \widehat{f_1}(F, G)_< = \widehat{f_2}(F, G)_< = (f_2(F), f_2(G))_{\prec_2}$ for all cubes $(F, G)_<$ in $\mathscr{F}(\mathrm{E}^1)$. This means that $f_1(e) = f_2(e)$ for all $e \in E$ such that there is a set F such that $F \diamond (F \sqcup e)$. Since E^1 is reachable, we have $f_1 = f_2$. □

Lemma 4. *$\mathscr{F}$ is a full functor.*

Proof. Let $\mathrm{g} = \langle g, \alpha \rangle : \mathscr{F}(\mathrm{E}^1) \to \mathscr{F}(\mathrm{E}^2)$ be a morphism of the category **oPS**. We need to show that there exists a morphism $\mathrm{f} = \langle f, \alpha \rangle : \mathrm{E}^1 \to \mathrm{E}^2$ of the category **SCS** such that $\mathscr{F}(\mathrm{f}) = \mathrm{g}$.

For all $e \in E^1$ we set $f(e) = p_2(g(F, F \sqcup e)) \setminus p_1(g(F, F \sqcup e))$, where p_i is the projection onto the ith element of the pair $g(F, F \sqcup e)$ ($i = 1, 2$). We first show that f is a mapping. Suppose that $f(e) = p_2(g(F, F \sqcup e)) \setminus p_1(g(F, F \sqcup e)) = \epsilon$ and $f(e) = p_2(g(G, G \sqcup e)) \setminus p_1(g(G, G \sqcup e)) = \varepsilon$. We need to show that $\epsilon = \varepsilon$. Since E^1 is a transitional Chu space, there exist paths

$$P = (\{\varnothing\} \diamond^{k_1}_{<_1} F_1 \diamond^{k_2}_{<_2} \cdots \diamond^{k_n}_{<_n} (F_n = F) \diamond F \sqcup \{e\}),$$
$$Q = (\{\varnothing\} \diamond^{m_1}_{\prec_1} \cdots \diamond^{m_l}_{\prec_l} (F \cap G) \diamond ((F \cap G) \sqcup \{e\}) \diamond^{m_{l+1}}_{\prec_{l+1}} \cdots \diamond^{m_r}_{\prec_r} (F \sqcup \{e\})).$$

Without loss of generality we can assume that $G \subseteq F$. Then

$$Q = (\varnothing \diamond^{m_1}_{\prec_1} \cdots \diamond^{m_l}_{\prec_l} G \diamond (G \sqcup \{e\}) \diamond^{m_{l+1}}_{\prec_{l+1}} \cdots \diamond^{m_r}_{\prec_r} (F \sqcup \{e\})).$$

Since the paths P and Q contain $\{e\}$, they are dihomotopic. Since E^1 is a transitional Chu space, there are 2-cubes $z_1, \ldots, z_n$ in $\mathscr{F}(\mathrm{E}^1)$ such that

$$(F, F \sqcup e) = d^{k_1}_{i_1}(z_1), d^{3-k_1}_{i_1}(z_1) = d^{k_2}_{i_2}(z_2), \ldots, d^{1-k_n}_{i_n}(z_n) = (G, G \sqcup e).$$

By induction on n and the obvious equalities

$$p_2(g(d^0_s(z_i))) \setminus p_1(g(d^0_s(z_i))) = p_2(g(d^1_s(z_i))) \setminus p_1(g(d^1_s(z_i))), \quad 1 \leqslant i \leqslant n,$$

we find

$$\epsilon = p_2(g(F, F \sqcup e)) \setminus p_1(g(F, F \sqcup e)) = \cdots = p_2(g(G, G \sqcup e)) \setminus p_1(g(G, G \sqcup e)) = \varepsilon.$$

We prove that $\mathrm{f} = \langle f, \alpha \rangle$ is a morphism of the category **SCS**. We prove condition (2) of Definition 3. Let $F(\diamond^1)^n_< G$ in E^1. Since E^1 is a transitional Chu space, there exists a path $\{\varnothing = \epsilon_0\} \diamond^1 \{\epsilon_1\} \diamond^1 \cdots \diamond^1 (\{\epsilon_1, \ldots, \epsilon_k\} = F)(\diamond^1)^n_< G$. We have

$$f(F) = \bigcup_{e \in F} f(e) = \bigcup_{j=1}^{k} p_2(g(\epsilon_1 \sqcup \ldots \sqcup \epsilon_{j-1}, \epsilon_1 \sqcup \ldots \sqcup \epsilon_j)) \setminus p_1(g(\epsilon_1 \sqcup \ldots \sqcup \epsilon_{j-1}, \epsilon_1 \sqcup \ldots \sqcup \epsilon_j))$$
$$= p_2(g(\epsilon_1 \sqcup \ldots \sqcup \epsilon_{k-1}, F)) = p_1(g(F, G)_<).$$

Similarly, $f(G) = p_2(g(F, G)_<)$. Since g is a morphism of the category **PS**, the dimension of the cube $g(F, G)_<$ in $\mathscr{F}(\mathrm{E}^2)$ is equal to n. Thus, $g(F, G)_< \in (\diamond^2)^n_\prec$. Consequently, $f(F)(\diamond^2)^n_\prec f(G)$ in E^2. Conditions (1) and (3) of Definition 3 are obviously satisfied.

Finally we show that $\mathscr{F}(\mathrm{f}) = \mathrm{g}$. By the definition of $\mathscr{F}$ and f, the second components of $\mathscr{F}(\mathrm{f})$ and g coincide. Let us prove that their first components $\widehat{f}$ and g are also equal. For an arbitrary cube $(F,G)_<$ in $\mathscr{F}(\mathrm{E}^1)$ we have $\widehat{f}(F,G)_< = (f(F), f(G))_\prec = (p_1(g(F,G)), p_2(g(F,G)))_\prec = g(F,G)_<$. □

Lemma 5. *$\mathscr{F}$ is a dense functor.*

Proof. Let M be an object of the category **oPS**. We construct an object E of the category **SCS** such that M is isomorphic to $\mathscr{F}(\mathrm{E})$.

We set $\mathrm{E} = \mathscr{G}(\mathrm{M})$ and consider a simply diconnected precubical set $\mathscr{F}(\mathscr{G}(\mathrm{M}))$. By the construction of $\mathscr{F}$ and $\mathscr{G}$, the cube $(F,G)_<$ belongs to $\mathscr{F}(\mathscr{G}(\mathrm{M}))$ if and only if there exists a unique cube $x_{(F,G,<)}$ in M such that $F = \pi(D^0(x_{(F,G,<)}))$, $G = \pi(D^1(x_{(F,G,<)}))$, and $<=<_{x_{(F,G,<)}}$. We set $\xi_M(x) = (\pi(D^0(x)), \pi(D^1(x)))_{<_x}$ for all cubes x in M and $\eta_M((F,G)_<) = x_{(F,G,<)}$ for all cubes $(F,G)_<$ in $\mathscr{F}(\mathscr{G}(\mathrm{M}))$. It is clear that $\xi_{\mathrm{M}} = \langle \xi_M, id\rangle : \mathrm{M} \to \mathscr{F}(\mathscr{G}(\mathrm{M}))$ and $\eta_{\mathrm{M}} = \langle \eta_M, id\rangle : \mathscr{F}(\mathscr{G}(\mathrm{M})) \to \mathrm{M}$ are morphisms of the category **PS**. We show that these morphisms are mutually invertible. Indeed,

$$\xi_M(\eta_M((F,G)_<)) = \xi_M(x_{(F,G,<)}) = (\pi(D^0(x_{(F,G,<)})), \pi(D^1(x_{(F,G,<)})))_{<_{x_{(F,G,<)}}} = (F,G)_<\,,$$

$$\eta_M(\xi_M(x)) = \eta_M((\pi(D^0(x)), \pi(D^1(x)))_{<_x}) = x_{(\pi(D^0(x)), \pi(D^1(x)), <_x)} = x.$$

The lemma is proved. □

Further, we show that $\mathbf{P}_L$-open morphisms are mapped by the functor $\mathscr{F} : \mathbf{SCS} \to \mathbf{oPS}$ into $\mathbf{cP}_L$-open morphisms and vice versa. In the sequel, we need the following facts. The isomorphisms $\xi_{\mathrm{M}} = \langle \xi_M, id\rangle : \mathrm{M} \to \mathscr{F}(\mathscr{G}(\mathrm{M}))$, where M is an object of the category **oPS** given in the proof of Lemma 5, and the isomorphisms $\varsigma_{\mathrm{E}} = \langle \varsigma_E, id\rangle : \mathrm{E} \to \mathscr{G}(\mathscr{F}(\mathrm{E}))$, given by $\varsigma_E(e) = \ll (F, F \sqcup e) \gg$ for all $e \in E$, can be extended to the natural isomorphisms $\xi : \mathbf{1_{oPS}} \to \mathscr{F} \circ \mathscr{G}$ and $\varsigma : \mathbf{1_{SCS}} \to \mathscr{G} \circ \mathscr{F}$. Indeed, we have

$$\xi_{\mathrm{M}^2} \circ f = \mathscr{F}(\mathscr{G}(\mathrm{f})) \circ \xi_{\mathrm{M}^1} \tag{1}$$

for an arbitrary morphism $\mathrm{f} : \mathrm{M}^1 \to \mathrm{M}^2$ of the category **oPS** and

$$\varsigma_{\mathrm{E}^2} \circ g = \mathscr{G}(\mathscr{F}(\mathrm{g})) \circ \varsigma_{\mathrm{E}^1} \tag{2}$$

for an arbitrary morphism $\mathrm{g} : \mathrm{E}^1 \to \mathrm{E}^2$ of the category **SCS**.

Proposition 3. *Suppose that E^1 is an object of the category $\mathbf{P}_L$, E^2 is an object of the category $\mathbf{SCS}_L$, $\mathrm{f} = \langle f, id\rangle : \mathrm{E}^1 \to \mathrm{E}^2$ is a morphism of the category $\mathbf{SCS}_L$, and $\mathrm{g} = \langle g, id\rangle : \mathscr{F}(\mathrm{E}^1) \to \mathscr{F}(\mathrm{E}^2)$ is a morphism of the category $\mathbf{oPS}_L$. Then f and g are injections.*

Proof. We prove that f is an injection by contradiction, opposing a path ending at a set of events E^1 to its image under the action f. Since the functor $\mathscr{F}$ is full, we conclude that g is an injection. □

A *cubical computation* in a simply diconnected precubical set M is a simply diconnected precubical set V such that $\mathscr{G}(\mathrm{V})$ is a computation in the transitional Chu space $\mathscr{G}(\mathrm{M})$.

Corollary 1. *Suppose that* E^1 *and* E^2 *are objects of the category* $\mathbf{SCS}_L$, $\mathscr{F}(\mathrm{f}) = \langle \widehat{f}, id\rangle : \mathscr{F}(\mathrm{E}^1) \to \mathscr{F}(\mathrm{E}^2)$ *is a morphism of the category* $\mathbf{oPS}_L$, *and* V *is a cubical computation in* $\mathscr{F}(\mathrm{E}^1)$. *Then the first component of the restriction* $\mathscr{F}(\mathrm{f}) \mid_{\mathrm{V}} : \mathrm{V} \to \mathscr{F}(\mathrm{E}^2)$ *is an injection.*

Proof. Since $\mathscr{G}(\mathrm{V})$ is an object of $\mathbf{P}_L$ and $\mathscr{F}$ is a full functor, from Proposition 3 it follows that the first component of $\mathscr{F}(\mathrm{f})_{\mathrm{V}} \circ \eta_{\mathrm{V}} : \mathscr{F}(\mathscr{G}(\mathrm{V})) \to \mathrm{V} \to \mathscr{F}(\mathrm{E}^2)$ is an injection. Since η_{V} is the inverse of ξ_{V}, the first component of $\mathscr{F}(\mathrm{f})_{\mathrm{V}}$ is also an injection. □

Theorem 4. *A morphism* $\mathrm{f} = \langle f, id\rangle : \mathrm{E}^1 \to \mathrm{E}^2$ *of the category* $\boldsymbol{SCS}_L$ *is* $\mathbf{P}_L$*-open if and only if* $\mathscr{F}(\mathrm{f})$ *is* $\mathbf{cP}_L$*-open.*

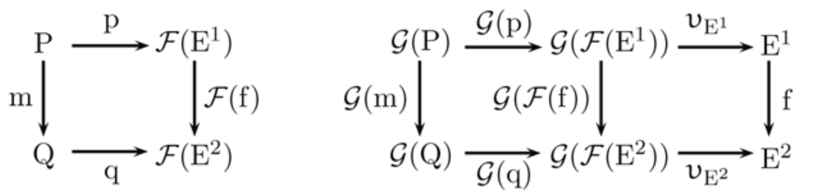
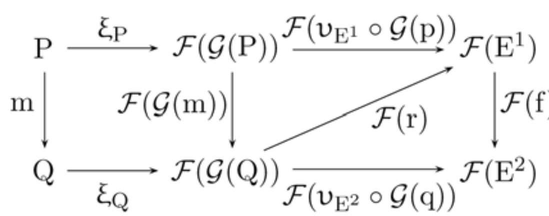


Figure 5. Diagrams for the morphisms $\mathscr{F}(\mathrm{f})$ and f of the categories $\mathbf{oPS}_L$ and $\mathbf{SCS}_L$ respectively.

Proof. *Necessity.* Assume that $\mathrm{m} : \mathrm{P} \to \mathrm{Q}$ is a morphism of the category $\mathbf{cP}_L$ and $\mathrm{p} : \mathrm{P} \to \mathscr{F}(\mathrm{E}^1)$, $\mathrm{q} : \mathrm{Q} \to \mathscr{F}(\mathrm{E}^2)$ are morphisms of the category $\mathbf{oPS}_L$ such that the diagram at the left in Figure 5 is commutative. Applying the functor $\mathscr{G}$ and using formula (2), we see that the diagram at the right in Figure 5 is also commutative, where $\upsilon_{\mathrm{E}^i} : \mathscr{G}(\mathscr{F}(\mathrm{E}^i)) \to \mathrm{E}^i$ is the inverse of ς_{E^i} given by $\upsilon_{\mathrm{E}^i}(\ll (F, F \sqcup e) \gg) = e$ for all events $\ll (F, F \sqcup e) \gg$ in $\mathscr{G}(\mathscr{F}(\mathrm{E}^i))$ $(i = 1, 2)$. Since $\mathscr{G}(\mathrm{P})$, $\mathscr{G}(\mathrm{Q})$ are objects of $\mathbf{P}_L$, and f is $\mathbf{P}_L$-open, there is a morphism $\mathrm{r} : \mathscr{G}(\mathrm{Q}) \to \mathrm{E}^1$ of $\mathbf{SCS}_L$ such that $\upsilon_{\mathrm{E}^1} \circ \mathscr{G}(\mathrm{p}) = \mathrm{r} \circ \mathscr{G}(\mathrm{m})$ and $\upsilon_{\mathrm{E}^2} \circ \mathscr{G}(\mathrm{q}) = \mathrm{f} \circ \mathrm{r}$. Applying the functor $\mathscr{F}$ to the diagram at the right in Figure 5 and using formula (1), we obtain the following commutative diagram:



We consider the equality $\mathscr{F}(\upsilon_{\mathrm{E}^1}) \circ \mathscr{F}(\mathscr{G}(\mathrm{p})) = \mathscr{F}(\mathrm{r}) \circ \mathscr{F}(\mathscr{G}(\mathrm{m}))$. By formula (1), we have

$$\mathscr{F}(\upsilon_{\mathrm{E}^1}) \circ \xi_{\mathscr{F}(\mathrm{E}^1)} \circ \mathrm{p} \circ \eta_{\mathrm{P}} = \mathscr{F}(\mathrm{r}) \circ \mathscr{F}(\mathscr{G}(\mathrm{m})), \tag{3}$$

where $\eta_{\mathrm{P}} : \mathscr{F}(\mathscr{G}(\mathrm{P})) \to \mathrm{P}$ is the inverse isomorphism of ξ_{P}. Since $\mathscr{F}$ is a faithful full functor, we have $\xi_{\mathscr{F}(\mathrm{E}^1)} = \mathscr{F}(\varsigma_{\mathrm{E}^1})$. Thus, multiplying (3) by ξ_{P} from the left and using the commutativity of the left square of the diagram, we find $\mathrm{p} = \mathscr{F}(\mathrm{r}) \circ \xi_{\mathrm{Q}} \circ \mathrm{m}$. Similarly, $\mathrm{q} = \mathscr{F}(\mathrm{f}) \circ \mathscr{F}(\mathrm{r}) \circ \xi_{\mathrm{Q}}$. Consequently, $\mathscr{F}(\mathrm{r}) \circ \xi_{\mathrm{Q}} : \mathrm{Q} \to \mathscr{F}(\mathrm{E}^1)$ satisfies the required equalities, i.e., $\mathscr{F}(\mathrm{f})$ is $\mathbf{cP}_L$-open.

Sufficiency. Suppose that $\mathrm{m} = \langle m, id\rangle : \widehat{\mathrm{P}} \to \widehat{\mathrm{Q}}$ is a morphism of the category $\mathbf{P}_L$ and $\mathrm{p} = \langle p, id\rangle : \widehat{\mathrm{P}} \to \mathrm{E}^1$, $\mathrm{q} = \langle q, id\rangle : \widehat{\mathrm{Q}} \to \mathrm{E}^2$ are morphisms of the category $\mathbf{SCS}_L$ such that the diagram at the left in Figure 6 is commutative. Hence the diagram at the right in Figure 6 is also commutative.

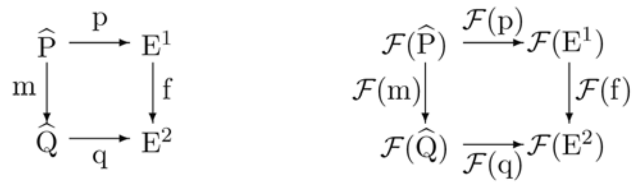


Figure 6. Diagrams for the morphisms f and $\mathscr{F}$(f) of the categories $\mathbf{SCS}_L$ and $\mathbf{oPS}_L$ respectively.

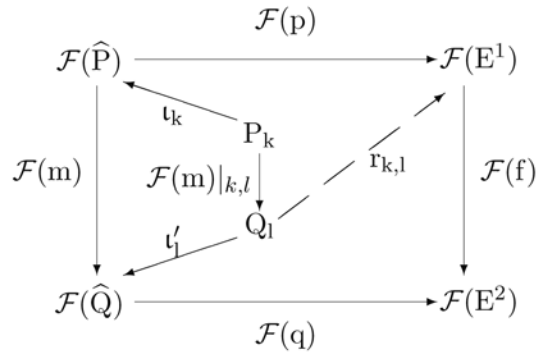


Figure 7. The extended diagram for the morphism $\mathscr{F}$(f) of the category $\mathbf{oPS}_L$.

Let $\{\mathrm{P_k}\}_{k\leqslant n^P}$ and $\{\mathrm{Q_l}\}_{l\leqslant n^Q}$ be collections of objects of the category $\mathbf{cP}_L$ such that $\iota_\mathrm{k}:\mathrm{P_k}\to\mathscr{F}(\widehat{\mathrm{P}})$ and $\iota'_\mathrm{l}:\mathrm{Q_l}\to\mathscr{F}(\widehat{\mathrm{Q}})$ are natural morphisms of inclusion and $(\widehat{Q},\widehat{Q})=\iota'_l(t_l)$, where t_l is a finite point[5] in $\mathrm{Q_l}$. For all $k\leqslant n^P$ and $l\leqslant n^Q$ we choose maximal cubical paths P_k and Q_1 for $\mathrm{P_k}$ and $\mathrm{Q_l}$ respectively. Let $S=\{(k,l)\mid \exists\mathscr{F}(\mathrm{m})|_{k,l}:\mathrm{P_k}\to\mathrm{Q_l},\ k\leqslant n^P,\ l\leqslant n^Q\}$, where $\mathscr{F}(\mathrm{m})|_{k,l}:\mathrm{P_k}\to\mathrm{Q_l}$ is the restriction of the morphism $\mathscr{F}(\mathrm{m})$ onto $\mathrm{P_k}$ from the range of $\mathrm{Q_l}$. We consider arbitrary $(k,l)\in S$. Then the left inner square of the diagram in Figure 7 is commutative. Since the outer square of the same diagram is also commtative, we have $\mathscr{F}(\mathrm{f})\circ(\mathscr{F}(\mathrm{p})\circ\iota_\mathrm{k})=(\mathscr{F}(\mathrm{q})\circ\iota'_\mathrm{l})\circ\mathscr{F}(\mathrm{m})|_{k,l}$. Since $\mathscr{F}(\mathrm{f})$ is a $\mathbf{cP}_L$-open morphism, there is a morphism $\mathrm{r_{k,l}}:\mathrm{Q_l}\to\mathscr{F}(\mathrm{E}^1)$ such that $\mathscr{F}(\mathrm{p})\circ\iota_\mathrm{k}=\mathrm{r_{k,l}}\circ\mathscr{F}(\mathrm{m})|_{k,l}$ and $\mathscr{F}(\mathrm{q})\circ\iota'_\mathrm{l}=\mathscr{F}(\mathrm{f})\circ\mathrm{r_{k,l}}$. As a rule, such a morphism is not unique, which is asserted by the following lemma.

Lemma 6. *For a fixed* $(k,l)\in S$ *there exists a set of morphisms* $\{\mathrm{r^V_{k,l}}:\mathrm{Q_l}\to\mathscr{F}(\mathrm{E}^1)\}_{\mathrm{V}\in A_k}$ *such that* $\mathrm{r^V_{k,l}}(\mathrm{Q_l})\subseteq\mathrm{V}$*, where*

$$A_k=\{\mathrm{V}\in A\mid\mathscr{F}(\mathrm{p})(\iota_\mathrm{k}(\mathrm{P_k}))\subseteq\mathrm{V}\},$$

$$A=\{\mathrm{V}\ \textit{ is a cubical computation in } \mathscr{F}(\mathrm{E}^1)\mid\widehat{f}(t_V)=\widehat{q}(\widehat{Q},\widehat{Q}),t_V\ \textit{ is a finite point in }\mathrm{V}\}.$$

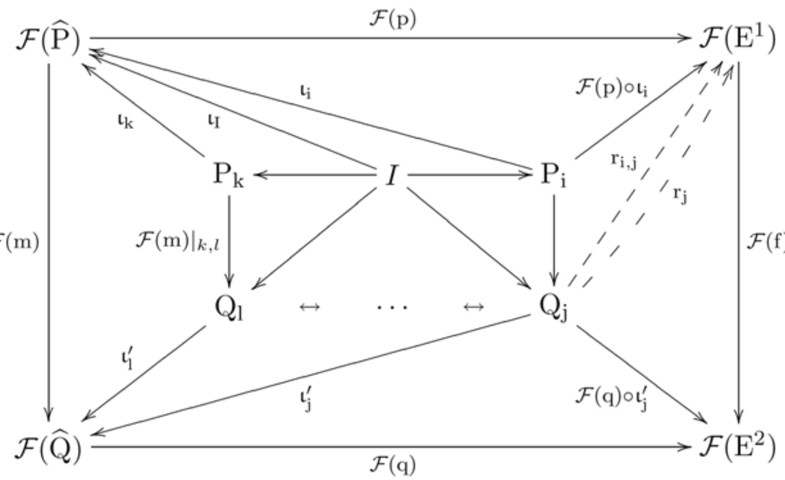
Furthermore, $\mathscr{F}(\mathrm{p})\circ\iota_\mathrm{k}=\mathrm{r^V_{k,l}}\circ\mathscr{F}(\mathrm{m})|_{k,l}$ *and* $\mathscr{F}(\mathrm{q})\circ\iota'_\mathrm{l}=\mathscr{F}(\mathrm{f})\circ\mathrm{r^V_{k,l}}$.

Proof. Indeed, for fixed $(k,l)\in S$ the existence of at least one $\mathrm{r^W_{k,l}}\in\{\mathrm{r^V_{k,l}}\}_{\mathrm{V}\in A_k}$ follows from the $\mathbf{cP}_L$-openness of the morphism $\mathscr{F}(\mathrm{f})$. If $|A_k|>1$, then we consider an element $\mathrm{U}\in A_k$ such that $\mathrm{U}\neq\mathrm{W}$. It is clear that there is a cubical path P_{t_U} in U that ends up with a finite point t_U of the cubical computation U. Since $\widehat{f}(t_U)=\widehat{q}(\widehat{Q},\widehat{Q})=\widehat{f}(r^W_{k,l}(t_l))$ and $\mathscr{F}(\mathrm{E}^2)$ is a simply diconnected precubical set, we have $\widehat{f}(P_{t_U})\overset{(\overline{s}_1,\overline{u}_1,\overline{v}_1)}{\longleftrightarrow}\cdots\overset{(\overline{s}_b,\overline{u}_b,\overline{v}_b)}{\longleftrightarrow}\widehat{f}(r^W_{k,l}(Q_l))$. Since $\mathscr{F}(\mathrm{f})$ is a $\mathbf{cP}_L$-open morphism, we can apply Theorem 1 b times to conclude that there exists a cubical path P'_U such that $P_{t_U}\overset{(\overline{s}_1,\overline{u}_1,\overline{v}_1)}{\longleftrightarrow}\cdots\overset{(\overline{s}_b,\overline{u}_b,\overline{v}_b)}{\longleftrightarrow}P'_U$, i.e., P'_U belongs to U and $\widehat{f}(P'_U)=\widehat{f}(r^W_{k,l}(Q_l))$.

[5] A point u in a precubical set M is said to be *finite* if there are no $v\in M_1$ such that $u=d^0_1(v)$.

It is clear that the mapping $\mathrm{r}_{\mathrm{k,l}}^{\mathrm{U}}$ defined by $r_{k,l}^{U}(Q_l) = P_U'$ can be extended to a morphism of the category $\mathbf{oPS}_L$ with the help of the equality $\widehat{f}(r_{k,l}^{U}(Q_l)) = \widehat{f}(r_{k,l}^{W}(Q_l)) = \widehat{q}(\iota_l'(Q_l))$. This equality also implies the commutativity of the lower inner square of the diagram in Figure 7 for $\mathrm{r}_{\mathrm{k,l}}^{\mathrm{U}}$. Hence $\mathscr{F}(\mathrm{f})\circ\mathrm{r}_{\mathrm{k,l}}^{\mathrm{U}}\circ\mathscr{F}(\mathrm{m})|_{k,l} = \mathscr{F}(\mathrm{q})\circ\iota_{\mathrm{l}}'\circ\mathscr{F}(\mathrm{m})|_{k,l}$. Consequently, by the commutativity of the left inner square and the outer square, we have $\mathscr{F}(\mathrm{f})|_{\mathrm{U}}\circ\mathrm{r}_{\mathrm{k,l}}^{\mathrm{U}}\circ\mathscr{F}(\mathrm{m})|_{k,l} = \mathscr{F}(\mathrm{f})|_{\mathrm{U}}\circ\mathscr{F}(\mathrm{p})\circ\iota_{\mathrm{k}}$ since $\mathrm{r}_{\mathrm{k,l}}^{\mathrm{U}}(\mathscr{F}(\mathrm{m})|_{k,l}(\mathrm{P_k})), \mathscr{F}(\mathrm{p})(\iota_{\mathrm{k}}(\mathrm{P_k})) \subseteq \mathrm{U}$. By Corollary 1, the upper inner square in the diagram in Figure 7 for $\mathrm{r}_{\mathrm{k,l}}^{\mathrm{U}}$ is also commutative. □

We consider the diagram



where I is an object of the category **oPS** consisting of a single cube of zero dimension. It is clear that there exist morphisms of inclusion $\iota_{\mathrm{I,k}} : \mathrm{I} \hookrightarrow \mathrm{P_k}$ and $\iota_{\mathrm{I,i}} : \mathrm{I} \hookrightarrow \mathrm{P_i}$. Since $I \in \{\mathrm{P_k}\}_{k\leqslant n^P}$, there exist morphisms $\{\mathrm{r}_{\mathrm{I,l}}^{\mathrm{V}}\}_{\mathrm{V}\in A_I}$ for all $l \leqslant n^Q$. Such morphisms are denoted by $\mathrm{r}_{\mathrm{l}}^{\mathrm{V}} = \mathrm{r}_{\mathrm{I,l}}^{\mathrm{V}}$. We note that $A_I = A$.

Lemma 7. *Let* $\mathrm{V} \in A$. *If* $y^l \in M_{\mathrm{Q_l}}$ *and* $y^j \in M_{\mathrm{Q_j}}$ *are such that* $\iota_l'(y^l) = \iota_j'(y^j) = y \in M_{\mathscr{F}(\widehat{\mathrm{Q}})}$, *then* $r_l^V(y^l) = r_j^V(y^j)$.

Proof. Since $\mathscr{F}(\widehat{\mathrm{Q}})$ is a simply diconnected precubical set, from the definition of Q_l and Q_j we conclude that $\iota_l'(Q_l)$ and $\iota_j'(Q_j)$ are dihomotopic. Hence it suffices to consider the case $\iota_l'(Q_l) \overset{(s,u,v)}{\longleftrightarrow} \iota_j'(Q_j)$. Let, for example, $\iota_{\mathrm{l}}'(\mathrm{Q_l}) \subseteq \iota_{\mathrm{j}}'(\mathrm{Q_j})$. Then there exists a morphism of inclusion $\iota_{\mathrm{l,j}}' : \mathrm{Q_l} \to \mathrm{Q_j}$. It is clear that $\iota_{\mathrm{l}}' = \iota_{\mathrm{j}}' \circ \iota_{\mathrm{l,j}}'$. By the commutativity of the lower interior squares of the diagram in Figure 7 for $\mathrm{r}_{\mathrm{l}}^{\mathrm{V}}$ and $\mathrm{r}_{\mathrm{j}}^{\mathrm{V}}$, we find $\mathscr{F}(\mathrm{f}) \circ \mathrm{r}_{\mathrm{j}}^{\mathrm{V}} \circ \iota_{\mathrm{l,j}}' = \mathscr{F}(\mathrm{f}) \circ \mathrm{r}_{\mathrm{l}}^{\mathrm{V}}$. Since $\mathrm{r}_{\mathrm{l}}^{\mathrm{V}}(\mathrm{Q_l})$, $\mathrm{r}_{\mathrm{j}}^{\mathrm{V}}(\iota_{\mathrm{l,j}}'(\mathrm{Q_l})) \subseteq \mathrm{V}$, the required assertion becomes obvious by Corollary 1. □

Since for every $y \in M_{\mathscr{F}(\widehat{\mathrm{Q}})}$ there exists $l \leqslant n^Q$ such that $y = \iota_l'(y^l)$, where $y^l \in M_{\mathrm{Q_l}}$, we conclude that, in view of Lemma 7, the equality $r^V \circ \iota_l' = r_l^V$ for all $l \leqslant n^Q$ yields a mapping $\mathrm{r}^{\mathrm{V}} = \langle r^V, id_L\rangle : \mathscr{F}(\widehat{\mathrm{Q}}) \to \mathscr{F}(\mathrm{E}^1)$ which is a morphism of the category **oPS**.

We consider arbitrary $\mathrm{P_k}$ such that $\iota_k(\tau_k) = (\widehat{P}, \widehat{P})$, where τ_k is a finite point of $\mathrm{P_k}$. For k we choose a number l such that $(k,l) \in S$. Since $\mathscr{F}(\mathrm{f})$ is a $\mathbf{cP}_L$-open morphism, there is a morphism $\mathrm{r}_{\mathrm{k,l}}^{\mathrm{V_0}}$ (with some $\mathrm{V}_0 \in A_k$) such that $\mathrm{r}_{\mathrm{k,l}}^{\mathrm{V_0}} \circ \mathscr{F}(\mathrm{m})|_{k,l} = \mathscr{F}(\mathrm{p}) \circ \iota_{\mathrm{k}}$ and $\mathscr{F}(\mathrm{q}) \circ \iota_{\mathrm{l}}' = \mathscr{F}(\mathrm{f}) \circ \mathrm{r}_{\mathrm{k,l}}^{\mathrm{V_0}}$. Then $\mathrm{V}_0 \in A_i$ for all $i \leqslant n^P$. Indeed, we extend the cubical path $\iota_i(P_i)$ to a cubical path $\overline{P}$ that is contained in $\mathscr{F}(\widehat{\mathrm{P}})$ and ends up with the point $(\widehat{P}, \widehat{P})$. Since $\overline{P} \sim \iota_k(P_k)$ and, consequently, $\widehat{p}(\overline{P}) \sim \widehat{p}(\iota_k(P_k))$, and the cubical path $\widehat{p}(\iota_k(P_k)) = r^{V_0}_{k,l}(\widehat{m}|_{k,l}(P_k))$ lies in the cubical computation V_0, we conclude that the cubical path $\widehat{p}(\overline{P})$, together with $\widehat{p}(\iota_i(P_i))$, lies in the cubical computation V_0. Consequently, $\mathscr{F}(\mathrm{p})(\iota_\mathrm{i}(\mathrm{P_i})) \subseteq \mathrm{V}_0$. Thus, there exists a morphism $\mathrm{r}^{\mathrm{V}_0}_{\mathrm{i,j}}$ for all i, j such that $(i, j) \in S$.

It is clear that $\mathrm{r}^{\mathrm{V}_0}_{\mathrm{k,l}} = \mathrm{r}^{\mathrm{V}_0}_{\mathrm{l}}$. Indeed, by the commutativity of lower inner squares of the diagram in Figure 7 for $\mathrm{r}^{\mathrm{V}_0}_{\mathrm{k,l}}$ and $\mathrm{r}^{\mathrm{V}_0}_{\mathrm{l}}$, we have $\mathscr{F}(\mathrm{f}) \circ \mathrm{r}^{\mathrm{V}_0}_{\mathrm{k,l}} = \mathscr{F}(\mathrm{f}) \circ \mathrm{r}^{\mathrm{V}_0}_{\mathrm{l}}$. Since $\mathrm{r}^{\mathrm{V}_0}_{\mathrm{k,l}}(\mathrm{Q_l}), \mathrm{r}^{\mathrm{V}_0}_{\mathrm{l}}(\mathrm{Q_l}) \subseteq \mathrm{V}_0$, we obtain the required result in view of Corollary 1.

By the above, we have $\mathrm{r}^{\mathrm{V}_0} \circ \mathscr{F}(\mathrm{m}) = \mathscr{F}(\mathrm{p})$ and $\mathscr{F}(\mathrm{f}) \circ \mathrm{r}^{\mathrm{V}_0} = \mathscr{F}(\mathrm{q})$. By formula (2), the required morphism $\upsilon_{\mathrm{E}^1} \circ \mathscr{G}(\mathrm{r}^{\mathrm{V}_0}) \circ \varsigma_{\widehat{\mathrm{Q}}} : \widehat{\mathrm{Q}} \to \mathrm{E}^1$ satisfies the equalities $\mathrm{p} = \upsilon_{\mathrm{E}^1} \circ \mathscr{G}(\mathscr{F}(\mathrm{p})) \circ \varsigma_{\widehat{\mathrm{P}}} = \upsilon_{\mathrm{E}^1} \circ \mathscr{G}(\mathrm{r}^{\mathrm{V}_0}) \circ \mathscr{G}(\mathscr{F}(\mathrm{m})) \circ \varsigma_{\widehat{\mathrm{P}}} = \upsilon_{\mathrm{E}^1} \circ \mathscr{G}(\mathrm{r}^{\mathrm{V}_0}) \circ \varsigma_{\widehat{\mathrm{Q}}} \circ \mathrm{m}$ and, similarly, $\mathrm{q} = \mathrm{f} \circ \upsilon_{\mathrm{E}^1} \circ \mathscr{G}(\mathrm{r}^{\mathrm{V}_0}) \circ \varsigma_{\widehat{\mathrm{Q}}}$. □

To conclude the paper, we formulate an important consequence of Theorems 2, 3, 4 and Assertion 3 in [15].

Theorem 5. *Two precubical sets labeled by the same set L of actions are* **cP***-bisimular if and only if their universal dicoverings represented as transitional Chu spaces, are* **P***-bisimular.*

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