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Comparing Equivalences on Precubical Sets and Spaces

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Abstract—We study equivalences of concurrent processes represented by objects of algebraic topology. We use methods of category theory and consider precubical sets (analogs of semisimplicial sets) and precubical spaces (analogs of cell complexes). In particular, we consider categories of these objects and construct subcategories of path-objects. We define open morphisms with respect to these subcategories and formulate criteria for a morphism to be open. We prove that the equivalence of precubical sets (spaces) based on open morphisms coincides with a behavioral equivalence of concurrent processes.

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1. INTRODUCTION

Notions and methods of algebraic topology and category theory have appeared recently in the concurrency theory. In [16], Pratt suggested to use precubical sets as a model of concurrent processes. Such sets are similar to semisimplicial sets of algebraic topology. They represent the cubical structure of elements of the set in terms of boundary mappings. On the other hand, the structure of such sets allows us to construct adequate models of concurrent processes. For example, Fajstrup [2, 3] proved that two processes are concurrent if and only if the paths representing these processes in precubical sets are homotopic. In the articles by Goubault and Jensen [8], Grandis [9], Fahrenberg [4], and Khusainov [12], the homological approach is used. The authors represent (pre)cubical sets as algebraic complexes, and study concurrent processes from the point of view of homologies of (pre)cubical sets. In his thesis [7], Goubault suggested another geometrical model of concurrency; namely, precubical spaces (topological spaces endowed with a differential structure that allows us to determine the duration of concurrent processes).

Various equivalences arise in identification of “similar” concurrent processes. In an attempt to compare and unify these equivalences, a number of authors developed category theoretical approaches. In [11], Joyal, Nielsen, and Winskel introduced the abstract notion of equivalence in terms of a special construction (a span of open morphisms). This approach helped to unify a series of definitions of behavioral equivalences for various models of concurrency, see [5, 11, 13, 14, 17]. In [10, 18], open morphisms are introduced and studied for timed extensions of several models of concurrent processes.

In the present article, we represent concurrent processes by objects of algebraic topology and apply methods of category theory for studying and comparing equivalences of these objects. We consider precubical sets (analogs of semisimplicial sets) and precubical spaces (analogs of cell complexes). Using a criterion for a morphism to be open, we prove that the equivalence of precubical sets based on open morphisms coincides with a behavioral equivalence of concurrent processes. We also construct and investigate adjoint functors between categories of precubical sets and spaces. These functors allow us to transfer the criterion for behavioral equivalence (formulated in category theoretical terms of open morphisms) to precubical spaces.

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2. PRECUBICAL SETS

2.1. *Category* $\mathrm{p}\mathbb{S}\mathrm{et}$. Precubical sets allow us to construct natural models of concurrent processes. Namely, the concurrent execution of n actions is represented by an n-dimensional cube, while their mutually exclusive execution is represented by the edges of this cube. For example, consider the precubical set in Fig. 1. The right-hand 2-cube (filled square) x represents the concurrent execution of a and b. The boundary of this cube consists of 1-cubes (segments) x_1, y_2, and y_1, x_2. Functions of two types represent this boundary; namely, the (source) boundary mappings d_1^0 and d_2^0 and the (target) boundary mappings d_1^1 and d_2^1. In some sense, $x_2 = d_1^1(x)$ and $y_2 = d_2^1(x)$ are copies of $x_1 = d_1^0(x)$ and $y_1 = d_2^0(x)$ respectively. This distinction between the boundary mappings determines the direction. For example, the 1-cube x_1 starts at the 0-cube (point) $d_1^0(x_1) = i_0$ and terminates at the 0-cube $d_1^1(x_1) = s$. The left-hand square represents the mutually exclusive execution of a and b. This situation is modeled by the precubical set constructed from 1-cubes x_1, y_2 and y_1, x_2. Both processes start at the initial point i_0.

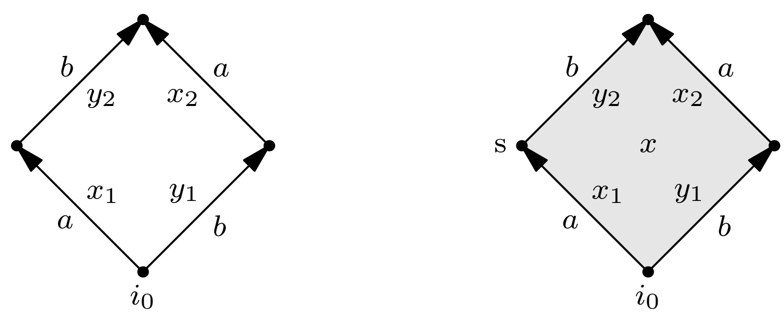


Fig. 1. Concurrent and mutually exclusive executions of a and b in a precubical set

We present the formal definition of a precubical set.

Definition 1. A (*labeled over a set L of actions*) *precubical set* (*with a distinguished point*) is a triple $\mathrm{M} = (M, i_0, l)_L$, where

- M is a *precubical set*, i.e., a collection of pairwise distinct sets $(M_n)_{n\geq 0}$ and boundary mappings

$$d_\lambda^0, d_\mu^1 : M_{n+1} \to M_n \quad \bigl(\lambda, \mu = 1 \dots n+1\bigr)$$

  satisfying the following cubical axioms: For all $1 \leq \lambda < \mu \leq n+2$ and $\alpha, \beta \in \{0,1\}$, the diagram

$$\begin{array}{ccc} M_{n+2} & \xrightarrow{d_\mu^\beta} & M_{n+1} \\ {\scriptstyle d_\lambda^\alpha}\big\downarrow & & \big\downarrow{\scriptstyle d_\lambda^\alpha} \\ M_{n+1} & \xrightarrow[d_{\mu-1}^\beta]{} & M_n \end{array}$$

  commutes;
- $i_0 \in M_0$ is a distinguished point, called the *initial* point;
- $l : M_1 \to L$ is a *labeling* function from the set of 1-cubes to the set L of actions such that $l\bigl(d_\lambda^0(x)\bigr) = l\bigl(d_\lambda^1(x)\bigr)$, where $\lambda = 1, 2$, for all $x \in M_2$.

Remark 1. We introduce the value $l(x)$ for every $x \in M_n$, $n \geq 0$, as follows. Put $l(x) = \varnothing$ for $n = 0$ and $l(x) = \bigl(l_1(x), \dots, l_n(x)\bigr)$ for $n > 1$, where

$$l_\lambda(x) = l\bigl(d_1^0 \circ \cdots \circ d_{\lambda-1}^0 \circ d_{\lambda+1}^0 \circ \cdots \circ d_n^0(x)\bigr) \text{ for all } 1 \leq \lambda \leq n.$$

Let $n \geq 0$ and let $0 \leq j \leq n$. We put

$$A(j,n) = \big\{(\gamma_1, \dots, \gamma_j, \theta_1, \dots, \theta_j) \;\big|\; \gamma_1, \dots, \gamma_j \in \{0,1\},\;\; 1 \leq \theta_1 < \dots < \theta_j \leq n\big\}.$$

For $(\Gamma, \Theta) \in A(j,n)$, we define the $(n-j)$-component of a cube $x \in M_n$ as follows:

$$D^{\Gamma}_{\Theta}(x) = \begin{cases} x & \text{if } j = 0, \\ d^{\gamma_1}_{\theta_1} \circ \dots \circ d^{\gamma_j}_{\theta_j}(x) & \text{otherwise.} \end{cases}$$

Let $A(n) = \bigcup_{0 \leq j \leq n} A(j,n)$ and let $D(x) = \bigcup_{(\Gamma,\Theta) \in A(n)} D^{\Gamma}_{\Theta}(x)$.

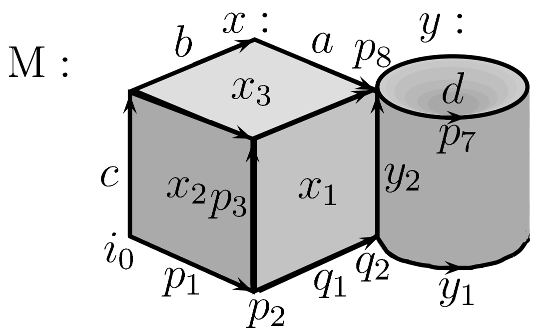


Fig. 2. Precubical set M

Example 1. We illustrate Definition 1. Consider the precubical set $\mathrm{M} = (M, i_0, l)_L$ with $L = \{a, b, c, d\}$ in Fig. 2. The set M contains the 3-cube x and the 2-cube y rolled up into a cylinder. The boundaries of x and y are defined as follows:

$$x_1 = d^1_1(x),\;\; x_2 = d^0_2(x),\;\; x_3 = d^1_3(x),\;\; y_1 = d^0_1(y),\;\; y_2 = d^0_2(y).$$

The initial point is $i_0 \in M_0$. The labeling function l is defined as follows: $l_1(x) = a$, $l_2(x) = b$, $l_3(x) = c$, and $l_2(y) = d$.

By a morphism we will mean a pair of functions taking cubes and actions of a precubical set into cubes and actions of another precubical set respectively and satisfying some additional requirements.

Definition 2. Let $\mathrm{M} = (M, i_0, l)_L$ and $\mathrm{M}' = (M', i'_0, l')_{L'}$ be precubical sets. Let $\mathrm{f} = \langle f, \sigma \rangle$, where $f = \bigcup f_n$ and $f_n : M_n \to M'_n$ and $\sigma : L \to L'$ are set mappings. We say that f is a *morphism* from M to M′ if the following conditions hold:

(1) $f_0(i_0) = i'_0$,
(2) $l' \circ f_n = \sigma \circ l$,
(3) $f_n \circ d^{\alpha}_{\lambda} = d^{\alpha}_{\lambda} \circ f_{n+1}$.

By the first condition, morphisms preserve the initial points. By the second condition, the actions are consistent. By the third condition, the boundaries of the cubes are consistent too.

Precubical sets with morphisms form the category $\mathrm{p}\mathbb{S}\mathrm{et}$, where the composition of morphisms

$$\mathrm{f} = \langle f, \sigma \rangle : \mathrm{M} \to \mathrm{M}' \quad \text{and} \quad \mathrm{g} = \langle g, \varrho \rangle : \mathrm{M}' \to \mathrm{M}''$$

is the morphism

$$\mathrm{g} \circ \mathrm{f} = \langle g \circ f, \varrho \circ \sigma \rangle : \mathrm{M} \to \mathrm{M}''$$

and the identity morphism consists of two identity mappings.

In the sequel, we will need a foliated structure on $\mathrm{p}\mathbb{S}\mathrm{et}$. Let $\mathrm{p}\mathbb{S}\mathrm{et}_L$ denote the subcategory of $\mathrm{p}\mathbb{S}\mathrm{et}$, whose objects are precubical sets labeled over a set L of actions and the second components of morphisms are identity mappings. Similar notation will be used for other categories considered in the article.

2.2. *Subcategory* $\mathbb{P}$. A *cubical path* in a precubical set M is a sequence

$$P = p_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} p_1 \dots p_{k-1} \xrightarrow{d^{\alpha_k}_{\lambda_k}} p_k$$

of cubes and boundary mappings such that $p_0 = i_0$ and

$$\begin{aligned} &\text{either } \ p_{s-1} = d^{\alpha_s}_{\lambda_s}(p_s) \ \ (\text{if } \ \alpha_s = 0) \\ &\text{or } \ p_s = d^{\alpha_s}_{\lambda_s}(p_{s-1}) \qquad (\text{if } \ \alpha_s = 1) \end{aligned}$$

for all $1 \le s \le k$.

Remark 2. Usually we omit boundary mappings in the notation and write $P = p_0 p_1 \dots p_{k-1} p_k$ instead of

$$P = p_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} p_1 \dots p_{k-1} \xrightarrow{d^{\alpha_k}_{\lambda_k}} p_k$$

if there is no ambiguity.

In the sequel, we denote by $\mathcal{CP}(\mathrm{M})$ $\big(\mathcal{CP}_{p_k}(\mathrm{M})\big)$ the set of all cubical paths (ending with a cube p_k) in a precubical set M.

The following assertion is obvious.

Assertion 1. *Let* $P = p_0 p_1 \dots p_{k-1} p_k$ *be a cubical path in a precubical set* $\mathrm{M} = (M, i_0, l)_L$. *Then* $\mathrm{M}' = (M', i_0, l')_L$, *where*

$$M'_n = \bigcup_{0 \le s \le k} D(p_s) \cap M_n \ (n \ge 0) \ \text{ and } \ l' = l\big|_{M'_1},$$

is a precubical set over L *and a precubical subset of* M. (*We will say that* M' *has the form of the cubical path* P *in* M.)

Let

$$\begin{aligned} P &= p_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} p_1 \dots p_{k-1} \xrightarrow{d^{\alpha_k}_{\lambda_k}} p_k, \\ P' &= p'_0 \xrightarrow{d^{\beta_1}_{\mu_1}} p'_1 \dots p'_{n-1} \xrightarrow{d^{\beta_n}_{\mu_n}} p'_n \end{aligned}$$

be cubical paths in a precubical set M. We say that P' is an *extension* of P and P is a *restriction* of P' (in symbols: $P \to P'$) if $n \ge k$ and the equalities $p_s = p'_s$, $\alpha_s = \beta_s$, and $\lambda_s = \mu_s$ hold for all $1 \le s \le k$. In particular, we write $P \xrightarrow{d^{\alpha}_{\lambda}} P'$ if $n = k + 1$, $\beta_{k+1} = \alpha$, and $\mu_{k+1} = \lambda$.

Following [6], we introduce the notion of the combinatorial homotopy for cubical paths in a precubical set M. The *homotopy* is the least equivalence on the set of cubical paths in M such that P and P' are equivalent if P and P' are s-*adjacent* (in symbols: $P \overset{s}{\longleftrightarrow} P'$), i.e., P' can be obtained from P by replacing, for $\lambda < \mu$ and $\alpha = 0, 1$,

$$\begin{aligned} &\text{either the segment } \xrightarrow{d^0_\lambda} p_s \xrightarrow{d^\alpha_\mu} \text{ by the segment } \xrightarrow{d^\alpha_{\mu-1}} p'_s \xrightarrow{d^0_\lambda} \text{ or vice versa;} \\ &\text{or the segment } \xrightarrow{d^\alpha_\mu} p_s \xrightarrow{d^1_\lambda} \text{ by the segment } \xrightarrow{d^1_\lambda} p'_s \xrightarrow{d^\alpha_{\mu-1}} \text{ or vice versa.} \end{aligned}$$

For every $P \in \mathcal{CP}(\mathrm{M})$, let $[P]$ denote the homotopy class of P.

Example 2. Recall the precubical set M from Example 1. The sequences

$$P = i_0 \xrightarrow{d^0_1} p_1 \xrightarrow{d^1_1} p_2 \xrightarrow{d^0_1} p_3 \xrightarrow{d^0_1} x_1 \xrightarrow{d^1_1} y_2 \xrightarrow{d^0_2} y \xrightarrow{d^1_1} p_7 \xrightarrow{d^1_1} p_8 \xrightarrow{d^0_1} p_7$$

and

$$Q = i_0 \xrightarrow{d^0_1} p_1 \xrightarrow{d^1_1} p_2 \xrightarrow{d^0_1} q_1 \xrightarrow{d^1_1} q_2 \xrightarrow{d^0_1} y_2 \xrightarrow{d^0_2} y \xrightarrow{d^1_1} p_7 \xrightarrow{d^1_1} p_8 \xrightarrow{d^0_1} p_7$$

in Fig. 2 are cubical paths in M. It is clear that P and Q are homotopic because

$$P \overset{4}{\longleftrightarrow} \left(i_0 \xrightarrow{d_1^0} p_1 \xrightarrow{d_1^1} p_2 \xrightarrow{d_1^0} q_1 \xrightarrow{d_2^0} x_1 \xrightarrow{d_1^1} y_2 \xrightarrow{d_2^0} y \xrightarrow{d_1^1} p_7 \xrightarrow{d_1^1} p_8 \xrightarrow{d_1^0} p_7 \right) \overset{5}{\longleftrightarrow} Q.$$

For a natural number N, we put

$$\boxplus^N = \begin{cases} \{0\} & \text{if } N = 0, \\ \left\{ (t_1, \dots, t_N) \;\middle|\; t_j \in \left\{0, \frac{1}{2}, 1\right\} \right\} & \text{otherwise.} \end{cases}$$

We partition $\boxplus^N$ into subsets of the form

$$\boxplus_n^N = \left\{ (t_1, \dots, t_N) \in \boxplus^N \;\middle|\; \left| \left\{ t_j = \frac{1}{2} \;\middle|\; 1 \le j \le N \right\} \right| = n \right\}, \text{ where } 0 \le n \le N.$$

Assume that $(t_1, \dots, t_N) \in \boxplus_n^N$, $1 \le j_1 < \dots < j_n \le N$, and $t_{j_i} = \frac{1}{2}$ for all $1 \le i \le n$. We define boundary mappings $\mathsf{d}_\lambda^\alpha : \boxplus_n^N \to \boxplus_{n-1}^N$. We put

$$\mathsf{d}_\lambda^\alpha(t_1, \dots, t_N) = (t_1, \dots, t_{j_\lambda - 1}, \alpha, t_{j_\lambda + 1}, \dots, t_N),$$

where $\alpha \in \{0, 1\}$, $1 \le \lambda \le n$, and $0 < n \le N$. It is clear that $\boxplus^N$ is a precubical set.

We construct a precubical set labeled over L whose initial point is $\boxplus^{\mathrm{N}}$. We put

$$\boxplus^{\mathrm{N}} = \begin{cases} \left(\boxplus^0, 0, \varnothing\right)_L & \text{if } N = 0, \\ \left(\boxplus^N, (0, \dots, 0), l^{\boxplus^{\mathrm{N}}}\right)_L & \text{otherwise.} \end{cases}$$

Here, $l^{\boxplus^{\mathrm{N}}}$ is a labeling function from $\boxplus_1^N$ to L such that $l^{\boxplus^{\mathrm{N}}}\left(\mathsf{d}_\lambda^0(p)\right) = l^{\boxplus^{\mathrm{N}}}\left(\mathsf{d}_\lambda^1(p)\right)$ for all $\lambda = 1, 2$ and $p \in \boxplus_2^N$.

A cubical path $P \in \mathcal{CP}_p\left(\boxplus^{\mathrm{N}}\right)$ is *consistent* with $\boxplus^N$ if either $N = 0$ or we have

$$D_\Theta^\Gamma(p) = (\underbrace{1, \dots, 1}_{N}), \text{ where } \Gamma = (\underbrace{1, \dots, 1}_{\dim p}), \ \Theta = (1, \dots, \dim p).$$

A *path-object* $\bar{\square}$ is a precubical set having the form of a cubical path $P \in \mathcal{CP}\left(\boxplus^{\mathrm{N}}\right)$, $N \ge 0$, that is consistent with $\boxplus^N$. Let $\mathbb{P}$ denote the full subcategory of path-objects of the category p$\mathbb{S}$et.

A cubical path P in a path-object $\bar{\square}$ is said to be *maximal* if $\bar{\square}$ has the form of P. Let $\mathcal{CP}_{\max}(\bar{\square})$ denote the set of all maximal cubical paths in $\bar{\square}$.

A morphism $\mathrm{m} = \langle m, 1_L \rangle : \bar{\square} \to \bar{\square}'$ of the subcategory $\mathbb{P}_L$ is called an *le-step* (*wi-step*) if $m(P) \xrightarrow{d_\lambda^\alpha} Q$ $\left(m(P) \overset{s}{\longleftrightarrow} Q\right)$ in $\bar{\square}'$ for suitable $P \in \mathcal{CP}_{\max}(\bar{\square})$ and $Q \in \mathcal{CP}_{\max}(\bar{\square}')$. Notice that every morphism of the category $\mathbb{P}_L$ is the composition of *le*-steps and *wi*-steps.

In the sequel, we will need the following definitions and facts about cubical paths. For the proof of the following lemma, see [15, Lemma 4.1].

Lemma 1. *Let* M *be an object of the category* p$\mathbb{S}$et$_L$. *For every cubical path P in* M, *there exist an object $\bar{\square}_{\dot{\mathrm{P}}}$ of the category $\mathbb{P}_L$ and a morphism $\pi = \langle \pi, 1_L \rangle : \bar{\square}_{\dot{\mathrm{P}}} \to$* M *of the category* p$\mathbb{S}et_L$ *satisfying the equality $\pi(\dot{P}) = P$ for a suitable $\dot{P} \in \mathcal{CP}_{\max}\left(\bar{\square}_{\dot{\mathrm{P}}}\right)$.*

We consider the notions of a λ-beginning and a λ-ending of a cubical path P in a precubical set M. For the proof of the following fact (see [6, Proposition 2]).

Lemma 2. *Let $P \in \mathcal{CP}(\mathrm{M})$ contain either a segment $\xrightarrow{d^0_{\lambda_s}} p_s \xrightarrow{d^1_{\lambda_{s+1}}}$, with $\lambda_s \neq \lambda_{s+1}$, or a segment $\xrightarrow{d^0_{\lambda_s}} p_s \xrightarrow{d^0_{\lambda_{s+1}}}$, or a segment $\xrightarrow{d^1_{\lambda_s}} p_s \xrightarrow{d^1_{\lambda_{s+1}}}$. Then there exist a unique cubical path P' in* M *with $P \overset{s}{\longleftrightarrow} P'$.*

Let $P \in \mathcal{CP}_{p_k}(\mathrm{M})$ and let $1 \le \lambda \le n$, $n = \dim p_k > 0$. By a λ-*beginning* of the cubical path P we mean a cubical path $d^0_\lambda(P) \in \mathcal{CP}(\mathrm{M})$ such that the diagram

$$\begin{array}{ccccccccc} & & & & & & & & d^0_\lambda(P) \\ & & & & & & & & \Big\downarrow d^0_\lambda \\ P & \overset{s}{\longleftrightarrow} & P_{s+1} & \overset{s+1}{\longleftrightarrow} & \dots & \overset{k-2}{\longleftrightarrow} & P_{k-1} & \overset{k-1}{\longleftrightarrow} & P_k \end{array}$$

commutes for a suitable $1 \le s \le k$. (For $s = k$, we mean the diagram $d^0_\lambda(P) \xrightarrow{d^0_\lambda} P$.) By a λ-*ending* of the cubical path P we mean a cubical path $d^1_\lambda(P) \in \mathcal{CP}(\mathrm{M})$ with $P \xrightarrow{d^1_\lambda} d^1_\lambda(P)$.

Lemma 3. *Let* M *be a precubical set, let $P \in \mathcal{CP}_{p_k}(\mathrm{M})$ be a cubical path, and let $1 \le \lambda \le n$, where* $\dim p_k = n > 0$. *Then there exists a unique cubical path $d^\alpha_\lambda(P) \in \mathcal{CP}(\mathrm{M})$, $\alpha = 0, 1$.*

Proof. The case in which $\alpha = 1$ is trivial. Assume that $\alpha = 0$. Let

$$P = p_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} p_1 \dots p_{k-1} \xrightarrow{d^{\alpha_k}_{\lambda_k}} p_k \ \ (\dim p_k = n > 0).$$

Distinct copies of the same action may occur in P (for example, if this action is concurrent to itself). For simplicity, we distinguish such actions by adding subscripts. Therefore, without loss of generality, we may assume that, in P, there is at most one copy of each actions.

It is clear that every segment $p_{s-1} \xrightarrow{d^{\alpha_s}_{\lambda_s}} p_s$ in P is either the start of the action $l_{\lambda_s}(p_s)$ (if $\alpha_s = 0$) or the termination of the action $l_{\lambda_s}(p_{s-1})$ (if $\alpha_s = 1$). Consider the cube p_k in P. It represents simultaneous execution of n distinct actions $l_1(p_k), \dots, l_n(p_k)$. By the definition of a cubical path, there exists a unique s such that $1 \le s \le k$ and the segment $p_{s-1} \xrightarrow{d^0_{\lambda_s}} p_s$ of P represents the start of the action $l_{\lambda_s}(p_s) = l_\lambda(p_k)$. We consider the segment $\xrightarrow{d^0_{\lambda_s}} p_s \xrightarrow{d^{\alpha_{s+1}}_{\lambda_{s+1}}}$ of P. If $\alpha_{s+1} = 0$ then, by Lemma 2, there exists a unique cubical path P_{s+1} in M such that $P \overset{s}{\longleftrightarrow} P_{s+1}$. Then P_{s+1} contains the segment $\xrightarrow{d^0_{\lambda^{s+1}_{s+1}}} p^{s+1}_s \xrightarrow{d^0_{\lambda^{s+1}_s}}$. It is easy to see that $l_{\lambda_s}(p_s) = l_{\lambda^{s+1}_s}(p_{s+1})$ in M. If $\alpha_{s+1} = 1$ and $\lambda_s \neq \lambda_{s+1}$ then the arguments are similar. If $\alpha_{s+1} = 1$ and $\lambda_s = \lambda_{s+1}$ then the action $l_{\lambda_s}(p_s)$ starts and terminates simultaneously. Since $l_{\lambda_s}(p_s) = l_\lambda(p_k)$ ends with p_k, we conclude that there is no termination of the action $l_{\lambda_s}(p_s) = l_\lambda(p_k)$ in P. We arrive at a contradiction. Repeating these arguments, we obtain a unique sequence of adjacent cubical paths of the form

$$P \overset{s}{\longleftrightarrow} \cdots \overset{k-1}{\longleftrightarrow} P_k \ \text{ in M, with } \ l_{\lambda_s}(p_s) = l_{\lambda^k_s}(p_k),$$

where P_k ends with the segment $p^k_{k-1} \xrightarrow{d^0_{\lambda^k_s}} p_k$. Since $l_\lambda(p_k) = l_{\lambda_s}(p_s) = l_{\lambda^k_s}(p_k)$ and, in P, each action occurs at most once, we have $\lambda = \lambda^k_s$. Therefore, there exists a unique cubical path $d^0_\lambda(P)$ satisfying the condition $d^0_\lambda(P) \xrightarrow{d^0_\lambda} P_k$. □

In conclusion, we present the following obvious fact.

Lemma 4. *Let* $\mathrm{f} = \langle f, \sigma \rangle : \mathrm{M} \to \mathrm{M}'$ *be a morphism of* pSet. *Then, for every cubical path $P = p_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} \cdots \xrightarrow{d^{\alpha_k}_{\lambda_k}} p_k \in \mathcal{CP}(\mathrm{M})$, the following assertions are valid*:

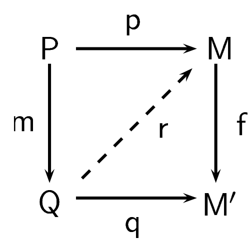
(1) $f(P) = f(p_0) \xrightarrow{d^{\alpha_1}_{\lambda_1}} \cdots \xrightarrow{d^{\alpha_k}_{\lambda_k}} f(p_k) \in \mathcal{CP}(\mathrm{M}')$;

(2) *if* $P \xrightarrow{d^\alpha_\lambda} P'$ *in* M *then* $f(P) \xrightarrow{d^\alpha_\lambda} f(P')$ *in* M';

(3) *if* $P \overset{s}{\longleftrightarrow} P'$ *in* M *then* $f(P) \overset{s}{\longleftrightarrow} f(P')$ *in* M'.

2.3. *Open morphisms of the category* p$\mathbb{S}$et. Let $\mathbf{M}$ be a category and let $\mathbf{P}$ be a subcategory of $\mathbf{M}$. Let $\mathbf{I} : \mathbf{P} \hookrightarrow \mathbf{M}$ denote the embedding functor. We recall the definition of an open morphism.

Definition 3. A morphism $\mathsf{f} : \mathsf{M} \to \mathsf{M}'$ of $\mathbf{M}$ is said to be $\mathbf{P}$-*open* if it possesses the right lifting property, i.e., for every morphism $\mathsf{m} : \mathsf{P} \to \mathsf{Q}$ of the category $\mathbf{P}$ and every commutative square of the category $\mathbf{M}$ as shown below



there exists a morphism $\mathsf{r} : \mathsf{Q} \to \mathsf{M}$ splitting this square into two commutative triangles.

Remark 3. Consider the definition of a $\mathbf{P}$-open morphism of the category $\mathbf{M}$ in terms of the comma category $\mathbf{I} \downarrow Id_{\mathbf{M}}$. A morphism $\mathsf{f} : \mathsf{M} \to \mathsf{M}'$ of $\mathbf{M}$ is $\mathbf{P}$-*open* if and only if every morphism

$$(\mathsf{m}, \mathsf{f}) : (\mathsf{P}, \mathsf{p}, \mathsf{M}) \to (\mathsf{Q}, \mathsf{q}, \mathsf{M}')$$

of $\mathbf{I} \downarrow Id_{\mathbf{M}}$ can be represented as the composition of the morphisms $(1_{\mathsf{Q}}, \mathsf{f})(\mathsf{m}, 1_{\mathsf{M}})$.

As is noticed in [11], using the notion of a $\mathbf{P}$-open morphism, we can define the notion of $\mathbf{P}$-equivalent objects of $\mathbf{M}$.

Definition 4. Objects X and Y of the category $\mathbf{M}$ are $\mathbf{P}$-*equivalent* if there exists a span of $\mathbf{P}$-open morphisms

$$\mathsf{X} \xleftarrow{f} \mathsf{Z} \xrightarrow{f'} \mathsf{Y}.$$

For a fixed set L, consider the category p$\mathbb{S}$et$_L$ and its subcategory $\mathbb{P}_L$. We prove the following criterion for a morphism of p$\mathbb{S}$et$_L$ to be $\mathbb{P}_L$-open.

Theorem 1. *A morphism* $\mathrm{f} = \langle f, 1_L \rangle : \mathrm{M} \to \mathrm{M}'$ *of the category* p$\mathbb{S}$et$_L$ *is* $\mathbb{P}_L$*-open if and only if, for every cubical path* $P \in \mathcal{CP}(\mathrm{M})$*, the following conditions hold*:

(a) *if* $f(P) \xrightarrow{d^\alpha_\lambda} Q'$ *in* M' *then* $P \xrightarrow{d^\alpha_\lambda} P'$ *in* M *and* $f(P') = Q'$;

(b) *if* $f(P) \overset{s}{\longleftrightarrow} Q'$ *in* M' *then* $P \overset{s}{\longleftrightarrow} P'$ *in* M *and* $f(P') = Q'$.

Proof. ($\Rightarrow$) Assume that $\mathrm{f} = \langle f, 1_L \rangle : \mathrm{M} \to \mathrm{M}'$ is a $\mathbb{P}_L$-open morphism. We only prove that condition (a) holds. The proof of condition (b) is similar. Let $P \in \mathcal{CP}(\mathrm{M})$ and let $f(P) \xrightarrow{d^\alpha_\lambda} Q'$ in M'. By Lemma 1, there exist objects $\bar{\square}_{\dot{\mathrm{P}}}$ and $\bar{\square}_{\dot{\mathrm{Q}}'}$ of the category $\mathbb{P}_L$ and morphisms $\pi = \langle \pi, 1_L \rangle : \bar{\square}_{\dot{\mathrm{P}}} \to \mathrm{M}$ and $\pi' = \langle \pi', 1_L \rangle : \bar{\square}_{\dot{\mathrm{Q}}'} \to \mathrm{M}'$ of the category p$\mathbb{S}$et$_L$ such that $\pi(\dot{P}) = P$ and $\pi'(\dot{Q}') = Q'$ for suitable maximal cubical paths $\dot{P} = \dot{p}_0 \ldots \dot{p}_k$ in $\bar{\square}_{\dot{\mathrm{P}}}$ and $\dot{Q}' = \dot{q}'_0 \ldots \dot{q}'_{k+1}$ in $\bar{\square}_{\dot{\mathrm{Q}}'}$. We define a mapping $\mathrm{m} = \langle m, 1_L \rangle : \bar{\square}_{\dot{\mathrm{P}}} \to \bar{\square}_{\dot{\mathrm{Q}}'}$. We put $m(D^\Gamma_\Theta(\dot{p}_s)) = D^\Gamma_\Theta(\dot{q}'_s)$ for all $(\Gamma, \Theta) \in A(\dim \dot{p}_s)$ and $0 \leq s \leq k$. It is easy to see that m is a morphism of the subcategory $\mathbb{P}_L$ and $\mathrm{f} \circ \pi = \pi' \circ \mathrm{m}$. Since f is a $\mathbb{P}_L$-open morphism, there exists a morphism $\mathrm{r} : \bar{\square}_{\dot{\mathrm{Q}}'} \to \mathrm{M}$ of the category p$\mathbb{S}$et$_L$ such that $\pi = \mathrm{r} \circ \mathrm{m}$ and $\pi' = \mathrm{f} \circ \mathrm{r}$. Hence, there exists a cubical path $r(\dot{Q}')$ in M. From the definitions of the morphism m and the cubical paths $\dot{P}$ and $\dot{Q}'$ together with the fact that $f(P) \xrightarrow{d^\alpha_\lambda} Q'$ in M' it follows that $m(\dot{P}) \xrightarrow{d^\alpha_\lambda} \dot{Q}'$ in $\bar{\square}_{\dot{\mathrm{Q}}'}$. By Lemma 4, we find that $r\big(m(\dot{P})\big) \xrightarrow{d_\lambda^\alpha} r(\dot{Q}')$ in M. Since $\pi = \mathrm{r} \circ \mathrm{m}$ and $\pi' = \mathrm{f} \circ \mathrm{r}$, we obtain $P = \pi(\dot{P}) = r\big(m(\dot{P})\big) \xrightarrow{d_\lambda^\alpha} r(\dot{Q}')$ in M and $f\big(r(\dot{Q}')\big) = \pi'(\dot{Q}') = Q'$.

($\Leftarrow$) Let $\mathrm{f} = \langle f, 1_L\rangle : \mathrm{M} \to \mathrm{M}'$ be a morphism of the category $\mathrm{p}\mathbb{S}\mathrm{et}_L$ such that the conditions of the theorem hold. We prove that f is $\mathbb{P}_L$-open. Assume that there exist morphisms $\pi : \bar{\Box} \to \mathrm{M}$ and $\pi' : \bar{\Box}' \to \mathrm{M}'$ of the category $\mathrm{p}\mathbb{S}\mathrm{et}_L$ and a morphism $\mathrm{m} : \bar{\Box} \to \bar{\Box}'$ of the subcategory $\mathbb{P}_L$ such that $\mathrm{f} \circ \pi = \pi' \circ \mathrm{m}$. We show that there exists a morphism $\mathrm{r} = \langle r, 1_L\rangle : \bar{\Box}' \to \mathrm{M}$ of the category $\mathrm{p}\mathbb{S}\mathrm{et}_L$ such that $\pi = \mathrm{r} \circ \mathrm{m}$ and $\pi' = \mathrm{f} \circ \mathrm{r}$. We only consider the case in which m is a wi-step (the case in which m is an le-step is similar). Since m is a wi-step, there exist $P \in \mathcal{CP}_{\max}(\bar{\Box})$ and $Q \in \mathcal{CP}_{\max}(\bar{\Box}')$ such that $m(P) \xleftrightarrow{s} Q = q_0 \dots q_k$ in $\bar{\Box}'$. By Lemma 4, we have $\pi'\big(m(P)\big) \xleftrightarrow{s} \pi'(Q)$ in M'. Since $f\big(\pi(P)\big) = \pi'\big(m(P)\big)$, we find that $\pi(P) \xleftrightarrow{s} P' = p_0 \dots p_k$ in M and $f(P') = \pi'(Q)$ (cf. the conditions of the theorem). We define a mapping $\mathrm{r} = \langle r, 1_L\rangle : \bar{\Box}' \to \mathrm{M}$. We put $r\big(D_\Theta^\Gamma(q_s)\big) = D_\Theta^\Gamma(p_s)$ for all $(\Gamma, \Theta) \in A(\dim q_s)$ and $0 \le s \le k$. It is easy to see that r is a morphism of the category $\mathrm{p}\mathbb{S}\mathrm{et}_L$ such that $\pi = \mathrm{r} \circ \mathrm{m}$ and $\pi' = \mathrm{f} \circ \mathrm{r}$. Since every morphism of $\mathbb{P}_L$ is representable as the composition of le-steps and wi-steps, we use induction on the number of such steps and obtain the required assertion for an arbitrary morphism. Thus, f is a $\mathbb{P}_L$-open morphism. □

2.4. *hhp-Bisimulation on precubical sets.* We introduce a behavioral equivalence (hhp-bisimulation) on precubical sets, which is an adaptation of the corresponding definition from [6]. We prove that this equivalence coincides with the $\mathbb{P}$-equivalence.

Definition 5. Let M' and M'' be precubical sets labeled over L.

Cubical paths

$$P = p_0 \xrightarrow{d_{\lambda_1}^{\alpha_1}} p_1 \dots p_{k-1} \xrightarrow{d_{\lambda_k}^{\alpha_k}} p_k \quad \text{in } \mathrm{M}'$$

and

$$Q = q_0 \xrightarrow{d_{\mu_1}^{\beta_1}} q_1 \dots q_{k-1} \xrightarrow{d_{\mu_k}^{\beta_k}} q_k \quad \text{in } \mathrm{M}''$$

are said to be *dl-connected* if $\alpha_s = \beta_s$, $\lambda_s = \mu_s$, and $l'(p_s) = l''(q_s)$ for all $1 \le s \le k$.

A binary relation $\mathcal{R}$ between cubical paths in M' and M'' is called an hhp-*bisimulation* between M' and M'' if, for each pair $(P, Q) \in \mathcal{R}$, the cubical paths P and Q are dl-connected and the following conditions hold:

(1) if $P \xrightarrow{d_\lambda^\alpha} P'$ in M' then $Q \xrightarrow{d_\lambda^\alpha} Q'$ in M'' and $(P', Q') \in \mathcal{R}$;

(2) if $Q \xrightarrow{d_\lambda^\alpha} Q'$ in M'' then $P \xrightarrow{d_\lambda^\alpha} P'$ in M' and $(P', Q') \in \mathcal{R}$;

(3) if $P' \xrightarrow{d_\lambda^\alpha} P$ in M' then $Q' \xrightarrow{d_\lambda^\alpha} Q$ in M'' and $(P', Q') \in \mathcal{R}$;

(4) if $Q' \xrightarrow{d_\lambda^\alpha} Q$ in M'' then $P' \xrightarrow{d_\lambda^\alpha} P$ in M' and $(P', Q') \in \mathcal{R}$;

(5) if $P \xleftrightarrow{s} P'$ in M' then $Q \xleftrightarrow{s} Q'$ in M'' and $(P', Q') \in \mathcal{R}$;

(6) if $Q \xleftrightarrow{s} Q'$ in M'' then $P \xleftrightarrow{s} P'$ in M' and $(P', Q') \in \mathcal{R}$.

Precubical sets M' and M'' are hhp-*bisimilar* if there exists an hhp-bisimulation between M' and M'' such that the initial points (regarded as cubical paths) are related.

Notice that the relation “to be hhp-bisimilar” for precubical sets is indeed an equivalence relation.

Example 3. To understand better the definition above, we discuss examples of hhp-bisimilar and non-hhp-bisimilar precubical sets.

We begin with the precubical sets depicted in Fig. 3. The boundary mappings are defined as follows:

$$d_1^0(x_1) = p_1, \ \ d_2^1(x_1) = p_3, \ \ d_1^0(x_2) = p_2, \ \ d_2^1(x_2) = p_4$$

in the left-hand precubical set and

$$d_1^0(y) = q_1, \;\; d_2^1(y) = q_2$$

in the right-hand precubical set. The initial points are s_0 and r_0. These semicubucal sets are hhp-bisimilar because the required hhp-bisimulation $\mathcal{R}$ can be easily constructed from the set

$$\{(P_1, Q_1), (P_2, Q_2), (P_1, Q_2), (P_2, Q_1)\}$$

with the help of conditions (1)–(6) of Definition 5. The cubical paths P_1, P_2, Q_1, and Q_2 have the following form:

$P_1 = s_0p_1s_1p_3s_3p_5s_5p_7s_7$ and $P_2 = s_0p_2s_2p_4s_4p_6s_6p_8s_8$ in the left-hand precubical set and $Q_1 = r_0q_1r_1q_2r_2q_3r_3q_4r_4$ and $Q_2 = r_0q_1r_1q_2r_2q_5r_5q_6r_4$ in the right-hand precubical set.

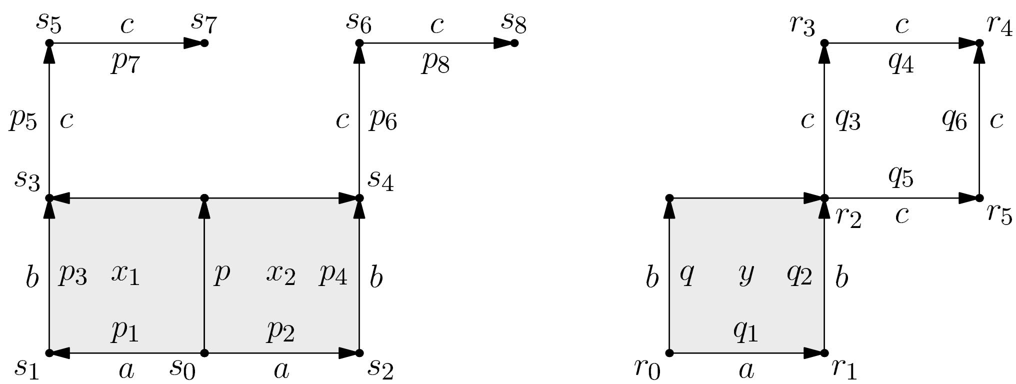


Fig. 3. hhp-Bisimilar precubical sets

Now we turn to the precubical sets depicted in Fig. 4. The boundary mappings are defined as follows:

$$d_1^0(x_1) = p_1, \;\; d_2^1(x_1) = p_5,$$
$$d_1^0(x_2) = p_1, \;\; d_2^1(x_2) = p_2,$$
$$d_2^0(x_3) = p_4, \;\; d_1^1(x_3) = p_3,$$
$$d_1^0(x_4) = p_5, \;\; d_2^0(x_4) = p_6$$

in the left-hand precubical set and

$$d_1^0(y_1) = q_1, \;\; d_2^1(y_1) = q_6,$$
$$d_1^0(y_2) = q_1, \;\; d_2^1(y_2) = q_2,$$
$$d_2^0(y_3) = q_4, \;\; d_1^1(y_3) = q_5,$$
$$d_1^0(y_4) = q_6, \;\; d_2^0(y_4) = q_7,$$
$$d_2^0(y_5) = q_2, \;\; d_1^1(y_5) = q_3$$

in the right-hand precubical set.

Assume that there exists an hhp-bisimulation $\widetilde{\mathcal{R}}$ between these precubical sets such that the initial points are equivalent with respect to $\widetilde{\mathcal{R}}$. Then, for the cubical path

$$Q = r_0 \xrightarrow{d_1^0} q_1 \xrightarrow{d_1^0} y_2 \xrightarrow{d_2^1} q_2 \xrightarrow{d_1^1} r_2 \xrightarrow{d_1^0} q_3 \xrightarrow{d_1^1} r_3$$

in the right-hand precubical set, there exists a cubical path P in the left-hand precubical set such that $(P, Q) \in \widetilde{\mathcal{R}}$. Since P and Q are dl-connected, the path P has either the form

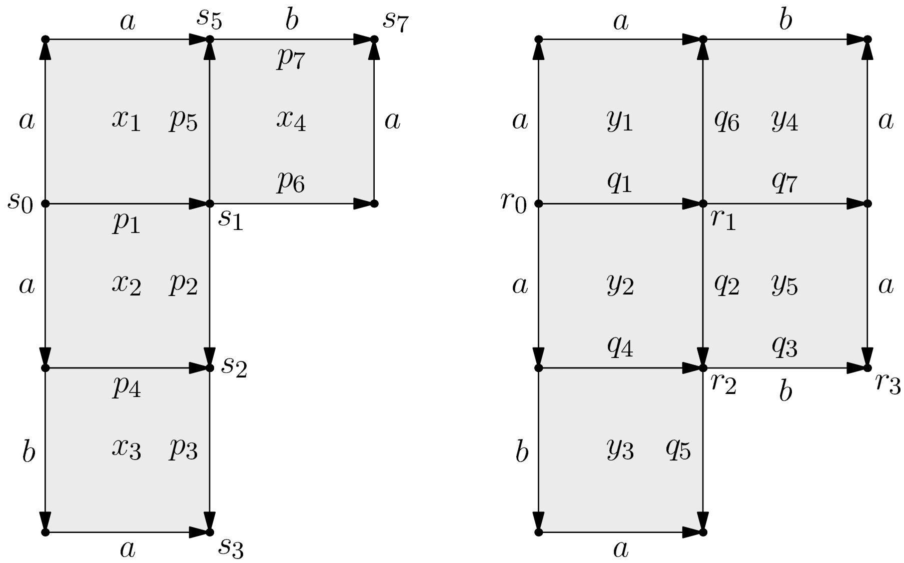


Fig. 4. Precubical sets that are not hhp-bisimilar

(1) $s_0 \xrightarrow{d_1^0} p_1 \xrightarrow{d_1^0} x_2 \xrightarrow{d_2^1} p_2 \xrightarrow{d_1^1} s_2 \xrightarrow{d_1^0} p_3 \xrightarrow{d_1^1} s_3$
or the form

(2) $s_0 \xrightarrow{d_1^0} p_1 \xrightarrow{d_1^0} x_1 \xrightarrow{d_2^1} p_5 \xrightarrow{d_1^1} s_5 \xrightarrow{d_1^0} p_7 \xrightarrow{d_1^1} s_7$.
Moreover, for the right-hand precubical set, we have

$$Q \xleftrightarrow{4} \left(Q' = r_0 \xrightarrow{d_1^0} q_1 \xrightarrow{d_1^0} y_2 \xrightarrow{d_2^1} q_2 \xrightarrow{d_2^0} y_5 \xrightarrow{d_1^1} q_3 \xrightarrow{d_1^1} r_3 \right).$$

Hence, in the left-hand precubical set, there exists a cubical path P' such that $P \xleftrightarrow{4} P'$ and $(P', Q') \in \widetilde{\mathcal{R}}$. In case (1), there exist no P' with $P \xleftrightarrow{4} P'$. In case (2), consider the path

$$P' = s_0 \xrightarrow{d_1^0} p_1 \xrightarrow{d_1^0} x_1 \xrightarrow{d_2^1} p_5 \xrightarrow{d_1^0} x_4 \xrightarrow{d_2^1} p_7 \xrightarrow{d_1^1} s_7.$$

We have $P \xleftrightarrow{4} P'$. However, we have $(P', Q') \notin \widetilde{\mathcal{R}}$ because P' and Q' are not dl-connected.

We prove that, for precubical sets labeled over a common set L, the relations “to be hhp-bisimilar” and “to be $\mathbb{P}_L$-equivalent” coincide.

Theorem 2. *Let* M' *and* M'' *be precubical sets labeled over the same set* L *of actions. Then the following conditions are equivalent*:

(1) M' *and* M'' *are* hhp-*bisimilar*;

(2) M' *and* M'' *are* $\mathbb{P}_L$-*equivalent*.

Proof. (2 ⇒ 1) Assume that M' and M'' are $\mathbb{P}_L$-equivalent. Then there exists a span $\mathrm{M}' \xleftarrow{\mathrm{f}'} \mathrm{M} \xrightarrow{\mathrm{f}''} \mathrm{M}''$, where M is an object of $\mathrm{p}\mathbb{S}\mathrm{et}_L$ and $\mathrm{f}' = \langle f', 1_L \rangle$ and $\mathrm{f}'' = \langle f'', 1_L \rangle$ are $\mathbb{P}_L$-open morphisms of $\mathrm{p}\mathbb{S}\mathrm{et}_L$. It is immediate from Definition 2, Lemma 4, and Theorem 1 that the relation

$$\mathcal{R} = \left\{ \big(f'(P), f''(P)\big) \;\middle|\; P \in \mathcal{CP}(\mathrm{M}) \right\}$$

is an hhp-bisimulation between M' and M''.

(1 ⇒ 2) Let M' and M'' be precubical sets labeled over the same set L of actions and let $\mathcal{R}$ be an hhp-bisimulation between M' and M'' with $(i_0', i_0'') \in \mathcal{R}$. We construct a span $\mathrm{M}' \xleftarrow{\mathrm{f}'} \mathrm{M} \xrightarrow{\mathrm{f}''} \mathrm{M}''$, where M

is an object of pSet_L and $\mathrm{f}' = \langle f', 1_L\rangle$ and $\mathrm{f}'' = \langle f'', 1_L\rangle$ are $\mathbb{P}_L$-open morphisms of pSet_L. For a pair $(P, Q) \in \mathcal{R}$, we define

$$\langle P, Q\rangle = \Big\{(P', Q') \;\Big|\; (P = P_0) \overset{s_1}{\longleftrightarrow} \dots \overset{s_m}{\longleftrightarrow} (P_m = P'),\ (Q = Q_0) \overset{s_1}{\longleftrightarrow} \dots \overset{s_m}{\longleftrightarrow} (Q_m = Q'),\ (P_j, Q_j) \in \mathcal{R}\ (1 \le j \le m,\ m \ge 0)\Big\}.$$

For $m = 0$, we mean $(P, Q) \in \langle P, Q\rangle$.

We define a structure $\langle \mathrm{M}', \mathrm{M}''\rangle = (M, i_0, l)_L$. We put

- $M_n = \big\{\langle P, Q\rangle \;\big|\; (P, Q) \in \mathcal{R},\ P \in \mathcal{CP}_p(\mathrm{M}'),\ \dim p = n\big\}$ and $\bar{d}^{\alpha}_{\lambda}\big(\langle P, Q\rangle\big) = \langle d^{\alpha}_{\lambda}(P), d^{\alpha}_{\lambda}(Q)\rangle$ for all $\langle P, Q\rangle \in M_n$ $(n > 0)$;
- $i_0 = \langle i'_0, i''_0\rangle$;
- $l\big(\langle P, Q\rangle\big) = l'(p)$ for all $\langle P, Q\rangle \in M_1$.

We show that $\langle \mathrm{M}', \mathrm{M}''\rangle$ is a precubical set.

Let $\langle P, Q\rangle \in M_{n+1}$, where $n \ge 0$. We prove that

$$\bar{d}^{0}_{\lambda}\big(\langle P, Q\rangle\big) = \langle d^0_\lambda(P), d^0_\lambda(Q)\rangle \in M_n.$$

Assume that the cubical path $d^0_\lambda(P)$ is obtained according to the relations

$$(P = P_s) \overset{s}{\longleftrightarrow} P_{s+1} \overset{s+1}{\longleftrightarrow} \cdots \overset{k-1}{\longleftrightarrow} P_k \ \text{ and } \ d^0_\lambda(P) \overset{d^0_\lambda}{\longrightarrow} P_k.$$

Since $(P, Q) \in \mathcal{R}$, we conclude that the cubical paths P and Q are dl-connected; hence, the cubical path $d^0_\lambda(Q)$ is obtained according to the relations

$$(Q = Q_s) \overset{s}{\longleftrightarrow} Q_{s+1} \overset{s+1}{\longleftrightarrow} \cdots \overset{k-1}{\longleftrightarrow} Q_k \ \text{ and } \ d^0_\lambda(Q) \overset{d^0_\lambda}{\longrightarrow} Q_k.$$

On the other hand, if $(P, Q) \in \mathcal{R}$ then, by condition (5) of Definition 5, there exist cubical paths $Q'_{s+1}, \dots, Q'_k \in \mathcal{CP}(\mathrm{M}'')$ such that

$$(Q = Q_s) \overset{s}{\longleftrightarrow} Q'_{s+1} \overset{s+1}{\longleftrightarrow} \cdots \overset{k-1}{\longleftrightarrow} Q'_k \ \text{ and } \ (P_j, Q'_j) \in \mathcal{R}$$

for all $(s + 1) \le j \le k$. By Lemma 2, we have $Q_j = Q'_j$ for all $(s + 1) \le j \le k$; hence, $(P_k, Q_k) \in \mathcal{R}$. By condition (3) of Definition 5, we have $\big(d^0_\lambda(P), d^0_\lambda(Q)\big) \in \mathcal{R}$, i.e., $\bar{d}^0_\lambda\big(\langle P, Q\rangle\big) \in M_n$. By condition (1) of Definition 5 for the pair $(P, Q) \in \mathcal{R}$, we have $\bar{d}^1_\lambda\big(\langle P, Q\rangle\big) \in M_n$.

We prove that $\langle \mathrm{M}', \mathrm{M}''\rangle$ satisfies the cubical axioms, i.e., if $\langle P, Q\rangle \in M_{n+2}$ with $n \ge 0$, $1 \le \lambda < \mu \le (n + 2)$, and $\alpha, \beta = 0, 1$ then we have

$$\bar{d}^{\alpha}_{\lambda}\Big(\bar{d}^{\beta}_{\mu}\big(\langle P, Q\rangle\big)\Big) = \bar{d}^{\beta}_{\mu-1}\Big(\bar{d}^{\alpha}_{\lambda}\big(\langle P, Q\rangle\big)\Big).$$

We only consider the case in which $\alpha = 0$ and $\beta = 1$. The remaining cases are similar. Assume that the cubical path $d^0_\lambda(P)$ is obtained according to the relations $P \overset{s}{\longleftrightarrow} \cdots \overset{k-1}{\longleftrightarrow} P_k$ and $d^0_\lambda(P) \overset{d^0_\lambda}{\longrightarrow} P_k$. Since the last cube p_k in P satisfies the cubical axioms, we obtain the diagram depicted in Fig. 5.

On the other hand, $d^1_\mu(P)$ is an extension of the cubical path P, i.e., $d^0_\lambda\big(d^1_\mu(P)\big)$ is obtained according to the relations

$$d^1_\mu(P) \overset{s}{\longleftrightarrow} \cdots \overset{k}{\longleftrightarrow} P_{k+1} \ \text{ and } \ d^0_\lambda\big(d^1_\mu(P)\big) \overset{d^0_\lambda}{\longrightarrow} P_{k+1}.$$

By Lemma 2, this sequence of adjacent cubical paths coincides with the lower sequence in Fig. 5, i.e., $P_{k+1} = \bar{P}_{k+1}$. Hence, $d^0_\lambda\big(d^1_\mu(P)\big) = d^1_{\mu-1}\big(d^0_\lambda(P)\big)$. Similar arguments prove that $d^0_\lambda\big(d^1_\mu(Q)\big) = d^1_{\mu-1}\big(d^0_\lambda(Q)\big)$. We conclude that

$$\bar{d}^{0}_{\lambda}\Big(\bar{d}^{1}_{\mu}\big(\langle P, Q\rangle\big)\Big) = \bar{d}^{1}_{\mu-1}\Big(\bar{d}^{0}_{\lambda}\big(\langle P, Q\rangle\big)\Big).$$

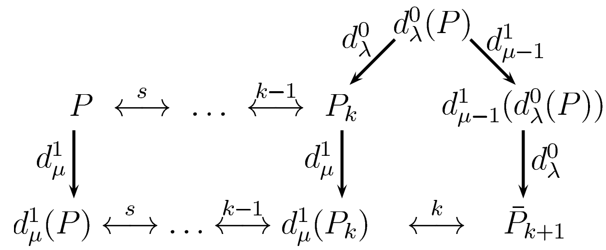


Fig. 5. Cubical paths in M′

Therefore, $\langle \mathrm{M}', \mathrm{M}'' \rangle$ is a precubical set.

We define mappings

$$\langle pr_1, 1_L \rangle : \langle \mathrm{M}', \mathrm{M}'' \rangle \to \mathrm{M}' \text{ and } \langle pr_2, 1_L \rangle : \langle \mathrm{M}', \mathrm{M}'' \rangle \to \mathrm{M}''.$$

We put

$$pr_1\big(\langle P, Q \rangle\big) = p, \quad pr_2\big(\langle P, Q \rangle\big) = q$$

for all $\langle P, Q \rangle \in M$ with $P \in \mathcal{CP}_p(\mathrm{M}')$ and $Q \in \mathcal{CP}_q(\mathrm{M}'')$. It is easy to verify that $\langle pr_1, 1_L \rangle$ and $\langle pr_2, 1_L \rangle$ are morphisms of $\mathrm{p}\mathbb{S}\mathrm{et}_L$. We use Theorem 1 and prove that $\langle pr_1, 1_L \rangle$ is a $\mathbb{P}_L$-open morphism (for the morphism $\langle pr_2, 1_L \rangle$, the proof is similar).

We consider an arbitrary cubical path $O = o_0 \dots o_k \in \mathcal{CP}\big(\langle \mathrm{M}', \mathrm{M}'' \rangle\big)$. By Lemma 4, we have $pr_1(O) \in \mathcal{CP}(\mathrm{M}')$ and $pr_2(O) \in \mathcal{CP}(\mathrm{M}'')$. Let

$$pr_1(O) = p_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} p_1 \dots p_{k-1} \xrightarrow{d^{\alpha_k}_{\lambda_k}} p_k,$$
$$pr_2(O) = q_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} q_1 \dots q_{k-1} \xrightarrow{d^{\alpha_k}_{\lambda_k}} q_k.$$

Using induction on the number of cubes in the cubical path O, it is easy to show that

$$o_s = \left\langle p_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} p_1 \dots p_{s-1} \xrightarrow{d^{\alpha_s}_{\lambda_s}} p_s,\ q_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} q_1 \dots q_{s-1} \xrightarrow{d^{\alpha_s}_{\lambda_s}} q_s \right\rangle$$

for all $0 \le s \le k$. By the construction of $\langle \mathrm{M}', \mathrm{M}'' \rangle$, we have $\big(pr_1(O), pr_2(O)\big) \in \mathcal{R}$. We show that condition (a) of Theorem 1 holds for the morphism $\langle pr_1, 1_L \rangle$ (similar arguments prove that condition (b) holds too). Assume that $pr_1(O) \xrightarrow{d^{\alpha_{k+1}}_{\lambda_{k+1}}} P'$ for some $P' \in \mathcal{CP}(\mathrm{M}')$. By condition (1) of Definition 5, there exists a cubical path $Q' \in \mathcal{CP}(\mathrm{M}'')$ such that $pr_2(O) \xrightarrow{d^{\alpha_{k+1}}_{\lambda_{k+1}}} Q'$ and $(P', Q') \in \mathcal{R}$. Put $o_{k+1} = \langle P', Q' \rangle$. By the construction of $\langle \mathrm{M}', \mathrm{M}'' \rangle$, we obtain

$$O' = o_0 \xrightarrow{d^{\alpha_1}_{\lambda_1}} o_1 \dots o_{k-1} \xrightarrow{d^{\alpha_k}_{\lambda_k}} o_k \xrightarrow{d^{\alpha_{k+1}}_{\lambda_{k+1}}} o_{k+1} \in \mathcal{CP}\big(\langle \mathrm{M}', \mathrm{M}'' \rangle\big) \text{ and } O \xrightarrow{d^{\alpha_{k+1}}_{\lambda_{k+1}}} O'.$$

It is clear that $pr_1(O') = P'$. □

3. PRECUBICAL SPACES

3.1. *Category* $\mathfrak{Space}^{\le}$. Precubical spaces were introduced in [7] as topological spaces endowed with a differential structure. This structure is determined by cubes that are realized in the space and a family of norms on the tangent bundle. For the formal definition of a precubical space, we need some notions and notation.

Consider the unit cube of dimension $n \ge 0$ in $\mathbb{R}^n$, i.e., let

$$\square^n = \begin{cases} \{0\} & \text{if } n = 0, \\ \big\{(t_1, \dots, t_n) \in \mathbb{R}^n \bigm| 0 \le t_i \le 1,\ 1 \le i \le n\big\} & \text{otherwise.} \end{cases}$$

Let $\mathring{\square}^n$ denote the topological interior of $\square^n$. We assume that $\mathring{\square}^0 = \{0\}$.

By boundary mappings

$$\delta_\lambda^\alpha : \square^n \to \square^{n+1} \quad \big(\lambda \in \{1, \ldots, n+1\},\ \alpha \in \{0,1\},\ n \geq 0,\big)$$

we mean continuous mappings with

$$\delta_\lambda^\alpha(t_1, \ldots, t_n) = (t_1, \ldots, t_{\lambda-1}, \alpha, t_\lambda, \ldots, t_n).$$

Such mappings satisfy the cubical axioms, i.e., we have $\delta_\lambda^\alpha \circ \delta_\mu^\beta = \delta_{\mu+1}^\beta \circ \delta_\lambda^\alpha$ for $\lambda \leq \mu$.

Consider a compactly generated Hausdorff space X. Recall that a topological space X is *compactly generated* if, for every $U \subset X$, the subset U is closed provided that the intersection $U \cap K$ is closed for every compact subset $K \subset X$. By a cube we mean a continuous mapping $x : \square^n \to X$ inducing a homomorphism from $\mathring{\square}^n$. The mapping x endows the set $x\big(\mathring{\square}^n\big)$ with a trivial structure of a differentiable manifold (see [1]). The definition of local coordinates on $x\big(\mathring{\square}^n\big)$, $n > 0$, is standard, i.e., we have $\big(x(t_1, \ldots, t_n)\big)_i = t_i$, $i = 1, \ldots, n$. The definition of the boundary of a cube is well-defined if the collection of cubes is closed with respect to boundary mappings. We require that, for every cube $x : \square^n \to X$, $n \geq 1$, and all $\lambda \in \{1, \ldots, n\}$ and $\alpha \in \{0,1\}$, the mapping $x \circ \delta_\lambda^\alpha : \square^{n-1} \to X$ be a cube too. As an example, consider the square $\square^2$, the segment $\square^1$, and the torus T depicted in Fig. 6. The mapping x_2 is a continuous mapping from $\square^2$ onto T; moreover, $x_2\big(\mathring{\square}^2\big)$ is a torus without the small circle $x_2(0,t)$, $0 \leq t \leq 1$, and the big circle $x_2(t,0)$, $0 \leq t \leq 1$. The mapping x_1 is a continuous mapping from $\square^1$ onto the small circle of T such that $x_1\big(\mathring{\square}^1\big)$ is the small circle without the intersection point of the circles. We have $x_1 = x_2 \circ \delta_1^0$.

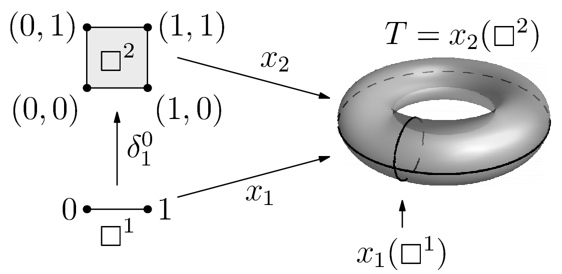


Fig. 6. Boundary of a cube

Let X_n, $n \geq 0$, denote the set of cubes whose domain is $\square^n$. We require that the space X be covered by its cubes. Namely, let X be equal to the disjoint union $\bigsqcup_{x \in X_n,\, n \geq 0} x\big(\mathring{\square}^n\big)$. Finally, we define norms $\|\cdot\|_u$ on the tangent spaces $T_uX =_{\text{def}} T_u x\big(\mathring{\square}^n\big)$, where $u \in x\big(\mathring{\square}^n\big)$, for each point $u \in X$. Put $F(u, \dot{u}) = \|\dot{u}\|_u$. This norm is consistent with the space if it is a continuous mapping from the tangent bundle TX to the half-line $\mathbb{R}^+$. We define a topology on TX in a standard way. Let $\mathcal{B}_X$ be a base of the topology on X. We define $\mathcal{B}_{TX}$ as follows:

$$V \in \mathcal{B}_{TX} \Leftrightarrow V = \bigsqcup_{x \in X_n^U,\, n \geq 0} \xi_x^{-1}(W_x, B_x).$$

Here $U \in \mathcal{B}_X$, $X_n^U = \big\{x \in X_n \;\big|\; U \cap x\big(\mathring{\square}^n\big) \neq \varnothing\big\}$, $\xi_x : Tx\big(\mathring{\square}^n\big) \to \mathring{\square}^n \times \mathbb{R}^n$ is the natural bijection defined for all $x \in X_n$ with $n \geq 0$, $W_x = x^{-1}\big(U \cap x\big(\mathring{\square}^n\big)\big)$, B_x is an open ball in $\mathbb{R}^n$ such that $B_x = pr_\lambda B_{\bar{x}}$ for all $\bar{x} \in X_{n+1}^U$ with $x = \bar{x} \circ \delta_\lambda^\alpha$, and the projection $pr_\lambda : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is defined by the rule

$$pr_\lambda(t_1, \ldots, t_{n+1}) = \big(t_1, \ldots, \widehat{t_\lambda}, \ldots, t_{n+1}\big) \text{ for all } (t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1}.$$

It is easy to verify that the set $\mathcal{B}_{TX}$ forms a base of the topology TX and is independent of the choice of $\mathcal{B}_X$.

We present the formal definition of a precubical space.

Definition 6. By a (*labeled over a set L of actions*) *precubical space* (*with a distinguished point*) we mean a tuple $\mathrm{X} = \big(X, i_0, l, \|\cdot\|\big)_L$, where

- X is a compactly generated Hausdorff space together with its representation by cubes; namely, we have

$$X = \bigsqcup_{x \in X_n,\, n \geq 0} x\big(\overset{\circ}{\square}{}^n\big),$$

where X_n consists of continuous mappings $x : \square^n \to X$ inducing homomorphisms from $\overset{\circ}{\square}{}^n$ such that $x \circ \delta_\lambda^\alpha \in X_{n-1}$ for all $\alpha = 0, 1$, $1 \leq \lambda \leq n$, and $n > 0$;
- i_0 is a distinguished point of X (called the *initial* point) and $i_0 = x(0)$ for a suitable $x \in X_0$;
- $l : X_1 \to L$ is a *labeling function* from the set of 1-cubes of X to the set L of actions such that $l\big(x \circ \delta_\lambda^0\big) = l\big(x \circ \delta_\lambda^1\big)$ for all $\lambda = 1, 2$ and $x \in X_2$;
- X is endowed with a family of norms $\|\cdot\|_u$ on each tangent space $T_u X =_{\mathrm{def}} T_u x\big(\overset{\circ}{\square}{}^n\big)$, $u \in x\big(\overset{\circ}{\square}{}^n\big)$, such that the formula $F(u, \dot{u}) = \|\dot{u}\|_u$ defines a continuous mapping from the tangent bundle TX (with the topology introduced above) to the half-line $\mathbb{R}^+$ (with the topology induced by the topology of $\mathbb{R}$).

Remark 4. We introduce the value $l(x)$ for every $x \in M_n$, $n \geq 0$, as follows:

$$l(x) = \begin{cases} \varnothing & \text{if } n = 0, \\ \big(l_1(x), \ldots, l_n(x)\big) & \text{if } n > 1. \end{cases}$$

Here

$$l_\lambda(x) = l\big(x \circ \delta_n^0 \circ \cdots \circ \delta_{\lambda+1}^0 \circ \delta_{\lambda-1}^0 \circ \cdots \circ \delta_1^0(x)\big) \ \text{ for all } \ 1 \leq \lambda \leq n.$$

Let $n \geq 0$ and let $0 \leq j \leq n$. For each $(\Gamma, \Theta) \in A(j, n)$, we define the $(n-j)$-component of a cube $x \in X_n$ as follows:

$$x \circ \Delta_\Theta^\Gamma = \begin{cases} x & \text{if } j = 0, \\ x \circ \delta_{\theta_j}^{\gamma_j} \circ \cdots \circ \delta_{\theta_1}^{\gamma_1} & \text{otherwise.} \end{cases}$$

Let

$$x \circ \Delta = \bigcup_{(\Gamma,\Theta) \in A(n)} x \circ \Delta_\Theta^\Gamma.$$

Since x is continuous, for every topological space X satisfying the first condition of Definition 6, the following assertion holds: If U is open in X then $x^{-1}(U)$ is open in $\square^n$ for every $x \in X_n$ and $n \geq 0$, where the topology on $\square^n \subseteq \mathbb{R}^n$ is induced by the standard topology on $\mathbb{R}^n$.

We say that X is a $\square$-*topological space* if its topology is defined as follows:

U is open in X if and only if $x^{-1}(U)$ is open in $\square^n$ for all $x \in X_n$ and $n \geq 0$.

Precubical sets allow us to construct models of concurrent processes. Precubical spaces additionally allow us to estimate the duration of a computation in the space X by using the norms on the tangent bundle TX. In fact, the tangent space consists of possible “directions” at a point u and the norms characterize infinitesimal lengths of computations at this point.

By a path (representing computation) in a precubical space we will mean a curve connecting two points in X. We present the formal definition. A continuous mapping $\gamma : [0,1] \to X$ is a *path* in a precubical space X if there exist open intervals $I_j = (\tau_{j-1}, \tau_j)$ and cubes $x_j \in X_{n_j}$, $1 \leq j \leq m$, such that $\tau_0 = 0$, $\tau_m = 1$, and, for each $1 \leq j \leq m$, the following conditions hold: The mapping $\gamma : I_j \to x_j(\mathring{\Box}^{n_j})$ is nondecreasing with respect to each coordinate of the cube x_j and the mapping $x_j^{-1} \circ \gamma : I_j \to \mathring{\Box}^{n_j}$ is differentiable for each $n_j > 0$. The *lenght* of a path γ (the duration of a computation) is defined in the natural way, i.e., $length(\gamma) = \int_0^1 \|\dot{\gamma}(s)\|_{\gamma(s)} ds$.

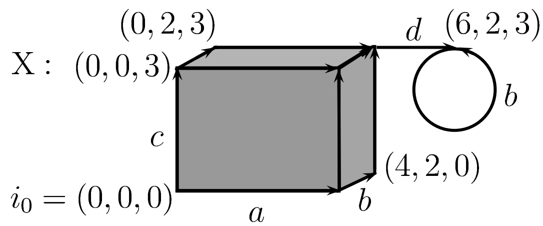


Fig. 7. Precubical space X

Example 4. The precubical space

$$\mathrm{X} = \Big(X = x(\Box_3) \cup x_1(\Box_1) \cup x_0(\Box_1), i_0, l, \|\cdot\|\Big)_L$$

is labeled over the set $L = \{a, b, c, d\}$ (see Fig. 7). This space is generated by the following cubes:

the 3-cube $x(t_1, t_2, t_3) = (4t_1, 2t_2, 3t_3)$ $\big((t_1, t_2, t_3) \in \Box_3\big)$,

the 1-cube $x_1(t) = (4 + 2t, 2, 3)$ $(t \in \Box_1)$,

the 1-cube $x_0(t) = \big(6 - \sin(2\pi t), 2, 2 + \cos(2\pi t)\big)$ $(t \in \Box_1)$.

These cubes are depicted in Fig. 7 as the cube, the segment, and the circle respectively. The initial point is $i_0 = (0, 0, 0)$. The labeling function is defined by the formulas

$$l_1(x) = a, \quad l_2(x) = b, \quad l_3(x) = c, \quad l(x_1) = d, \quad l(x_0) = b.$$

The norm $\|\cdot\|_u$, $u \in X$, is induced by the Euclidean norm in $\mathbb{R}^3$. Notice that the interior of the cube is the union of all paths (computations) such that the actions a, b, and c are executed simultaneously. The length of the path along the unidimensional boundary of the 3-cube with action a is equal to 4. For the actions b and c, the corresponding lengths are equal to 2 and 3 respectively. Therefore, the lengths of paths in the cube starting with $(0, 0, 0)$ and ending with $(4, 2, 3)$ vary from $\sqrt{4^2 + 2^2 + 3^2}$ to $4 + 2 + 3$.

A morphism establishes a correspondence between a topological space and the set of actions of one precubical space and a topological space and the set of actions of another precubical space and satisfies some additional requirements.

Definition 7. Let

$$\mathrm{X} = \big(X, i_0^{\mathrm{X}}, l^{\mathrm{X}}, \|\cdot\|^{\mathrm{X}}\big)_{L^{\mathrm{X}}} \quad \text{and} \quad \mathrm{Y} = \big(Y, i_0^{\mathrm{Y}}, l^{\mathrm{Y}}, \|\cdot\|^{\mathrm{Y}}\big)_{L^{\mathrm{Y}}}$$

be precubical spaces. A mapping $\mathrm{f} = \langle f, \sigma\rangle$, where $f : X \to Y$ is a continuous mapping and $\sigma : L^{\mathrm{X}} \to L^{\mathrm{Y}}$ is a set mapping, is called a *morphism* from X to Y if the following conditions hold:

(1) $f(i_0^{\mathrm{X}}) = i_0^{\mathrm{Y}}$;

(2) for every mapping $x \in X_n$ with $n \geq 0$, there exists a mapping $y \in Y_n$ such that

 (a) the diagram in Fig. 8 commutes,

 (b) $l^{\mathrm{Y}}(y) = \sigma\big(l^{\mathrm{X}}(x)\big)$;

(3) $\big\|d_u f(\dot{u})\big\|^{\mathrm{Y}}_{f(u)} \leq \|\dot{u}\|^{\mathrm{X}}_u$ for all $\dot{u} \in T_u X$ and $u \in X$.

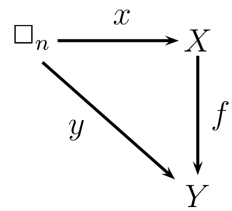


Fig. 8. Diagram relating a cube $x \in X_n$ to a cube $y \in Y_n$ via f

Remark 5. Since the diagram in Fig. 8 commutes, the differential $df : TX \to TY$ of the mapping f is described as follows: For $u \in x(\mathring{\Box}^n)$, $f(u) \in y(\mathring{\Box}^n)$, and $\dot{u} \in T_u X$, we have

$$d_u f(\dot{u}) = \xi_y^{-1} \circ d(y^{-1} \circ f \circ x) \circ \xi_x(u, \dot{u}) = \xi_y^{-1} \circ \xi_x(u, \dot{u}).$$

Condition (1) of Definition 7 means that each morphism takes the initial point into the initial point. Condition (2) means that each morphism takes an n-cube in X into an n-cube in Y; hence, the actions of the cubes are consistent. Condition (3) means that df is a nonexpanding mapping, i.e., the length of the image of a path in X does not exceed the length of this path. If the equality $\|d_u f(\dot{u})\|_{f(u)} = \|\dot{u}\|_u$ holds for all $\dot{u} \in T_u X$ and $u \in X$ then df is an isometry, i.e., it preserves the length of each path.

Precubical spaces and morphisms between them form the category $\mathfrak{Space}^{\leq}$, where the composition of morphisms $\mathrm{f} = \langle f, \sigma \rangle : \mathrm{X} \to \mathrm{Y}$ and $\mathrm{g} = \langle g, \varrho \rangle : \mathrm{Y} \to \mathrm{Z}$ is the morphism $\mathrm{g} \circ \mathrm{f} = \langle g \circ f, \varrho \circ \sigma \rangle : \mathrm{X} \to \mathrm{Z}$ and the identity morphism is the pair of identity mappings.

We consider an auxiliary property of mappings between topological spaces.

Lemma 5. *Let X and Y be topological spaces satisfying condition* (1) *of Definition* 6. *Let $f : X \to Y$ be a mapping satisfying condition* (2)(a) *of Definition* 7. *If X is a $\Box$-topological space then $f : X \to Y$ is a continuous mapping*; *moreover, $df : TX \to TY$ is continuous too.*

Proof. We first show that the mapping $f : X \to Y$ is continuous. Consider an open set V in Y. Since each mapping $y : \Box^n \to Y$ is continuous, the set $y^{-1}(V)$ is open in $\Box^n$ for each $y \in Y_n$ with $n \geq 0$. In particular, the set $x^{-1} \circ f^{-1}(V)$ is open in $\Box^n$ for each $x \in X_n$ with $n \geq 0$. Since X is a $\Box$-topological space, the set $f^{-1}(V)$ is open in X.

Now we prove that the mapping $df : TX \to TY$ is continuous. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases of the topologies on X and Y respectively. Consider an arbitrary set $\widetilde{V}$ in the base $\mathcal{B}_{TY}$ of the topology on TY, i.e., let

$$\widetilde{V} = \bigsqcup_{y \in Y_n^V,\, n \geq 0} \xi_y^{-1}(W_y, B_y),$$

where $V \in \mathcal{B}_Y$, $Y_n^V = \{y \in Y_n \mid V \cap y(\mathring{\Box}^n) \neq \varnothing\}$, $\xi_y : Ty(\mathring{\Box}^n) \to \mathring{\Box}^n \times \mathbb{R}^n$ is the natural bijection, $W_y = y^{-1}(V \cap y(\mathring{\Box}^n))$, and B_y is an open ball in $\mathbb{R}^n$; moreover, assume that $y = \bar{y} \circ \delta_k^m$ implies $B_y = pr_k B_{\bar{y}}$.

We need to show that $(df)^{-1}(\widetilde{V})$ is an open set in TX. We have

$$(df)^{-1}(\widetilde{V}) = \bigsqcup_{y \in Y_n^V,\, n \geq 0} (df)^{-1}\big(\xi_y^{-1}(W_y, B_y)\big) = \bigsqcup_{\substack{x \in \{x \mid f \circ x \in Y_n^V\},\\ n \geq 0}} \xi_x^{-1}(W_{f \circ x}, B_{f \circ x}).$$

Since $f : X \to Y$ is continuous and $V \in \mathcal{B}_Y$, we conclude that $f^{-1}(V)$ is an open set in X, i.e., we have $f^{-1}(V) = \bigcup_\alpha U_\alpha$, where $U_\alpha \in \mathcal{B}_X$. It is easy to verify that $(df)^{-1}(\widetilde{V}) = \bigcup_\alpha \widetilde{U}_\alpha$, where

$$\widetilde{U}_\alpha = \bigsqcup_{x_\alpha \in X_n^{U_\alpha},\, n \geq 0} \xi_{x_\alpha}^{-1}\Big(x_\alpha^{-1}\big(U_\alpha \cap x_\alpha(\mathring{\Box}^n)\big), B_{(f \circ x_\alpha)}\Big) \in \mathcal{B}_{TX}.$$

Thus, $(df)^{-1}(\widetilde{V})$ is an open set in TX. $\Box$

3.2. *Connections between* pSet *and* $\mathfrak{Space}^{\leq}$. We construct mappings between the categories pSet and $\mathfrak{Space}^{\leq}$. These mappings will allow us to transfer the results of Theorems 1 and 2 (see Subsections 2.3 and 2.4) to precubical spaces.

In his thesis [7], Goubault mentioned adjoint functors

$$\mathcal{T} : \text{p}\mathbb{S}\text{et} \to \mathfrak{Space} \quad \text{and} \quad \mathcal{F}t : \mathfrak{Space} \to \text{p}\mathbb{S}\text{et}$$

between the categories pSet and $\mathfrak{Space}$. The objects of $\mathfrak{Space}$ are precubical spaces (see Definition 6) and the morphisms of $\mathfrak{Space}$ are mappings satisfying conditions (1) and (2) of Definition 7. We consider these functors in connection with the categories pSet and $\mathfrak{Space}^{\leq}$.

Proposition 1. 1. *We define a mapping* $\mathcal{T} : \text{p}\mathbb{S}\text{et} \to \mathfrak{Space}^{\leq}$. *For an object* $(M, i_0^{\mathrm{M}}, l^{\mathrm{M}})_L$, *we put*

$$\mathcal{T}\big((M, i_0^{\mathrm{M}}, l^{\mathrm{M}})_L\big) = \big(X, i_0^{\mathrm{X}}, l^{\mathrm{X}}, \|\cdot\|^{\mathrm{X}}\big)_L,$$

where

- $X = \bigsqcup_{x \in M_n,\, n \geq 0} (x, \square^n) \,/ \equiv$ *is endowed with the topology of the quotient space induced by the topology of the direct sum on the space* $\bigsqcup_{x \in M_n,\, n \geq 0} (x, \square^n)$, *where each* $(x, \square^n)$ *inherits the standard topology of* $\mathbb{R}^n$ *and* $\equiv$ *is defined as follows*:

  $$\big(d_\lambda^\alpha(x), \square^{n-1}\big) \equiv \big(x, \delta_\lambda^\alpha(\square^{n-1})\big)\Big);$$

  we denote

  $$X_n = \big\{(x, \cdot) : \square^n \to X \;\big|\; x \in M_n\big\};$$

- $i_0^{\mathrm{X}} = (i_0^{\mathrm{M}}, \square^0)$;
- $l^{\mathrm{X}}(x, \cdot) = l^{\mathrm{M}}(x)$ *for all* $(x, \cdot) \in X_1$;
- $\|\dot{t}\|_{(x,t)} = \max_{1 \leq i \leq n} |\dot{t}_i|$ *for all* $\dot{t} = (\dot{t}_1, \ldots, \dot{t}_n) \in \mathbb{R}^n = T_{(x,t)}\big(x, \overset{\circ}{\square}{}^n\big)$ *and all* $(x, t) \in X$ (*recall that this norm is called a Chebyshev norm*).

For a morphism $\langle f, \sigma \rangle : \mathrm{M} \to \mathrm{M}'$, *we put*

$$\mathcal{T}\big(\langle f, \sigma \rangle\big) = \langle \bar{f}, \sigma \rangle,$$

where $\bar{f}(x, t) = \big(f(x), t\big)$ *for all points* (x, t) *in* $\mathcal{T}(\mathrm{M})$. *Then* $\mathcal{T}$ *is a functor* (*usually called the geometric realization functor*).

2. *We define a mapping* $\mathcal{F}t : \mathfrak{Space}^{\leq} \to \text{p}\mathbb{S}\text{et}$. *For an object* $\big(X, i_0^{\mathrm{X}}, l^{\mathrm{X}}, \|\cdot\|^{\mathrm{X}}\big)_L$, *we put*

$$\mathcal{F}t\Big(\big(X, i_0^{\mathrm{X}}, l^{\mathrm{X}}, \|\cdot\|^{\mathrm{X}}\big)_L\Big) = \big(M, i_0^{\mathrm{M}}, l^{\mathrm{M}}\big)_L,$$

where

- $M_n = X_n$ *for* $n \geq 0$ *and* $d_i^m(x) = x \circ \delta_i^m$ *for all* $x \in X_n$ *and* $n \geq 1$;
- $i_0^{\mathrm{M}} = x_0$, *where* $x_0(0) = i_0^{\mathrm{X}}$;
- $l^{\mathrm{M}} = l^{\mathrm{X}}$.

For a morphism $\langle f, \sigma \rangle : \mathrm{X} \to \mathrm{X}'$, *we put*

$$\mathcal{F}t\big(\langle f, \sigma \rangle\big) = \langle \check{f}, \sigma \rangle,$$

where $\check{f}(x) = f \circ x$ *for all cubes* x *in* $\mathcal{F}t(\mathrm{X})$. *Then* $\mathcal{F}t$ *is a functor* (*usually called the forgetful functor*).

Proof. A morphism of the category $\mathfrak{Space}^{\leq}$ is a morphism of the category $\mathfrak{Space}$ additionally satisfying condition (3) of Definition 7. Hence, it suffices to prove that this condition holds for the mapping $\langle \bar{f}, \sigma \rangle = \mathcal{T}\big(\langle f, \sigma \rangle\big)$, where $\langle f, \sigma \rangle$ is a morphism of the category pSet. It is clear that the inequality $\big\|d_{(x,t)} \bar{f}(\dot{t})\big\|_{\bar{f}(x,t)} \leq \|\dot{t}\|_{(x,t)}$ becomes an identity because the vectors are the same and the norms are Chebyshev norms. ∎

We mention a difference from the article [7]. Namely, the functors $\mathcal{T}$ and $\mathcal{F}t$ between the categories p$\mathbb{S}$et and $\mathfrak{Space}^{\leq}$ need not be adjoint. Indeed, the requirements on the norms of the precubical space in the image of $\mathcal{T}$ are too strong and need not agree with condition (3) of Definition 7. We weaken this requirement. We construct functors between the comma category $\mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}} \downarrow \mathcal{F}t$ and the category of morphisms

$$\overrightarrow{\mathfrak{Space}^{\leq}} = \mathcal{I}d_{\mathfrak{Space}^{\leq}} \downarrow \mathcal{I}d_{\mathfrak{Space}^{\leq}},$$

which allows us to prove analogs of Theorems 1 and 2 for precubical spaces.

Proposition 2. 1. *We define a mapping*

$$\mathcal{G} : \mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}} \downarrow \mathcal{F}t \to \overrightarrow{\mathfrak{Space}^{\leq}}.$$

For objects, put

$$\mathcal{G}\big(\mathrm{M}, \mathrm{f} = \langle f, \sigma\rangle, \mathrm{Y}\big) = \widehat{\mathrm{f}} = \langle \widehat{f}, \sigma\rangle : \mathcal{T}_{\mathrm{f},\mathrm{Y}}(\mathrm{M}) \to \mathrm{Y},$$

where $\widehat{f}(x,t) = f(x)(t)$ *for all points* (x,t) *in* $\mathcal{T}_{\mathrm{f},\mathrm{Y}}(\mathrm{M})$ *and the structure*

$$\mathcal{T}_{\mathrm{f},\mathrm{Y}}(\mathrm{M}) = \big(X, i_0^{\mathrm{X}}, l^{\mathrm{X}}, \|\cdot\|^{\mathrm{X}}\big)_L$$

consists of X, i_0^{X}, *and* l^{X} *as defined in condition* (1) *of Assertion* 1 *and the norm defined by the rule*

$$\|\cdot\|^{\mathrm{X}}_{(x,t)} = \big\|d_{(x,t)}\widehat{f}(\cdot)\big\|^{\mathrm{Y}}_{\widehat{f}(x,t)}$$

for all $(x,t) \in X$. *For morphisms, put*

$$\mathcal{G}\Big(\big(\mathrm{g} = \langle g, \sigma_g\rangle, \mathrm{h} = \langle h, \sigma_h\rangle\big) : (\mathrm{M}, \mathrm{f}, \mathrm{Y}) \to (\mathrm{M}', \mathrm{f}', \mathrm{Y}')\Big)$$
$$= \big(\bar{\mathrm{g}} = \langle \bar{g}, \sigma_g\rangle, \mathrm{h} = \langle h, \sigma_h\rangle\big) : \widehat{\mathrm{f}} \to \widehat{\mathrm{f}}',$$

where $\bar{g}(x,t) = \big(g(x), t\big)$ *for all points* $(x,t) \in X$. *Then* $\mathcal{G}$ *is a functor.*

2. *The functor* $\mathcal{F}t$ *induces a mapping*

$$\mathcal{F} : \overrightarrow{\mathfrak{Space}^{\leq}} \to \mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}} \downarrow \mathcal{F}t.$$

For objects, we have

$$\mathcal{F}(\mathrm{f} : \mathrm{X} \to \mathrm{Y}) = \big(\mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{f}), Y\big).$$

For morphisms, we have

$$\mathcal{F}(\mathrm{g}, \mathrm{h}) = \big(\mathcal{F}t(\mathrm{g}), \mathrm{h}\big).$$

Then $\mathcal{F}$ *is a functor.*

Proof. We prove the first assertion only. The second assertion is obvious. We show that X is a $\square$-topological space. Consider a mapping $(x,\cdot) : \square^n \to X$ in X_n. It is clear that it can be represented as $\phi \circ \iota_x \circ \sigma_x$, where

$\sigma_x : \square^n \to (x, \square^n)$ is the identity mapping,
$\iota_x : (x, \square^n) \to \coprod_{x \in M_n,\, n\geq 0} (x, \square^n)$ is the inclusion mapping,
$\phi : \coprod_{x \in M_n,\, n\geq 0} (x, \square^n) \to X$ is the quotient mapping.

By the definitions of the corresponding topologies, we have

$$U \text{ is open in } X \Leftrightarrow \phi^{-1}(U) \text{ is open in } \coprod_{x \in M_n,\, n\geq 0} (x, \square^n)$$
$$\Leftrightarrow \iota_x^{-1}\big(\phi^{-1}(U)\big) \text{ is open in } (x, \square^n)\ \forall x \in M_n\ (n \geq 0)$$

$$\Leftrightarrow \sigma_x^{-1}\Big(\iota_x^{-1}\big(\phi^{-1}(U)\big)\Big) \text{ is open in } \square^n \; \forall x \in M_n \; (n \geq 0),$$

i.e., we have

$$(x,\cdot)^{-1}(U) \text{ is open in } \square^n \; \forall (x,\cdot) \in X_n \; (n \geq 0).$$

We conclude that X is a $\square$-topological space. In view of the definition of $\widehat{f}$, condition (2)(a) of Definition 7 holds. By Lemma 5, the mapping $\widehat{f}$ is continuous. It is clear that the mapping

$$\langle \widehat{f}, \sigma \rangle : \mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}) \to \mathrm{Y}$$

satisfies the remaining conditions of Definition 7. We need to prove that $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M})$ is a precubical space (hence, $\langle \widehat{f}, \sigma \rangle$ is an object of the category $\overrightarrow{\mathfrak{Space}}$). By Assertion 1 (1), it suffices to prove that the norm $\|\cdot\|^{\mathrm{X}}$ is continuous on TX. By the construction of $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M})$, we have $\|\cdot\|^{\mathrm{X}} = \|\cdot\|^{Y} \circ d\widehat{f}$. By Lemma 5, the mapping $d\widehat{f}$ is continuous. Since Y is a precubical space, the norm $\|\cdot\|^{Y}$ is continuous. The norm $\|\cdot\|^{\mathrm{X}}$ is the composition of two continuous mappings; hence, it is continuous too. We have proven that $\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M})$ is an object of the category $\mathfrak{Space}^{\leq}$; hence, $\widehat{\mathrm{f}} = \langle \widehat{f}, \sigma \rangle : \mathcal{T}_{\mathrm{f,Y}}(\mathrm{M}) \to \mathrm{Y}$ is an object of the category $\overrightarrow{\mathfrak{Space}}^{\leq}$.

Let $(\mathrm{g}, \mathrm{h}) : (\mathrm{M}, \mathrm{f}, \mathrm{Y}) \to (\mathrm{M}', \mathrm{f}', \mathrm{Y}')$ be a morphism of the category $\mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}} \downarrow \mathcal{F}t$, i.e., let the equality $\mathrm{f}' \circ \mathrm{g} = \mathrm{h} \circ \mathrm{f}$ hold. Then $\widehat{f'} \circ \bar{g} = h \circ \widehat{f}$. It is easy to see that $\bar{\mathrm{g}}$ is a morphism of $\mathfrak{Space}^{\leq}$ and $(\bar{\mathrm{g}}, \mathrm{h})$ is a morphism of $\overrightarrow{\mathfrak{Space}}^{\leq}$. ☐

Proposition 3. *The functors $\mathcal{F}$ and $\mathcal{G}$ are adjoint. Moreover, the counit of adjunction $\varepsilon : \mathcal{F}\mathcal{G} \to \mathcal{I}d_{\mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}} \downarrow \mathcal{F}t}$ is the natural isomorphism.*

Proof. For an arbitrary object $(\mathrm{M}, \mathrm{f}, \mathrm{Y})$ of the category

$$\mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}} \downarrow \mathcal{F}t,$$

the counit of adjunction ε is defined as follows:

$$\varepsilon_{(\mathrm{M,f,Y})} = \Big(\langle \varepsilon_{(M,f,Y)}, 1_L \rangle, 1_{\mathrm{Y}}\Big) : \Big(\mathcal{F}t\big(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M})\big), \mathcal{F}t(\widehat{\mathrm{f}}), Y\Big) \to (\mathrm{M}, \mathrm{f}, \mathrm{Y}),$$

where $\varepsilon_{(M,f,Y)}(x,\cdot) = x$ for all cubes $(x,\cdot)$ in $\mathcal{F}t\big(\mathcal{T}_{\mathrm{f,Y}}(\mathrm{M})\big)$. Consider the morphism

$$\varepsilon^{-1}_{(\mathrm{M,f,Y})} = \Big(\langle \varepsilon^{-1}_{(M,f,Y)}, 1_L \rangle, 1_{\mathrm{Y}}\Big)$$

of the category $\mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}} \downarrow \mathcal{F}t$ such that $\varepsilon^{-1}_{(M,f,Y)}(x) = (x,\cdot)$ for all $x \in M$. This morphism turns ε into the natural isomorphism.

For an arbitrary object $\mathrm{f} : \mathrm{X} \to \mathrm{Y}$ of the category $\overrightarrow{\mathfrak{Space}}^{\leq}$, the unit of adjunction $\eta : \mathcal{I}d_{\overrightarrow{\mathfrak{Space}}^{\leq}} \to \mathcal{G}\mathcal{F}$ is defined as follows:

$$\eta_{\mathrm{f}} = \big(\langle \eta_f, 1_L \rangle, 1_{\mathrm{Y}}\big) : (\mathrm{f} : \mathrm{X} \to \mathrm{Y}) \to \Big(\widehat{\mathcal{F}t(\mathrm{f})} : \mathcal{T}_{\mathcal{F}t(\mathrm{f}),\mathrm{Y}}\big(\mathcal{F}t(\mathrm{X})\big) \to \mathrm{Y}\Big),$$

where $\eta_f\big(x(t)\big) = (x, t)$ for all $x(t) \in X$. ☐

3.3. *Subcategory $\mathfrak{P}^{\leq}$.* The functor $\mathcal{F}t$ allows us to forget that cubes in a precubical space X are continuous mappings and regard them as elements of a discrete set. The definitions of a cubical path, an extension and a restriction of a cubical path, the s-adjointness, the homotopy, and the dl-connectedness are easily reformulated for (continuous) precubical spaces. Indeed, for a cube p, it suffices to replace each expression of the form $d^{\alpha}_{\lambda}(p)$ by an expression of the form $p \circ \delta^{\alpha}_{\lambda}$. If P is a cubical path in a precubical space X, then $P_{\mathcal{F}t}$ denotes the corresponding cubical path in the precubical set $\mathcal{F}t(\mathrm{X})$. Let $\mathcal{CP}(\mathrm{X})$ $\big(\mathcal{CP}_p(\mathrm{X})\big)$ denote the set of all cubical paths (ending with a cube p) in a precubical space X. A point u in a precubical space X is said to be *reachable* if there exists a cubical path $P \in \mathcal{CP}_p(\mathrm{X})$ such that $u \in p(\mathring{\square}^n)$ and $p \in X_n$.

The following assertion is obvious.

Assertion 2. *Let $P = p_0 p_1 \dots p_{k-1} p_k$ be a cubical path in a precubical space* $\mathrm{X} = \big(X, i_0, l, \|\cdot\|\big)_L$. *Let*

$$\mathrm{X}' = \Big(X', i_0, l\big|_{X_1'}, (\|\cdot\|)\big|_{X'}\Big)_L,$$

where

$$X' = \bigsqcup_{x \in X_n',\, n \geq 0} x(\overset{\circ}{\Box}{}^n) \subseteq X \quad \left(X_n' = \bigcup_{0 \leq s \leq k} \big((p_s \circ \Delta) \cap X_n\big)\right)$$

is endowed with the topology of a subset. Then X', *is a precubical space labeled over L. Moreover,* X' *is a precubical subspace of the precubical space* X. (*We will say that* X' *has the form of the cubical path P in* X.)

For a natural N, we put

$$\Box^{\mathrm{N}} = \begin{cases} \big(\{0\}, (0), \varnothing, \|\cdot\| \equiv 0\big)_L & \text{if } N = 0, \\ \big(\Box^N, (0, \dots, 0), l, \|\cdot\|\big)_L & \text{otherwise.} \end{cases}$$

The topology of the space

$$\Box^N = \bigsqcup_{\Delta_\Theta^\Gamma \in \Box_n^N,\, 0 \leq n \leq N} \Delta_\Theta^\Gamma(\overset{\circ}{\Box}{}^n) \quad \left(\Box_n^N = \bigcup_{(\Gamma, \Theta) \in A(N-n, N)} \Delta_\Theta^\Gamma\right)$$

is induced by the standard topology of $\mathbb{R}^N$, l is a labeling function satisfying the equality $l(x \circ \delta_\lambda^0) = l(x \circ \delta_\lambda^1)$ for all $1 \leq \lambda \leq 2$ and $x \in \Box_2^N$, and $\|\cdot\|$ is a continuous norm on the tangent bundle $T\Box^N$. Then $\Box^{\mathrm{N}}$ is a precubical space.

A cubical path $P \in \mathcal{CP}_p(\Box^{\mathrm{N}})$ is *consistent* with $\Box^N$ if either $N = 0$ or the equality

$$p \circ \Delta_\Theta^\Gamma(\overset{\circ}{\Box}{}^0) = (1, \dots, 1)$$

holds, where $\Gamma = (1, \dots, 1)$ and $\Theta = (1, \dots, \dim p)$.

A *path-object* is a precubical space $\widetilde{\Box}$ having the form of a suitable cubical path $P \in \mathcal{CP}(\Box^{\mathrm{N}})$, $N \geq 0$, that is consistent with $\Box^N$. Let $\mathfrak{P}^{\leq}$ denote the full subcategory of path-objects of the category $\mathfrak{Space}^{\leq}$.

We present a series of obvious facts.

Lemma 6. *Let $N \geq 0$. Then $\Box^N$ is a $\Box$-topological space.*

Lemma 7. *Let* $\mathrm{f} = \langle f, \sigma \rangle : \mathrm{X} \to \mathrm{Y}$ *be a morphism of the category* $\mathfrak{Space}^{\leq}$. *For every cubical path* $P = p_0 \xrightarrow{\delta_{\lambda_1}^{\alpha_1}} \cdots \xrightarrow{\delta_{\lambda_k}^{\alpha_k}} p_k \in \mathcal{CP}(\mathrm{X})$, *the following assertions hold*:

(1) $f(P) = (f \circ p_0) \xrightarrow{\delta_{\lambda_1}^{\alpha_1}} \cdots \xrightarrow{\delta_{\lambda_k}^{\alpha_k}} (f \circ p_k) \in \mathcal{CP}(\mathrm{Y})$;

(2) *if* $P \xrightarrow{\delta_{\lambda_{k+1}}^{\alpha_{k+1}}} P'$ *in* X *then* $f(P) \xrightarrow{\delta_{\lambda_{k+1}}^{\alpha_{k+1}}} f(P')$ *in* Y;

(3) *if* $P \xleftrightarrow{s} P'$ *in* X *then* $f(P) \xleftrightarrow{s} f(P')$ *in* Y.

3.4. *Connections between* $\mathbb{P}$ *and* $\mathfrak{P}^{\leq}$. We consider several connections between objects of the subcategories $\mathbb{P}$ and $\mathfrak{P}^{\leq}$.

Lemma 8. *Let* $\tilde{\Box}$ *be an объект of* $\mathfrak{P}^{\leq}$. *Then there exists an object* $\tilde{\tilde{\Box}}$ *of* $\mathbb{P}$ *such that the mapping* $\chi_{\tilde{\Box}} : \tilde{\tilde{\Box}} \to \mathcal{F}t(\tilde{\Box})$ *is an isomorphism of* p$\mathbb{S}$et.

Proof. Let $\tilde{\Box}$ have the form of a cubical path

$$\widetilde{P} = \tilde{p}_0 \xrightarrow{\delta^{\alpha_1}_{\lambda_1}} \tilde{p}_1 \dots \tilde{p}_{k-1} \xrightarrow{\delta^{\alpha_k}_{\lambda_k}} \tilde{p}_k$$

in the precubical space $\Box^{\mathrm{N}}$ labeled over L. Each cube in $\Box^{\mathrm{N}}$ has the form Δ^{Γ}_{Θ}, where $(\Gamma, \Theta) \in A(N)$. We may assume that (Γ_s, Θ_s) corresponds to $\tilde{p}_s$ and $\dim \tilde{p}_s = n_s$ for $0 \leq s \leq k$. In $\boxplus^N$, we consider the cubes

$$p_s = D^{\Gamma_s}_{\Theta_s}\left(\frac{1}{2}, \dots, \frac{1}{2}\right) \ (0 \leq s \leq k).$$

Let $\tilde{\tilde{\Box}}$ denote the precubical set having the form of the cubical path

$$P = p_0 \xrightarrow{\mathsf{d}^{\alpha_1}_{\lambda_1}} p_1 \dots p_{k-1} \xrightarrow{\mathsf{d}^{\alpha_k}_{\lambda_k}} p_k$$

in the precubical set $\boxplus^{\mathrm{N}}$ labeled over L. (If $\widetilde{P}$ is consistent with $\Box^N$ then P is consistent with $\boxplus^N$. Hence, the labeling function $l^{\boxplus^{\mathrm{N}}}$ is completely determined by the cubes p_s, i.e., we have $l^{\boxplus^{\mathrm{N}}}(p_s) = l^{\tilde{\Box}}(\tilde{p}_s)$ for all $1 \leq s \leq k$). Thus, $\tilde{\tilde{\Box}}$ is an object of $\mathbb{P}$.

We construct a mapping $\chi_{\tilde{\Box}} = \langle \chi_{\tilde{\Box}}, 1_L \rangle : \tilde{\tilde{\Box}} \to \mathcal{F}t(\tilde{\Box})$ and its inverse $\chi^{-1}_{\tilde{\Box}} = \langle \chi^{-1}_{\tilde{\Box}}, 1_L \rangle$. We put

$$\chi_{\tilde{\Box}}\big(D^{\Gamma}_{\Theta}(p_s)\big) = \tilde{p}_s \circ \Delta^{\Gamma}_{\Theta} \quad \text{and} \quad \chi^{-1}_{\tilde{\Box}}\big(\tilde{p}_s \circ \Delta^{\Gamma}_{\Theta}\big) = D^{\Gamma}_{\Theta}(p_s)$$

for all $(\Gamma, \Theta) \in A(n_s)$ and $0 \leq s \leq k$. It is clear that these mappings are isomorphisms of p$\mathbb{S}$et. $\Box$

Lemma 9. *Let* $\bar{\Box}$ *be an object of* $\mathbb{P}$ *and let* $\pi = \langle \pi, \sigma \rangle : \bar{\Box} \to \mathcal{F}t(\mathrm{X})$ *be a morphism of* p$\mathbb{S}$et. *Then there exists an object* $\overset{\simeq}{\Box}$ *of* $\mathfrak{P}^{\leq}$ *such that the mapping* $\zeta_{\bar{\Box}} : \overset{\simeq}{\Box} \to \mathcal{T}_{\pi,\mathrm{X}}(\bar{\Box})$ *is an isomorphism of* $\mathfrak{Space}^{\leq}$.

Proof. Let $\bar{\Box}$ have the form of a cubical path

$$P = p_0 \xrightarrow{\mathsf{d}^{\alpha_1}_{\lambda_1}} p_1 \dots p_{k-1} \xrightarrow{\mathsf{d}^{\alpha_k}_{\lambda_k}} p_k$$

in the precubical set $\boxplus^{\mathrm{N}}$ labeled over L. Let $p_s = D^{\Gamma_s}_{\Theta_s}\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ for a suitable $(\Gamma_s, \Theta_s) \in A(N)$ and let $\dim p_s = n_s$ for $0 \leq s \leq k$. In $\Box^N$, we consider the cubes

$$\tilde{p}_s = \Delta^{\Gamma_s}_{\Theta_s} \ (0 \leq s \leq k).$$

We define a structure $\overset{\simeq}{\Box} = \left(\overset{\simeq}{\Box}, (0, \dots, 0), l^{\overset{\simeq}{\Box}}, \| \cdot \|^{\overset{\simeq}{\Box}}\right)_L$, where

- $\overset{\simeq}{\Box} = \displaystyle\bigsqcup_{x \in \overset{\simeq}{\Box}_n,\, 0 \leq n \leq N} x(\overset{\circ}{\Box}{}^n) \ \left(\overset{\simeq}{\Box}_n = \left(\bigcup_{0 \leq s \leq k} \tilde{p}_s \circ \Delta\right) \cap \Box^N_n\right);$
- $l^{\overset{\simeq}{\Box}}(\tilde{p}_s) = l^{\bar{\Box}}(p_s)$ for all $1 \leq s \leq k$;

- $\|\cdot\|_u^{\tilde{\tilde{\Box}}} = \left\|d_u\zeta_{\bar{\Box}}(\cdot)\right\|_{\zeta_{\bar{\Box}}(u)}^{\mathcal{T}_{\pi,\mathrm{X}}(\bar{\Box})}$, where $\zeta_{\bar{\Box}}(u) = \zeta_{\bar{\Box}}\left(\tilde{p}_s\big(\Delta^{\Gamma}_{\Theta}(t)\big)\right) = \big(D^{\Gamma}_{\Theta}(p_s), t\big)$ for some $t \in \mathring{\Box}^n$ with $u = \tilde{p}_s\big(\Delta^{\Gamma}_{\Theta}(t)\big)$ and all $u \in \tilde{\tilde{\Box}}$.

Repeating the proof of Assertion 2(1) and taking into account Lemma 6, we conclude that $\tilde{\tilde{\Box}}$ is a precubical space. Moreover, the cubical path $\widetilde{P} = \tilde{p}_0 \xrightarrow{\delta^{\alpha_1}_{\lambda_1}} \tilde{p}_1 \ldots \tilde{p}_{k-1} \xrightarrow{\delta^{\alpha_k}_{\lambda_k}} \tilde{p}_k$ is consistent with $\Box^N$ and $\tilde{\tilde{\Box}}$ has the form of $\widetilde{P}$ in the precubical space $\Box^{\mathrm{N}}$ labeled over L. Notice that the labeling function $l^{\Box^{\mathrm{N}}}$ is completely determined by the function $l^{\tilde{\tilde{\Box}}}$. For $\|\cdot\|^{\Box^{\mathrm{N}}}$, we may take an arbitrary continuous extension of $\|\cdot\|^{\tilde{\tilde{\Box}}}$ to the tangent bundle $T\Box^N$. Therefore, $\tilde{\tilde{\Box}}$ is an object of the category $\mathfrak{P}^{\leq}$.

The inverse of the mapping $\zeta_{\bar{\Box}} = \langle\zeta_{\bar{\Box}}, 1_L\rangle$ is defined by the formula

$$\zeta_{\bar{\Box}}^{-1}\big(D^{\Gamma}_{\Theta}(p_s), t\big) = \tilde{p}_s\big(\Delta^{\Gamma}_{\Theta}(t)\big) \ \text{ for all } \ \big(D^{\Gamma}_{\Theta}(p_s), t\big) \ \text{ in } \ \mathcal{T}_{\pi,\mathrm{X}}(\bar{\Box}).$$

It is clear that both mappings are isomorphisms of the category $\mathfrak{Space}^{\leq}$. ∎

The embedding functors

$$\mathcal{I} : \mathbb{P} \hookrightarrow \mathrm{p}\mathbb{S}\mathrm{et} \quad \text{and} \quad \mathcal{I}_{\mathcal{T}} : \mathfrak{P}^{\leq} \hookrightarrow \mathfrak{Space}^{\leq}$$

induce the embedding functors

$$\mathcal{I} \downarrow \mathcal{F}t \hookrightarrow \mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}} \downarrow \mathcal{F}t \quad \text{and} \quad \mathcal{I}_{\mathcal{T}} \downarrow \mathcal{I}d_{\mathfrak{Space}^{\leq}} \hookrightarrow \overrightarrow{\mathfrak{Space}}^{\leq}.$$

The following assertion is immediate from Lemmas refTcP-cP and 9.

Proposition 4. 1. *We define a mapping* $\widetilde{\mathcal{G}} : \mathcal{I} \downarrow \mathcal{F}t \to \mathcal{I}_{\mathcal{T}} \downarrow \mathcal{I}d_{\mathfrak{Space}^{\leq}}$. *For objects, put*

$$\widetilde{\mathcal{G}}\big(\bar{\Box}, \pi, \mathrm{X}\big) = \Big(\tilde{\tilde{\Box}}, \widehat{\pi} \circ \zeta_{\bar{\Box}}, \mathrm{X}\Big).$$

For morphisms, put

$$\widetilde{\mathcal{G}}\Big((\mathrm{m}, \mathrm{f}) : \big(\,\bar{\Box}, \pi, \mathrm{X}\big) \to \big(\,\bar{\Box}', \pi', \mathrm{Y}\big)\Big) = \Big(\zeta_{\bar{\Box}'}, 1_{\mathrm{Y}}\Big)^{-1} \mathcal{G}(\mathrm{m}, \mathrm{f})\Big(\zeta_{\bar{\Box}}, 1_{\mathrm{X}}\Big).$$

Then $\widetilde{\mathcal{G}}$ *is a functor.*

2. *We define a mapping* $\widetilde{\mathcal{F}} : \mathcal{I}_{\mathcal{T}} \downarrow \mathcal{I}d_{\mathfrak{Space}^{\leq}} \to \mathcal{I} \downarrow \mathcal{F}t$. *For objects, put*

$$\widetilde{\mathcal{F}}(\tilde{\Box}, \pi, \mathrm{X}) = (\bar{\tilde{\Box}}, \mathcal{F}t(\pi) \circ \chi_{\tilde{\Box}}, \mathrm{X}).$$

For morphisms, put

$$\widetilde{\mathcal{F}}\Big((\mathrm{m}, \mathrm{f}) : \big(\tilde{\Box}, \pi, \mathrm{X}\big) \to \big(\tilde{\Box}', \pi', \mathrm{Y}\big)\Big) = \big(\chi_{\tilde{\Box}'}, 1_{\mathrm{Y}}\big)^{-1} \mathcal{F}(\mathrm{m}, \mathrm{f})\big(\chi_{\tilde{\Box}}, 1_{\mathrm{X}}\big).$$

Then $\widetilde{\mathcal{F}}$ *is a functor.*

Proposition 5. *The functors* $\widetilde{\mathcal{F}}$ *and* $\widetilde{\mathcal{G}}$ *are adjoint. Moreover, the counit of adjunction* $\tilde{\varepsilon} : \widetilde{\mathcal{F}}\widetilde{\mathcal{G}} \to \mathcal{I}d_{\mathcal{I}\downarrow\mathcal{F}t}$ *is the natural isomorphism.*

Proof. Consider an arbitrary object $(\bar{\Box},\pi,\mathrm{X})$ of $\mathcal{I}\downarrow\mathcal{F}t$. Since the precubical spaces $\overset{\approx}{\Box}$ and $\bar{\Box}$ coincide, the counit of adjunction $\tilde{\varepsilon}$ is determined by the equality

$$\tilde{\varepsilon}_{(\bar{\Box},\pi,\mathrm{X})}=\Big(\big\langle 1_{\bar{\Box}},1_L\big\rangle,1_{\mathrm{X}}\Big):\Big(\overset{\approx}{\Box},\mathcal{F}t\big(\widehat{\pi}\circ\zeta_{\bar{\Box}}\big)\circ\chi_{\overset{\simeq}{\Box}},\mathrm{X}\Big)\to\big(\bar{\Box},\pi,\mathrm{X}\big).$$

Since

$$\pi\circ\big\langle 1_{\bar{\Box}},1_L\big\rangle=\mathcal{F}t\big(\widehat{\pi}\circ\zeta_{\bar{\Box}}\big)\circ\chi_{\overset{\simeq}{\Box}}\circ 1_{\mathcal{F}t(\mathrm{X})},$$

the mapping $\tilde{\varepsilon}_{(\bar{\Box},\pi,\mathrm{X})}$ is a morphism of $\mathcal{I}\downarrow\mathcal{F}t$. It is clear that the identity transformation $\tilde{\varepsilon}$ is the natural isomorphism.

Consider an arbitrary object $(\tilde{\Box},\pi,\mathrm{X})$ of $\mathcal{I}_{\mathcal{T}}\downarrow\mathcal{I}d_{\mathfrak{Space}^{\leq}}$. Since the precubical spaces $\tilde{\Box}$ and $\overset{\approx}{\Box}$ differ by their norms only, the unit of adjunction $\tilde{\eta}:\mathcal{I}d_{\mathcal{I}_{\mathcal{T}}\downarrow\mathcal{I}d_{\mathfrak{Space}^{\leq}}}\to\widetilde{\mathcal{G}}\widetilde{\mathcal{F}}$ is determined by the equality

$$\tilde{\eta}_{(\tilde{\Box},\pi,\mathrm{X})}=\Big(\big\langle\eta_{(\tilde{\Box},\pi,X)},1_L\big\rangle,1_{\mathrm{X}}\Big):\big(\tilde{\Box},\pi,\mathrm{X}\big)\to\Big(\overset{\approx}{\Box},\mathcal{F}t\widehat{(\pi)\circ\chi_{\tilde{\Box}}}\circ\zeta_{\overset{\approx}{\Box}},\mathrm{X}\Big),$$

where $\tilde{\eta}_{(\tilde{\Box},\pi,X)}(u)=u$ for all $u\in\tilde{\Box}$. In view of the equality

$$\mathcal{F}t\widehat{(\pi)\circ}\chi_{\tilde{\Box}}\circ\zeta_{\overset{\approx}{\Box}}\circ\tilde{\eta}_{(\tilde{\Box},\pi,\mathrm{X})}=\pi\circ 1_{\mathrm{X}},$$

the mapping $\big\langle\tilde{\eta}_{(\tilde{\Box},\pi,X)},1_L\big\rangle$ is a morphism of $\mathfrak{P}^{\leq}$ and the mapping $\tilde{\eta}_{(\tilde{\Box},\pi,\mathrm{X})}$ is a morphism of $\mathcal{I}_{\mathcal{T}}\downarrow\mathcal{I}d_{\mathfrak{Space}^{\leq}}$. □

3.5. *Open morphisms of the category* $\mathfrak{Space}^{\leq}$. We consider the categories $\mathfrak{Space}^{\leq}_L$ and $\mathfrak{P}^{\leq}_L$ and define $\mathfrak{P}^{\leq}_L$-open morphisms and the $\mathfrak{P}^{\leq}_L$-equvalence between objects of $\mathfrak{Space}^{\leq}_L$. We prove a criterion for a morphism of $\mathfrak{Space}^{\leq}_L$ to be open. In the proof, we use the following facts about open morphisms.

Proposition 6. 1. *A morphism* $\mathrm{f}:\mathrm{X}\to\mathrm{Y}$ *of* $\mathfrak{Space}^{\leq}_L$ *is* $\mathfrak{P}^{\leq}_L$*-open if and only if* $\mathcal{F}t(\mathrm{f})$ *is a* $\mathbb{P}_L$*-open morphism of* $\mathrm{p}\mathbb{S}\mathrm{et}_L$ *and* $d_u f$ *is an isometry for every reachable point* $u\in X$.

2. *If a morphism* $\mathrm{f}:\mathrm{M}\to\mathcal{F}t(\mathrm{Y})$ *of* $\mathrm{p}\mathbb{S}\mathrm{et}_L$ *is* $\mathbb{P}_L$*-open then the morphism* $\mathcal{G}(\mathrm{M},\mathrm{f},\mathrm{Y})$ *of* $\mathfrak{Space}^{\leq}_L$ *is* $\mathfrak{P}^{\leq}_L$*-open.*

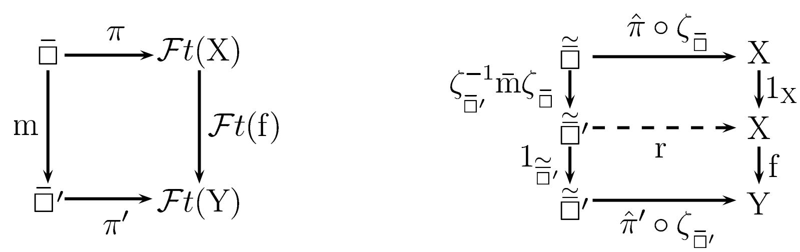


Fig. 9. Diagrams for the morphism $\mathcal{F}t(\mathrm{f})$ of $\mathrm{p}\mathbb{S}\mathrm{et}_L$ and the morphism f of $\mathfrak{Space}^{\leq}_L$

Proof. We prove the first assertion.

($\Rightarrow$) Assume that the left-hand diagram in Fig. 9 commutes, i.e.,

$$\big(\mathrm{m},\mathcal{F}t(\mathrm{f})\big):\big(\bar{\Box},\pi,\mathcal{F}t(\mathrm{X})\big)\to\big(\bar{\Box}',\pi',\mathcal{F}t(\mathrm{Y})\big)$$

is a morphism of $(\mathcal{I}\downarrow\mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}})_L$. In $(\mathcal{I}\downarrow\mathcal{F}t)_L$, this morphism assumes the form

$$(\mathrm{m},\mathrm{f}):\big(\bar{\Box},\pi,\mathrm{X}\big)\to\big(\bar{\Box}',\pi',\mathrm{Y}\big).$$

By Assertion 4, the formula

$$\widetilde{\mathcal{G}}(\mathrm{m}, \mathrm{f}) = \left(\zeta^{-1}_{\bar{\square}'}\bar{\mathrm{m}}\zeta_{\bar{\square}}, \mathrm{f}\right)$$

defines a morphism of $\left(\mathcal{I}_{\mathcal{T}} \downarrow \mathcal{I}d_{\mathfrak{Space}^{\leq}}\right)_L$. Since f is a $\mathfrak{P}^{\leq}_L$-open morphism of $\mathfrak{Space}^{\leq}_L$, we conclude that

$$\widetilde{\mathcal{G}}(\mathrm{m}, \mathrm{f}) = \left(1_{\overset{\simeq}{\square}'}, \mathrm{f}\right)\left(\zeta^{-1}_{\bar{\square}'}\bar{\mathrm{m}}\zeta_{\bar{\square}}, 1_{\mathrm{X}}\right),$$

see Remark 3. By Assertion 5, the counit of adjunction $\tilde{\varepsilon} : \widetilde{\mathcal{F}}\widetilde{\mathcal{G}} \to \mathcal{I}d_{\mathcal{I}\downarrow\mathcal{F}t}$ is the natural isomorphism; moreover, this is the identity mapping in view of the construction. We conclude that

$$\begin{aligned}(\mathrm{m}, \mathrm{f}) = \widetilde{\mathcal{F}}\widetilde{\mathcal{G}}(\mathrm{m}, \mathrm{f}) &= \widetilde{\mathcal{F}}\left(1_{\overset{\simeq}{\square}'}, \mathrm{f}\right)\widetilde{\mathcal{F}}\left(\zeta^{-1}_{\bar{\square}'}\bar{\mathrm{m}}\zeta_{\bar{\square}}, 1_{\mathrm{X}}\right) \\ &= \left(1_{\overset{\simeq}{\square}'}, \mathrm{f}\right)\left(\chi^{-1}_{\overset{\simeq}{\square}'}\mathcal{F}\left(\zeta^{-1}_{\bar{\square}'}\bar{\mathrm{m}}\zeta_{\bar{\square}}\right)\chi_{\overset{\simeq}{\square}}, 1_{\mathrm{X}}\right) \\ &= \left(1_{\overset{\simeq}{\square}'}, \mathrm{f}\right)\left(\mathrm{m}, 1_{\mathrm{X}}\right)\end{aligned}$$

in the category $(\mathcal{I} \downarrow \mathcal{F}t)_L$. Thus, in $\left(\mathcal{I} \downarrow \mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}}\right)_L$, we have

$$\left(\mathrm{m}, \mathcal{F}t(\mathrm{f})\right) = \left(1_{\overset{\simeq}{\square}'}, \mathcal{F}t(\mathrm{f})\right)\left(\mathrm{m}, 1_{\mathcal{F}t(\mathrm{X})}\right),$$

i.e., $\mathcal{F}t(\mathrm{f})$ is a $\mathbb{P}_L$-open morphism.

If $u \in X$ is a reachable point then $u = r(v)$ for a suitable morphism $\mathrm{r} = \langle r, 1_L\rangle : \overset{\simeq}{\square}' \to \mathrm{X}$ from the right-hand diagram in Fig. 9 and a suitable point $v \in \overset{\simeq}{\square}'$. Then, for every vector $\dot{u} \in T_u X$, there exists a vector $\dot{v} \in T_v\overset{\simeq}{\square}'$ such that $\dot{u} = d_v r(\dot{v})$. Therefore, we have

$$\begin{aligned}\|\dot{u}\|^{\mathrm{X}}_u = \left\|d_v r(\dot{v})\right\|^{\mathrm{X}}_{r(v)} &\leq \left\|\dot{v}\right\|^{\overset{\simeq}{\square}'}_v \\ &= \left\|d_v\left(\widehat{\pi}' \circ \zeta_{\bar{\square}'}\right)(\dot{v})\right\|^{\mathrm{Y}}_{\widehat{\pi}'\left(\zeta_{\bar{\square}'}(v)\right)} \\ &= \left\|d_v(f \circ r)(\dot{v})\right\|^{\mathrm{Y}}_{f(r(v))} \leq \left\|d_v r(\dot{v})\right\|^{\mathrm{X}}_{r(v)} = \|\dot{u}\|^{\mathrm{X}}_u.\end{aligned}$$

Thus, $d_u f$ is an isometry.

($\Leftarrow$) The proof is similar to the reasoning above. Existence of the unit of adjunction

$$\tilde{\eta} : \mathcal{I}d_{\mathcal{I}_{\mathcal{T}}\downarrow\mathcal{I}d_{\mathfrak{Space}^{\leq}}} \to \widetilde{\mathcal{G}}\widetilde{\mathcal{F}}$$

is used in the proof.

The proof of the second assertion is similar to the reasoning above. We use the facts that the counit of adjunction

$$\varepsilon : \mathcal{F}\mathcal{G} \to \mathcal{I}d_{\mathcal{I}d_{\mathrm{p}\mathbb{S}\mathrm{et}}\downarrow\mathcal{F}t}$$

is the natural isomorphism (see Assertion 3) and the unit of adjunction

$$\tilde{\eta} : \mathcal{I}d_{\mathcal{I}_{\mathcal{T}}\downarrow\mathcal{I}d_{\mathfrak{Space}^{\leq}}} \to \widetilde{\mathcal{G}}\widetilde{\mathcal{F}}$$

exists. ∎

The following theorem provides us with a criterion for a morphism of $\mathfrak{Space}^{\leq}_L$ to be $\mathfrak{P}^{\leq}_L$-open.

Theorem 3. *A morphism*

$$\mathrm{f} = \langle f, 1_L\rangle : \mathrm{X} \to \mathrm{Y}$$

of the category $\mathfrak{Space}_L^{\leq}$ *is* $\mathfrak{P}_L^{\leq}$*-open if and only if, for every cubical path* $P \in \mathcal{CP}(\mathrm{X})$*, the following conditions hold*:

(1) *if* $f(P) \xrightarrow{\delta_\lambda^\alpha} Q'$ *in* Y *then* $P \xrightarrow{\delta_\lambda^\alpha} P'$ *in* X *and* $f(P') = Q'$;

(2) *if* $f(P) \overset{s}{\longleftrightarrow} Q'$ *in* Y *then* $P \overset{s}{\longleftrightarrow} P'$ *in* X *and* $f(P') = Q'$;

(3) $d_u f$ *is an isometry for every reachable point* $u \in X$.

The *proof* is immediate from Theorem 1 and Assertion 6(1). □

3.6. thhp-*Bisimulation on precubical spaces.* We introduce an analog of the notion of an hhp-bisimulation for precubical spaces. Using the notion of the $\mathfrak{P}$-equivalence, we characterize the arising equivalence.

We introduce the notion of a thhp-bisimulation.

Definition 8. Let X and Y be precubical spaces labeled over a set L. Cubical paths $P = p_0 \dots p_k$ in X and $Q = q_0 \dots q_k$ in Y are said to be *isometric* if, for every s with $1 \leq s \leq k$, we have

$$\left\|d_t p_s(\dot{t})\right\|^{\mathrm{X}}_{p_s(t)} = \left\|d_t q_s(\dot{t})\right\|^{\mathrm{Y}}_{q_s(t)}$$

for all $\dot{t} \in T_t(\overset{\circ}{\square}{}^{n_s})$ and $t \in (\overset{\circ}{\square}{}^{n_s})$, where $n_s = \dim p_s$.

A binary relation $\mathcal{R}$ between cubical paths in X and Y is called a thhp-*bisimulation* between X and Y if, for every pair $(P, Q) \in \mathcal{R}$, the cubical paths P and Q are dl-connected and isometric and the following conditions hold:

(1) if $P \xrightarrow{\delta_\lambda^\alpha} P'$ in X then $Q \xrightarrow{\delta_\lambda^\alpha} Q'$ in Y and $(P', Q') \in \mathcal{R}$;

(2) if $Q \xrightarrow{\delta_\lambda^\alpha} Q'$ in Y then $P \xrightarrow{\delta_\lambda^\alpha} P'$ in X and $(P', Q') \in \mathcal{R}$;

(3) if $P' \xrightarrow{\delta_\lambda^\alpha} P$ in X then $Q' \xrightarrow{\delta_\lambda^\alpha} Q$ in Y and $(P', Q') \in \mathcal{R}$;

(4) if $Q' \xrightarrow{\delta_\lambda^\alpha} Q$ in Y then $P' \xrightarrow{\delta_\lambda^\alpha} P$ in X and $(P', Q') \in \mathcal{R}$;

(5) if $P \overset{s}{\longleftrightarrow} P'$ in X then $Q \overset{s}{\longleftrightarrow} Q'$ in Y and $(P', Q') \in \mathcal{R}$;

(6) if $Q \overset{s}{\longleftrightarrow} Q'$ in Y then $P \overset{s}{\longleftrightarrow} P'$ in X and $(P', Q') \in \mathcal{R}$.

We say that precubical spaces X and Y are thhp-*bisimilar* if there exists a thhp-bisimulation between them such that the initial points (regarded as cubical paths) are related.

It is clear that the relation "to be thhp-bisimilar" is an equivalence relation. We show how to construct a thhp-bisimulation from an hhp-bisimulation.

Lemma 10. *A relation* $\mathcal{R}$ *is a* thhp-*bisimulation between precubical spaces* X *and* Y *if and only if the relation*

$$\mathcal{R}_{\mathcal{F}t} = \left\{(P_{\mathcal{F}t}, Q_{\mathcal{F}t}) \;\middle|\; (P, Q) \in \mathcal{R}\right\}$$

is an hhp-*bisimulation between the precubical sets* $\mathcal{F}t(\mathrm{X})$ *and* $\mathcal{F}t(\mathrm{Y})$ *and, for each pair* $(P, Q) \in \mathcal{R}$*, the cubical paths* P *and* Q *are isometric.*

The *proof* is obvious. □

Example 5. Let X be the precubical space depicted in the left-hand picture of Fig. 3. We have

$$\mathrm{X} = \Big(X = x_1(\square_2) \cup x_2(\square_2) \cup p_5(\square_1) \cup p_6(\square_1) \cup p_7(\square_1) \cup p_8(\square_1),\ i_0^{\mathrm{X}},\ l^{\mathrm{X}},\ \|\cdot\|^{\mathrm{X}}\Big)_L.$$

This precubical space is labeled over the set $L = \{a, b, c\}$ of actions. The space X is generated by the 2-cubes

$$x_1(t_1, t_2) = (-t_1, t_2), \quad x_2(t_1, t_2) = (t_1, t_2) \ \big((t_1, t_2) \in \square_2\big)$$

and the 1-cubes

$$p_5(t) = (-1, 1 + t), \quad p_7(t) = (-1 - t, 2),$$
$$p_6(t) = (1, 1 + t), \quad p_8(t) = (1 + t, 2) \ (t \in \square_1)$$

and is endowed with the topology of a subspace of $\mathbb{R}^2$. The initial point is $i_0^{\mathrm{X}} = s_0 = (0, 0)$. The labeling function l^{X} is determined by the equalities

$$l_2^{\mathrm{X}}(x_1) = a, \quad l_1^{\mathrm{X}}(x_1) = b, \quad l_2^{\mathrm{X}}(x_2) = a,$$
$$l^{\mathrm{X}}(p_5) = l^{\mathrm{X}}(p_6) = l^{\mathrm{X}}(p_7) = l^{\mathrm{X}}(p_8) = c.$$

The norm $\|\cdot\|_u^{\mathrm{X}}$, $u \in X$, is induced by the Euclidean norm on $\mathbb{R}^2$. For every ε with $1 \le \varepsilon \le 2$, let Y_ε denote the precubical space

$$\mathrm{Y}_\varepsilon = \Big(Y_\varepsilon = y(\square_2) \cup q_3(\square_1) \cup q_4(\square_1) \cup q_5(\square_1) \cup q_6(\square_1),\ i_0^{\mathrm{Y}_\varepsilon},\ l^{\mathrm{Y}_\varepsilon},\ \|\cdot\|^{\mathrm{Y}_\varepsilon}\Big)_L$$

depicted in the right-hand picture of Fig. 3. The space Y_ε is generated by the 2-cube

$$y(t_1, t_2) = (t_1, \varepsilon t_2) \ \big((t_1, t_2) \in \square_2\big)$$

and the 1-cubes

$$q_3(t) = (1, \varepsilon + t), \quad q_4(t) = (1 + t, 1 + \varepsilon),$$
$$q_5(t) = (1 + t, \varepsilon), \quad q_6(t) = (2, \varepsilon + t) \ (t \in \square_1)$$

and is endowed with the topology of a subspace of $\mathbb{R}^2$. The initial point is $i_0^{\mathrm{Y}_\varepsilon} = r_0 = (0, 0)$. The labeling function $l^{\mathrm{Y}_\varepsilon}$ is determined by the equalities

$$l_2^{\mathrm{Y}_\varepsilon}(y) = a, \quad l_1^{\mathrm{Y}_\varepsilon}(y) = b,$$
$$l^{\mathrm{Y}_\varepsilon}(q_3) = l^{\mathrm{Y}_\varepsilon}(q_4) = l^{\mathrm{Y}_\varepsilon}(q_5) = l^{\mathrm{Y}_\varepsilon}(q_6) = c.$$

The norm $\|\cdot\|_v^{\mathrm{Y}_\varepsilon}$, $v \in Y_\varepsilon$, is induced by the Euclidean norm on $\mathbb{R}^2$.

It is easy to see that the precubical spaces X and Y_1 are thhp-bisimilar. Indeed, consider the relation $\widehat{\mathcal{R}}$ with $\widehat{\mathcal{R}}_{\mathcal{F}t} = \mathcal{R}$, where $\mathcal{R}$ is the hhp-bisimulation from Example 3. By the definition of the norm $\|\cdot\|^{\mathrm{Y}_1}$, the cubical paths related by $\widehat{\mathcal{R}}$ are isometric. By Lemma 10, we conclude that $\widehat{\mathcal{R}}$ is a thhp-bisimulation.

We show that, for $1 < \varepsilon \le 2$, the precubical spaces X and Y_ε are not thhp-bisimilar. Assume that there exists a thhp-bisimulation $\widetilde{\mathcal{R}}$ between X and Y_ε relating their initial points. Consider the cubical path $P = s_0 p x_1$ in X. There exists a cubical path Q in Y_ε such that $(P, Q) \in \widetilde{\mathcal{R}}$. Since P and Q are dl-connected, we find that Q is of the form $r_0 q y$. However, the cubical paths P and Q are not isometric because

$$\big\|d_t p(\dot{t})\big\|_{p(t)} = \|\dot{t}\|_t \ne \varepsilon \|\dot{t}\|_t = \big\|d_t q(\dot{t})\big\|_{q(t)} \ \text{ if } 1 < \varepsilon \le 2.$$

In conclusion, we formulate an analog of Theorem 2 for precubical spaces.

Theorem 4. *Let* X *and* Y *be precubical spaces labeled over the same set* L *of actions. Then the following conditions are equivalent*:

(1) X *and* Y *are* thhp-*bisimilar*;

(2) X *are* Y *are* $\mathfrak{P}_L^{\le}$-*equivalent.*

Proof. $(2 \Rightarrow 1)$ Assume that X and Y are $\mathfrak{P}_L^{\leq}$-equivalent. Then there exists a span $\mathrm{X} \xleftarrow{\mathrm{f_X}} \mathrm{Z} \xrightarrow{\mathrm{f_Y}} \mathrm{Y}$, where Z is an object of $\mathfrak{Space}_L^{\leq}$ and $\mathrm{f_X} = \langle f_X, 1_L\rangle$ and $\mathrm{f_Y} = \langle f_Y, 1_L\rangle$ are $\mathfrak{P}_L^{\leq}$-open morphisms of $\mathfrak{Space}_L^{\leq}$. It is immediate from Definition 7, Theorem 3, and Lemma 7 that the relation

$$\mathcal{R} = \Big\{ \big(f_X(P), f_Y(P)\big) \mid P \in \mathcal{CP}(\mathrm{M}) \Big\}$$

is a thhp-bisimulation between X and Y relating their initial points.

$(1 \Rightarrow 2)$ Assume that X and Y are precubical spaces labeled over a set L of actions and $\mathcal{R}$ is a thhp-bisimulation between X and Y relating their initial points. We present a span $\mathrm{X} \xleftarrow{\mathrm{f_X}} \mathrm{Z} \xrightarrow{\mathrm{f_Y}} \mathrm{Y}$, where Z is an object of $\mathfrak{Space}_L^{\leq}$ and $\mathrm{f_X} = \langle f_X, 1_L\rangle$ and $\mathrm{f_Y} = \langle f_Y, 1_L\rangle$ are $\mathfrak{P}_L^{\leq}$-open morphisms of $\mathfrak{Space}_L^{\leq}$. By Lemma 10, we find that $\mathcal{R}_{\mathcal{F}t}$ is an hhp-bisimulation between the precubical sets $\mathcal{F}t(\mathrm{X})$ and $\mathcal{F}t(\mathrm{Y})$ relating their initial points. Following the arguments from the proof of Theorem 2, we find a span

$$\mathcal{F}t(\mathrm{X}) \xleftarrow{\mathrm{pr}_1} \mathrm{M} \xrightarrow{\mathrm{pr}_2} \mathcal{F}t(\mathrm{Y}),$$

where $\mathrm{M} = \big\langle \mathcal{F}t(\mathrm{X}), \mathcal{F}t(\mathrm{Y})\big\rangle$ is an object of $\mathrm{p}\mathbb{S}\mathrm{et}_L$ and $\mathrm{pr}_1 = \langle pr_1, 1_L\rangle$ and $\mathrm{pr}_2 = \langle pr_2, 1_L\rangle$ are $\mathbb{P}_L$-open morphisms of $\mathrm{p}\mathbb{S}\mathrm{et}_L$. By Assertion 6 (2), the mappings

$$\mathcal{G}(\mathrm{M}, \mathrm{pr}_1, \mathrm{X}) : \mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{M}) \to \mathrm{X} \quad \text{and} \quad \mathcal{G}(\mathrm{M}, \mathrm{pr}_2, \mathrm{Y}) : \mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\mathrm{M}) \to \mathrm{Y}$$

are $\mathfrak{P}_L^{\leq}$-open morphisms of $\mathfrak{Space}_L^{\leq}$.

We show that $\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{M}) = \mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\mathrm{M})$. It suffices to prove that the norms $\|\cdot\|^1$ and $\|\cdot\|^2$ of these precubical spaces coincide. Let Z be the common topological space of the precubical spaces $\mathcal{T}_{\mathrm{pr}_1,\mathrm{X}}(\mathrm{M})$ and $\mathcal{T}_{\mathrm{pr}_2,\mathrm{Y}}(\mathrm{M})$. By Theorem 3, the $\mathbb{P}_L$-open morphisms $\mathcal{G}(\mathrm{M}, \mathrm{pr}_1, \mathrm{X})$ and $\mathcal{G}(\mathrm{M}, \mathrm{pr}_2, \mathrm{Y})$ are isometric. The cubical paths P and Q are isometric too. By the construction of M, for all points

$$w = \big(\langle P_{\mathcal{F}t}, Q_{\mathcal{F}t}\rangle, t\big) \in Z$$

$$\Big(P \in \mathcal{CP}_{p_k}(\mathrm{X}),\ Q \in \mathcal{CP}_{q_k}(\mathrm{Y}),\ \dim p_k = \dim q_k = n,\ (P,Q) \in \mathcal{R},\ t \in \overset{\circ}{\square}_n\Big)$$

and all vectors $\dot{t} \in T_w Z$, we have

$$\begin{aligned}\|\dot{t}\|^1_w &= \big\|d_w \widehat{pr}_1(\dot{t})\big\|^{\mathrm{X}}_{\widehat{pr}_1(w)} = \big\|d_t p_k(\dot{t})\big\|^{\mathrm{X}}_{p_k(t)} \\ &= \big\|d_t q_k(\dot{t})\big\|^{\mathrm{Y}}_{q_k(t)} = \big\|d_w \widehat{pr}_2(\dot{t})\big\|^{\mathrm{Y}}_{\widehat{pr}_2(w)} = \|\dot{t}\|^2_w.\end{aligned}$$ □

Remark 6. Notice that the results of the article remain valid if we replace the category $\mathfrak{Space}^{\leq}$ by either the category $\mathfrak{Space}$ or the category $\mathfrak{Space}^{=}$ whose objects are precubical spaces and morphisms are mappings from Definition 7, where the first components are isometries.

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