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THE GEOMETRICAL PROBLEM OF ELECTRICAL IMPEDANCE TOMOGRAPHY IN THE DISK

© **V. A. Sharafutdinov**
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Abstract: The geometrical problem of electrical impedance tomography consists of recovering a Riemannian metric on a compact manifold with boundary from the Dirichlet-to-Neumann operator (DN-operator) given on the boundary. We present a new elementary proof of the uniqueness theorem: A Riemannian metric on the two-dimensional disk is determined by its DN-operator uniquely up to a conformal equivalence. We also prove an existence theorem that describes all operators on the circle that are DN-operators of Riemannian metrics on the disk.

Keywords: electrical impedance tomography, Dirichlet-to-Neumann operator, conformal map

1. Introduction

Electrical impedance tomography (EIT) deals with determining the electric conductivity of a medium by making voltage and current measurements at the boundary of the medium. We will discuss the anisotropic version of the problem where the conductivity is a second rank symmetric tensor. The problem is posed mathematically as follows. Let $(\gamma^{ij}(x))_{i,j=1}^n$ be a positive definite symmetric matrix sufficiently regularly depending on $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently good boundary. The boundary value problem

$$\begin{cases} \sum\limits_{i,j=1}^n \frac{\partial}{\partial x_i}\big(\gamma^{ij}\frac{\partial u}{\partial x_j}\big) = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = f \end{cases} \tag{1.1}$$

is uniquely solvable for a sufficiently regular function f on $\partial\Omega$. Not intending for maximal generality, we restrict ourselves to considering smooth γ, $\partial\Omega$, and f. Throughout the paper, the word “smooth” is used as the synonym of “C^∞-smooth.” The *Dirichlet-to-Neumann operator*

$$\Lambda_\gamma : C^\infty(\partial\Omega) \longrightarrow C^\infty(\partial\Omega)$$

is defined by

$$\Lambda_\gamma f = \sum_{i,j=1}^n \nu_i \gamma^{ij} \left.\frac{\partial u}{\partial x_j}\right|_{\partial\Omega},$$

where ν is the unit outward normal to $\partial\Omega$ and u is the solution to (1.1). The physical EIT problem consists of recovering γ from the given operator Λ_γ. Here the adjective “physical” is added in order to make distinction between this problem and the geometrical version of the problem to be discussed below. But the distinction is relevant only in the two-dimensional case, as we will see.

To transform the problem to a geometric form, we observe a similarity between the differential operator on (1.1) and the Riemannian Laplacian. Given a Riemannian manifold (M, g), the Laplacian $\Delta_g : C^\infty(M) \to C^\infty(M)$ is expressed in local coordinates by the formula

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij}\frac{\partial u}{\partial x^j}\right), \tag{1.2}$$

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where $(g^{ij}) = (g_{ij})^{-1}$ and $\det g = \det(g_{ij})$. In dimensions $n \geq 3$, equation (1.1) can be represented as $\Delta_g u = 0$ for an appropriately chosen Riemannian metric. Indeed, given a matrix function (γ^{ij}), we have to find (g_{ij}) satisfying

$$\sqrt{\det g}\, g^{ij} = \gamma^{ij}. \tag{1.3}$$

Therefore, $(\det g)^{\frac{n-2}{2}} = \det(\gamma^{ij})$. Hence, $g^{ij} = (\det \gamma^{ij})^{\frac{2}{n-2}} \gamma^{ij}$ for $n \geq 3$. For $n = 2$, the condition $\det\gamma = 1$ is necessary and sufficient for the solvability of (1.3).

Let (M, g) be a compact Riemannian manifold with nonempty boundary ∂M. We include the assumption of smoothness of M, ∂M, and g into the definition of Riemannian manifold. By g_∂ we denote the Riemannian metric on ∂M induced by g. The DN-operator

$$\Lambda_g : C^\infty(\partial M) \longrightarrow C^\infty(\partial M) \tag{1.4}$$

is defined by

$$\Lambda_g f = \left.\frac{\partial u}{\partial \nu}\right|_{\partial M}, \tag{1.5}$$

where ν is the unit outward normal to the boundary and u is the solution to the boundary value problem

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u|_{\partial M} = f. \end{cases} \tag{1.6}$$

We list the main properties of the DN-operator. To this end we introduce the Hilbert spaces $L^2_g(M)$ and $L^2_{g_\partial}(\partial M)$ by means of the inner products

$$(u, v)_{L^2_g} = \int\limits_M u\bar{v}\, dV_g \quad \text{for } u, v \in C(M),$$

$$(u, v)_{L^2_{g_\partial}} = \int\limits_{\partial M} u\bar{v}\, dV_{g_\partial} \quad \text{for } u, v \in C(\partial M),$$

where dV_g and dV_{g_∂} are volume forms of g and g_∂ respectively. Recall the Green's formula for the Laplacian

$$(\Delta_g u, u')_{L^2_g} - (u, \Delta_g u')_{L^2_g} = \left(\left.\frac{\partial u}{\partial \nu}\right|_{\partial M}, u'|_{\partial M} \right)_{L^2_{g_\partial}} - \left(u|_{\partial M}, \left.\frac{\partial u'}{\partial \nu}\right|_{\partial M} \right)_{L^2_{g_\partial}} \quad (u, u' \in C^\infty(M)). \tag{1.7}$$

The operator Λ_g is formally self-dual and positive:

$$(\Lambda_g f, f')_{L^2_{g_\partial}} = (f, \Lambda_g f')_{L^2_{g_\partial}}, \quad (\Lambda_g f, f)_{L^2_{g_\partial}} \geq 0 \quad (f, f' \in C^\infty(\partial M)). \tag{1.8}$$

Indeed, for $f, f' \in C^\infty(\partial M)$, let u be a solution to (1.6) and let u' be a solution to the problem obtained from (1.6) by replacing f with f'. The left-hand side of (1.7) equals to zero and we arrive at the first formula of (1.8). To prove positivity we set $u := |u|^2/2$ and $u' = 1$ in (1.7):

$$\frac{1}{2} \int\limits_M \Delta |u|^2\, dV_g = \frac{1}{2} \int\limits_{\partial M} \frac{\partial |u|^2}{\partial \nu}\, dV_{g_\partial} = \mathrm{Re} \int\limits_{\partial M} \bar{u} \frac{\partial u}{\partial \nu}\, dV_{g_\partial} = (f, \Lambda_g f)_{L^2_{g_\partial}}$$

for the solution u to (1.6). Since u is a harmonic function, $\Delta |u|^2 = 2|\mathrm{grad}_g u|^2$. This, together with the previous formula, gives

$$(\Lambda_g f, f)_{L^2_{g_\partial}} = \|\mathrm{grad}_g u\|^2_{L^2_g}. \tag{1.9}$$

For a connected M, the kernel of Λ_g is the one-dimensional space consisting of constant functions as follows from (1.9). The range of (1.4) coincides with the space $C_0^\infty(\partial M) = \{h \in C^\infty(\partial M) \mid \int_{\partial M} h\, dV_{g_\partial} = 0\}$ consisting of functions with zero mean. Indeed, $\int_{\partial M} h\, dV_{g_\partial} = 0$ is the necessary and sufficient condition for solvability of the boundary value problem

$$\Delta_g u = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial M} = h.$$

Therefore the inverse operator

$$\Lambda_g^{-1} : C_0^\infty(\partial M) \to C_0^\infty(\partial M) \tag{1.10}$$

is well defined in the case of M connected.

As is known [1], Λ_g is a first order pseudodifferential operator. Therefore, for M connected, (1.10) is extendable to the compact operator

$$\Lambda_g^{-1} : L^2_{g_\partial,0}(\partial M) \to L^2_{g_\partial,0}(\partial M), \tag{1.11}$$

where $L^2_{g_\partial,0}(\partial M) = \{f \in L^2_{g_\partial}(\partial M) \mid \int_{\partial M} f\, dV_{g_\partial} = 0\}$.

The geometrical EIT problem is posed as follows: We assume the Riemannian manifold $(\partial M, g_\partial)$ to be known as well as Λ_g. Given $(\partial M, g_\partial, \Lambda_g)$, the problem is to recover (M, g). For $n = \dim M \geq 3$, the geometrical problem is equivalent to the physical EIT problem above. For $n = 2$, these problems are different. In what follows, the present paper discusses only the geometrical problem, mostly for $n = 2$.

Remark. A slightly different definition of DN-operator is used by the authors of [1]. Instead of (1.5), they introduce the operator

$$\Lambda'_g f = (\mathrm{grad}_g u) \lrcorner\, dV_g|_{\partial M},$$

taking values in the space of $(n-1)$-forms. Here dV_g is the volume form of g. The pair $(\partial M, \Lambda'_g)$ stands as the data for the inverse problem where ∂M is considered as a manifold with no prescribed metric. These two formulations of the problem are equivalent. Indeed, the metric g_∂ can be recovered from the principal symbol of Λ'_g as is shown in [1]. If g_∂ is known, then Λ_g and Λ'_g can be expressed through each other.

The following ambiguity in the solution to the problem is obvious: If $\varphi : M \to M$ is an arbitrary diffeomorphism of the manifold onto itself fixing the boundary, $\varphi|_{\partial M} = \mathrm{Id}$, then $g' = \varphi^* g$ satisfies $g'_\partial = g_\partial$ and $\Lambda_{g'} = \Lambda_g$. The equality $g' = \varphi^* g$ means that $\langle v, w\rangle_{g'} = \langle (d_p\varphi)v, (d_p\varphi)w\rangle_g$ for every $p \in M$ and all vectors v and w belonging to the tangent space T_pM. Hereafter $\langle\cdot,\cdot\rangle_g$ stands for the inner product of tangent vectors in the sense of g, and $d_p\varphi : T_pM \to T_{\varphi(p)}M$ is the differential of φ. Let us note that $\varphi : (M, g') \to (M, g)$ is an isometry of Riemannian manifolds, and so the ambiguity is quite natural from the geometric viewpoint.

The conjecture exists that this ambiguity exhausts the nonuniqueness of the solution to the problem in dimensions ≥ 3: a connected compact Riemannian manifold of dimension ≥ 3 with a nonempty boundary is determined by the data $(\partial M, g_\partial, \Lambda_g)$ uniquely up to an isometry identical on the boundary. We emphasize that the above is a conjecture so far which is proved only in the real analytic case [2].

There is one more ambiguity in the two-dimensional case. We recall that the Laplacian on a two-dimensional Riemannian manifold possesses the following conformal invariancy: If $g' = \rho g$ for $0 < \rho \in C^\infty(M)$, then $\Delta_{g'} = \rho^{-1}\Delta_g$. If ρ satisfies the boundary condition $\rho|_{\partial M} = 1$, then $\Lambda_{\rho g} = \Lambda_g$.

For a smooth map $\varphi : N \to N'$ between two manifolds, we denote by $\varphi^* : C^\infty(N') \to C^\infty(N)$ the operator defined by $\varphi^* u = u \circ \varphi$. The two above-mentioned ambiguities exhaust the nonuniqueness of a solution to the problem in the two-dimensional case. Namely, the following is valid:

Theorem 1.1. *Let (M, g) and (M', g') be two-dimensional connected compact Riemannian manifolds with nonempty boundaries and let $\varphi : (\partial M, g_\partial) \to (\partial M', g'_\partial)$ be an isometry preserving the DN-map,* *i.e., such that the following square commutes:*

$$
\begin{array}{ccc}
C^\infty(\partial M) & \xleftarrow{\varphi^*} & C^\infty(\partial M') \\
{\scriptstyle\Lambda_g}\downarrow & & \downarrow{\scriptstyle\Lambda_{g'}} \\
C^\infty(\partial M) & \xleftarrow[\varphi^*]{} & C^\infty(\partial M').
\end{array}
\tag{1.12}
$$

Then φ can be extended to a diffeomorphism $\Phi : M \to M'$ such that $\Phi|_{\partial M} = \varphi$ and $\Phi^* g' = \rho g$ for some function $0 < \rho \in C^\infty(M)$ satisfying the boundary condition $\rho|_{\partial M} = 1$.

The two proofs of the theorem are known: one by Lassas–Uhlmann [2] and the other by Belishev [3]. The first proof is based on the following observation: The DN-operator allows us to determine values of the Green's function $G(x, y)$ at $x, y \in \widetilde{M} \setminus M$, where $\widetilde{M}$ is the extension of M obtained by gluing a collar. Given the Green's function on $\widetilde{M} \setminus M$, one constructs an analytic sheaf whose linear connection component can be identified with M. This proof gives a strengthened version of Theorem 1.1 which, roughly speaking, claims that the knowledge of the DN-map on any open subset of the boundary is sufficient for the same statement; see details in [2]. The same arguments by Lassas–Uhlmann give the corresponding uniqueness theorem for real analytic manifolds of dimension ≥ 3.

Belishev's proof is based on Gelfand's theorem that claims: a compact topological space X can be uniquely, up to a homeomorphism, recovered from the Banach algebra $C(X)$ of continuous functions. A similar statement is true for the Banach algebra $\mathscr{A}(X)$ of holomorphic functions in the case of a complex manifold X. Belishev notes that, under the hypotheses of Theorem 1.1, the DN-map allows us to construct a Banach algebra that is isometric to $\mathscr{A}(M)$.

In the present paper, we give the third alternative proof of Theorem 1.1, but only under the additional assumption of simply connectedness of manifolds M and M'. Note that just this case is needed for the proof of the boundary distance rigidity of a simple two-dimensional manifold [4]. Unlike [2, 3], our proof is quite elementary, i.e., it is based on the standard facts of complex analysis and differential geometry of surfaces. So far, we cannot extend our arguments to the nonsimply connected case.

A two-dimensional connected and simply connected compact manifold with nonempty boundary is diffeomorphic to the disk. Without loss of generality we can assume both M and M' in Theorem 1.1 to coincide with the unit disk of $\mathbb{R}^2$ and φ to be the identical mapping of the boundary circle. Thus, we will prove the following

Theorem 1.2. *Let g and g' be two Riemannian metrics on the disk*

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$$

which induce the same arc length on the circle

$$\gamma = \partial D = \{(x, y) \mid x^2 + y^2 = 1\}.$$

If $\Lambda_g = \Lambda_{g'}$, then there exists a diffeomorphism $\Phi : D \to D$ of the disk onto itself such that $\Phi|_\gamma = \mathrm{Id}$ and $\Phi^* g' = \rho g$ for some $0 < \rho \in C^\infty(D)$ satisfying $\rho|_\gamma = 1$.

The proof of Theorem 1.2 is presented in the next section. The scheme of the proof is as follows: First, on using the conformal invariance of the Laplacian, we reduce the question to the case of g and g' both flat. A Riemannian metric on a two-dimensional manifold is flat if its Gaussian curvature is identically zero. A flat metric on the disk determines uniquely, up to rotation and parallel translation, an isometric immersion of the disk into the Euclidean plane. In the simplest case, the immersion is an embedding and identifies (D, g) with (M, e), where M is a domain in $\mathbb{R}^2$ bounded by a simple closed smooth curve and e is the standard Euclidean metric on $\mathbb{R}^2$. If the immersion corresponding to g' is also an embedding, then Theorem 1.2 reduces to the partial case of Theorem 1.1 where M and M' are simply connected domains in $\mathbb{R}^2$ bounded by smooth closed curves and each of the metrics g and g' coincides with e. In this case, if (1.12) commutes, then the diffeomorphism $\varphi : \partial M \to \partial M'$ extends to a conformal map $\Phi : M \to M'$. The latter fact is proved by applying the DN-map to $x|_{\partial M}$ and $y|_{\partial M}$, where (x, y) are

Cartesian coordinates on the plane. It remains to use the following statement: If a conformal map is continuous up to the boundary and preserves the arc length of the boundary curve, then it is composition of a rotation and parallel translation. All of these arguments with small modifications apply to the case of immersed disks.

The proof of Theorem 1.2 splits into several lemmas in the next section. Each of these lemmas (with exception of Lemma 2.2) is not new but represents a known statement in the form fit to our purposes. Nevertheless, we present all proofs for the reader's convenience.

Let us now discuss the question of the existence of a solution to the two-dimensional EIT problem. For a Riemannian metric g on the unit disk D, the induced metric g_∂ on the circle $\gamma = \partial D = \{e^{i\theta} \mid \theta \in \mathbb{R}\} \subset \mathbb{C} = \mathbb{R}^2$ is uniquely determined by its length form that will be denoted by ds_g. In turn, the 1-form ds_g on γ is uniquely determined by g under the additional requirement of the positiveness with respect to the standard orientation of the circle, i.e., by the inequality $(ds_g)(d/d\theta) > 0$ that will be assumed to be satisfied in what follows. Let e be the standard Euclidean metric on the disk $D \subset \mathbb{R}^2$.

Theorem 1.3. *Assume a smooth* 1*-form* ω *on the circle* $\gamma = \{e^{i\theta}\}$ *to be positive, i.e.,* $\omega(d/d\theta) > 0$*, and let* $A : C^\infty(\gamma) \to C^\infty(\gamma)$ *be a linear operator. The necessary and sufficient condition for the existence of a Riemannian metric* g *on* D *satisfying*

$$ds_g = \omega, \quad \Lambda_g = A, \tag{1.13}$$

is the existence of an orientation preserving diffeomorphism $\varphi : \gamma \to \gamma$ *such that the following diagram commutes:*

$$\begin{array}{ccc} C^\infty(\gamma) & \xrightarrow{\varphi^*} & C^\infty(\gamma) \\ A \downarrow & & \downarrow a^{-1}\Lambda_e \\ C^\infty(\gamma) & \xrightarrow[\varphi^*]{} & C^\infty(\gamma). \end{array} \tag{1.14}$$

Here the function $0 < a \in C^\infty(\gamma)$ *is defined by*

$$\varphi^*\omega = a\, d\theta. \tag{1.15}$$

The proof is presented in Section 3. It bases on Riemann's theorem that claims the existence of a conformal map between two simply connected bounded domains. Let us explain why this statement can be considered as an existence theorem. We consider (1.13) as a system of equations in the desired metric g. Theorem 1.3 gives an explicit description of pairs (ω, A) such that the system is solvable. Indeed, let Ω be the space of all smooth positive 1-forms on the circle γ, and let Φ be the set of all orientation preserving diffeomorphisms of γ. For $\omega \in \Omega$ and $\varphi \in \Phi$, formula (1.15) and the commutativity of (1.14) uniquely determine the linear operator $A = A(\omega, \varphi)$. Thus, the system is solvable if and only if (ω, A) belongs to the family

$$\{(\omega, A(\omega, \varphi)) \mid (\omega, \varphi) \in \Omega \times \Phi\}.$$

To the author's knowledge, only one result was obtained before in EIT which gives some sufficient existence conditions [5]. From the constructive viewpoint, Theorem 1.3 reduces the two-dimensional problem of recovering a metric g to the one-dimensional problem of finding a diffeomorphism φ.

In the author's opinion, the significance of the present work consists first of all in the reduction of the two-dimensional EIT problem to the question of the classification of flat metrics which in turn is closely related to the classical theory of conformal maps. Conformal maps were used before in the two-dimensional EIT problem (see, for example, [6]). But the combination of conformal maps with flat metrics is used for the first time. By the author's opinion, such a combination constitutes the most conceptual approach to the problem. We will also briefly discuss two possible approaches to finding an effective procedure for recovering a metric from its DN-map; see Problems 2.6 and 3.2 and the comments nearby.

2. Flat Metrics on the Disk

Recall that D stands for the closed unit disk on the plane $\mathbb{R}^2$, and $\gamma = \partial D$ is the unit circle.

Lemma 2.1. *Let g be a Riemannian metric on D. There exists a function $0 < \rho \in C^\infty(D)$ satisfying $\rho|_\gamma = 1$ and such that ρg is a flat metric.*

PROOF. We look for the function in the form $\rho = e^{2\varphi}$. Recall the formula relating the Gaussian curvature K of g to the Gaussian curvature K_φ of $e^{2\varphi}g$: $K_\varphi = e^{-2\varphi}(K - \Delta_g\varphi)$. The formula can be easily derived from the Gauss formula that expresses the Gaussian curvature through coefficients of the first quadratic form [7]. Thus,

$$\Delta_g\varphi = K \text{ in } D, \quad \varphi|_\gamma = 0.$$

There exists a unique solution to the problem. □

Note that simple connectedness is not needed for Lemma 2.1, i.e., a similar statement is valid for every two-dimensional compact Riemannian manifold with boundary. Lemma 2.1 reduces Theorem 1.2 to the following statement:

Lemma 2.2. *Let g and g' be two flat Riemannian metrics on D that induce the same arc length on γ. If $\Lambda_g = \Lambda_{g'}$, then there exists a diffeomorphism $\Phi : D \to D$ such that $\Phi|_\gamma = \mathrm{Id}$ and $\Phi^*g' = g$.*

The circle γ plays an important role while considering the DN-operator. But γ is not relevant in some of our auxiliary statements. To avoid the discussion of boundary points in these statements, it is comfortable to consider D as a closed subset of the open disk $D_\varepsilon = \{(x, y) \mid x^2 + y^2 < 1 + \varepsilon\}$ for some $\varepsilon > 0$.

Lemma 2.3. *Each flat metric on D can be extended to a flat metric on D_ε for some $\varepsilon > 0$.*

PROOF. Let g be a flat metric on D and let $\gamma(v)$ be the parametrization of γ by the arc length v in the metric g. We consider $\gamma(v)$ as a smooth L-periodic function of the variable $v \in \mathbb{R}$, where L is the length of γ. Given v, let $\beta_v : [0, \varepsilon) \to D$ be the geodesic of g starting at $\gamma(v)$ perpendicularly to γ and parametrized by the arc length. The map $E : [0, \varepsilon) \times \mathbb{R} \to D$, $E(u, v) = \beta_v(u)$ is smooth and, for a sufficiently small $\varepsilon > 0$, the numbers u and $v \,(\mathrm{mod}\, L)$ can be considered as coordinates defined in some neighborhood $U \subset D$ of γ. This is the so-called semigeodesic coordinate system with basis curve γ. The metric g is expressed in these coordinates by the formula

$$ds_g^2 = du^2 + G^2(u, v)dv^2 \tag{2.1}$$

with some smooth positive function $G(u, v)$. The function satisfies the condition

$$G(0, v) = 1, \tag{2.2}$$

since v is the arc length on γ. The Gaussian curvature of (2.1) is expressed by the formula [7]

$$K = -\frac{1}{G}\frac{\partial^2 G}{\partial u^2}. \tag{2.3}$$

Since g is a flat metric, G satisfies $G_{uu} = 0$. Together with (2.2), this yields $G(u, v) = 1 + uk(v)$. Thus (2.1) takes the form

$$ds_g^2 = du^2 + (1 + uk(v))^2 dv^2 \tag{2.4}$$

with some smooth L-periodic function $k(v)$.

We extend E to a smooth L-periodic in the second variable map $E : (-\varepsilon, \varepsilon) \times \mathbb{R} \to \mathbb{R}^2$ and redefine g for u negative by the same formula (2.4) which makes sense for $1 + uk(v) > 0$. The variables u and $v \,(\mathrm{mod}\, L)$ constitute a coordinate system in some neighborhood $U' \subset \mathbb{R}^2$ of γ, and g extends to the neighborhood by (2.4). □

If $w : D_\varepsilon \to \mathbb{R}^2$ is a smooth immersion, and e is the standard Euclidean metric on $\mathbb{R}^2$; then w^*e is a flat metric on D_ε. It turns out that all flat metrics on the disk can be obtained in this way.

Lemma 2.4. *Let g be a flat metric on D_ε. There exists a Riemannian immersion $w : (D_\varepsilon, g) \to (\mathbb{R}^2, e)$.*

PROOF. By a *Riemannian immersion* we mean a smooth map $w : D_\varepsilon \to \mathbb{R}^2$ whose differential $d_p w : (T_p D_\varepsilon, g) \to (\mathbb{R}^2, e)$ is an isometry for all $p \in D_\varepsilon$. We will first prove that, for every point $p \in D_\varepsilon$, there exist a neighborhood $U \subset D_\varepsilon$ of p and an isometric embedding $w : (U, g) \to (\mathbb{R}^2, e)$. To this end we choose a geodesic $c : (-\delta, \delta) \to D_\varepsilon$ passing through p and parametrized by the arc length. Define a semigeodesic coordinate system with the basis curve c in a neighborhood U of the point p. The metric g is expressed by (2.4) in the chosen coordinates. The function $k(v)$ on (2.4) coincides with the geodesic curvature of the curve $c(v)$ as follows from the standard formula for the geodesic curvature [7]. Since c is a geodesic, $k \equiv 0$ and (2.4) takes the form $ds_g^2 = du^2 + dv^2$. Thus, the coordinates (u, v) realize the isometric embedding $w : (U, g) \to (\mathbb{R}^2, e)$.

Now, the proof is complete by the standard arguments of the classical analytic continuation theory [8]. Namely, let us use the term a *Euclidean element* for a pair (U, w), where $D_\varepsilon \supset U$ is an open disk and $w : (U, g) \to (\mathbb{R}^2, e)$ is an orientation preserving isometric embedding. If (U_1, w_1) and (U_2, w_2) are two Euclidean elements such that $U_1 \cap U_2 \neq \varnothing$, then we have the unique Euclidean element (U_2, w_2') satisfying $w_2'|_{U_1 \cap U_2} = w_1|_{U_1 \cap U_2}$ and $w_2' = T \circ w_2$, where T is the composition of a rotation and parallel translation. The Euclidean element (U_2, w_2') is called the *continuation* of the Euclidean element (U_1, w_1) through the domain $U_1 \cap U_2$. By induction on k we prove that if $(U_1, w_1), (U_2, w_2), \dots, (U_k, w_k)$ is a chain of Euclidean elements satisfying $U_i \cap U_{i+1} \neq \varnothing$ $(1 \le i < k)$, then there exists a unique chain of Euclidean elements $(U_1, w_1), (U_2, w_2'), \dots, (U_k, w_k')$ such that every next element is the continuation of the previous one.

Finally, repeating word by word the corresponding arguments of analytic continuation theory, we define a continuation of a Euclidean element (U, w) along a continuous curve $c : [0, 1] \to D_\varepsilon$, $c(0) \in U$, and prove that such a continuation exists, is unique, and does not change under a continuous deformation of the curve c. Therefore, for the simply connected disk D_ε, the following analog of the monodromy theorem is valid: Given a Euclidean element (U, w), there exists a unique Riemannian immersion $(D_\varepsilon, g) \to (\mathbb{R}^2, e)$ which coincides with w on U. $\square$

Let us recall that a *complex structure* on a two-dimensional manifold M is a maximal atlas $\mathscr{C} = \{(U_\alpha, \varphi_\alpha)\}$ consisting of open sets $U_\alpha \subset M$, $\bigcup_\alpha U_\alpha = M$ and homeomorphisms φ_α of U_α onto some open subsets of $\mathbb{R}^2 = \mathbb{C}$ such that the transformation functions $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are holomorphic. A complex structure on M determines the complex structure on the tangent spaces $T_p M$, i.e., the multiplication of tangent vectors by complex numbers such that each $T_p M$ becomes a one-dimensional complex vector space. Conversely, if the tangent spaces $T_p M$ are equipped with a complex structure depending smoothly on $p \in M$, then M is equipped with the *almost complex structure.*

Let $(M, \mathscr{C})$ and $(M', \mathscr{C}')$ be one-dimensional complex manifolds and let $B \subset M$ $(B' \subset M')$ be a closed domain bounded by a simple closed smooth curve $\beta = \partial B$ $(\beta' = \partial B')$. Let $\mathscr{C}|_{\operatorname{int} B}$ $(\mathscr{C}'|_{\operatorname{int} B'})$ be the restriction of the complex structure $\mathscr{C}$ to the interior $\operatorname{int} B$ of B (restriction of $\mathscr{C}'$ to $\operatorname{int} B'$). By Kellog's theorem [9] (see also [10, Theorem 1.8]), every holomorphism (conformal map) $\varphi : (\operatorname{int} B, \mathscr{C}|_{\operatorname{int} B}) \to (\operatorname{int} B', \mathscr{C}'|_{\operatorname{int} B'})$ is extendable to the diffeomorphism (that is denoted by the same letter) of closed domains $\varphi : B \to B'$. We call the latter map the *holomorphism of closed domains* and denote it by $\varphi : (B, \mathscr{C}|_{\operatorname{int} B}) \to (B', \mathscr{C}'|_{\operatorname{int} B'})$.

Lemma 2.5. *Let g be a flat metric on the disk D_ε $(\varepsilon > 0)$ and let $w : (D_\varepsilon, g) \to (\mathbb{C}, e)$ be a Riemannian immersion. There exists a unique complex structure $\mathscr{C}_g$ on D_ε such that $w : (D_\varepsilon, \mathscr{C}_g) \to \mathbb{C}$ is a holomorphic function. The following statements are valid for this complex structure.*

(i) *The multiplication by the imaginary unit i in the tangent space $T_p D_\varepsilon$ coincides with the rotation of $T_p D_\varepsilon$ by the angle $\pi/2$ in the positive direction, where the angle is measured in the sense of g and the orientation of $D_\varepsilon \subset \mathbb{R}^2$ coincides with the standard orientation of $\mathbb{R}^2$.*

(ii) *For real u and v, the function $f = u + iv : (D_\varepsilon, \mathscr{C}_g) \to \mathbb{C}$ is holomorphic if and only if u and v are harmonic functions, i.e. $\Delta_g u = \Delta_g v = 0$, and $\operatorname{grad}_g v = i \operatorname{grad}_g u$. In this case (u, v) is said to be a pair of conjugate harmonic functions. For every harmonic function, there exists a conjugate harmonic function.*

(iii) *If g' is a second flat metric on D_ε and $f : (D_\varepsilon, \mathscr{C}_g) \to (D_\varepsilon, \mathscr{C}_{g'})$ is a holomorphism, then $f^*g' = e^\rho g$ for some function ρ harmonic in (D_ε, g).*

Proof. Since w is an immersion, we can choose a neighborhood $U_p \subset D_\varepsilon$ of every point $p \in D_\varepsilon$ such that $w_p = w|_{U_p}$ is a diffeomorphism of U_p onto some open set of $\mathbb{C}$. Since $w_p : (U_p, g) \to (\mathbb{C}, e)$ is an isometry, transformation functions are linear, i.e., $\big(w_q \circ w_p^{-1}\big)(z) = a_{pq}z + b_{pq}$ for some $a_{pq}, b_{pq} \in \mathbb{C}$ with $|a_{pq}| = 1$. Thus, $\mathscr{C}'_g = \{(U_p, w_p)\}_{p \in D_\varepsilon}$ is a holomorphic atlas on D_ε and determines a complex structure $\mathscr{C}_g$. If w is holomorphic with respect to some complex structure on D_ε, then the structure contains the atlas $\mathscr{C}'_g$ and therefore coincides with $\mathscr{C}_g$.

The differential $d_pw : (T_pD_\varepsilon, g) \to (\mathbb{C}, e)$ is a linear isometry commuting with the multiplication by i. This implies (i).

Some holomorphic coordinates $z = x + iy$ can be introduced in a neighborhood of an arbitrary point of the complex manifold $(D_\varepsilon, \mathscr{C}_g)$ so that g takes the form

$$ds_g^2 = dx^2 + dy^2. \tag{2.5}$$

In these coordinates $\Delta_g = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the almost complex structure is expressed by the formula $i(a\partial/\partial x + b\partial/\partial y) = -b\partial/\partial x + a\partial/\partial y$, and the property of a function $f = u + iv$ of being holomorphic is expressed by the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x. \tag{2.6}$$

This implies the first part of (ii).

Let us prove that each harmonic function $u \in C^\infty(D_\varepsilon)$ has a conjugate harmonic function. Define the 1-form ω on D_ε by $\omega(X) = \langle X, i\,\mathrm{grad}_g u\rangle_g$ for $X \in T_pD_\varepsilon$. Let us check that ω is a closed form. Indeed, $\omega = -u_y dx + u_x dy$ in local coordinates satisfying (2.5) and the closeness condition for ω coincides with the harmonicity condition $\Delta_g u = 0$. Since D_ε is simply connected, ω is an exact form, i.e., $\omega = dv$ for some $v \in C^\infty(D_\varepsilon)$. The last equation coincides with Cauchy–Riemann equations (2.6) in coordinates satisfying (2.5). Therefore v is harmonic and conjugate to u.

Finally, we prove (iii). If f is a holomorphism, then $d_pf : T_pD_\varepsilon \to T_{f(p)}D_\varepsilon$ is not equal to zero for any $p \in D_\varepsilon$ and commutes with the multiplication by i. With the help of (i), this implies the existence of a real $\rho(p)$ such that the map

$$e^{\rho(p)}d_pf : (T_pD_\varepsilon, g) \to (T_{f(p)}D_\varepsilon, g') \tag{2.7}$$

is a linear isometry. The function $\rho(p)$ is smooth. Indeed, if ω and ω' are the area forms of the metrics g and g' respectively, then $f^*\omega' = e^{2\rho}\omega$. This implies the smoothness of ρ since ω and ω' are smooth 2-forms. The property of (2.7) of being an isometry is equivalent to $f^*g' = e^\rho g$. The metric f^*g' is flat since g' is flat. For a flat metric g, the statement "$e^\rho g$ is a flat metric" is equivalent to the harmonicity of ρ as we have seen in the proof of Lemma 2.1. □

Proof of Lemma 2.2. Let g and g' be two flat metrics on D inducing the same arc length on γ. We extend them to flat metrics on D_ε ($\varepsilon > 0$) and denote the extensions by g and g' again. Let $\gamma(s)$, $0 \le s \le L$, be the natural parametrization of the circle γ, where s is the arc length in g and g', which is chosen such that the circle runs in the positive direction while s increases. Let $\tau = d\gamma/ds$ be the unit tangent vector to γ and let $\nu = \nu(s)$ ($\nu' = \nu'(s)$) be the unit outward normal to D with respect to g (g'). Assume the DN-maps of metrics g and g' to coincide and denote their common value by $\Lambda : C^\infty(\gamma) \to C^\infty(\gamma)$.

Let $w : (D_\varepsilon, g) \to (\mathbb{C}, e)$ and $w' : (D_\varepsilon, g') \to (\mathbb{C}, e)$ be the Riemannian immersions existing by Lemma 2.4, and let $\mathscr{C}_g$ and $\mathscr{C}_{g'}$ be the corresponding complex structures on D_ε. We consider w and w' as holomorphic functions $w = u + iv : (D_\varepsilon, \mathscr{C}_g) \to \mathbb{C}$ and $w' = u' + iv' : (D_\varepsilon, \mathscr{C}_{g'}) \to \mathbb{C}$ with real u, v, u', v'.

Observe that (u, v) is a pair of conjugate harmonic functions on (D_ε, g). Therefore the following Cauchy–Riemann equations hold

$$\frac{\partial u}{\partial \tau}\bigg|_\gamma = -\frac{\partial v}{\partial \nu}\bigg|_\gamma, \quad \frac{\partial u}{\partial \nu}\bigg|_\gamma = \frac{\partial v}{\partial \tau}\bigg|_\gamma. \tag{2.8}$$

By the definition of DN-map, the functions

$$\varphi = u|_\gamma \in C^\infty(\gamma), \quad \psi = v|_\gamma \in C^\infty(\gamma) \tag{2.9}$$

satisfy

$$\Lambda\varphi = \Lambda_g \varphi = \left.\frac{\partial u}{\partial \nu}\right|_\gamma, \quad \Lambda\psi = \Lambda_g\psi = \left.\frac{\partial v}{\partial \nu}\right|_\gamma.$$

Together with (2.8), these equations give

$$\Lambda\varphi = \dot{\psi}, \quad \Lambda\psi = -\dot{\varphi}, \tag{2.10}$$

where the dot means the differentiation with respect to s.

Let $U, V \in C^\infty(D)$ be the solutions to the boundary value problems

$$\begin{cases} \Delta_{g'} U = 0 & \text{in } D, \\ U|_\gamma = \varphi, \end{cases} \qquad \begin{cases} \Delta_{g'} V = 0 & \text{in } D, \\ V|_\gamma = \psi. \end{cases} \tag{2.11}$$

Then

$$\Lambda\varphi = \Lambda_{g'}\varphi = \left.\frac{\partial U}{\partial \nu'}\right|_\gamma, \quad \Lambda\psi = \Lambda_{g'}\psi = \left.\frac{\partial V}{\partial \nu'}\right|_\gamma.$$

This, together with (2.10), gives

$$\left.\frac{\partial U}{\partial \nu'}\right|_\gamma = \dot{\psi} = \left.\frac{\partial V}{\partial \tau}\right|_\gamma, \quad \left.\frac{\partial V}{\partial \nu'}\right|_\gamma = -\dot{\varphi} = -\left.\frac{\partial U}{\partial \tau}\right|_\gamma. \tag{2.12}$$

Thus, (U, V) is a pair of harmonic functions in (D, g') satisfying the Cauchy–Riemann equations on γ. This implies that U and V are conjugate harmonic functions. Indeed, if $\widetilde{V}$ is a conjugate harmonic function to U, then (2.12) implies $dV(\gamma(s))/ds = d\widetilde{V}(\gamma(s))/ds$. Hence $(V - \widetilde{V})|_\gamma = \text{const}$ and $V - \widetilde{V} = \text{const}$ in D. Thus, the function $W = U + iV \in C^\infty(D)$ is holomorphic in the interior part of the disk D that is considered as a closed subset of the complex manifold $(D_\varepsilon, \mathscr{C}_{g'})$.

We combine the holomorphic functions w and W into the diagram

$$\begin{array}{ccccc} & D & \subset & (D_\varepsilon, \mathscr{C}_g) \\ & \Phi \uparrow & & \downarrow w \\ (D_\varepsilon, \mathscr{C}_{g'}) \supset & D & \xrightarrow[W]{} & \mathbb{C}\,. \end{array} \tag{2.13}$$

Observe that the restrictions of w and W to γ coincide. Indeed, by (2.9) and (2.11) $w(\gamma(s)) = W(\gamma(s)) = \varphi(s) + i\psi(s)$. This implies the coincidence of the sets $w(D)$ and $W(D)$. Indeed, let $\beta(s) = \varphi(s) + i\psi(s)$. Applying the argument principle [11] to the holomorphic function $w - z$, we see that $z \in \mathbb{C} \setminus \operatorname{Ran}\beta$ belongs to $w(D)$ if and only if the index of the closed curve β with respect to z is not equal to zero (in this case the index is positive). The same statement is true for $W(D)$. Hence $w(D) = W(D)$.

Since w is a local homeomorphism, the coincidence of the images of D under the maps w and W implies the existence and uniqueness of a continuous map Φ making (2.13) commutative and satisfying the boundary condition

$$\Phi|_\gamma = \operatorname{Id}. \tag{2.14}$$

Indeed, for every continuous curve $\delta : [a, b] \to D$, $\delta(a) \in \gamma$, there exists a unique continuous curve $\delta' : [a, b] \to D$ such that $W \circ \delta = w \circ \delta'$, $\delta(a) = \delta'(a)$. By the monodromy theorem, this proves the existence and uniqueness of Φ satisfying (2.13)–(2.14).

We observe that Φ sends interior points of D again to interior points since the holomorphic function W sends interior points of D to interior points of $\operatorname{Ran} W$.

The equality $w \circ \Phi = W$ and local invertibility of w imply the holomorphy of Φ. Indeed, locally $\Phi = w^{-1} \circ W$, where w^{-1} is a local inverse of w.

Applying the argument principle to the holomorphic function Φ and using the boundary condition (2.14), we infer that Φ is a bijective map. Moreover, the differential of Φ is nondegenerate at all points since a holomorphic function cannot be injective in a neighborhood of its critical point. Thus, Φ is a holomorphism.

Applying the last statement of Lemma 2.5, we obtain $\Phi^* g' = e^\rho g$ for some function ρ harmonic in (D, g'). Since $\Phi|_\gamma = \mathrm{Id}$, the function ρ vanishes on γ and therefore is identically zero on D. We have thus constructed the diffeomorphism $\Phi : D \to D$ satisfying $\Phi|_\gamma = \mathrm{Id}$ and $\Phi^* g' = g$. $\square$

It is important to mention that only a small part of the hypothesis $\Lambda_g = \Lambda_{g'}$ is used in our proof of Theorem 1.2. Indeed, we have used only the equality $\Lambda_g \beta = \Lambda_{g'} \beta$ for $\beta(s) = \varphi(s) + i\psi(s) = w(\gamma(s))$; see (2.9). The following question arises: Can the function $\beta(s)$ be effectively recovered from the data $(\gamma, g_\partial, \Lambda_g)$ of our problem? Instead of $\beta(s)$, it is more comfortable to use the geodesic curvature $k(s)$ of the boundary circle γ with respect to a flat metric g which is parametrized by the arc length in the metric g. We have already used this function in (2.4). Since $w : (D, g) \to (\mathbb{R}^2, e)$ is a Riemannian immersion, $k(s)$ coincides with the Euclidean curvature of the plane curve β. A plane curve is determined by its curvature uniquely up to rotation and parallel translation. The explicit expression of $\beta(s)$ through $k(s)$ is given by the formula [7]

$$\beta(s) = \left(x_0 + \int_0^s \sin \alpha(\sigma)\, d\sigma\right) + i\left(y_0 + \int_0^s \cos \alpha(\sigma)\, d\sigma\right), \tag{2.15}$$

where

$$\alpha(s) = \alpha_0 + \int_0^s k(\sigma)\, d\sigma. \tag{2.16}$$

Here (x_0, y_0) and α_0 are arbitrary constants responsible for the rotation and parallel translation. Thus, the problem of the effective reconstruction of a metric on the disk from its DN-map can be reduced to the following question.

Problem 2.6. *Can the geodesic curvature $k(s)$ of the boundary circle γ with respect to a flat metric g on the disk D be effectively expressed in terms of the DN-map Λ_g?*

The question is not easy as is seen from the following. Repeating the calculations of [1] for a flat metric, one can see that the full symbol of Λ_g is independent of $k(s)$. This means that the information on $k(s)$ is encoded in the smoothing part of Λ_g.

By (2.15)–(2.16), the closeness condition for $\beta(s)$, $0 \le s \le L$, is expressed by the equalities

$$\int_0^L k(s)\, ds = 2\pi n \quad (0 < n \in \mathbb{Z}), \tag{2.17}$$

$$\int_0^L \sin \alpha(s)\, ds = 0, \qquad \int_0^L \cos \alpha(s)\, ds = 0. \tag{2.18}$$

In the case of a positive answer to Problem 2.6, (2.17)–(2.18) would give some interesting quantization conditions for the DN-operator.

Finally, we note that not every closed smooth curve on the plane bounds an immersed disk. The interesting question arises: Under which conditions on the curvature $k(s)$ of a plane curve $\beta(s)$ does there exist an immersion w of the disk D to $\mathbb{R}^2$ such that $\beta = w \circ \gamma$, where $\gamma = \partial D$? An answer to the question was obtained in the recent paper [12] by Sabitov but in a very implicit form. Some new quantization conditions for the DN-map can be obtained in this way.

3. The Existence Theorem

Here we present the proof of Theorem 1.3. Recall that, for a Riemannian metric g on the unit disk D, the corresponding length form on $\gamma = \partial D = \{e^{i\theta}\}$ is denoted by ds_g and is normalized by the condition $(ds_g)(d/d\theta) > 0$; e is the standard Euclidean metric on D.

Proof of necessity. Given a smooth positive 1-form ω on γ and a linear operator $A : C^\infty(\gamma) \to C^\infty(\gamma)$, assume the existence of a Riemannian metric g on the disk D such that $ds_g = \omega$ and $\Lambda_g = A$. Without loss of generality g can be assumed to be a flat metric. We extend g to a flat metric on D_ε for some $\varepsilon > 0$ and consider D as a closed domain in the complex manifold $(D_\varepsilon, \mathscr{C}_g)$, where $\mathscr{C}_g$ is the complex structure existing by Lemma 2.5. On the other hand, D is a closed domain in $(D_\varepsilon, \mathscr{C}_e)$, where $\mathscr{C}_e$ is the standard complex structure. By Riemann's theorem, there exists a holomorphism of closed domains $\Phi : (D, \mathscr{C}_e|_{\mathrm{int}\, D}) \to (D, \mathscr{C}_g|_{\mathrm{int}\, D})$. We set $\varphi = \Phi|_\gamma$.

Let us prove the commutativity of (1.14). Given a real function $f \in C^\infty(\gamma)$, let $u \in C^\infty(D)$ be the solution to the problem

$$\Delta_g u = 0 \quad \text{in } D, \quad u|_\gamma = f$$

and let $v \in C^\infty(D)$ be a harmonic function conjugate to u with respect to g. Then

$$(Af)(\gamma(s)) = (\Lambda_g f)(\gamma(s)) = \frac{dv(\gamma(s))}{ds},$$

where $\gamma(s)$ is the parametrization of γ by the arc length s in g. Setting $\gamma(s) = \varphi(e^{i\theta})$ in this formula, we have

$$(Af)(\varphi(e^{i\theta})) = \left.\frac{dv(\gamma(s))}{ds}\right|_{\gamma(s)=\varphi(e^{i\theta})}. \tag{3.1}$$

Since $w = u + iv$ is a holomorphic function in $(D, \mathscr{C}_g|_{\mathrm{int}\, D})$, the function $W = w \circ \Phi = U + iV$ is holomorphic in $(D, \mathscr{C}_e|_{\mathrm{int}\, D})$. Let $F = \varphi^* f = U|_\gamma$. Then

$$\Lambda_e(\varphi^* f) = \Lambda_e F = \frac{dV(e^{i\theta})}{d\theta}. \tag{3.2}$$

Differentiate the equality $V(e^{i\theta}) = v(\varphi(e^{i\theta}))$ to obtain

$$\frac{dV(e^{i\theta})}{d\theta} = a(e^{i\theta})\, \left.\frac{dv(\gamma(s))}{ds}\right|_{\gamma(s)=\varphi(e^{i\theta})},$$

where $a = (\varphi^*(ds_g))/d\theta = (\varphi^*\omega)/d\theta$ satisfies (1.15). Together with (3.1)–(3.2), this gives

$$(Af) \circ \varphi = a^{-1} \Lambda_e(\varphi^* f),$$

that is equivalent to the commutativity of (1.14). $\square$

To prove the sufficiency we need the following

Lemma 3.1. *Assume we are given an orientation preserving diffeomorphism $\Phi : D \to D$ of the unit disk onto itself and a function $0 < \rho \in C^\infty(D)$. Set $\varphi = \Phi|_\gamma : \gamma \to \gamma$ and $a = \rho|_\gamma \in C^\infty(\gamma)$. Define the Riemannian metric g on the disk D by $\Phi^* g = \rho e$, where e is the standard Euclidean metric. Then*

$$\varphi^*(ds_g) = a\, d\theta \tag{3.3}$$

and the following diagram commutes:

$$\begin{array}{ccc} C^\infty(\gamma) & \xrightarrow{\varphi^*} & C^\infty(\gamma) \\ \Lambda_g \downarrow & & \downarrow a^{-1}\Lambda_e \\ C^\infty(\gamma) & \xrightarrow[\varphi^*]{} & C^\infty(\gamma). \end{array} \tag{3.4}$$

Proof. Put $g' = \Phi^* g = \rho e$. Since $\Phi : (D, g') \to (D, g)$ is an isometry of Riemannian manifolds,

$$\varphi^*(ds_g) = ds_{g'} \tag{3.5}$$

and the diagram commutes:

$$\begin{array}{ccc} C^\infty(\gamma) & \xrightarrow{\varphi^*} & C^\infty(\gamma) \\ \Lambda_g \downarrow & & \downarrow \Lambda_{g'} \\ C^\infty(\gamma) & \xrightarrow[\varphi^*]{} & C^\infty(\gamma). \end{array} \tag{3.6}$$

The length forms $ds_e = d\theta$ and $ds_{g'}$ of e and $g' = \rho e$ are related by

$$ds_{g'} = a\, d\theta, \quad a = \rho|_\gamma.$$

With the help of (3.5), this implies (3.3).

Let us demonstrate that

$$\Lambda_{g'} = a^{-1}\Lambda_e, \tag{3.7}$$

i.e., (3.4) and (3.6) coincide. Indeed, for $f \in C^\infty(\gamma)$, let $u \in C^\infty(D)$ be the solution to the problem

$$\Delta_e u = 0 \quad \text{in } D, \quad u|_\gamma = f.$$

Then

$$\Lambda_e f = \left.\frac{\partial u}{\partial \nu_e}\right|_\gamma, \tag{3.8}$$

where ν_e is the unit outer normal to γ in the Euclidean metric. Since $\Delta_{g'} = \rho^{-1}\Delta_e$, the function u solves also the problem

$$\Delta_{g'} u = 0 \quad \text{in } D, \quad u|_\gamma = f$$

and so

$$\Lambda_{g'} f = \left.\frac{\partial u}{\partial \nu'}\right|_\gamma, \tag{3.9}$$

where ν' is the unit outer normal to γ in g'. Since $g' = \rho e$, the normal vectors are related by $\nu' = a^{-1}\nu_e$ and (3.9) can be rewritten as

$$\Lambda_{g'} f = a^{-1} \left.\frac{\partial u}{\partial \nu_e}\right|_\gamma.$$

Comparing the last equality with (3.8), we obtain (3.7). □

Proof of sufficiency in Theorem 1.3. Assume we are given a smooth positive 1-form ω on the circle γ and a linear operator $A : C^\infty(\gamma) \to C^\infty(\gamma)$. Assume the existence of an orientation preserving diffeomorphism $\varphi : \gamma \to \gamma$ such that (1.14) commutes, where $a \in C^\infty(\gamma)$ is defined by (1.15). We extend φ to a diffeomorphism $\Phi : D \to D$ and extend a to a positive function $\rho \in C^\infty(D)$. By Lemma 3.1, (3.3) holds for $g = (\Phi^{-1})^*(\rho e)$ and (3.4) commutes. From (1.15) and (3.9) we obtain that $\varphi^*(ds_g) = \omega$. The commutativity of (1.14) and (3.4) implies $A = \Lambda_g$. □

Since (3.4) commutes, the eigenvalue spectra of Λ_g and $a^{-1}\Lambda_e$ coincide. Therefore Theorem 1.3 leads to the following inverse spectral problem: Can a function $0 < a \in C^\infty(\gamma)$ be recovered from the known eigenvalue spectrum of $a^{-1}\Lambda_e$? In the case of the positive answer to the question, the diffeomorphism φ in (1.14) is recovered by the integrating equation (1.15). To make the question well posed, we observe that the operators $a^{-1}\Lambda_e$ and $a'^{-1}\Lambda_e$ are isospectral if $a' = a \circ T$, where $T : \gamma \to \gamma$ is an arbitrary isometry of $\gamma = \{e^{i\theta}\}$ onto itself, i.e., either $T(e^{i\theta}) = e^{i(\theta_0 + \theta)}$ or $T(e^{i\theta}) = e^{i(\theta_0 - \theta)}$. In this case the functions a and a' are said to be equivalent.

Problem 3.2. *Can a function $0 < a \in C^\infty(\gamma)$ be recovered up to equivalency from the known eigenvalue spectrum of $a^{-1}\Lambda_e$?*

We notice finally that the DN-operator of the Euclidean metric can be written as

$$\Lambda_e = \left(-\frac{d^2}{d\theta^2}\right)^{1/2}.$$

Indeed, as follows from the definition, $\Lambda_e(e^{in\theta}) = |n|e^{in\theta}$ for every integer n.

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V. A. Sharafutdinov
Sobolev Institute of Mathematics and Novosibirsk State University, Novosibirsk, Russia
E-mail address: `sharaf@math.nsc.ru`