Siberian Mathematical Journal, Vol. 52, *No.* 6, *pp.* 1104–1107, 2011
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A CONDITION FOR ASYMPTOTIC FINITE-DIMENSIONALITY OF AN OPERATOR SEMIGROUP

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Abstract: Let X be a Banach space and let $T : X \to X$ be a power bounded linear operator. Put $X_0 = \{x \in X \mid T^n x \to 0\}$. Assume given a compact set $K \subset X$ such that $\liminf_{n\to\infty} \rho\{T^n x, K\} \le \eta < 1$ for every $x \in X$, $\|x\| \le 1$. If $\eta < \frac{1}{2}$, then $\operatorname{codim} X_0 < \infty$. This is true in X reflexive for $\eta \in [\frac{1}{2}, 1)$, but fails in the general case.

Keywords: asymptotically finite-dimensional operator semigroup

1. Definitions, Formulations, and Available Results

In this paper, X is a Banach space (real or complex), $T : X \to X$ is a power bounded linear operator, i.e. $\forall n \in \mathbb{N}\ \|T^n\| \le C < \infty$.

Put $X_0 = \{x \in X \mid T^n x \underset{n\to\infty}{\longrightarrow} 0\}$. This subspace of X is closed. The operator T is called *asymptotically finite-dimensional* if $\operatorname{codim} X_0 < \infty$.

The purpose of the paper is to prove asymptotic finite-dimensionality provided that we can take $\eta < \frac{1}{2}$ under the following condition (B_X is the unit ball of X):

$$\text{there exists a compact set } \ K \subset X: \quad \forall x \in B_X \quad \liminf_{n\to\infty} \rho(T^n x, K) \le \eta < 1. \tag{1}$$

In some studies the issues of asymptotic finite-dimensionality were investigated under more rigid conditions than (1). Let us list them: they are connected with substituting the phrase "$\liminf \dots \le \eta$" in (1) by one of the following: (a) "$\lim \dots = 0$"; (b) "$\limsup \dots \le \eta$"; (c) "$\liminf \dots = 0$."

The issues of sufficiency of these conditions for asymptotic finite-dimensionality are also analogous to one-parameter operator semigroups.

In case (a), T is asymptotically finite-dimensional and even *splittable*; i.e. $X = X_0 \oplus L$, where L is a finite-dimensional T-invariant subspace. For Markov semigroups in L_1 this was proved in [1]; in [2] this was proved for positive operators in Banach lattices; for arbitrary X this was established in [3, 4]. For the Frobenius–Perron operators, a condition analogous to condition (a) was already studied in [5]. In case (b), asymptotic finite-dimensionality is established in [6]; also see [7] where, among other things, condition (b) is studied for arbitrary abelian operator semigroups. In the earlier works, a version of condition (b) was investigated in [8] for Markov operators in L_1; then in the Banach lattices in [9–11]. It appears that, in the context of Markov operators, condition (b) has been studied in the so far most general form in [12] for the so-called Lotz–Rabiger nets. Sufficiency of condition (c) for asymptotic finite-dimensionality was proved in [13].

In [14] there appeared condition (1), and Question 1.3.33 was asked: Will T be asymptotically finite-dimensional in this case? In [15] a positive answer is given in the case of X reflexive (note that in this case T is splittable). In [16] we answered in the negative to the question from [14] by presenting some isometries of the space $C(M)$ satisfying condition (1) with K a point and $\eta = \frac{1}{2}$. In particular, if c is the Banach space of converging sequences, $\lambda_n \in \mathbb{C}, |\lambda_n| \equiv 1$, $\lambda_n \to \lambda$ and $\{\lambda, \lambda_1, \lambda_2, \dots\}$ is the Kronecker

The author was supported by the State Maintenance Program for the Leading Scientific Schools of the Russian Federation (Grant NSh 6613.2010.1) and the Integration Grant of the Siberian Division of the Russian Academy of Sciences (No. 30).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 52, No. 6, pp. 1389–1393, November–December, 2011. Original article submitted November 15, 2010.

 0037-4466/11/5206–1104

set, then the multiplication operator $T : c \to c$, $(Tx)_n = \lambda_n x_n$ is an isometry satisfying condition (1) for $\eta = \frac{1}{2}$ and a singleton K.

In this work we demonstrate that for $\eta < \frac{1}{2}$ the operator T is asymptotically finite-dimensional. Thus, there is no gap between positive and negative answers to Question 1.3.33 of [14] in the form it was stated.

2. The Main Theorem

Theorem. *Let T satisfy* (1). *If $\eta < \frac{1}{2}$ then T is asymptotically finite-dimensional, i.e.* $\operatorname{codim} X_0 < \infty$.

PROOF. After factorizing by X_0, we obtain an operator on the factor space X/X_0 with the factor norm, which again satisfies condition (1) with the same η with respect to the compact set $\overline{K} = K + X_0 \subset X/X_0$; it can be verified that $(X/X_0)_0 = \{0\}$. Thus, it suffices to prove that if $\eta < \frac{1}{2}$ and $X_0 = 0$ then $\dim X < \infty$.

We prove first that the condition $\eta < \frac{1}{2}$ guarantees uniform apartness from zero of the iterates $\|T^n x\|$ (Lemma 1): $\forall n \ \frac{1-2\eta}{C}\|x\| \le \|T^n x\| \le C\|x\|$.

It is clear from this estimate that the norm $\|x\|_1 = \limsup_{n\to\infty} \|T^n x\|$ on X is equivalent to the original. In the new norm, T is an isometry. But condition (1) may no longer be satisfied for the norm.

In Lemma 2 we derive condition (2) from (1). The former is rather lengthy; but it is obviously preserved under renorming:

$$\forall \varepsilon > 0 \ \exists m \in \mathbb{N} : \forall F = \{x_1, \dots, x_m\} \subset B_X$$

$$\exists x_{i_1}, \dots, x_{i_p} \in F, \ i_1 < \dots < i_p, \ n_1 \le \dots \le n_p \in \mathbb{N}, \ \Big\| \sum_{j=1}^{p} \pm T^{n_j} x_{i_j} \Big\| \le \varepsilon. \tag{2}$$

(The complete formulations and proofs of the lemmas are given in the final part of the article.)

Let us suppose that an isometry T of an infinite-dimensional space X satisfies condition (2) and obtain a contradiction. Consider a certain $\varepsilon < \frac{1}{2}$ and take some $m = m_\varepsilon \in \mathbb{N}$ as in condition (2). Every isometry admits proper T-invariant subspaces (for X complex this result goes back to [17], for X real this was shown in [18]) and the chains of such subspaces of arbitrary length, in particular, of length greater than m: it is this fact that leads to a contradiction. Consider such a chain $X = L_1 \supset L_2 \supset \dots \supset L_m \supset L_{m+1}$. For each $i \in \{1, \dots, m\}$ we choose an ε-perpendicular to L_{i+1} in the space L_i, i.e. a vector $x_i \in L_i$ such that $\|x_i\| = 1$ and $\rho(x_i, L_{i+1}) > 1 - \varepsilon$. Put $F = \{x_1, \dots, x_m\}$.

According to condition (2), there can be found indices $i_1 < i_2 < \dots < i_p \in \{1, \dots, m\}$ and exponents $n_1 < n_2 < \dots < n_p \in \mathbb{N}$ such that for some selection of the signs $\pm$ we have

$$\|T^{n_1} x_{i_1} \pm T^{n_2} x_{i_2} + \dots \pm T^{n_p} x_{i_p}\|_1 \le \varepsilon. \tag{3}$$

Because T is an isometry and n_1 is the smallest exponent, it follows from (3) that

$$\|x_{i_1} \pm T^{n_2 - n_1} x_{i_2} \pm \dots \pm T^{n_p - n_1} x_{i_p}\|_1 \le \varepsilon. \tag{4}$$

All but first summands in (4) lie in L_{i_2}; whereas the first summand is $(1-\varepsilon)$ away from L_{i_2}. Therefore, the norm in (4) cannot be less than $1 - \varepsilon$. This is a contradiction. Thus, $\dim X < \infty$ (even $< m$). The theorem is proved.

3. Formulations and Proofs of Lemmas 1 and 2

Lemma 1. *If $\eta < \frac{1}{2}$, then $\forall x \in X \ \forall n \ \|T^n x\| \ge \frac{1-2\eta}{C}\rho(x, X_0)$. In particular, if $\eta < \frac{1}{2}$ and $X_0 = 0$, then*

$$\forall x \in X \ \forall n \quad \|T^n x\| \ge \frac{1-2\eta}{C}\|x\|.$$

The proof will be conducted in three stages. Let us fix $\alpha, 2\eta < \alpha < 1$.

1. Given $x \in B_X$ there exist arbitrary large pairs $n_1 < n_2$, with n_2 arbitrarily larger than n_1, such that $\|T^{n_1}x - T^{n_2}x\| \le \alpha\|x\|$.

Indeed, by compactness of K some iterates $T^{n_i}(\frac{x}{\|x\|})$ of $\frac{x}{\|x\|} \in B_X$ come $\frac{\alpha}{2}$-close to one and the same element from K; thus, they are α-close to one another. The rest of the claim follows from the homogeneity of the norm and operator.

2. Take $\varepsilon > 0$ and $x \in X$. If for some n $\|T^n x\| \le \varepsilon$, then there can be found a vector x_1 such that $\|x - x_1\| \le C\varepsilon$, and $\|T^m x_1\| \le \alpha\varepsilon$ for some m.

In fact, step 1) applied to $T^n x$ allows us to notice the following: there exist $m_1 > n$, $m_2 > m_1 + n$ such that $\|T^{m_1}x - T^{m_2}x\| \le \alpha\varepsilon$. Put $x_1 = x - T^{m_2 - m_1}x$. Then

$$\|x - x_1\| = \|T^{m_2-m_1}x\| \le C\|T^n x\| \le C\varepsilon, \quad \|T^{m_1}x_1\| = \|T^{m_1}x - T^{m_2}x\| \le \alpha\varepsilon.$$

3. If for some n $\|T^n x\| \le \varepsilon$, then $\rho(x, X_0) \le \frac{1}{1-\alpha}C\varepsilon$.

Indeed, we construct x_1 in the same way as in 2): $\|x - x_1\| \le C\varepsilon$ and for a certain m_1 $\|T^{m_1}x_1\| \le \alpha\varepsilon$. Applying 2) to x_1 this time, we construct x_2, $\|x_1 - x_2\| \le C\alpha\varepsilon$, and for some m_2 $\|T^{m_2}x_2\| \le \alpha^2\varepsilon$. By continuous application of the reasoning of 2) to the newly obtained vectors, we get some sequence of vectors x_k such that

$$\|x_k - x_{k-1}\| \le C\alpha^{k-1}\varepsilon, \quad \|T^{m_k}x_k\| \le \alpha^k\varepsilon.$$

The sequence x_k converges to a vector x_∞ in X_0. But then

$$\rho(x, X_0) \le \|x - x_\infty\| \le (1 + \alpha + \alpha^2 + \cdots)C\varepsilon = \frac{1}{1-\alpha}C\varepsilon.$$

Eliminating ε and reversing the inequalities, we rewrite statement 3) in equivalent form: for all n $\|T^n x\| \ge \frac{1-\alpha}{C}\rho(x, X_0)$. The rest of the claim is obvious.

Remark. An impression can be created that the condition $X_0 = 0$ already guarantees the uniform apartness from zero of the orbits $T^n x$, even if we do not require (1) to hold. But this is not so. Consider the space $l_2(\mathbb{Z})$ of two-sided sequences. A weighted right shift operator $T : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$ will be defined by the formula $(Tx)_n = \begin{cases} \frac{x_{n-1}}{2}, & n \le 0 \\ x_{n-1}, & n > 0 \end{cases}$. Clearly, $\|T\| = 1$, $X_0 = 0$, and, however, $\forall\varepsilon > 0$ $\exists x \|T^n x\| \le_{n\to\infty} \varepsilon\|x\|$.

It is not inconceivable that the uniform apartness from X_0 from below holds for $\frac{1}{2} \le \eta < 1$ as well. But the following lemma is no longer true for $\eta \ge \frac{1}{2}$.

Lemma 2. *Let an operator $T : X \to X$ satisfy condition* (1) *and $\eta < \frac{1}{2}$. Then for each $\varepsilon > 0$ there exists $m = m_\varepsilon \in \mathbb{N}$ such that for every m-element set $F = \{x_1, x_2, \dots, x_m\} \subset B_X$ we have*

$$\exists x_{i_1}, \dots, x_{i_p} \in F \ (i_1 < \cdots < i_p), \quad n_1 \le \cdots \le n_p \in \mathbb{N}, \quad \Big\| \sum_{j=1}^{p} \pm T^{n_j} x_{i_j} \Big\| \le \varepsilon$$

for an appropriate specification of the signs $\pm$.

Proof. Take the same α as in Lemma 1: $2\eta < \alpha < 1$ and find a finite m_α. It will turn out later that we may take $m_{(\alpha^n)} = m_\alpha^n$. This will complete the proof, because α^n can be made arbitrarily small.

Put $\delta = \frac{\alpha}{2} - \eta > 0$. In the compact set K there exists a finite δ-net $\{y_1, y_2, \dots, y_s\}$; we let $m = m_\alpha = s + 1$.

Take $F = \{x_1, x_2, \dots, x_m\} \subset B_X$. For each $i \in \{1, \dots, m\}$ there are arbitrarily large exponents n in the orbit $\{T^n x_i \ n \in \mathbb{N}\}$ for which $T^n x_i$ are η-close to K; among these exponents, in turn, there can be found an infinite set $N(i)$ such that for every $n \in N(i)$ the vector $T^n x_i$ is $(\eta + \delta)$-close to a certain fixed element $y_{j(i)}$ of the net, $j(i) \in \{1, \dots, s\}$. (Cf. with the first step of the proof of Lemma 1.) Since $m > s$, for some $i_1 < i_2$ $j(i_1) = j(i_2) = j$ and for all $n_1 \in N(i_1)$ and $n_2 \in N(i_2)$ we have the estimate

$$\|T^{n_1}x_{i_1} - T^{n_2}x_{i_2}\| \le \|T^{n_1}x_{i_1} - y_j\| + \|y_j - T^{n_2}x_{i_2}\| \le 2(\eta + \delta) = \alpha.$$

By homogeneity of the norm and linear operator, we get

$$\|x_1\|, \dots, \|x_m\| \le r \Rightarrow \ \exists i_1 < i_2 \quad n_1 < n_2 : \|T^{n_1}x_{i_1} - T^{n_2}x_{i_2}\| \le \alpha r. \tag{5}$$

Thus, the lemma is proved for $\varepsilon = \alpha$. Let us move from α to α^2. As m_{α^2} we will take $m^2 = m_\alpha^2$. Consider an arbitrary collection F of m^2 vectors from the unit ball. Subdivide it into m successive ordered collections $F_1, \dots, F_m$, each consisting of m vectors. According to what was already proved, it is possible to choose in every F_j a pair of vectors $x_{i_{(j,1)}}$ and $x_{i_{(j,2)}}$, $i_{(j,1)} < i_{(j,2)}$ and find exponents $n_{(j,1)} \le n_{(j,2)}$ in such a way that

$$y_j = T^{n_{(j,1)}} x_{i_{(j,1)}} - T^{n_{(j,2)}} x_{i_{(j,2)}}, \quad \|y_1\|, \dots, \|y_m\| \le \alpha. \tag{6}$$

Now, according to (5), we select only from these $y_1, \dots, y_m$ some vectors y_{j_1}, y_{j_2}, $j_1 < j_2$ such that

$$\exists k_1 < k_2 \quad \|T^{k_1} y_{j_1} - T^{k_2} y_{j_2}\| \le \alpha^2. \tag{7}$$

After inserting into (7) the expressions for y_j from (6), we get

$$\|T^{k_1}(T^{n_{(j_1,1)}} x_{i_{(j_1,1)}} - T^{n_{(j_1,2)}} x_{i_{(j_1,2)}}) - T^{k_2}(T^{n_{(j_2,1)}} x_{i_{(j_2,1)}} - T^{n_{(j_2,2)}} x_{i_{(j_2,2)}})\| \le \alpha^2.$$

The number k_2 should be selected so larger than k_1 that $k_1 + n_{(j_1,2)} \le k_2 + n_{(j_2,1)}$. After removing the parentheses, we will obtain a sum of four vectors, whose exponents and indices do not decrease, as it is required by the conclusion of the lemma under proof.

By repeating the reasoning initiated after formula (5), we infer that the conclusion of the lemma is true for $\alpha^3, \alpha^4, \dots$. As m_{α^n} we take m_α^n (here, in the corresponding expression "$\sum \pm \cdots$" there will be 2^n summands). Lemma 2 is proved.

References

1. *Lasota A., Li T. Y., and Yorke J. A.*, "Asymptotic periodicity of the iterates of Markov operators," Trans. Amer. Math. Soc., **286**, 751–764 (1984).
2. *Bartoszek W.*, "Asymptotic periodicity of the iterates of positive contractions on Banach lattices," Studia Math., **91**, No. 3, 179–188 (1988).
3. *Vu Quoc Phong*, "Asymptotic almost-periodicity and compactifying representations of semigroups," Ukrainian Math. J., **38**, No. 6, 576–579 (1986).
4. *Sine R.*, "Constricted systems," Rocky Mountain J. Math., **21**, 1373–1383 (1991).
5. *Lasota A. and Yorke J. A.*, "Exact dynamical systems and the Frobenius–Perron operator," Trans. Amer. Math. Soc., **273**, No. 1, 375–384 (1982).
6. *Emel′yanov E. Yu. and Wolff M. P. H.*, "Quasi-constricted linear operators on Banach spaces," Studia Math., **144**, No. 2, 169–179 (2001).
7. *Emel′yanov E. Yu. and Wolff M. P. H.*, "Quasi constricted linear representations of abelian semigroups on Banach spaces," Math. Nachr., **233–234**, 103–110 (2002).
8. *Komornik J. and Lasota A.*, "Asymptotic decomposition of Markov operators," Bull. Polish Acad. Sci. Math., **35**, No. 5–6, 321–327 (1987).
9. *Räbiger F.*, "Attractors and asymptotic periodicity of positive operators on Banach lattices," Forum Math., **7**, No. 6, 665–683 (1995).
10. *Emel′yanov E. Yu. and Wolff M. P. H.*, "Mean ergodicity on Banach lattices and Banach spaces," Arch. Math. (Basel), **72**, No. 3, 214–218 (1999).
11. *Gorokhova S. G. and Emel′yanov É. Yu.*, "A sufficient condition for order boundedness of an attractor for a positive mean ergodic operator in a Banach lattice," Siberian Adv. in Math., **9**, No. 3, 78–85 (1999).
12. *Emel′yanov É. Yu. and Erkursun N.*, "Lotz–Räbiger's nets of Markov operators in L_1-spaces," J. Math. Anal. Appl., **371**, 777–783 (2010).
13. *Storozhuk K. V.*, "An extension of the Vu–Sine theorem and compact-supercyclicity," J. Math. Anal. Appl., **332**, No. 2, 1365–1370 (2007).
14. *Emel′yanov E. Yu.*, Non-Spectral Asymptotic Analysis of One-Parameter Operator Semigroups, Birkhäuser, Basel (2007) (Oper. Theory, Advances Appl.; 173).
15. *Storozhuk K. V.*, "Slowly changing vectors and the asymptotic finite-dimensionality of an operator semigroup," Siberian Math. J., **50**, No. 4, 737–740 (2009).
16. *Storozhuk K. V.*, "Isometries with dense windings of the torus in $C(M)$," Funct. Anal. Appl. (to be published).
17. *Godement R.*, "Théorèmes taubériens et théorie spectrale," Ann. Sci. Éc. Norm. Supér., III Sér., **64**, 119–138 (1947).
18. *Storozhuk K. V.*, "Symmetric invariant subspaces of complexifications of linear operators," Math. Notes (to be published).

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