ISSN 0001-4346, Mathematical Notes, 2012, Vol. 91, No. 4, pp. 597–599. © Pleiades Publishing, Ltd., 2012.
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SHORT COMMUNICATIONS

Symmetric Invariant Subspaces of Complexifications of Linear Operators

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Received December 24, 2010; in final form, May 15, 2011

DOI: 10.1134/S0001434612030376

Keywords: *Banach space, complexification of a linear operator, spectral projection, invariant subspace, analytic continuation.*

1. INTRODUCTION

Let $T\colon X \to X$ be a linear operator on a complex Banach space. We denote its spectrum and resolvent by $\sigma(T)$ and $R(\lambda, T)$, respectively.

Assume that $\sigma(T)$ is not connected and that F and $\sigma \setminus F$ are open-closed parts of the spectrum. Let us surround F by a contour γ. The range $[F]$ and the null space $[\sigma \setminus F]$ of the spectral projection operator $P = (1/(2\pi i)) \int_\gamma R(\lambda, T)\, d\lambda$ are invariant subspaces, and $\sigma(T|_{[F]}) = F$.

Assume that the spectrum is connected. Then, when cutting away a subset F of the spectrum by a contour γ, one should multiply the resolvent by a weight function g that is small in a neighborhood of the intersection $\gamma \cap F$,

$$f(T) = \frac{1}{2\pi i} \int_\gamma R(\lambda, T) g(\lambda)\, d\lambda.$$

In this way one can construct spectral subspaces under conditions on the resolvent growth (see [1] and [2]) which are necessarily satisfied if the powers $T^{\pm n}$ have tempered growth. For example, the nonquasianalyticity condition

$$\sum_{n=-\infty}^{\infty} \frac{\ln \|T^n\|}{1+n^2} < \infty$$

guarantees that the spectrum is separable [3].

Now let $T\colon X \to X$, and let X be real. The spectral projection corresponding to a symmetric component of the spectrum of the complexified operator $T_{\mathbb{C}}\colon X_{\mathbb{C}} \to X_{\mathbb{C}}$ gives a symmetric invariant subspace $L_{\mathbb{C}}$. Its real part $L \subset X$ is T-invariant; e.g., see [4, Theorem 5.3].

However, even if the spectrum $T_{\mathbb{C}}$ is connected, one can readily obtain a symmetric $T_{\mathbb{C}}$-subspace by the method outlined above. Namely, one must integrate over an $\mathbb{R}$-*symmetric* contour with a *symmetric* function g. The "real part" of the range $f(T_{\mathbb{C}})$ will then be a T-invariant subspace of X. In what follows, we present the "realification" of one spectral method in more detail. By way of application, we obtain a theorem whose complex counterpart is proved in [5]. The last assertion of the theorem is often used in the complex case and goes back to Theorem J in [6].

Theorem 1. *Let X be a real Banach space, and let $T\colon X \to X$ be an invertible linear operator such that $\|T^n\| = O(|n|^k)$, $k < \infty$, as $n \to \pm\infty$. If $\dim X > 2$, then T has an invariant subspace. In particular, a linear isometry $T\colon X \to X$ of a real space has an invariant subspace if $\dim X > 2$.*

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The results in [4] are closest in spirit to the present paper. Note that, in contrast to Wermer's and Godement's results, the Aronszajn–Smith and Lomonosov theorems on invariant subspaces of compact operators were generalized to the real case earlier; see [7] and references therein.

Needless to say, not all invariant subspaces can be obtained by spectral methods. There exist operators of Volterra type whose restriction to the invariant subspaces has the same spectrum as the original operator, $\sigma(T|_L) = \sigma(T) = [0, 1]$; see [8]. Generally speaking, if $T_{\mathbb{C}}$ has invariant subspaces, it is not known whether there exist symmetric invariant subspaces; the assertion that they exist is equivalent to Conjecture 3 in [7].

2. DETAILED DEFINITIONS AND PROOF OF THEOREM 1

An *invariant subspace* is a *proper closed* subspace of X taken to itself by the operator $T\colon X \to X$.

Let X be a complex Banach space, and let $x \in X$. The mapping $\lambda \mapsto R(\lambda, T)x$ is a holomorphic function defined outside $\sigma(T)$ and ranging in X. If this mapping admits a maximal single-valued analytic continuation to some set $\rho(x)$, then the set $\sigma(x) := \mathbb{C} \setminus \rho(x) \subset \sigma(T)$ is called the *local spectrum* of x, and the continuation itself is called the *local resolvent* of x.

Now let X be real. The *complexification* of X is the space $X_{\mathbb{C}}$ whose elements have the form $z = (x + iy)$; the vectors $x, y \in X$ are naturally called the real part $(\mathrm{Re}(z))$ and the imaginary part $(\mathrm{Im}(z))$ of z. The space $X_{\mathbb{C}}$ is equipped with the conjugation J, $J(x + iy) = (x - iy)$. The norm on $X_{\mathbb{C}}$ is as follows:

$$\|z\|^2 = \max\{\|\mathrm{Re}(\lambda z)\|^2 + \|\mathrm{Im}(\lambda z)\|^2 \mid \lambda \in \mathbb{C},\ |\lambda| = 1\}.$$

This norm is equivalent to the direct sum norm on $X \oplus X$. An operator $T\colon X \to X$ is complexified as follows: $T_{\mathbb{C}}(x + iy) = (Tx + iTy)$. Clearly, $(T^n)_{\mathbb{C}} = T_{\mathbb{C}}^n$.

A subset $F \subset \mathbb{C}$ is said to be *symmetric* if it is symmetric with respect to the real axis, i.e., if $F = \overline{F}$. Likewise, a subset $Z \subset X_{\mathbb{C}}$ is said to be *symmetric* if $J(Z) = Z$. It is easily seen that if $Z \subset X_{\mathbb{C}}$ is a symmetric subspace, then $Z = L_{\mathbb{C}}$, where $L = \mathrm{Re}\, Z = \mathrm{Im}\, Z$.

Lemma. *Let $T_{\mathbb{C}}\colon X_{\mathbb{C}} \to X_{\mathbb{C}}$ be the complexification of an operator $T\colon X \to X$. Then the spectrum $\sigma(T_{\mathbb{C}}) \subset \mathbb{C}$ is symmetric. If the operator $T_{\mathbb{C}}$ has a local resolvent, then $\sigma(J(z)) = \overline{\sigma(z)}$ for each $z \in X_{\mathbb{C}}$.*

Proof. One can readily verify that $R(\overline{\lambda}, T_{\mathbb{C}}) = J \circ R(\lambda, T_{\mathbb{C}}) \circ J$, and hence the spectrum of $T_{\mathbb{C}}$ is symmetric. (This is Lemma 4.1 in [4].) Next, if $z \in X_{\mathbb{C}}$ and f is an analytic continuation of the resolvent $R(\lambda, T)z$, then the function

$$\lambda \mapsto J \circ (f(\lambda))(J(z))$$

is an analytic continuation of the resolvent $R(\overline{\lambda}, T_{\mathbb{C}})z$. Hence the maximal continuations coincide, and $\sigma(J(z)) = \overline{\sigma(z)}$. The proof of the lemma is complete. ☐

Proof of Theorem 1. Let F be a symmetric circular arc containing part of the spectrum of the operator $T_{\mathbb{C}}$, and let $[F] \subset X_{\mathbb{C}}$ be the subspace formed by the vectors whose local spectrum is contained in F. Since the spectrum of $T_{\mathbb{C}}$ is separable [3], it follows that $[F]$ is $T_{\mathbb{C}}$-invariant. Now the lemma implies that this subspace is symmetric. It is easily seen that $\mathrm{Re}[F] \subset X$ is a T-invariant subspace.

It may happen that the spectrum $T_{\mathbb{C}}$ contains at most two points $\eta, \overline{\eta} \in \Lambda$, and so there does not exist a symmetric arc F containing *part* of the spectrum. In this case, the separability of the spectrum and the restriction on the growth of $\|T^{\pm n}\|$ permit one to use the Gelfand–Hille theorem [10], [9] (cf. the proof of [5, Theorem 3]) and readily conclude that

$$((T_{\mathbb{C}} - \eta)(T_{\mathbb{C}} - \overline{\eta}))^{k+1} = 0;$$

hence the closure of the range of the operator

$$T_{\mathbb{C}}^2 - aT_{\mathbb{C}} + bI = (T_{\mathbb{C}} - \eta)(T_{\mathbb{C}} - \overline{\eta})$$

does not coincide with the entire space $X_{\mathbb{C}}$. The coefficients a and b are real, and consequently the closure of the range of the operator $T^2 - aT + bI$ does not coincide with the entire space X; therefore, X is either an invariant subspace or zero. In the latter case, each nonzero vector $x \in X$ generates at most a two-differential invariant subspace.

Consider the case of an isometry T. If it is bijective, then $\|T^{\pm n}\| = 1$ for each $n \in \mathbb{N}$ and the proof is complete. If, on the contrary, $TX \neq X$, then TX is the desired closed invariant subspace. The proof of Theorem 1 is complete. □

Remark 1. Although we have used the rather general results from [3], the separability of the spectral space $[F]$ under the assumptions of our theorem could be proved by reference to Leaf's paper [2] or even to Dunford's original results [1] (see also [11, Chap. XVI, Sec. 5, Corollary 8]).

Remark 2. It is mentioned at the end of [3] that Werner established the existence of invariant subspaces "of course provided that the spectrum contains more than one point" (translated from the Russian). This is not completely true. The one-point case is separately studied in the proof of Theorem 3 in [5], and it is this argument that we have used in the proof of Theorem 1 for the case of a two-point spectrum. If the spectrum of $T_{\mathbb{C}}$ is separable (for example, the operator is nonquasianalytic) but the powers $\|T^n\|$ grow faster than polynomially, then three points of the spectrum are needed to obtain a symmetric invariant subspace by our method.

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