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Isometries with Dense Windings of the Torus in $C(M)$

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Abstract. Let $C(M)$ be the space of all continuous functions on $M \subset \mathbb{C}$. We consider the multiplication operator $T : C(M) \to C(M)$ defined by $Tf(z) = zf(z)$ and the torus $O(M) = \{f : M \to \mathbb{C}, \|f\| = \|\frac{1}{f}\| = 1\}$. If M is a Kronecker set, then the T-orbits of the points of the torus $\frac{1}{2}O(M)$ are dense in $\frac{1}{2}O(M)$ and are $\frac{1}{2}$-dense in the unit ball of $C(M)$.

Key words: Kronecker set, asymptotically finite-dimensional operator.

Let X be a Banach space, and let $T\colon X \to X$ be a linear operator such that $\|T^n\| \leqslant C < \infty$ for any $n \in \mathbb{N}$. We set $X_0 = \{x \in X \mid T^n x \to_{n\to\infty} 0\}$. The operator T is said to be *asymptotically finite-dimensional* if $\operatorname{codim} X_0 < \infty$.

Suppose that there exists a compact set $K \subset X$ such that the orbits of the elements of the unit ball approach K in some sense. What conditions on this "some sense" do guarantee the *asymptotically finite dimensionality* of T? Similar questions can also be posed for operator semigroups. Below we list some conditions and results.

1. If $\lim_{n\to\infty} \rho(T^n x, K) = 0$ for any $x \in B_X$, then T is asymptotically finite-dimensional and even *decomposable*, i.e., $X = X_0 \oplus L$, where L is a finite-dimensional T-invariant subspace. This was proved in [1] for Markov semigroups in L_1, in [2] for positive operators in Banach lattices, and in [3] and [4] for arbitrary X.

2. If $\limsup_{n\to\infty} \rho(T^n x, K) \leqslant \eta < 1$ for any $x \in B_X$ (i.e., the compact set K "is attracting but, possibly, not strongly"), then T is asymptotically finite-dimensional [5].

3. If $\liminf_{n\to\infty} \rho(T^n x, K) = 0$ for any $x \in B_X$ (i.e., the set K "is attracting only sometimes"), then the operator T is still asymptotically finite-dimensional [6].

4_η. Suppose that $\liminf_{n\to\infty} \rho(T^n x, K) \leqslant \eta < 1$ for any $x \in B_X$ (i.e., the set K "is attracting only sometimes and not strongly"). In [7, Problem 1.3.33], the question of whether T is asymptotically finite-dimensional in this case was posed. In [8] we gave a positive answer to this question for reflexive X (note that, in this case, T is decomposable).

In this note, we give a negative answer to the question of [7, Problem 1.3.33] in the general case. We show that, for each 0-dimensional compact set M, there exist linear *isometries* of the space $C(M)$ satisfying condition $4_{1/2}$ with an attractive *point* $K = \{p\}$.

Note that the "attraction force" $\eta = 1/2$ cannot be diminished: the operators satisfying 4_η with $\eta < 1/2$ are always asymptotically finite-dimensional [9].

Recall that $C(M)$ is the space of continuous functions on a compact set M. In the sequel, we assume that all functions are continuous and M is always a compact set. Let $D \subset \mathbb{C}$ denote the disk of radius 1 and $\Lambda = \partial D$, the unit circle. The unit ball $B_{C(M)} \subset C(M)$ consists of all functions of the form $f\colon M \to D$. By the *torus* $O(M) \subset C(M)$ we mean the set of all functions $f\colon M \to \Lambda$.

Lemma 1. *Let $M \subset \Lambda$. The torus of radius $1/2$ is $(1/2+\varepsilon)$-dense in the unit ball $B_{C(M)}$, i.e., for any $\varepsilon > 0$ and $f\colon M \to D$, there exists a function $\tilde f\colon M \to \Lambda$ such that $\|f - \tilde f/2\| < 1/2 + \varepsilon$.*

Proof. If $f(t) \neq 0$ for any t, then we set $\tilde f = f/|f|$; in the general case, we first "move" f away from zero and then normalize (this is not difficult). This completes the proof of the lemma.

Topological remark. If $f, g\colon \Lambda \to \Lambda$ and $\|f - g\| < 2$, then the maps f and g are homotopic. Thus, if $\deg f \neq \deg g$, then $\|f - g\| = 2$. Therefore, if the interior of a set $M \subset \mathbb{C}$ is nonempty, then Lemma 1 is false. For example, if $M = D$, then we have $\|\operatorname{id}|_D - f/2\| \geqslant 3/2$ for every $f\colon D \to \Lambda$.

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Let us define an operator $T\colon C(M) \to C(M)$ by the formula $(Tf)(t) = tf(t)$, $t \in M$.

A compact set $M \subset \Lambda$ is called a *Kronecker set* if every continuous function $f\colon M \to \Lambda$ can be uniformly approximated by characters of Λ (i.e., by functions of the form $t \to t^n$, $n \in \mathbb{Z}$; it suffices to take $n \in \mathbb{N}$). It is convenient for us to reformulate this definition as follows.

Lemma 2. *A compact set M is a Kronecker set if and only if the orbit $\{f, Tf, T^2 f, \dots\}$ is dense in $O(M)$ for every $f \in O(M)$.*

Proof. The operator T is an isometry. Hence it suffices to prove the lemma under the assumption $f \equiv 1 \in O(M)$. In this case, $T^n f(t) = t^n$. The rest is obvious.

Clearly, T can be replaced by T^{-1} in Lemma 2.

Theorem 1. *Suppose that $M \subset \mathbb{C}$ and $T\colon C(M) \to C(M)$ is the operator of multiplication by t, that is, $(Tf)(t) = tf(t)$. If $M \subset \Lambda$ is a Kronecker set, then, for every $f \in B_{C(M)}$ and every $\tilde f \in O(M)$, there is a sequence of powers $m_k \to \infty$ such that $\liminf_{k\to\infty} \|T^{m_k} f - \tilde f/2\| \leqslant 1/2$.*

Proof. The theorem is an easy consequence of Lemmas 1 and 2. It should only be noted that $\|T^{m_k} f - \frac{\tilde f}{2}\| = \|f - T^{-m_k} \frac{\tilde f}{2}\|$.

Theorem 2. *Let M be a zero-dimensional compact set. Then there is a homeomorphism $g\colon M \to g(M) \subset \Lambda$ such that $g(M)$ is a Kronecker set and, for the multiplication operator $T\colon C(M) \to C(M)$, $(Tf)t = g(t)f(t)$, the assertion of Theorem* 1 *is true*; *moreover,* $\mathrm{Sp}(T) = g(M)$.

Proof. Any zero-dimensional metric compact set M is homeomorphic to a subset of any perfect set, for example, of a perfect Kronecker set (such sets exist; see, e.g., [10]). But closed subsets of a Kronecker set are always Kronecker. The rest is easy.

Example. Let c be the Banach space of convergent sequences. Suppose that $\lambda_n \in \mathbb{C}$, $\lambda_n \to \lambda$. We can identify c with $C(M)$ for $M = \{\lambda, \lambda_1, \lambda_2, \dots\}$ by considering convergent sequences $(f_n) \in c$ as functions $f \in C(M)$, $f(\lambda_n) = f_n$; we have $f(\lambda) = \lim f_n$. Consider the operator $T\colon c \to c$ defined by $(Tf)_n = \lambda_n f_n$. If $\{\lambda, \lambda_1, \lambda_2, \dots\}$ is a Kronecker set, then T is an isometry satisfying condition $4_{1/2}$ for any singleton $K \in \frac{O(M)}{2}$.

Note that, although the operators from c to c of the form $(Tf)_n = \lambda_n f_n$, $\lambda_n \to \lambda \neq 0$, have no complete system of finite-dimensional subspaces, all of them are nevertheless scalarly almost periodic [11].

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