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THE CARATHÉODORY–RASHEVSKY–CHOW THEOREM FOR THE NONHOLONOMIC LIPSCHITZ DISTRIBUTIONS

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Abstract: It is proved that if a k-dimensional Lipschitz distribution H in $\mathbb{R}^{k+1}$ is nonholonomic in a connected domain, then every pair of points can be joined by an H-polygonal path.

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1. Introduction and an Overview of the Results of the Article

Let H be a smooth k-dimensional distribution of linear subspaces on $\mathbb{R}^n$ or in the tangent bundle of a smooth manifold M^n. Denote by $[H, H]$ the distribution generated by the commutators (the Lie brackets) of all possible smooth vector fields lying in H (in what follows, these fields will be called *H-fields* or *horizontal fields*, and their trajectories will be called *horizontal curves*).

If H is involutive, i.e. $[H, H] \subset H$, then H is integrable; i.e., H is tangent to k-dimensional integral manifolds (the Frobenius Theorem). In the nonintegrable case, the iterated commutators of the distribution H form the increasing chain of distributions

$$H_1 := H,\ H_2 := \operatorname{Lin}\{H, [H, H]\},\ \dots,\ H_m := \operatorname{Lin}\{H_{m-1}, [H_{m-1}, H]\}. \tag{1}$$

If the maximal subspace H_m coincides with $\mathbb{R}^n$, then the distribution H is called *totally nonintegrable* or *totally nonholonomic*. If H is generated by sufficiently smooth vector fields $X_1, \dots, X_k$, then the distributions H_l in (1) are generated by the iterated commutators of the form $[X_{i_1}, [X_{i_2}, \dots, [X_{i_{s-1}}, X_{i_s}] \dots]]$ of the length s at most l. The condition of total nonintegrability for these H is called the *Hörmander condition* for these fields.

The Rashevsky–Chow Theorem [1, 2] claims: If H is totally nonholonomic, then every pair of points can be joined by H-trajectories, i.e., the piecewise smooth curves tangent to the distribution H.

The work [3] by Carathéodory contains a result according to which the orbits of a nonintegrable analytic distribution of codimension 1 coincide with the entire manifold. The Rashevsky–Chow Theorem is a natural geometric generalization of Carathéodory's results. The main result of our article is the Carathéodory Theorem in the Lipschitz case (Theorem 1).

In particular, this result yields the Rashevsky–Chow Theorem for two Lipschitz vector fields in $\mathbb{R}^3$. In [4] this theorem is proved for the vector fields of a special form (they depend on two variables). Note that our proof, although rather general, does not provide any estimates for the Carnot–Carathéodory metric of the type of estimates obtained in [4], because it relies on topological methods. Instead, the topological nature of the proof allows us to generalize the formulation of the theorem in the two directions:

First, it is possible to relax the Lipschitz condition on H, just demanding some nice properties of the solutions of Cauchy problems for the vector fields generating H (Theorem 2). From Theorem 2 it follows, in particular, the Rashevsky–Chow Theorem proved in [5] for the vector fields of a special form with the measurable coefficients with respect to some variables. Second, we demonstrate that in the case

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of codimension greater than 1, it can be stated that the orbit of a nonintegrable k-distribution contains a topological $(k+1)$-dimensional cell (Theorem 3). The last theorem also holds for the corresponding distributions in the Banach space.

In the end of the article the Orbit Theorem [6, 7] is briefly discussed, and its application is given to proving one of the versions of the Rashevsky–Chow C^1-Theorem from [8]. Some arguments (for example, the Orbit Lemma) allows us to conjecture that a version of the Orbit Theorem for Lipschitz distributions is valid which is analogous to the theorems of Stefan and Sussmann. So far it is only a hypothesis.

Let us emphasize that Theorems 1–3 have a qualitative character. In the studies containing specific estimates for the length of horizontal curves, we have to use the iterated commutators. But the smoothness of the vector fields decreases when they are commuted; thus, the initial distribution has to be sufficiently smooth. If the smoothness is insufficient, then the approximations of commutators by the ε-commutators of the form (3) and (4) are utilized (see below). Moreover, the results remain applicable for the smooth problems. Such approach enables us also to demonstrate some "quantitative analogs" of the smooth theorems about the Carnot–Carathéodory spaces and the distances there (see [9–14]).

2. Definitions and the Geometric Preparatory Work

We say that *we get from a point p to a point q moving in the direction of a field X in time ε* and write $p \xrightarrow{X,\varepsilon} q$, if $q = u(\varepsilon)$, where u is the solution of the Cauchy problem

$$u(0) = p, \quad \dot{u}(t) = X(u(t)). \tag{2}$$

A vector field will be called a *horizontal field* or *H-field*, if at each point it belongs to H. A *horizontal trajectory* or *H-trajectory* will denote a piecewise smooth curve tangent to H. Finally, the *H-orbit* of a point p is the set of points, such that we can get from p moving along H-trajectories. Denote the orbit of p by $O(p)$. The symbol $O_\varepsilon(p)$ will denote the set of points q, H-reachable from p at time at most ε. These sets will be referred to as *local orbits*.

We will call a k-dimensional distribution in $\mathbb{R}^n$ *integrable at p*, if there exists a topological k-dimensional disk $B^k \subset \mathbb{R}^n$, containing a local orbit of p. (It will be clear in the sequel that in this case the disk can be considered to be C^1-smooth; but it is irrelevant for the purposes of the present article.) Correspondingly, we will call H *nonintegrable* or *nonholonomic*, if H is integrable at no point. The last definition agrees with the standard.

Theorem 1 (the Carathéodory–Rashevsky–Chow Theorem for the Lipschitz distributions of hyperplanes). *Let H be a k-dimensional nonholonomic Lipschitz distribution in a connected domain $U \subset \mathbb{R}^n$, $n = k + 1$. Then every two points $p, q \in U$ are H-connectable.*

Let us describe the geometric idea of the proof of the "smooth" Rashevsky–Chow Theorem; this will be useful in the nonsmooth case.

The commutator $[X, Y]$ of the vector fields can be obtained using the four-segment curves with the help of (3) or, more generally, (4):

$$[X, Y] := \lim_{\varepsilon \to 0} \frac{p_{XYXY}(\varepsilon)}{\varepsilon^2}, \quad p \xrightarrow{X,\varepsilon} p_X \xrightarrow{Y,\varepsilon} p_{XY} \xrightarrow{X,-\varepsilon} p_{XYX} \xrightarrow{Y,-\varepsilon} p_{XYXY}, \tag{3}$$

$$[X, Y] = \lim_{s,t \to 0} \frac{p_{XYXY}(s,t)}{st}, \quad p \xrightarrow{X,s} p_X \xrightarrow{Y,t} p_{XY} \xrightarrow{X,-s} p_{XYX} \xrightarrow{Y,-t} p_{XYXY}. \tag{4}$$

If the vector $[X, Y]$ does not lie in the plane of the vectors X and Y, then the endpoints of the trajectories of the type (3), (4) will form, for ε small, a trajectory that is transversal to this plane; whereas the polygonal trajectories going alternately along X and Y will fill already a certain "three-dimensional" set in a neighborhood of p. If we iterate such "ε-commutators," we will sweep out the sets of dimensions 4 and 5; and, finally, a neighborhood of the point p will be covered by the H-polygonal curves.

Let us proceed to the Lipschitz case.

Let $U \subset \mathbb{R}^n$, and let H be a k-dimensional Lipschitz distribution in $\mathbb{R}^n$. Show that the Lipschitz fields, generating H, can be chosen in a very special fashion.

Lemma 1. *If $H(p)$ is transversal to the $(n-k)$-dimensional linear subspace $x_1 = x_2 = \cdots = x_k = 0$, then in a neighborhood of p the distribution H can be defined by the Lipschitz fields $X_1, \dots, X_k$ which are the rows of the $k \times n$ matrix (where $h_1, \dots, h_k$ are Lipschitz functions from U to $\mathbb{R}^{n-k}$):*

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & h_1 \\ 0 & 1 & 0 & \cdots & 0 & h_2 \\ \cdots & \cdots & & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & h_k \end{pmatrix}. \tag{5}$$

Proof. Take as a vector X_i, $i = 1, 2, \dots, k$, the directional vector of the line obtained by intersection of H with the $(n-k+1)$-dimensional plane parallel to the corresponding coordinate axes of $\mathbb{R}^n$.

Clearly, H-connectivity is an equivalence. The cosets of the equivalence are disjoint. The theorem will follow immediately from the connectedness of U, if we manage to demonstrate that each coset is open. Thus, to prove Theorem 1 it suffices to prove the following lemma.

Lemma 2. *Let $k+1 = n$. For each Lipschitz distribution generated by the rows of matrix* (5) *in* $\mathbb{R}^n$ *and which is nonholonomic at $p \in U$, the H-orbit of p is a neighborhood of p.*

3. Proof of Lemma 2

First, suppose that $n = 3$ and $k = 2$. The three-dimensional situation includes all of the needed geometric ideas which are sufficient for the general case.

We assume that $p = (0, 0, 0)$. Taking Lemma 1 into account, we suppose that the distribution H in U is generated by the vector fields $X = (1, 0, h_1)$ and $Y = (0, 1, h_2)$. It is possible to assume that the functions $h_{1,2}$ in U are bounded by the constant $M = \max\{|h_1|, |h_2|\}$.

Let ε be so small that the vertical prism $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \times [-7M\varepsilon, 7M\varepsilon]$ lies in U. The small thickness of this prism (as compared with its height) prevents the polygonal curves, appearing in the proof, from leaving the prism through the bases too early. The reader can fill in details.

Let us denote the base of the prism $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$ in the plane $0xy$ by "$\square$."

Define the mappings Ψ_p^{XY}, $\Theta_p^{XY} : \square \to U$ by the points on the following integral polygonal curve (compare with (3)):

$$p \xrightarrow{X,x} * \xrightarrow{Y,y} \Psi_p^{XY}(x,y) \xrightarrow{X,-x} * \xrightarrow{Y,-y} \Theta_p^{XY}(x,y), \tag{6}$$

$$p \xrightarrow{Y,y} * \xrightarrow{X,x} \Psi_p^{YX}(x,y) \xrightarrow{Y,-y} * \xrightarrow{X,-x} \Theta_p^{YX}(x,y). \tag{7}$$

The mappings $\Psi_p^{**}(x,y)$ preserve the first two coordinates.

The visual structure of the mapping $\Psi_p^{XY} : \square \to U$ is shown in Fig. 1. From $p = (0, 0, 0)$ we move along the field X in the plane $y = 0$ until the abscissa becomes equal to the number x, then we turn and go along the field Y in the plane $x = \text{const}$ until the ordinate gets equal to y. Thus we arrive at the point $\Psi_p^{XY}(x,y)$. Similarly, to get to the point Ψ_p^{YX}, we have to go first along the field Y, and then, along X.

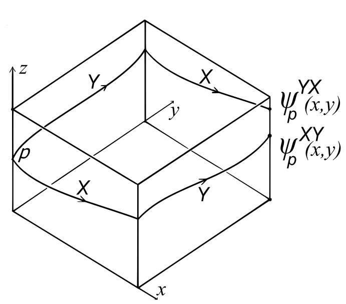

Fig. 1

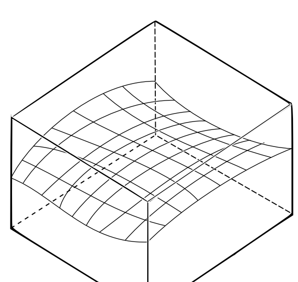

Fig. 2

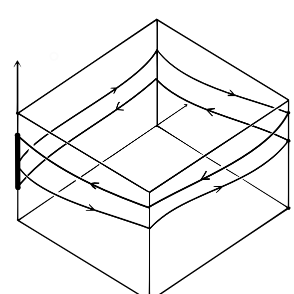

Fig. 3

It follows from the theorems of existence, uniqueness, and continuous dependence on the parameter for the solution of the Cauchy problem for an ordinary differential equation that the mappings Ψ_p^{**} are correctly defined and continuous. It is clear that $X = \frac{\partial}{\partial x}\Psi_p^{YX}$ and $Y = \frac{\partial}{\partial y}\Psi_p^{XY}$.

If $\Psi_p^{XY}(x,y) = \Psi_p^{YX}(x,y)$ for all $(x,y) \in \Box$, then the image of $\Box$ under $\Psi_p = \Psi_p^{XY} = \Psi_p^{YX}$ will be a 2-manifold M. The vectors X and Y, tangent to M, are the partial derivatives Ψ_p. Since they are continuous, the manifold is differentiable (and belongs to the class C^1), whereas the mapping $\Psi_p^{XY} : \Box \to M$ is its C^1-parametrization. These are the so-called coordinates of the second kind. The corresponding coordinate grid is shown in Fig. 2. Clearly, $H = TM$. The last statement means that the distribution H is integrable at the point p.

Thus, if H is nonintegrable, then there exists a point (x^0, y^0) in $\Box$ at which $\Psi_p^{XY} \neq \Psi_p^{YX}$.

The image of $\Box$ under the action of Θ_p^{**} is connected and lies on the axis $0z$ (Fig. 3). Therefore, each point of the vertical segment with the endpoints $\Theta_p^{XY}(x^0,y^0)$ and $\Theta_p^{YX}(x^0,y^0)$ is reachable from $p = (0,0,0)$ by means of a four-segment horizontal polygonal curve of the kind (6) or (7).

So far we have been fixing $p = (0,0,0)$. We have a possibility of moving it along a vertical segment. Consider the following mapping

$$(x,y,t) \mapsto \big(\Psi_{(0,0,t)}^{XY}(x,y)\big). \tag{8}$$

It is continuous and injective in some neighborhood of $(0,0,0)$, while $(0,0,0)$ is mapped to itself. By Brouwer's Theorem on the embedding of a domain, the image of this mapping is a neighborhood of $(0,0,0)$. At the same time, it is clear that the image of this mapping consists of the points achievable from $(0,0,0)$ with the help of 6-segment H-polygonal curves going alternately along X and along Y. For $n = 2$ the proof of the lemma is complete. The proof for $k > 2$ does not involve additional ideas. Let us present it, while omitting simple estimates. We assume again that p is the origin. Consider an arbitrary neighborhood U of p. Let ε be small. By $\Box_k$ we denote the set $\{x_1, x_2, \ldots, x_k \mid |x_i| \leq \varepsilon, i = 1, \ldots, k\} \subset \mathbb{R}^k$.

Various permutations $I = (i_1, i_2, \ldots, i_k)$ and $J = (j_1, j_2, \ldots, j_k)$ of the set $1, 2, \ldots, k$ define the families of mappings Ψ^I and $\Theta^{IJ} : \Box_k \to U$, which put in correspondence to the collection $r = (x_1, \ldots, x_k) \in \Box_n$ the endpoint of the trajectory, similar to (6) or (7):

$$p \overset{X_{i_1}, x_{i_1}}{\longrightarrow} \ldots \overset{X_{i_k}, x_{i_k}}{\longrightarrow} \Psi_p^I(r) \overset{X_{j_1}, -x_{j_1}}{\longrightarrow} \ldots \overset{X_{j_k}, -x_{j_k}}{\longrightarrow} \Theta_p^{IJ}(r). \tag{9}$$

The images of the mappings Θ^{IJ} are connected, lie on the "vertical" axis $0x_{n=(k+1)}$ and contain the point p. Thus, the image either consists of p or includes a "vertical" segment with the endpoints $(0, 0, \ldots, 0, \pm\delta)$.

If the mappings Ψ^I coincide with one another for all permutations I; i.e. they do not depend on the order of "switching" of the vector fields $X_1, X_2, \ldots, X_k$, then the corresponding mapping $\Psi : \Box_n \to U$ defines a C^1-parametrization of a k-dimensional integral manifold M^k which is swept out by H-trajectories. In this case H is integrable.

Suppose that there exist $r \in \Box_k$ and two permutations I and J such that $\Psi^I(r) \neq \Psi^J(r)$. Then the image of $\Box_k$ under the mapping Θ^{IJ} contains a line segment S of the axis Ox_{k+1}, since this image is linearly connected, but is not reduced to a singleton. As it is easy to see, the mapping $(x_1, \ldots, x_k, z) \mapsto \Psi^I_{(0,\ldots,0,z)}(x_1, \ldots, x_k)$ is injective for small $z \in S$. It remains to apply Brouwer's Theorem. Lemma 2 and Theorem 1 are proved.

4. Two Appendices: Relaxing the Lipschitz Condition and the Condition That codim = 1

I. In the proof of Lemma 2, the condition for the fields $X_i$to be Lipschitz was used only to ensure nice properties of the solutions of (2) for X_i. Thus it is possible to relax the formulation of Theorem 1, just demanding these properties to hold. For example, let us call a continuous vector field X *nice at a* *neighborhood* of p, if in some neighborhood U of p the Cauchy problem $u(0) = r$, $\dot{u}(t) = X(u(t))$ has a unique solution continuous in (r, t).

Let us call a family of curves in a domain U *nice in a neighborhood of* p, if this family is a family of integral curves of a certain nonzero vector field which is nice in a neighborhood of p.

Let us call a k-dimensional distribution H in a domain $U \subset \mathbb{R}^n$ *nice*, if for each point p the intersection of H with every $(n-k+1)$-dimensional plane Π, transversal to $H(p)$, form in Π a family of curves nice in a neighborhood of p. Clearly, the Lipschitz distributions are nice. The next theorem is proved in the same way as Theorem 1. The reader can again estimate alone the height of the "prism" $\square_k \times [-?, ?]$, using some simple considerations of compactness and uniform continuity.

Theorem 2. *Let H be a nice k-dimensional distribution in $\mathbb{R}^{k+1}$ generated by continuous vector fields $X_1, \dots, X_k$. If H is nonholonomic in a connected domain U, then every pair of points $p, q \in U$ are H-connectable.*

II. Let $m = n - k$ be the codimension of H. So far m has been equal to 1. In the proof of Lemma 2, some injective mapping $(x_1, \dots, x_k, z) \mapsto \Psi^I_{(0,\dots,0,z)}(x_1, \dots, x_k)$ was constructed acting on the set $\square_k \times S$, where S is a "vertical" segment. This segment was obtained as a nondegenerate image of a certain nonconstant mapping $\Theta^{IJ}(\square_k) \subset \mathbb{R}^{m=1}$. The existence of S was a key geometric property in the proof of Lemma 2 and Theorem 1.

If $m > 1$, then a linearly connected set in $\mathbb{R}^m$ also contains some topological segment S. It seems to be obvious on the intuitive level; but the proof is substantive (for example, see [15, § 50, the remark to Theorem 2]). Let $\gamma : [0,1] \to S$ be a parametrization of this segment. The mapping of the compact set $\square_k \times [0,1]$ in $\mathbb{R}^n$, acting by the formula $(x_1, \dots, x_k, z) \mapsto \Psi^I_{\gamma(z)}(x_1, \dots, x_k)$, is injective (compare with the end of the proof of Lemma 2). Thus, we have

Theorem 3. *Let H be a k-dimensional Lipschitz distribution in $\mathbb{R}^n$. If H is nonintegrable at a point p, then the local H-orbits of p contain a homeomorphic image of a $k+1$-dimensional cell.*

Remark. It is possible to take distributions in a Banach space.

5. C^1-Orbits and the Lipschitz Orbits

In [6] Sussmann demonstrated that the orbit of an arbitrary system of C^∞-smooth vector fields is a smooth injective image of some C^∞-manifold. In the work [7] by Stefan the corresponding result was obtained for the C^q-category, where $q \geq 1$. The Stefan–Sussmann Theorem in its general form does not provide any estimates, but is a "guiding and directing" force for the proof of the theorems of the type of the Rashevsky–Chow Theorem. For example, let us prove with its help a result due to Basalaev and Vodopyanov from [8] in a slightly different formulation.

Corollary of the Orbit Theorem. *Let H be a distribution on a smooth manifold M^n, generated by C^1-fields $X_1, \dots, X_s$. Suppose that for each $i \geq 1$ a distribution H_{i+1} is obtained by taking the linear span of all C^1-fields from H_i and their single commutators. Suppose that all $H_1, H_2, \dots, H_N$ are C^1-smooth and $H_N(p) = TM$. Then the orbit of a point p under the action of the system of fields $X_1, \dots, X_s$ is a neighborhood of p in M. Moreover, there exists a natural number L such that a neighborhood of p is swept out by the polygonal curves, consisting of no more than L links.*

Proof. By the Orbit Theorem, we see that the $\{X_1, \dots, X_s\}$-orbit of p is an injective smooth image of some C^1-manifold $\widetilde{M}^l$ under the action of a smooth injective mapping $f : \widetilde{M}^l \to M^n$. The shifts along the vector fields X_i leave the orbit invariant, and hence (for example, see [7, Lemma 5.1]) the commutators of these fields are also tangent to the orbit. Therefore, all distributions H_i, restricted to the orbit, will be at most l-dimensional. But $H_N(p) = TM$, and so $l = n$, and the orbit covers a set with a nonempty interior in $\mathbb{R}^n$. It remains to show the existence of a finite number L. For each $j = 1, 2, \dots$ let us consider the set K_j, swept out by the polygonal curves, consisting of j links of the length at most j. These sets are compact, and their union is the entire orbit. It follows from Baire's Theorem that the interior of a certain K_L is nonempty. The corollary is proved.

If $n = k + 1$, then it follows from Theorem 1 that H-orbits of a nonintegrable Lipschitz distribution are open in $\mathbb{R}^n$. If $k + 1 < n$, then, between the integrability and total nonintegrability, there arise some intermediate “not-quite-integrable” situations. In the smooth case, H-orbits have the dimension of the maximal subspace H_N in the chain of iterated commutators (1). Obviously, the analysis of the nonsmooth case cannot rely on (1). Many works (for example, see the bibliography in [14]) are devoted to studying the H-orbits of the distributions with low smoothness. In [14] there is introduced a “condition of s-integrability” ($k \le s \le n$): H-orbits are s-dimensional C^1-manifolds.

The author thinks that, as in the Orbit Theorem, the global H-orbit of a Lipschitz distribution is an immersed manifold. For the author, the main argument in support of this is the following

Orbit Lemma. *The orbit of an arbitrary family of Lipschitz vector fields is topologically uniform.*

Proof. The family of local Lipschitz isomorphisms, generated by Lipschitz H-fields, acts transitively on the orbit.

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